Fiber Bundles and Connections

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2012

Contents

1 Introduction

The present text was used as a reference for Master courses given by the author at the University of Metz and at the University of Luxembourg. The part on principal bundles might need a revision. Since the notes grew gradually over a number of years, some references might have been lost or forgotten; in this case, the author would like to apologize and would be glad to add those references (in particular, online encyclopedias such as nLab and Wikipedia were used).

2 Fiber bundles

2.1 Definition and first remarks

We will see that many basic concepts in Theoretical Physics can be interpreted in terms of fiber bundles. In the main, a fiber bundle is a manifold that locally looks like a product manifold. Well-known examples are the tangent and the cotangent bundles. Here the precise definition of a fiber bundle. Let us mention that all the manifolds below are smooth, finite-dimensional, Hausdorff, and second countable.

Definition 1. *Let E and M be two manifolds and $\pi : E \to M$ a smooth surjective map from E onto M. The manifold E is called a fiber bundle over the base manifold M—with projection π—if and only if it is locally trivial, i.e. for any $x \in M$, there is an open neighborhood U in M, a manifold F, and a diffeomorphism*

$$\varphi : \pi^{-1}(U) \to U \times F, \tag{1}$$

such that for any $p \in \pi^{-1}(U)$, we have

$$\varphi(p) = (\pi(p), \phi(p)).$$

Some remarks are necessary.

1. A fiber bundle E with base manifold M and projection π will be denoted in the following by $\pi : E \to M$ or by (E, M, π).

2. The diffeomorphism φ is called a trivialization of E over U. It allows to identify the part of E over U with the product manifold $U \times F$.

3. The map
$$\phi : \pi^{-1}(U) \to F$$
is nothing but $\mathrm{pr}_2 \circ \varphi$, where pr_2 is the projection on the second factor.

4. Since $\pi = \mathrm{pr}_1 \circ \varphi$ on $\pi^{-1}(U)$, we see that π is a submersion.

5. For any $x \in M$, the preimage $E_x = \pi^{-1}(x)$ is called the fiber of E at x and is a closed embedded submanifold of E.

6. For any $x \in U$, we have
$$\phi_x \in \mathrm{Diff}(E_x, F),$$
where ϕ_x is the restriction of ϕ to the fiber E_x. If the base manifold M is connected, which happens in particular if E is connected, all the fibers are diffeomorphic to a unique and same manifold F, called the typical fiber of E.

7. Of course any product manifold $M \times F$ is a (trivial) fiber bundle over M with typical fiber F and projection pr_1.

8. The notion of (global or local) section of a fiber bundle $\pi : E \to M$ will not be recalled here. All sections below are assumed to be smooth. The set of global sections will be denoted by $\mathrm{Sec}(E)$ and the set of local sections over an open subset $U \subset M$ by $\mathrm{Sec}(E_U)$. We will see that the existence of global sections depends on the global geometry of the bundle.

2.2 Transition functions

Let $\pi : E \to M$ be a fiber bundle. Take an open covering $(U_\alpha)_{\alpha\in\mathfrak{A}}$ of M such that E is trivial over each U_α. Let

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F_\alpha$$

be a trivialization of E over U_α. For any $\alpha, \beta \in \mathfrak{A}$, the restriction of this diffeomorphism φ_α to $\pi^{-1}(U_\alpha \cap U_\beta)$ is a diffeomorphism

$$\varphi_{\beta\alpha} : \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times F_\alpha.$$

This means that the portion of E over $U_\alpha \cap U_\beta$ has the same differential structure as $(U_\alpha \cap U_\beta) \times F_\alpha$. Of course it has also the same structure as $(U_\alpha \cap U_\beta) \times F_\beta$. So the structures of $(U_\alpha \cap U_\beta) \times F_\alpha$ and $(U_\alpha \cap U_\beta) \times F_\beta$ coincide, more precisely,

$$\psi_{\beta\alpha} = \varphi_{\alpha\beta} \circ \varphi_{\beta\alpha}^{-1} : (U_\alpha \cap U_\beta) \times F_\alpha \to (U_\alpha \cap U_\beta) \times F_\beta$$

is a diffeomorphism, called transition function. The information how the portions $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ are glued together is encoded in these functions. These transition functions have the following properties. For any $\alpha, \beta, \gamma \in \mathfrak{A}$,

$$\begin{aligned} \psi_{\alpha\alpha} &= \mathrm{id}, \\ \psi_{\alpha\beta} \circ \psi_{\beta\alpha} &= \mathrm{id}, \\ \psi_{\alpha\beta} \circ \psi_{\beta\gamma} \circ \psi_{\gamma\alpha} &= \mathrm{id}\,. \end{aligned} \tag{2}$$

In the last equation appropriate restrictions of all the factors are understood. This property can be rewritten in form

$$\psi_{\beta\gamma} \circ \psi_{\gamma\alpha} = \psi_{\beta\alpha}. \tag{3}$$

Its meaning is obvious. If we glue $\pi^{-1}(U_\alpha)$ with $\pi^{-1}(U_\gamma)$ in the way prescribed by $\psi_{\gamma\alpha}$, and $\pi^{-1}(U_\gamma)$ with $\pi^{-1}(U_\beta)$ as prescribed by $\psi_{\beta\gamma}$, then portion $\pi^{-1}(U_\alpha)$ has been glued with portion $\pi^{-1}(U_\beta)$ in accordance with $\psi_{\beta\gamma} \circ \psi_{\gamma\alpha}$. The result has of course to coincide with the direct prescription encoded in $\psi_{\beta\alpha}$. So Equation (3) is a compatibility condition. It actually contains the three properties (2).

Take now an open covering $(U_\alpha)_{\alpha\in\mathfrak{A}}$ of a manifold M and assume that with any U_α is associated a manifold F_α. Roughly speaking, if the differential structures of $U_\alpha \times F_\alpha$ and $U_\beta \times F_\beta$ coincide over $U_\alpha \cap U_\beta$, i.e. if there are transition diffeomorphisms

$$\psi_{\beta\alpha} : (U_\alpha \cap U_\beta) \times F_\alpha \to (U_\alpha \cap U_\beta) \times F_\beta$$

that verify compatibility condition (3), and if we glue the trivial fiber bundles $\mathrm{pr}_1 : U_\alpha \times F_\alpha \to U_\alpha$ together as encoded in the $\psi_{\beta\alpha}$, we get a fiber bundle over M that is locally diffeomorphic to the $U_\alpha \times F_\alpha$.

Exercise 1. The prototype of a nontrivial fiber bundle is the Möbius strip. We now construct this bundle using the just described method. Take $M = S^1$ and let $U_1 =]0, 2\pi[$ and $U_2 =]-\pi, \pi[$ be an open covering of S^1. Let F_1 and F_2 be the open subset $]-1, 1[$ of $\mathbb{R}$. If we glue the portions $U_1 \times F_1$ and $U_2 \times F_2$ over $U_1 \cap U_2 =]0, \pi[\cup]\pi, 2\pi[$ as described by the transition function

$$\psi_{21} : (U_1 \cap U_2) \times F_1 \ni (x, f) \to \left\{ \begin{array}{l} (x, f), \text{ if } x \in]0, \pi[\\ (x, -f), \text{ if } x \in]\pi, 2\pi[\end{array} \right\} \in (U_1 \cap U_2) \times F_2,$$

we get a Möbius strip. If we just glue by the identity map, we get a cylinder, which is of course a trivial bundle.

3 Vector bundles

3.1 Definitions and remarks

The most important fiber bundles in Physics are vector bundles and principal bundles. In this section we briefly recall the key-facts of vector bundles. Roughly speaking, a vector bundle is just a fiber bundle $\pi : E \to M$, such that the restrictions of the mappings $\phi : \pi^{-1}(U) \to F$ to the E_x $(x \in U)$,

$$\phi_x : E_x \to F,$$

are vector space isomorphisms. It is of course understood that the fibers E_x $(x \in M)$ and the manifolds F are r-dimensional vector spaces over the field $\mathbb{K}$ of real or complex numbers. Here the precise definition of a vector bundle.

Definition 2. *Let E and M be two manifolds and $\pi : E \to M$ a smooth surjective map from E onto M. The manifold E is a vector bundle over the base manifold M—with projection π—if and only if the fibers $E_x = \pi^{-1}(x)$ $(x \in M)$ are r-dimensional vector spaces over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C})$ and for any $x \in M$, there is an open neighborhood U in M, a $\mathbb{K}$-vector space F of dimension r, and a diffeomorphism*

$$\varphi : \pi^{-1}(U) \to U \times F, \tag{4}$$

such that for any $p \in \pi^{-1}(U)$, we have

$$\varphi(p) = (\pi(p), \phi(p))$$

and for any $x \in U$, the restriction

$$\phi_x : E_x \to F$$

is a vector space isomorphism.

Some remarks:

1. It is obvious that ϕ_x is one-to-one.

2. In the following we assume that dimension r is constant over M. We systematically identify the vector spaces F with $\mathbb{K}^r$ and speak of a vector bundle of constant rank r and typical fiber $\mathbb{K}^r$. A bundle of rank 1 is called a $\mathbb{K}$-line bundle.

3. Well-known examples of vector bundles are the tangent and cotangent bundles, TM and T^*M, as well as the tensor bundle $\otimes TM = \cup_{x\in M} \oplus_{p,q\in\mathbb{N}} \otimes^p_q T_xM$, where $\otimes^p_q T_xM$ is the space of p-contravariant and q-covariant tensors of vector space T_xM.

4. The set $\mathrm{Sec}(E)$ of sections of a vector bundle $\pi : E \to M$ has an obvious $\mathbb{K}$-vector space structure and also a $C^\infty(M, \mathbb{K})$-module structure. Any vector bundle has a global section, namely the zero-section, $s_0 : M \ni x \to 0 \in E_x \subset E$.

5. Note that any transition function of any fiber bundle, say $\psi_{\beta\alpha} = \varphi_{\alpha\beta} \circ \varphi^{-1}_{\beta\alpha} : (U_\alpha \cap U_\beta) \times F_\alpha \to (U_\alpha \cap U_\beta) \times F_\beta$, can be viewed as a family

$$\theta_{\beta\alpha}(x) \in \mathrm{Diff}(F_\alpha, F_\beta) \quad (x \in U_\alpha \cap U_\beta)$$

of diffeomorphisms. The relationship between $\psi_{\beta\alpha}$ and the $\theta_{\beta\alpha}(x)$ is of course

$$\psi_{\beta\alpha}(x, f) = (x, (\theta_{\beta\alpha}(x))\,(f)),$$

for any $(x, f) \in (U_\alpha \cap U_\beta) \times F_\alpha$. In our vector bundle setting, we have $F_\alpha = F_\beta = \mathbb{K}^r$, so that we can look at transition functions as families of isomorphisms $\theta_{\beta\alpha}(x) \in \mathrm{Isom}(\mathbb{K}^r, \mathbb{K}^r)$ $(x \in U_\alpha \cap U_\beta)$, or, better, as families of nonsingular matrices

$$\theta_{\beta\alpha}(x) \in \mathrm{GL}(r, \mathbb{K}) \quad (x \in U_\alpha \cap U_\beta).$$

This way of looking at transition functions as (smooth) mappings $\theta_{\beta\alpha}$ from non-empty intersections $U_\alpha \cap U_\beta$ into a Lie group G will be of importance in the case of principal bundles.

3.2 Local frames

Let $\pi : E \to M$ be a $\mathbb{K}$-vector bundle of rank r. Choose a trivialization $\varphi : \pi^{-1}(U) \to U \times \mathbb{K}^r$ over an open subset $U \subset M$, as well as a basis $(e_1, \ldots, e_r)$ of vector space $\mathbb{K}^r$. Since for any $x \in U$, the induced map $\phi_x : E_x \to \mathbb{K}^r$ is a vector space isomorphism, the vectors

$$\sigma_i(x) = \phi_x^{-1}(e_i) \in E_x \quad (i \in \{1, \ldots, r\})$$

define r sections $\sigma_i \in \mathrm{Sec}(E_U)$ $(i \in \{1, \ldots, r\})$, such that their values at any point $x \in U$ form a basis of the corresponding fiber E_x.

We say that such local sections

$$\sigma_i \in \mathrm{Sec}(E_U) \quad (i \in \{1, \ldots, r\})$$

that induce a basis of E_x over any point $x \in U$, define a frame of E over U.

Since any section $s \in \mathrm{Sec}(E)$ then locally reads

$$s|_U = \sum_i s^i \sigma_i \quad (s^i \in C^\infty(U, \mathbb{K})),$$

this frame is also a basis of the $C^\infty(U, \mathbb{K})$-module $\mathrm{Sec}(E_U)$.

Actually the data of a trivialization over U is equivalent to the data of a frame over U. Indeed, if σ_i is such a frame, then

$$\varphi : \pi^{-1}(U) \ni p \to (\pi(p), k) \in U \times \mathbb{K}^r,$$

where $k = (k^1, \ldots, k^r)$ is defined by

$$p = \sum_i k^i \sigma_i \left(\pi(p)\right),$$

is a trivialization.

Note also that the choice of a trivialization φ of E over U (or of a frame) induces a vector space isomorphism

$$\mathrm{Sec}(E_U) \ni s|_U = \sum_i s^i \sigma_i \to (s^1, \ldots, s^r) =: s^\varphi \in C^\infty(U, \mathbb{K}^r).$$

This isomorphism allows to identify the space of sections over U with the space of functions on U valued in the typical fiber. Function s^φ is called the local form of section s in the chosen trivialization φ.

In view of the preceding explanations, the following proposition is obvious.

Proposition 1. *A vector bundle of rank* 1 *is trivial if and only if it admits a nowhere vanishing global section.*

3.3 Operations on vector bundles

Operations on vector bundles will not be studied in detail. Explanations are confined to some basic facts.

Vector space notions such as the dual of a vector space, the direct sum of two spaces, or the tensor product, canonically extend to vector bundles. If $\pi : E \to M$ and $\pi' : E' \to M$ are two $\mathbb{K}$-vector bundles over the same base manifold M, we obtain the dual bundle $\pi^* : E^* \to M$, the sum bundle $\pi^{\oplus} : E \oplus E' \to M$, and the tensor product bundle $\pi^{\otimes} : E \otimes E' \to M$, by assigning to any $x \in M$, the dual $E_x^* = \mathcal{L}(E_x, \mathbb{K})$ of the fiber at x, the direct sum $E_x \oplus E'_x$ of those fibers, and the tensor product $E_x \otimes E'_x$ respectively. It is also possible to define tensor product bundles such as $\otimes_q^p E$, $\bigwedge^p E$, ... These bundle operations are well-known for $E = TM$.

Exercise 2. Show that a trivialization φ of a vector bundle E over an open subset U of the base manifold, canonically induces a trivialization φ^* (resp. $\varphi^{\otimes}$) over U of the bundle E^* (resp. $\otimes_q^p E$). In the following we often write φ instead of φ^* or $\varphi^{\otimes}$. Note that a similar result holds for local frames. Here the (simplified) notations are σ_i, σ^j, and $\sigma_{i_1 \ldots i_p}^{j_1 \ldots j_q}$. Moreover, if $T \in \mathrm{Sec}(\otimes_q^p E)$, $\xi^1, \ldots, \xi^p \in \mathrm{Sec}(E^*)$, and $s_1, \ldots, s_q \in \mathrm{Sec}(E)$, we have on U:

$$T(\xi^1, \ldots, \xi^p, s_1, \ldots, s_q) = T^{\varphi^{\otimes}}((\xi^1)^{\varphi^*}, \ldots, (\xi^p)^{\varphi^*}, s_1^{\varphi}, \ldots, s_q^{\varphi}). \tag{5}$$

Hint : It suffices to observe that

$$\begin{aligned}\left(T(\xi^1, \ldots, \xi^p, s_1, \ldots, s_q)\right)|_U &= T|_U(\xi^1|_U, \ldots, \xi^p|_U, s_1|_U, \ldots, s_q|_U) \\ &= T^{\varphi^{\otimes}}((\xi^1)^{\varphi^*}, \ldots, (\xi^p)^{\varphi^*}, s_1^{\varphi}, \ldots, s_q^{\varphi}).\end{aligned}$$

4 Connections on vector bundles

4.1 Characterization of tensor fields

In the following, out of comfort, we confine ourselves to real vector bundles (of constant rank).

Remember that the tensor product $F_1 \otimes \ldots \otimes F_p$ of finite-dimensional $\mathbb{R}$-vector spaces is canonically isomorphic to the space $\mathcal{L}_p(F_1^* \times \ldots \times F_p^*, \mathbb{R})$ of p-linear forms on the dual spaces. So the first part of the following theorem is clear since the involved map T is defined pointwise.

Theorem 1. *Any tensor field $T \in \mathrm{Sec}(\otimes_q^p TM)$ $(p, q \in \mathbb{N})$ over a manifold M can be interpreted as a $(p+q)$-linear map*

$$T : \left(\Omega^1(M)\right)^p \times (\mathcal{X}(M))^q \ni (\eta^1, \ldots, \eta^p, X_1, \ldots, X_q) \to T(\eta^1, \ldots, \eta^p, X_1, \ldots, X_q) \in C^\infty(M),$$

which is also multilinear for the multiplication by functions $f \in C^\infty(M)$. Conversely, any such map can be viewed as a tensor field of type (p, q).

Proof. Let $x \in M$ and try to define a tensor $T_x \in \otimes^p_q T_xM \simeq \mathcal{L}_{p+q}((T^*_xM)^p \times (T_xM)^q, \mathbb{R})$. The point is that the $C^\infty(M)$-linearity will entail that the value of $T(\eta^1, \ldots, \eta^p, X_1, \ldots, X_q)$ at x only depends on the values at x of the arguments. So T induces a tensor T_x at any point x.

We first prove that T is a local operator, i.e. that if some argument vanishes in an open subset $U \subset M$, then the value of T on the arguments vanishes at any $x \in U$. This can be done by using a smooth function α that vanishes in some neighborhood V of x in U and has value 1 outside U.

Now we are able to show that if an argument, say η^1, vanishes at x, then the value of T vanishes at x. Indeed, let $(x^1, \ldots, x^n)$ be a coordinate system of M in a neighborhood U of x and let $\eta^1|_U = \sum_i \eta^1_i dx^i$, $\eta^1_i \in C^\infty(U)$. Choose global functions $f_i \in C^\infty(M)$ and global 1-forms $\omega^i \in \Omega^1(M)$ such that $f_i = \eta^1_i$ and $\omega^i = dx^i$ in a neighborhood $V \subset U$ of x (it suffices to multiply η^1_i and dx^i with a smooth function that is compactly supported in U and has value 1 in V). Since we then have $\eta^1 = \sum_i f_i\omega^i$ in V and since T is local, we get $T(\eta^1, \ldots, \eta^p, X_1, \ldots, X_q) = \sum_i f_i T(\omega^i, \ldots, \eta^p, X_1, \ldots, X_q)$ in V. It suffices now to remark that $f_i(x) = \eta^1_i(x) = 0$. ∎

This result can be extended in many ways, but the underlying philosophy remains unchanged. Thus, for any vector bundle $\pi : E \to M$, a $C^\infty(M)$-multilinear map

$$T : (\mathrm{Sec}(\mathrm{E}^*))^p \times (\mathrm{Sec}(E))^q \to C^\infty(M)$$

for instance, can be viewed as a tensor field $T \in \mathrm{Sec}(\otimes^p_q E)$, and a $C^\infty(M)$-linear map

$$T : \mathrm{Sec}(TM) \to \mathrm{Sec}(E),$$

i.e. a $C^\infty(M)$-bilinear map

$$T : \mathrm{Sec}(TM) \times \mathrm{Sec}(E^*) \to C^\infty(M)$$

can be interpreted as a tensor field $T \in \mathrm{Sec}(T^*M \otimes E)$.

4.2 Definition and existence

Let us start with an intuitive approach and work in $M = \mathbb{R}^3$ as in elementary Classical Mechanics. We examine the temperature or any other scalar field $\mathfrak{s} : \mathbb{R}^3 \to \mathbb{R}$. If X is a vector field of $\mathbb{R}^3$, it is clear that, at any point x of $\mathbb{R}^3$, the change of $\mathfrak{s}$ in the direction of X, i.e.

$$(\nabla_X \mathfrak{s})(x) := \frac{\mathfrak{s}(x + hX_x) - \mathfrak{s}(x)}{h} \quad (h \in \mathbb{R}^*, h \simeq 0),$$

should 'not vary significantly' or should 'vary in a simple way', if we replace X by a vector field fX, $f \in C^\infty(\mathbb{R}^3)$, which has 'the same direction'. It immediately follows from the essential aspect of the differential in elementary Physics (small change computed at the first order) that

$$\nabla_X \mathfrak{s} = (d\mathfrak{s})(X) = L_X \mathfrak{s}.$$

Hence,

$$\nabla_{fX}\mathfrak{s} = f\nabla_X\mathfrak{s},$$

i.e. the directional derivative is $C^\infty(\mathbb{R}^3)$-linear with respect to X. In other words, see Subsection 4.1, the value at a point x of the directional derivative $\nabla_X\mathfrak{s}$, $\mathfrak{s} \in C^\infty(M) = \mathrm{Sec}(\otimes_0^0 TM)$, does not depend on the values of X in a whole neighborhood of x, but only depends on X_x. This property should be viewed as a natural requirement when looking for a good notion of directional derivative. Observe that for other tensor fields $T \in \mathrm{Sec}(\otimes_q^p TM)$, $(p, q) \neq (0, 0)$, e.g. for vector fields $Y \in \mathrm{Sec}(TM)$ or differential 1-forms $\omega \in \mathrm{Sec}(T^*M)$, the Lie derivative with respect to X is a first order differential operator in both arguments (see local forms of L_XY and $L_X\omega$ in the course 'Differential Geometry'). Therefore, these Lie derivatives are not function linear in X and they depend on the values of X in some neighborhood of x. In this respect the Lie derivative is 'not a good' directional derivative.

Let us now try to define the directional derivative—we will call it *covariant derivative*—of a section $s \in \mathrm{Sec}(E)$ of an arbitrary vector bundle $\pi : E \to M$ (functions are sections of a trivial line bundle). Indeed, if we compute this derivative of s again at a point $x \in M$ in the direction of a vector field $X \in \mathcal{X}(M)$, we have to compare as above the value of s at a nearby point of x in the direction of X with the value of s at x. But these values are vectors in different fibers. To compute their difference, we have to transport the second in the fiber of the first.

In the case of the Lie derivative this was done by means of the flow of X and that is why the Lie derivative at x depends on the values of X in some neighborhood of x.

Here we would like to transfer the second vector "without change" to the fiber of the first. So we need a rule of parallel transport of a vector from one fiber into another. We feel that such a "connection" between fibers should allow to define the covariant derivative of a section s in the direction of a vector field X. It can be proven that, conversely, a covariant derivative induces a "connection". Actually covariant derivatives, parallel transports and connections are tightly related concepts. In the frame of connections on vector bundles, i.e. of linear connections, most authors identify a connection with its covariant derivative. In the beginning we adopt the same approach and use the words "covariant derivative" and "connection" as synonyms. Let us emphasize that there is no canonical way of choosing a connection on an arbitrary vector bundle. So we define a connection or covariant derivative axiomatically and show that such objects always exist.

In view of the above remarks the following definition seems natural.

Definition 3. *A connection or covariant derivative on a vector bundle $\pi : E \to M$ is a bilinear map*

$$\nabla : \mathcal{X}(M) \times \mathrm{Sec}(E) \ni (X, s) \to \nabla_X s \in \mathrm{Sec}(E),$$

such that

$$\nabla_{fX}s = f\nabla_X s, \tag{6}$$

and

$$\nabla_X(fs) = (L_X f)\, s + f\nabla_X s, \tag{7}$$

for any $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, *and* $s \in \mathrm{Sec}(E)$.

It follows from Subsection 4.1 that this definition can be rewritten in the following form.

Definition 4. *A connection or covariant derivative on a vector bundle* $\pi : E \to M$ *is a linear map*

$$\nabla : \mathrm{Sec}(E) \ni s \to \nabla s \in \mathrm{Sec}(T^*M \otimes E),$$

such that

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for any $f \in C^\infty(M)$ *and any* $s \in \mathrm{Sec}(E)$.

Remarks:

1. As seen in Subsection 4.1, the $C^\infty(M)$-linearity of ∇ with respect to X means that the value $(\nabla_X s)\,(x)$ $(x \in M)$ only depends on the value X_x.

2. Any covariant derivative is a local operator with respect to X and to s. For X this is clear from the preceding observation. For s it can be proved in the usual way by choosing a function α as described in Theorem 1.

3. The space $\mathrm{Sec}(T^*M \otimes E)$ is the space of differential 1-forms on M valued in E.

4. For $E = \otimes^p_q TM$ $(p, q \in \mathbb{N})$, the derivative ∇ maps $\mathrm{Sec}(\otimes^p_q TM)$ into $\mathrm{Sec}(\otimes^p_{q+1} TM)$. Hence the name "covariant derivative".

5. If $E = M \times \mathbb{R}^r$ is the trivial bundle, sections $s \in \mathrm{Sec}(E)$ are just functions $s \in C^\infty(M, \mathbb{R}^r) = C^\infty(M) \otimes \mathbb{R}^r$. As seen above, the natural way of defining ∇ is to set $\nabla = d$, where d is the de Rham differential. It is easily checked that this actually defines a covariant derivative in the sense of the preceding definition. This covariant derivative is called the trivial or canonical connection of the considered trivial bundle and will be denoted by ∇^0. Note that it is true that there is no canonical connection on an arbitrary vector bundle, but that it is nevertheless natural to have a canonical connection on a trivial bundle.

6. Any affine combination $\sum_i f_i \nabla^i$ $(f_i \in C^\infty(M), \sum_i f_i = 1)$ of covariant derivatives ∇^i on E is a covariant derivative on E. The condition $\sum_i f_i = 1$ is needed in the proof of (7).

Theorem 2. *Covariant derivatives exist on any vector bundle.*

Proof. We use the usual notations. Let $(f_\alpha)_{\alpha\in\mathfrak{A}}$ be a locally finite partition of unity subordinate to an open covering of M by local trivializations $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathfrak{A}}$ of E. We first construct a covariant derivative locally over U_α. Since the bundle is trivial over U_α, we choose the trivial connection (or better, we transport the trivial connection from $U_\alpha \times \mathbb{R}^r$ to $\pi^{-1}(U_\alpha)$), i.e. we set

$$(\nabla^\alpha_X s)^{\varphi_\alpha} = (ds^{\varphi_\alpha})(X),$$

for $X \in \mathcal{X}(U_\alpha)$ and $s \in \mathrm{Sec}(E_{U_\alpha})$. It suffices now to glue these local covariant derivatives by means of the partition of unity. Hence we set

$$\nabla_X s = \sum_\alpha f_\alpha \, \nabla^\alpha_{X|_{U_\alpha}} s|_{U_\alpha},$$

for any $X \in \mathcal{X}(M)$ and any $s \in \mathrm{Sec}(E)$. Since the partition of unity is locally finite, the sum in the r.h.s. is locally finite. As this sum is also an affine combination of covariant derivatives, it defines a covariant derivative on E. ∎

Proposition 2. *The set of connections of a vector bundle $\pi : E \to M$ is an affine space modelled on the space* $\mathrm{Sec}(T^*M \otimes \mathrm{End}(E))$ *of differential* 1*-forms on M valued in the endomorphism bundle of E.*

Proof. If ∇, ∇' are two covariant derivatives, the map

$$\nabla - \nabla' : \mathcal{X}(M) \times \mathrm{Sec}(E) \ni (X, s) \to \nabla_X s - \nabla'_X s \in \mathrm{Sec}(E)$$

is $C^\infty(M)$-bilinear. So $\nabla - \nabla' \in \mathrm{Sec}(T^*M \otimes E^* \otimes E)$. Conversely, if $\Omega \in \mathrm{Sec}(T^*M \otimes E^* \otimes E)$, the sum $\nabla + \Omega$ is obviously a covariant derivative. ∎

4.3 Local forms and extensions

Let ∇ be a covariant derivative on a vector bundle $\pi : E \to M$ and let (U, φ) be a trivialization over $U \subset M$. If $\sigma_i \in \mathrm{Sec}(E_U)$ is the corresponding local frame, any section $s \in \mathrm{Sec}(E)$ reads

$$s|_U = \sum_i s^i \sigma_i, \tag{8}$$

where $s^i \in C^\infty(U)$. Since ∇ is local in each argument, it restricts to a covariant derivative on E_U, which we also denote by ∇. Hence, in U,

$$\nabla_X s = \sum_i \left(L_X s^i\right) \sigma_i + \sum_i s^i \nabla_X \sigma_i,$$

for any $X \in \mathcal{X}(M)$. If we decompose the derivatives $\nabla_X \sigma_i$ in the frame σ_i, we get

$$\nabla_X s = \sum_i \left(L_X s^i\right) \sigma_i + \sum_{i,j} \mathcal{A}(X)^i_j \, s^j \sigma_i,$$

where $\mathcal{A}(X)^i_j$ is the ith component of $\nabla_X \sigma_j$. Therefore, the local form of $\nabla_X s$ in the chosen trivialization reads

$$(\nabla_X s)^\varphi = L_X(s^\varphi) + \mathcal{A}(X)s^\varphi.$$

This result entails that $\mathcal{A}(fX) = f\mathcal{A}(X)$, for any smooth f, so that $\mathcal{A}$ is actually a differential 1-form in U valued in $\mathrm{gl}(r,\mathbb{R})$. Hence the result:

Theorem 3. *If ∇ is a connection on a vector bundle $\pi : E \to M$ of rank r and (U,φ) is a trivialization of E over an open subset U of M, there is a differential 1-form $\mathcal{A}$ on U valued in $\mathrm{gl}(r,\mathbb{R})$, called the connection 1-form in the chosen trivialization, such that*

$$(\nabla_X s)^\varphi = L_X(s^\varphi) + \mathcal{A}(X)s^\varphi,$$

for any $s \in \mathrm{Sec}(E)$ and any $X \in \mathcal{X}(M)$.

In other words, locally, in a trivialization, a covariant derivative is characterized by its connection 1-form.

Remark 1. *A covariant derivative ∇ on a vector bundle $\pi : E \to M$ induces a covariant derivative, still denoted by ∇, on every tensor bundle associated with E, e.g. on $\otimes^p_q E$, $\bigwedge^p E$, $\bigvee^p E$, ...*

Notations are the same than above. Connection ∇ extends "by derivation", for instance to a connection on E^* and on $\otimes^p_q E$. Indeed, if $\xi, \xi^1, \ldots, \xi^p \in \mathrm{Sec}(E^*)$, $s, s_1, \ldots, s_q \in \mathrm{Sec}(E)$, $X \in \mathcal{X}(M)$, and $T \in \mathrm{Sec}(\otimes^p_q E)$, we set

$$(\nabla_X \xi)(s) = L_X\left(\xi(s)\right) - \xi\left(\nabla_X s\right)$$

and

$$\begin{aligned}(\nabla_X T)\left(\xi^1, \ldots, \xi^p, s_1, \ldots, s_q\right) = \; & L_X\left(T(\xi^1, \ldots, \xi^p, s_1, \ldots, s_q)\right) \\ & - \textstyle\sum_i T(\xi^1, \ldots, \nabla_X \xi^i, \ldots, \xi^p, s_1, \ldots, s_q) \\ & - \textstyle\sum_j T(\xi^1, \ldots, \xi^p, s_1, \ldots, \nabla_X s_j, \ldots, s_q).\end{aligned}$$

It is easily checked that the r.h.s. of the first (resp. the second) of the two preceding equations is $C^\infty(M)$-linear with respect to s (resp. with respect to the arguments $\xi^1, \ldots, \xi^p, s_1, \ldots, s_q$), and that the mapping

$$\nabla : \mathcal{X}(M) \times \mathrm{Sec}(E^*) \to \mathrm{Sec}(E^*)$$

$$(\text{resp. } \nabla : \mathcal{X}(M) \times \mathrm{Sec}(\otimes^p_q E) \to \mathrm{Sec}(\otimes^p_q E)),$$

obtained in this way, has properties (6) and (7).

Exercise 3. Prove that the above canonical definition of ∇ on $\otimes^p_q E$ entails that for a decomposable T, i.e. for $T = t_1 \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes \eta^q$ ($t_i \in \mathrm{Sec}(E)$, $\eta^j \in \mathrm{Sec}(E^*)$), we have

$$\begin{aligned}\nabla_X T = \; & \textstyle\sum_i t_1 \otimes \ldots \otimes \nabla_X t_i \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes \eta^q \\ & + \textstyle\sum_j t_1 \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes \nabla_X \eta^j \otimes \ldots \otimes \eta^q,\end{aligned}$$

i.e. that ∇ extends by derivation to the tensor bundle $\otimes^p_q E$ (to begin with, consider the case $T = t \otimes \eta$, and compute the values on (ξ, s) of the LHS and RHS just using the preceding definitions). Show also that if $T \in \mathrm{Sec}(\otimes^p_q E)$, $U \in \mathrm{Sec}(\otimes^r_s E)$, and $X \in \mathcal{X}(M)$, we have

$$\nabla_X (T \otimes U) = (\nabla_X T) \otimes U + T \otimes (\nabla_X U)\,.$$

Proposition 3 can now be extended as follows:

Proposition 3. *If a connection ∇ on a vector bundle $\pi : E \to M$ of rank r is characterized in a trivialization (U, φ) of E over an open subset U of M by a 1-form $\mathcal{A}$, the induced connection ∇ on $\otimes^p_q E$ has in the induced trivialization—also denoted by (U, φ)—the local form*

$$(\nabla_X T)^\varphi = L_X(T^\varphi) + \rho(\mathcal{A}(X))T^\varphi,$$

for any $T \in \mathrm{Sec}(\otimes^p_q E)$ and any $X \in \mathcal{X}(M)$. Here ρ is the canonical action of the Lie algebra $\mathrm{gl}(r, \mathbb{R})$ on $\otimes^p_q \mathbb{R}^r$. The action $\rho(\mathcal{A}(X))T^\varphi$ of $\mathcal{A}(X) \in C^\infty(U, \mathrm{gl}(r, \mathbb{R}))$ on $T^\varphi \in C^\infty(U, \otimes^p_q \mathbb{R}^r)$ is computed pointwise. In particular we have for $\xi \in \mathrm{Sec}(E^*)$,

$$(\nabla_X \xi)^\varphi = L_X\,(\xi^\varphi) - {}^t(\mathcal{A}(X))\,\xi^\varphi.$$

Exercise 4. Let us first recall that the action $\rho(A)T$ of $A \in \mathrm{gl}(r, \mathbb{R})$ on $T \in \otimes^p_q \mathbb{R}^r$ is given by

$$\begin{aligned}(\rho(A)T)\,(\xi^1, \ldots, \xi^p, x_1, \ldots, x_q) = \;& \textstyle\sum_i T(\xi^1, \ldots, {}^t\!A\,\xi^i, \ldots, \xi^p, x_1, \ldots, x_q)\\ & -\textstyle\sum_j T(\xi^1, \ldots, \xi^p, x_1, \ldots, Ax_j, \ldots, x_q),\end{aligned}$$

where $x_k \in \mathbb{R}^r$ and $\xi^\ell \in (\mathbb{R}^r)^*$. It follows from this equation that

$$\begin{aligned}\rho(A)\,(t_1 \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes \eta^q) = \;& \textstyle\sum_i t_1 \otimes \ldots \otimes At_i \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes \eta^q\\ & -\textstyle\sum_j t_1 \otimes \ldots \otimes t_p \otimes \eta^1 \otimes \ldots \otimes {}^t\!A\,\eta^j \otimes \ldots \otimes \eta^q\end{aligned},$$

with $t_k \in \mathbb{R}^r$ and $\eta^\ell \in (\mathbb{R}^r)^*$.

Prove Proposition 3, first for $\otimes^p_q E = E^*$, then for an arbitrary (p, q). *Suggestion:* Use Equation (5).

4.4 Classical results

We are now ready to recover well-known formulas from Classical Mechanics. Let ∇ be a connection on a vector bundle $\pi : E \to M$, characterized in a trivialization (U, φ) of E by a connection 1-form $\mathcal{A}$. Remember that the local form of ∇ in this trivialization is

$$(\nabla_X s)^\varphi = L_X(s^\varphi) + \mathcal{A}(X)s^\varphi,$$

($X \in \mathcal{X}(M)$, $s \in \mathrm{Sec}(E)$). If we choose local coordinates $(x^1, \ldots, x^n)$ in U, vector field X locally reads $X = \sum_i X^i \partial_i$ ($X^i \in C^\infty(U)$, $\partial_i = \partial_{x^i}$). Hence, if e_i denotes the canonical basis of $\mathbb{R}^r$, we have

$$\mathcal{A}(X)s^\varphi = \sum_{ijk} X^i \mathcal{A}(\partial_i)^j_k s^k e_j.$$

Set $\Gamma^j_{ik} = \mathcal{A}(\partial_i)^j_k \in C^\infty(U)$. For $E = TM$ these functions are known as **Christoffel's symbols**. It is well-known that these connection coefficients are not the components of a tensor field. If we take $X = \partial_i$ and set $\nabla_i s = \nabla_{\partial_i} s$, we get

$$(\nabla_i s)^j = \partial_i s^j + \Gamma^j_{i\ell} s^\ell,$$

where we have used Einstein's convention. For the covariant derivative of $\xi \in \mathrm{Sec}(T^*M)$ and $T \in \mathrm{Sec}(\otimes^p_q TM)$, we find

$$(\nabla_i \xi)_j = \partial_i \xi_j - \Gamma^\ell_{ij} \xi_\ell$$

and

$$(\nabla_i T)^{j_1 \ldots j_p}_{k_1 \ldots k_q} = \partial_i \, T^{j_1 \ldots j_p}_{k_1 \ldots k_q} + \sum_{\ell m} \Gamma^{j_\ell}_{im} T^{j_1 \ldots m \ldots j_p}_{k_1 \ldots k_q} - \sum_{\ell m} \Gamma^m_{ik_\ell} T^{j_1 \ldots j_p}_{k_1 \ldots m \ldots k_q}$$

respectively, with self-explaining notations.

Exercise 5. Prove the two preceding classical results. *Hint:* For the second, note that

$$(\nabla_i T)^{j_1 \ldots j_p}_{k_1 \ldots k_q} = (\nabla_i T)(\varepsilon^{j_1}, \ldots, \varepsilon^{j_p}, e_{k_1}, \ldots, e_{k_q}),$$

that

$$(\nabla_i \varepsilon^{j_\ell})_m = -\Gamma^k_{im} \delta^{j_\ell}_k, \quad \nabla_i \varepsilon^{j_\ell} = -\Gamma^{j_\ell}_{im} \varepsilon^m,$$

and similarly for $\nabla_i e_{k_\ell}$.

Exercise 6. Prove that Christoffel's symbols are not tensorial. *Hint:* Consider the case $E = TM$ and change local coordinates in M, $x \leftrightarrow y$. Denote the local frame ∂_{x^i} (resp., ∂_{y^j}) associated with x (resp., y) by σ_i (resp., τ_j), and observe that Christoffel's symbols may be defined by $\nabla_{\sigma_i} \sigma_j = \Gamma^k_{ij} \sigma_k$ (resp. $\nabla_{\tau_i} \tau_j = \Gamma'^k_{ij} \tau_k$). Note finally that tensoriality would mean that $\Gamma^k_{ij} = A^k_a A'^b_i A'^c_j \Gamma'^a_{bc}$, where $A^i_j = \partial_{y^j} x^i$ and $A'^i_j = \partial_{x^j} y^i$, and prove that we have in fact

$$\Gamma^k_{ij} = \partial_{y^a} x^k \partial_{x^i} y^b \partial_{x^j} y^c \Gamma'^a_{bc} + \partial_{x^i} \partial_{x^j} y^a \partial_{y^a} x^k \ .$$

4.5 Curvature of a connection

4.5.1 Definition

The notions of curvature and torsion are well known from elementary Geometry. In our current general framework, we define the curvature of a connection in an abstract way.

Definition 5. *Let ∇ be a connection on a vector bundle $\pi : E \to M$. The bilinear map*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{L}(\mathrm{Sec}(E), \mathrm{Sec}(E)),$$

defined for $X, Y \in \mathcal{X}(M)$ and $s \in \mathrm{Sec}(E)$ by

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s,$$

is called the curvature of the connection ∇.

Some remarks:

1. Note that R is a trilinear map $R : \mathrm{Sec}(TM) \times \mathrm{Sec}(TM) \times \mathrm{Sec}(E) \to \mathrm{Sec}(E)$, which is also $C^\infty(M)$-linear in each argument and skew-symmetric in the first two arguments. Hence

$$R \in \mathrm{Sec}(\bigwedge{}^2 T^*M \otimes E^* \otimes E),$$

   i.e. the curvature of a connection on a vector bundle E with base manifold M is a differential 2-form on M valued in the endomorphism bundle of E.

2. The curvature tensor R of a connection ∇ is often denoted by R^∇.

3. We immediately see that the curvature tensor of the trivial connection on a trivial bundle vanishes, since this covariant derivative with respect to a vector field X is just the Lie derivative with respect to X. This observation is in fact a motivation for the above abstract definition of the curvature of a connection.

4.5.2 Local form and components

We know that a connection is locally, in a trivialization, characterized by its connection 1-form $\mathcal{A}$ – valued in matrices. Since the curvature of a connection is a differential 2-form valued in the endomorphism bundle, it is natural to expect that, in a trivialization, it will be given by a differential 2-form $\mathcal{F}$ – valued in matrices. Our goal is to find the link between $\mathcal{A}$ and $\mathcal{F}$.

It is clear that the de Rham differential can be extended, for any manifold M, to $\Omega(M) \otimes \mathrm{gl}(r, \mathbb{R})$. Just set

$$d(\alpha \otimes A) = (d\alpha) \otimes A$$

($\alpha \in \Omega(M), A \in \mathrm{gl}(r, \mathbb{R})$). Note that we apply here the universal property of tensor product.

The tensor product of the graded commutative associative algebra $(\Omega(M), \wedge)$ and the associative algebra $(\mathrm{gl}(r, \mathbb{R}), \cdot)$, i.e. the vector space $\Omega(M) \otimes \mathrm{gl}(r, \mathbb{R})$ equipped with the product

$$(\alpha \otimes A) \Box (\beta \otimes B) = (\alpha \wedge \beta) \otimes (A \cdot B)$$

($\alpha \in \Omega(M)$, $\beta \in \Omega(M)$, and $A, B \in \mathrm{gl}(r, \mathbb{R})$), is a new associative algebra. Moreover, the multiplication $\Box$ respects the canonical grading of $\Omega(M) \otimes \mathrm{gl}(r, \mathbb{R})$. Hence, the graded commutator, defined for $\mathcal{A} \in \Omega^a(M) \otimes \mathrm{gl}(r, \mathbb{R})$ and $\mathcal{B} \in \Omega^b(M) \otimes \mathrm{gl}(r, \mathbb{R})$ by

$$[\![\mathcal{A}, \mathcal{B}]\!] = \mathcal{A} \Box \mathcal{B} - (-1)^{ab} \mathcal{B} \Box \mathcal{A},$$

is a graded Lie bracket. It is easily checked that

$$[\![\alpha \otimes A, \beta \otimes B]\!] = (\alpha \wedge \beta) \otimes [A, B],$$

where $[.,.]$ is the commutator of matrices.

We now compute the local form of the curvature R of connection ∇ in a trivialization (U, φ) of a vector bundle $\pi : E \to M$ of rank r. Let $X, Y \in \mathcal{X}(M)$, $s \in \mathrm{Sec}(E)$ and denote by $\mathcal{A}$ the 1-form that characterizes ∇ in (U, φ). Using Prop. 3, we get

$$(R(X,Y)s)^\varphi = (L_X(\mathcal{A}(Y)) - L_Y(\mathcal{A}(X)) - \mathcal{A}([X,Y]) + \mathcal{A}(X)\mathcal{A}(Y) - \mathcal{A}(Y)\mathcal{A}(X))\, s^\varphi,$$

since $L_X(\mathcal{A}(Y)s^\varphi) = (L_X \mathcal{A}(Y))\, s^\varphi + \mathcal{A}(Y) L_X(s^\varphi)$. Hence

$$(R(X,Y)s)^\varphi = ((d\mathcal{A} + [\mathcal{A}, \mathcal{A}])(X,Y))\, s^\varphi,$$

where $[\mathcal{A}, \mathcal{A}](X,Y) = [\mathcal{A}(X), \mathcal{A}(Y)]$. Let us try to write this result using the above defined Lie bracket $[\![.,.]\!]$. First note that if E^i_j is the canonical basis of $\mathrm{gl}(r, \mathbb{R})$, connection 1-form $\mathcal{A} \in \Omega^1(U) \otimes \mathrm{gl}(r, \mathbb{R})$ reads

$$\mathcal{A} = \sum_{ij} \mathcal{A}^j_i \otimes E^i_j,$$

with $\mathcal{A}^j_i \in \Omega^1(U)$. To simplify notation, we write $\mathcal{A} = \sum_k \mathcal{A}(k) \otimes E(k)$. For any $X, Y \in \mathcal{X}(U)$,

$$\begin{aligned} [\![\mathcal{A}, \mathcal{A}]\!](X,Y) &= \textstyle\sum_{k,\ell} (\mathcal{A}(k) \wedge \mathcal{A}(\ell))(X,Y)[E(k), E(\ell)] \\ &= \textstyle\sum_{k,\ell} (\mathcal{A}(k)(X)\mathcal{A}(\ell)(Y) - \mathcal{A}(\ell)(X)\mathcal{A}(k)(Y))\,(E(k)E(\ell) - E(\ell)E(k)) \\ &= 2[\mathcal{A}, \mathcal{A}](X,Y). \end{aligned}$$

Finally,

$$(R(X,Y)s)^\varphi = \left(\left(d\mathcal{A} + \frac{1}{2}[\![\mathcal{A}, \mathcal{A}]\!]\right)(X,Y)\right) s^\varphi.$$

It is clear that

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\![\mathcal{A}, \mathcal{A}]\!] = d\mathcal{A} + [\mathcal{A}, \mathcal{A}] \qquad (9)$$

is an element of $\Omega^2(U) \otimes \mathrm{gl}(r, \mathbb{R})$. This differential 2-form $\mathcal{F}$ on U valued in $\mathrm{gl}(r, \mathbb{R})$ is the local form of the curvature in the considered trivialization. We refer to it as the **curvature 2-form** in the trivialization (U, φ). The link (9) between the connection 1-form and the curvature 2-form is called "**Cartan's structure equation**".

Exercise 7. Remember that $\mathcal{A} \in \mathrm{Sec}(T^*U \otimes E_U^* \otimes E_U) = \Omega^1(U) \otimes \mathrm{gl}(r, \mathbb{R})$ is a twice covariant and once contravariant tensor field and has thus components $\mathcal{A}(\partial_i)^k_j = \Gamma^k_{ij}$ (if we choose coordinates $(x^1, \ldots, x^n)$ in the domain U of the trivialization φ and denote by ∂_i the corresponding frame of TM). On the other hand, since $\mathcal{F} = R|_U \in \mathrm{Sec}(\bigwedge^2 T^*U \otimes E_U^* \otimes E_U)$ is 3 times covariant and once contravariant, it is characterized by its components $R^\ell_{ijk} = (\mathcal{F}(\partial_i, \partial_j))^\ell_k$. Structure equation (9) shows that the components of $\mathcal{A}$ and $\mathcal{F}$ are related by

$$R^\ell_{ijk} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^\ell_{im}\Gamma^m_{jk} - \Gamma^\ell_{jm}\Gamma^m_{ik}.$$

4.5.3 Intuitive approach to curvature

Let us try to understand Def. 5. In order to achieve this goal, we have to work intuitively.

Choose on the sphere $M = S^2 \subset \mathbb{R}^3$ two half great circles, a horizontal one, $\mathcal{C}_h$, and a vertical one, $\mathcal{C}_v$, that have two common points p and q. We consider the tangent bundle $E = TM = T(S^2)$. Take now a vector Y_p tangent to $\mathcal{C}_h$ at p. Parallel transport of Y_p along $\mathcal{C}_h$ and $\mathcal{C}_v$, from tangent space to tangent space, is canonical. The reader easily figures out that the two resulting vectors at q, let us denote them by $\overset{\sim h}{Y}_q$ and $\overset{\sim v}{Y}_q$, are opposite.

Since the parallel transport of a vector in the flat space $\mathbb{R}^2$ along two different paths—connecting the same points—leads to the same resulting vector, we understand that the difference $\overset{\sim h}{Y}_q - \overset{\sim v}{Y}_q$ characterizes the curvature (in the common sense of the word). So, if Def. 5 actually makes sense, the computation of this difference should yield a result tightly connected with the curvature (in the sense of Def. 5).

The objective of the next heuristic exercise is to compute this difference.

Exercise 8. We work locally in a tangent bundle endowed with a connection ∇, consider a vector field Y and an infinitesimal parallelogram $x = (x^i)$, $x + \varepsilon = (x^i + \varepsilon^i)$, $x + \delta = (x^i + \delta^i)$, and $x + \varepsilon + \delta = (x^i + \varepsilon^i + \delta^i)$ in the base manifold.

1. Remember that $(\nabla_i Y)^j = \partial_i Y^j + \Gamma^j_{ik} Y^k$, view the l.h.s. of this equation as the jth component of
$$(\nabla_i Y)_x = \frac{Y_{x+\eta e_i} - \tilde{Y}_{x+\eta e_i}}{\eta},$$
where η is some small non-vanishing real number, e_i the canonical basis of $\mathbb{R}^n$, and $\tilde{Y}_{x+\eta e_i}$ the vector Y_x parallel transported into the tangent space at $x + \eta e_i$. Prove that
$$(\tilde{Y}_{x+\eta e_i})^j = Y^j_x - \Gamma^j_{ik}(x) Y^k_x \eta.$$

2. Compute the jth component of the parallel transport $\overset{\sim\delta}{\overset{\sim\varepsilon}{Y}}_{x+\varepsilon+\delta}$ of Y_x to $x + \varepsilon$ and then to $x + \varepsilon + \delta$, and compute the jth component of $\overset{\sim\varepsilon}{\overset{\sim\delta}{Y}}_{x+\varepsilon+\delta}$, i.e. of the parallel transport of Y_x, first to $x + \delta$, then to $x + \delta + \varepsilon$.

3. Show that the difference is given by

$$(\overset{\sim\delta}{\underset{\sim\varepsilon}{Y}}_{x+\varepsilon+\delta})^j - (\overset{\sim\varepsilon}{\underset{\sim\delta}{Y}}_{x+\varepsilon+\delta})^j = R^j_{ik\ell}\delta^i\varepsilon^k Y^\ell.$$

   Suggestion: Use a first order expansion of Christoffel's symbols and suppress the terms of order bigger than 2 in ε, δ.

4.5.4 Exterior covariant derivative

If a vector bundle $\pi : E \to M$ is endowed with a covariant derivative, it is possible to extend the de Rham differential to differential forms on M valued in E.

Theorem 4. *Let $\pi : E \to M$ be a vector bundle endowed with a connection ∇. There is a unique linear differential operator of order 1,*

$$d^\nabla : \mathrm{Sec}(\bigwedge{}^p\, T^*M \otimes E) \to \mathrm{Sec}(\bigwedge{}^{p+1}\, T^*M \otimes E),$$

such that for any $\alpha \in \mathrm{Sec}(\bigwedge^p T^*M)$ and any $s \in \mathrm{Sec}(E)$

$$d^\nabla(\alpha \otimes s) = (d\alpha) \otimes s + (-1)^p \alpha \wedge \nabla s. \tag{10}$$

Remarks:

1. We will not prove existence and uniqueness of this operator (see proof of existence of the de Rham differential).

2. For the trivial connection on the trivial line bundle $E = M \times \mathbb{R}$, we recover the usual de Rham differential. Indeed, in this case, the operator d^∇ acts between ordinary differential forms, and the characterizing property reduces to $d^\nabla(f\alpha) = d(f\alpha)$, where $f \in \mathrm{Sec}(E) = C^\infty(M)$, since the Lie derivative verifies $L_X f = (df)(X)$.

3. For any $\mathcal{E} \in \mathrm{Sec}(\bigwedge^p T^*M \otimes E)$ and any $X_0, \ldots, X_p \in \mathcal{X}(M)$, we have

$$\begin{aligned}\left(d^\nabla \mathcal{E}\right)(X_0, \ldots, X_p) = \; & \textstyle\sum_i (-1)^i \nabla_{X_i} \left(\mathcal{E}(X_0, \ldots, \hat{X}_i, \ldots, X_p)\right) \\ & + \textstyle\sum_{i<j} (-1)^{i+j} \mathcal{E}([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p),\end{aligned} \tag{11}$$

   where $\hat{X}_k$ means that field X_k is omitted. This formula extends Cartan's formula for the de Rham differential (that we recover for $E = M \times \mathbb{R}$ and $\nabla = \nabla^0$, since in this case $d^\nabla = d$). The proof of the preceding generalization uses Equation (10), Cartan's formula, the 'shuffle definition' of the wedge product, as well as Equation (7).

4. For $p = 0$, the source space of operator d^∇ is $\mathrm{Sec}(E)$ and $d^\nabla = \nabla$. Moreover we have, with self-explaining notations,

$$(d^\nabla)^2(X, Y)s = \left(d^\nabla(\nabla s)\right)(X, Y) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = R^\nabla(X, Y)s,$$

i.e.

$$R^\nabla = d^\nabla \circ \nabla.$$

Note that there is no reason to think about d^∇ as a differential.

5. If $\Omega \in \mathrm{Sec}(T^*M \otimes E^* \otimes E)$, the sum $\nabla + \Omega$ is also a connection. The curvatures of $\nabla + \Omega$ and ∇ verify

$$R^{\nabla+\Omega} = R^\nabla + d^\nabla \Omega + [\Omega, \Omega]. \tag{12}$$

Note that this result shows that $R^\nabla = 0$ does not imply that $R^{\nabla+\Omega} = 0$. This corroborates the already mentioned fact that the curvature is a property of the chosen connection (and a priori not of the considered vector bundle or base manifold).

Exercise 9. (1) Prove Cartan's 'magic' equation (11). (2) Prove the relation (12).

Hint: Remember that connection ∇ on E can be extended to the endomorphism bundle $\mathrm{End}(E) = E^* \otimes E$ of E. Hence $d^\nabla \Omega$ makes sense and, for $X, Y \in \mathcal{X}(M)$ and $s \in \mathrm{Sec}(E)$, we have

$$\nabla_X(\Omega(Y))(s) = \nabla_X(\Omega(Y)(s)) - (\Omega(Y))(\nabla_X s).$$

4.5.5 Vanishing curvature and triviality

To understand one of the basic results of the theory of connections, we must define the notions of isomorphism of vector bundles and equivalence of connections. These definitions are quite obvious.

Definition 6. *If (E, M, π) and (E', M', π') are two vector bundles, a morphism from (E, M, π) into (E', M', π') is a pair (F, f) of smooth maps*

$$F \in C^\infty(E, E') \text{ and } f \in C^\infty(M, M'),$$

such that

$$\pi' \circ F = f \circ \pi$$

and for any $x \in M$ the restriction

$$F_x : E_x \to E'_{f(x)}$$

is linear.

Note that the compatibility condition of the morphism maps with the projections entails that F maps fiber E_x into fiber $E'_{f(x)}$.

Example. If $f \in C^\infty(M, M')$, then $Tf \in C^\infty(TM, TM')$ is a vector bundle morphism.

Definition 7. *A vector bundle isomorphism is a vector bundle morphism that has an inverse vector bundle morphism.*

Let us recall that if $f \in \mathrm{Diff}(M, M')$ and $X \in \mathcal{X}(M)$, then $f_* X = Tf \circ X \circ f^{-1} \in \mathcal{X}(M')$ is called the pushforward of the vector field X by the diffeomorphism f. This concept can be extended to vector bundle isomorphisms and sections. If $(F, f) : E \to E'$ is an isomorphism and $s \in \mathrm{Sec}(E)$ a section, then $F \circ s \circ f^{-1} \in \mathrm{Sec}(E')$.

Definition 8. *Let (E, M, π) and (E', M', π') be two vector bundles endowed with connections ∇ and ∇' respectively. These covariant derivatives are equivalent connections on isomorphic bundles, if there is a vector bundle isomorphism (F, f) between (E, M, π) and (E', M', π') that is also a connection equivalence, i.e. that verifies for any $X \in \mathcal{X}(M)$ and any $s \in \mathrm{Sec}(E)$,*

$$\nabla_X s = F^{-1} \circ \nabla'_{f_* X} \left(F \circ s \circ f^{-1} \right) \circ f. \tag{13}$$

We are now able to understand the following fundamental result.

Theorem 5. *If the curvature tensor of a connection ∇ on a vector bundle (E, M, π) vanishes and if the base manifold M is simply connected, then the bundle E is isomorphic with the trivial bundle $M \times \mathbb{R}^r$, where r is the rank of E, and connection ∇ is equivalent with the trivial connection ∇^0 on this trivial bundle. If M is not necessarily simply connected, connection ∇ is locally equivalent to the trivial connection on the trivial bundle.*

The proof of this important theorem is quite complicated and cannot be given here.

If, in this general framework, the curvature of a connection on an arbitrary vector bundle vanishes, we say that the connection is flat (and not that the bundle or the base manifold are flat). The situation is different for Riemannian manifolds. Indeed, on the tangent bundle of a Riemannian manifold there exists a privileged connection, the so-called Levi-Civita connection. If the curvature of this canonical connection vanishes, the considered Riemannian manifold is said to be flat.

4.6 Torsion of a connection on a manifold

4.6.1 Definition

Take a connection ∇ on a manifold M, i.e. on the tangent bundle (TM, M, π) of M. For such a connection on a tangent bundle it is possible to compare $\nabla_X Y$ and $\nabla_Y X$, $X, Y \in \mathcal{X}(M)$. If ∇^0 is the trivial connection on the trivial bundle $(T\mathbb{R}^n, \mathbb{R}^n, \pi) \simeq (\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n, \pi)$, we have

$$\nabla^0_X Y - \nabla^0_Y X = \sum_i (L_X Y^i - L_Y X^i) e_i = [X, Y],$$

where e_i is the canonical basis of $\mathbb{R}^n$.

Definition 9. *Let ∇ be a connection on a manifold M. The bilinear skew-symmetric map*

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

defined for any $X, Y \in \mathcal{X}(M)$ by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

is called the torsion of the connection ∇.

The torsion T (or T^∇) is $C^\infty(M)$-linear with respect to each argument and can therefore be viewed as a $(1,2)$-tensor field on M. We even have

$$T \in \mathrm{Sec}(\bigwedge{}^2 T^*M \otimes TM),$$

so that the torsion of a connection on M is a differential 2-form on M valued in TM.

4.6.2 Local form and components

Choose a local coordinate system $(x^1, \ldots, x^n)$ of M in an open subset U. Since

$$T_{ij}^k = (T|_U(\partial_i, \partial_j))^k = (\nabla_i \partial_j)^k - (\nabla_j \partial_i)^k - \left(\nabla_{[\partial_i, \partial_j]}\right)^k,$$

the relation between the components Γ_{ij}^k of the connection 1-form $\mathcal{A}$ and the components T_{ij}^k of the $(1,2)$-tensor field T are given by

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

4.6.3 Vanishing torsion

If $\Omega \in \mathrm{Sec}(\otimes_2^1 TM)$, the sum $\nabla + \Omega$ is another connection on M. We immediately see that

$$T^{\nabla+\Omega}(X,Y) = T^\nabla(X,Y) + \Omega(X,Y) - \Omega(Y,X). \tag{14}$$

If we choose $\Omega = -\frac{1}{2}T^\nabla \in \mathrm{Sec}(\bigwedge{}^2 T^*M \otimes TM)$, we get $T^{\nabla - \frac{1}{2}T^\nabla} = 0$.

Proposition 4. *Any connection can be corrected in such a way that the resultant connection has vanishing torsion. In particular, any manifold admits a connection with vanishing torsion.*

The following theorem is given without proof and should be compared with Theo. 5.

Theorem 6. *Let M be a manifold of dimension n. A connection on M with vanishing curvature and vanishing torsion is locally equivalent to the trivial connection on $\mathbb{R}^n$.*

4.6.4 Levi-Civita connection, symplectic connections

If we deal with a manifold M endowed with some additional structure, such as a symplectic or a pseudo-Riemannian structure, it is natural to put reasonable restrictions on connections on M.

Remember from Mechanics that the standard Euclidean metric of the ambiant space is given by

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\ ,$$

whereas the pseudo-Euclidean metric in Minkowski spacetime is defined by

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - c^2(\mathrm{d}t)^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - (\mathrm{d}x^4)^2\ .$$

The first (resp., second) metric is represented by the diagonal matrix $(1,1,1)$ (resp., $(1,1,1,-1)$) and is thus a positive definite (resp., nondegenerate) symmetric bilinear form on $\mathbb{R}^3$ (resp., $\mathbb{R}^4$). If we consider a symmetric bilinear form $g_x : T_xM \times T_xM \to \mathbb{R}$ on each tangent space T_xM, $x \in M$, of a manifold M, we get an assignment $g : M \ni x \mapsto g_x \in S^2T_x^*M$, i.e., assuming smoothness, we obtain a tensor field $g \in \mathrm{Sec}(S^2T^*M)$. If, for each $x \in M$, g_x is positive definite (resp., nondegenerate), the pair (M, g) is a *Riemannian manifold* (resp., *a pseudo-Riemannian manifold*). These types of manifold will be studied in more detail below.

Consider now for instance a pseudo-Riemannian metric g on M. We know that a connection ∇ on M, i.e. on TM, induces a connection, still denoted by ∇, on any tensor bundle $\otimes^p_qTM$. A tensor field $T \in \mathrm{Sec}(\otimes^p_qTM)$ is said to be parallel or covariantly constant if $\nabla T = 0$. Let now $X, Y \in \mathcal{X}(M)$ be two parallel vector fields. It is natural to ask that their inner product $g(X, Y)$ be constant, i.e. that for any $Z \in \mathcal{X}(M)$

$$0 = \nabla_Z(g(X, Y)) = (\nabla_Z g)(X, Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = (\nabla_Z g)(X, Y).$$

This condition is fulfilled if we require that the metric is covariantly constant.

The following fundamental theorem of Riemannian Geometry is well-known.

Theorem 7. *On any pseudo-Riemannian manifold (M, g) there exists a unique torsion-free connection ∇ such that $\nabla g = 0$. This privileged connection is called the* Levi-Civita connection *of (M, g).*

Let us recall that a symplectic manifold is in some sense the antisymmetric counterpart of a pseudo-Riemannian manifold, i.e. that a *symplectic manifold* is a manifold M endowed with a *closed* differential 2-form $\omega \in \Omega^2(M) = \mathrm{Sec}(\bigwedge^2 T^*M)$, such that for any $x \in M$ the skew-symmetric bilinear form ω_x on T_xM is nondegenerate. We will give more information on symplectic manifolds in the next section.

Proposition 5. *On any symplectic manifold (M, ω) there exist torsionfree connections ∇ such that $\nabla\omega = 0$. The set of these connections is an affine space modeled on* $\mathrm{Sec}(S^3T^*M)$. *Any such connection is called a* symplectic connection.

Proof. The construction is the same as in the proof of Theo. 2, except that we choose local trivializations of TM induced by Darboux charts $(U, \psi = (p_1, \ldots, p_n, q^1, \ldots, q^n))$ of M. Let us recall that in such a chart the symplectic form ω reads

$$\omega|_U = \sum_i dp_i \wedge dq^i,$$

i.e. that

$$\omega|_U = \psi^* \omega_0,$$

where ω_0 is the canonical symplectic form of $\mathbb{R}^{2n}$.

Remember now that in order to construct a connection (see the proof of Theo. 2) we transport the trivial connection ∇^0 from $\psi(U) \times \mathbb{R}^{2n}$ to TU. Indeed, (ψ_*, ψ), where ψ_* is the tangent map of ψ, is an isomorphism between these tangent bundles TU and $\psi(U) \times \mathbb{R}^{2n}$. Hence equivalence condition (13) shows that connection transport is defined for any $X, Y \in \mathcal{X}(U)$ by

$$\psi_* (\nabla_X Y) = \nabla^0_{\psi_* X} \psi_* Y.$$

Here $\psi_* Y = \psi_* \circ Y \circ \psi^{-1}$ is the push-forward of Y by diffeomorphism ψ.

As $T^{\nabla^0} = 0$ and $\nabla^0 \omega_0 = 0$, the equivalent connection ∇ and the equivalent form $\omega|_U$ inherit the same properties (the precise verification of this fact is a good exercise).

Up till now we have constructed a local connection ∇ in U or better, since (U, ψ) runs through an atlas, ∇^α in U_α, such that $T^{\nabla^\alpha} = 0$ and $\nabla^\alpha \omega|_{U_\alpha} = 0$. As in the proof of Theo. 2, we now glue these connections ∇^α together using a partition of unity f_α. It is easily seen that the global connection, defined for any $X, Y \in \mathcal{X}(M)$ by

$$\nabla_X Y = \sum_\alpha f_\alpha \nabla^\alpha_{X|_{U_\alpha}} Y|_{U_\alpha},$$

is also torsionfree and parallel.

Let us prove that the symplectic connections form an affine space modelled on $\mathrm{Sec}(S^3 T^* M)$. If ∇' is another connection on M, we have $\Omega = \nabla' - \nabla \in \mathrm{Sec}(\otimes^1_2 TM)$. Since

$$(\nabla'_Z \omega)(X, Y) = L_Z (\omega(X, Y)) - \omega(\nabla'_Z X, Y) - \omega(X, \nabla'_Z Y)$$

and similarly for ∇, we get

$$(\nabla'_Z \omega)(X, Y) = (\nabla_Z \omega)(X, Y) - \omega(\Omega(Z, X), Y) - \omega(X, \Omega(Z, Y)),$$

for any $X, Y, Z \in \mathcal{X}(M)$. Hence, connection ∇' is parallel if and only if

$$\omega(\Omega(Z, X), Y) = \omega(\Omega(Z, Y), X),$$

for any $X, Y, Z \in \mathcal{X}(M)$. In other words, the tensor field $\tilde{\Omega} \in \mathrm{Sec}(\otimes^3 T^* M)$, defined by

$$\tilde{\Omega}(X, Y, Z) = \omega(\Omega(X, Y), Z),$$

has to be symmetric with respect to the two last arguments. On the other hand, it follows from Eq. (14) that connection ∇' is torsionfree if and only if Ω is symmetric, i.e. if and only if $\tilde{\Omega}$ is symmetric with respect to the first two arguments. It now suffices to remember that the 'musical map'

$$\flat : \mathcal{X}(M) \ni X \to -i_X\omega \in \Omega^1(M)$$

is a vector space isomorphism (we refer the non-informed reader to the next section). ∎

4.7 Symplectic manifolds

4.7.1 Hamilton's equations

Let us recall Hamilton's equations studied in Mechanics. Consider a dynamical system characterized by a Hamiltonian $H = H(q,p)$, where $q = (q^1, \ldots, q^n)$ (resp., $p = (p_1, \ldots, p_n)$) denote the (generalized) coordinates or positions (resp., the (generalized) velocities or momenta). The space (q,p), say $\mathbb{R}^{2n}$ to simplify, is referred to as the phase space of physical states. The motions of the considered system $(q(t), p(t))$ are given by Hamilton's equations

$$\dot{q} = \partial_p H,\ \dot{p} = -\partial_q H\ .$$

If we set $x = (q,p)$ and denote by ω the so-called symplectic unit

$$\omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix},$$

the matrix product

$$\omega\ (\partial_x H)\tilde{} = \omega \begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} =: X_x$$

is a vector field of the phase space. The motions are now given by

$$d_t x = X_{x(t)}\ ,$$

i.e. by the flow of this vector field. In the following, we will deduce these dynamics from the symplectic structure of the phase space.

4.7.2 Symplectic manifolds

Symplectic manifolds have been defined above as manifolds M endowed with a closed differential 2-form $\omega \in \Omega^2(M)$ that is nowhere degenerate. The latter condition means that, for any $x \in M$, the skew-symmetric bilinear form $\omega_x : T_xM \times T_xM \to \mathbb{R}$ is nondegenerate, i.e. that there is no nonzero vector in T_xM that is 'orthogonal' to each vector of T_xM. This in turn means that the linear map $\omega_x : T_xM \to T_x^*M$ is bijective. Indeed, as its source and target spaces have the same dimension, it suffices to prove injectivity, which immediately follows from the nondegeneracy. Hence, the representing matrix of ω_x is an invertible antisymmetric matrix, so that the dimension of M is necessarily even – odd skew-symmetric matrices are not invertible. Another immediate consequence of the bijectivity of ω_x is that the map

$$\flat : \mathcal{X}(M) \ni X \to -i_X\omega \in \Omega^1(M)$$

is a vector space isomorphism. Its inverse is denoted by $\sharp$. The notation comes from the fact that $\flat$ (resp., $\sharp$) lowers (resp., raises) the indices of the components, just as in music.

We give now two well-known examples of symplectic manifolds.

Example 1. Let e_i (resp., ε^i) be the standard basis (resp., the dual basis) of $\mathbb{R}^{2n}$ (resp., $(\mathbb{R}^{2n})^*$). The 2-form

$$\omega := \sum_{i=1}^{n} \varepsilon^i \wedge \varepsilon^{n+i} \in \wedge^2(\mathbb{R}^{2n})^* \subset C^\infty(\mathbb{R}^{2n}, \wedge^2(\mathbb{R}^{2n})^*) \simeq \mathrm{Sec}(\wedge^2 T^*\mathbb{R}^{2n}) = \Omega^2(\mathbb{R}^{2n})$$

is a symplectic form on $\mathbb{R}^{2n}$, called the *canonical symplectic form* of $\mathbb{R}^{2n}$.

Exercise 10. Prove that the constant form ω is symplectic. *Hint* : Show that the matrix of the components $\omega_{k\ell} = \omega(e_k, e_\ell)$ of the covariant 2-tensor $\omega \in \otimes^2(R^{2n})^*$ in the canonical basis is the symplectic unit.

Example 2. Let $\mathbb{R}^{2n}$ be the above-considered phase space with coordinates (q, p). The 1-form

$$\alpha = p\,\mathrm{d}q := \sum_{i=1}^{n} p_i\,\mathrm{d}q_i \in \Omega^1(\mathbb{R}^{2n}) \tag{15}$$

is referred to as the *Liouville 1-form*. Obviously, the canonical symplectic form ω and the Liouville form α are related by $\omega = -\,\mathrm{d}\alpha$ – remember that in a Euclidean space $\mathbb{R}^N$ with coordinates x^i, we have $\partial_{x^i} \simeq e_i$ and $\mathrm{d}x^i \simeq \varepsilon^i$, so that

$$\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i\ . \tag{16}$$

We will extend α (and thus ω) to an arbitrary cotangent bundle $P = T^*Q$, where Q is an n-dimensional manifold. Let $\pi : P = T^*Q \to Q$ be the projection of this bundle. To define a 1-form $\alpha \in \Omega^1(P)$, it suffices to define, for any $x \in P = T^*Q$, i.e. for any linear form $x : T_{\pi(x)}Q \to \mathbb{R}$, an element $\alpha_x \in T^*_xP$, i.e. a linear form $\alpha_x : T_xP \to \mathbb{R}$. Since $T_x\pi : T_xP \to T_{\pi(x)}Q$, it is natural to set $\alpha_x = x \circ T_x\pi$. This differential 1-form $\alpha \in \Omega^1(P)$ is the Liouville form of the cotangent bundle P and the differential 2-form $\omega = -\,\mathrm{d}\alpha \in \Omega^2(P)$ is the standard symplectic form of P.

Exercise 11. Prove that the coordinate form of α is given by (15) and that ω is actually symplectic. *Hint* : Denote the coordinates of $x \in P = T^*Q$ by (q, p), note that the linear form $x \in T^*_qQ$ is represented by the matrix $p = (p_1, \ldots, p_n)$, show that $T_x\pi \simeq (\mathbf{1}, 0)$ and that $\alpha_x \simeq (p, 0)$, so that $\alpha = p\,\mathrm{d}q + 0\,\mathrm{d}p = p\,\mathrm{d}q$.

Let us mention without proof Darboux's theorem (also called Darboux-Weinstein theorem) that states that locally any symplectic form is of the type (16).

Theorem 8. *For any symplectic manifold (M,ω) and any point $x \in M$, there is a neighborhood U of x with coordinate functions (q,p), in which ω reads*

$$\omega|_U = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i \ .$$

If (M,ω) is a symplectic manifold, there exists a map

$$X : C^\infty(M) \ni f \to X_f \in \mathcal{X}(M)$$

that associates to each function f of M a vector field X_f, called the *Hamiltonian vector field* of f. This vector field is defined as the unique solution of the equation $\flat(X) = \mathrm{d}f$, or, more explicitly,

$$i_X\omega = -\,\mathrm{d}f \ .$$

Exercise 12. Show that the local form (in a coordinate chart $(U,(x^1,\ldots,x^n))$) of X_f is given by

$$X_f = -\omega^{ik}\partial_{x^i}f\ \partial_{x^k} \ ,$$

where ω^{ki} is the entry (k,i) of the inverse ω^{-1} of the matrix ω (depending smoothly on $x \in U$) representing the isomorphism $\omega_x : T_xM \to T_x^*M$.

The function algebra $C^\infty(M)$ of any symplectic manifold (M,ω) is endowed with a Poisson-Lie algebra bracket $\{-,-\}$ defined, for any $f,g \in C^\infty(M)$, by

$$\{f,g\} = \omega(X_f,X_g) \in C^\infty(M) \ .$$

Let us recall that a Poisson-Lie bracket is a Lie bracket on an associative algebra, which is a derivation with respect to the associative multiplication. In our case this means that, for any $f,g,h \in C^\infty(M)$, we have

$$\{f, g\cdot h\} = \{f,g\}\cdot h + g\cdot\{f,h\} \ .$$

Exercise 13. Prove that the local form of $\{f,g\}$ is given by

$$\{f,g\}|_U = -\omega^{ij}\partial_{x^i}f\ \partial_{x^j}g \ ,$$

so that, if U is the domain of a Darboux chart with coordinates (q,p), the Poisson bracket reads

$$\{f,g\}|_U = \sum_{i=1}^{n}\left(\partial_{q^i}f\ \partial_{p_i}g - \partial_{p_i}f\ \partial_{q^i}g\right) \ ,$$

which is the well-known Poisson bracket from elementary Mechanics.

4.7.3 Link with Hamilton's equations

With this new formalism at hand a (Hamiltonian) dynamical system can be viewed as follows. It is a triplet (M, ω, H) made up by the phase space (M, ω) in which the system 'lives' (a cotangent bundle endowed with its canonical symplectic structure or, more generally, any other symplectic manifold) and by a Hamiltonian function $H \in C^\infty(M)$, which characterizes the considered system (e.g. its total energy). The dynamics of this system (its motions $\alpha(t)$) is then defined as the flow of the Hamiltonian vector field $X_H \in \mathcal{X}(M)$:

$$\mathrm{d}_t\alpha = X_H(\alpha(t)) \; .$$

Exercise 14. Show that in Darboux coordinates we thus recover Hamilton's equations

$$\dot{q} = \partial_p H = \{q, H\}, \; \dot{p} = -\partial_q H = \{p, H\} \; .$$

4.8 Riemannian manifolds

To be completed.

4.9 Parallel transport and Ehresmann connection

In an affine space (A, V) there is a canonical concept of parallel transport. As a result, we consider a same (free) vector together with different origins. Since (the endpoints of) these vectors are elements of the affine space in which we chose various origins $a, b \ldots$, they are vectors of V, or, better of $T_aA, T_bA \ldots$ Hence, there is a natural parallel transport map, or identity map, between two tangent spaces T_aA and T_bA.

Let G be a Lie group and let $g, h \in G$. The right translation $\rho_{g^{-1}h} : G \to G$ is known to be a diffeomorphism, so that its derivative $T\rho_{g^{-1}h} : T_gG \to T_hG$ is an isomorphism. The latter is a natural 'identity map' or parallel transport between the fibers of TG.

For an arbitrary manifold M (resp., vector bundle E), such an identity map or parallel transport between fibers T_mM and $T_{m'}M$ (resp., E_m and $E_{m'}$) is usually not canonical.

However, in specific situations a parallel transport can be canonically defined.

As first example, consider for instance the sphere $S \subset \mathbb{R}^3$. The Euclidean structure of $\mathbb{R}^3$ then allows to define a natural parallel transport in S. Indeed, let $s, s' \in S$ be two nearby points. There exists a unique 'segment' (of shortest length) in S that connects s, s'. This 'segment' is a smooth curve c, $c(0) = s, c(1) = s'$, whose velocities are well-defined. Set $v_0 = d_tc|_{t=0} \in T_sS$ and $v_1 = d_tc|_{t=1} \in T_{s'}S$. It is clear that there is a unique map $\tau : T_sS \to T_{s'}S$ that sends v_0 to v_1 and is an orientation preserving isometry (think about angles). This map τ is a natural parallel transport between two neighboring points. It is easily seen that the parallel transport of a tangent vector to a great circle along half of this circle and along half of the orthogonal great circle do not lead to the same result. This dependence of the parallel transport of the curve along which it is performed is due to the curvature.

A second example is given by a vector bundle together with a chosen connection. Let $\pi : E \to M$ be a vector bundle and let $c : [0,1] \to M$ be a curve in M that connects $x_0 = c(0)$ with $x_1 = c(1)$. In this section we explain that if the bundle is endowed with a connection ∇, it is possible to parallel transport the vectors of E_{x_0} to E_{x_1} along the curve c.

We denote the velocity vector of c at t by $d_t c = c_{*t}(1)$ and call lift of c any curve $\gamma : [0,1] \to E$ such that $\gamma(t) \in E_{c(t)}$ for all t. It is natural to say that a lift γ of c such that

$$\text{“}(\nabla_{d_t c}\gamma)_{c(t)} = 0\text{”}, \tag{17}$$

is parallel along c. Of course mathematical correctness of this condition will be improved. We will also see that for any $p \in E_{x_0}$ there is a unique parallel lift γ of c the source point of which is $\gamma(0) = p$. The (linear) map

$$T_c : E_{x_0} \ni p = \gamma(0) \to \gamma(1) \in E_{x_1}$$

is the parallel transport along c induced by connection ∇.

Let us be more precise. To improve condition (17), we need the following quite understandable result.

Let (E, M, π) be a fiber bundle, M' a manifold, and $f \in C^\infty(M', M)$. The disjoint union

$$\begin{aligned} E' &= \cup_{x' \in M'} \left(\{x'\} \times E_{f(x')}\right) \\ &= \{(x', p) \in M' \times E : \pi(p) = f(x')\} \end{aligned}$$

is a submanifold of $M' \times E$. The triplet (E', M', π'), where $\pi' = \mathrm{pr}_1|_{E'}$, is a fiber bundle, such that fiber $E'_{x'}$ is canonically diffeomorphic with fiber $E_{f(x')}$ for all $x' \in M'$.

If $F = \mathrm{pr}_2|_{E'}$, the pair (F, f) is a fiber bundle morphism from (E', M', π') into (E, M, π). Moreover, if (E, M, π) is a vector bundle, the bijection $F|_{E'_{x'}} : E'_{x'} \to E_{f(x')}$ allows, for any $x' \in M'$, to pull the linear structure of $E_{f(x')}$ back to $E'_{x'}$. This turns the fiber bundle (E', M', π') into a vector bundle, called pullback of E by f and denoted f^*E.

Sections of f^*E can be viewed as maps $s' : M' \to E$, such that $s'(x') \in E_{f(x')}$, $x' \in M'$. If $f : M' \to M$ is a curve $c : I \to M$, where I is an interval of $\mathbb{R}$, the above considered lift γ of c is nothing but a section of c^*E. Hence, to rewrite parallelism condition (17), we need a connection on the pullback bundle.

Proposition 6. *If (E, M, π) is a vector bundle endowed with a connection ∇, and if $f : M' \to M$ is an immersion, there exists on the pullback bundle (E', M', π') a unique connection ∇', such that for any $s \in \mathrm{Sec}(E)$, any $x' \in M'$, and any $v' \in T_{x'}M'$, we have*

$$(\nabla'_{v'}(s \circ f))_{x'} = \left(\nabla_{f_{*x'}v'} s\right)_{f(x')}.$$

Remarks:

1. Note that if (F, f), see above, were a vector bundle isomorphism between (E', M', π') and (E, M, π), we could transfer connection ∇ of E to E':

$$\nabla'_{X'} s' = F^{-1} \circ \nabla_{f_* X'} \left(F \circ s' \circ f^{-1}\right) \circ f.$$

   If we write this equation for $s' = F^{-1} \circ s \circ f$, evaluate both sides at x', and remark that, due to our way of thinking about sections of $f^* E = E'$, function F is just identity, we get exactly the characteristic property of the preceding proposition.

2. Connection ∇' is the connection induced by f and will be denoted by ∇^f.

Proposition 7. *Let (E, M, π) be a vector bundle, ∇ a connection of this bundle, I an open interval of $\mathbb{R}$, and $c : I \to M$ an immersion* [1]*. For any $t_0 \in I$ and any $p \in E_{c(t_0)}$, there is a unique section $\gamma \in \mathrm{Sec}(c^* E)$, such that $\gamma(t_0) = p$ and $\nabla^c \gamma = 0$.*

Proof. Let φ be a trivialization of E over $U \ni c(t_0)$ and let σ_i be the corresponding frame. If $J = c^{-1}(U)$, a section $\gamma \in \mathrm{Sec}(c^* E)$ locally reads

$$\gamma|_J = \sum_i \gamma^i (\sigma_i \circ c|_J).$$

Hence the local form of equation $\nabla^c \gamma = 0$ is

$$\begin{aligned} 0 = (\nabla^c_{d_t} \gamma)_t = & \left(\nabla^c_{d_t} \textstyle\sum_i \gamma^i (\sigma_i \circ c|_J)\right)_t \\ & = \textstyle\sum_i (\gamma^i)'(t) \sigma_i(c(t)) + \sum_i \gamma^i(t) \left(\nabla^c_{d_t} (\sigma_i \circ c|_J)\right)_t \\ & = \textstyle\sum_i (\gamma^i)'(t) \sigma_i(c(t)) + \sum_i \gamma^i(t) \left(\nabla_{d_t c} \sigma_i\right)_{c(t)} \\ & = \textstyle\sum_i (\gamma^i)'(t) \sigma_i(c(t)) + \sum_{ij} \gamma^i(t) \left(\mathcal{A}_{c(t)}(d_t c)\right)^j_i \sigma_j(c(t)), \forall t \in J, \end{aligned} \tag{18}$$

where $\mathcal{A}$ is the connection 1-form of ∇ in (U, φ). Finally,

$$(\gamma^i)'(t) + \sum_j \left(\mathcal{A}_{c(t)}(d_t c)\right)^i_j \gamma^j(t) = 0, \forall i \in \{1, \ldots, r\}, t \in J. \tag{19}$$

This linear differential equation with initial condition $\gamma^i(t_0) = p^i, \forall i \in \{1, \ldots, r\}$, where p^i are the components of p, has a unique solution in J. These unique local solutions define a unique global section $\gamma \in \mathrm{Sec}(c^* E)$, such that $\gamma(t_0) = p$ and $\nabla^c \gamma = 0$. ∎

Some remarks:

1. Eq. (19) is the local form of parallelism condition $\nabla^c\gamma=0$.

2. Observe that the aforementioned parallel transport T_c along c induced by ∇ is linear.

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[1] This theorem can be extended to larger sets of curves.

3. The question how the parallel transport T_c depends on the considered curve or loop c leads to the notion of holonomy. It will not be examined here.

4. Let p be an arbitrary element of E. Since the fiber $E_{\pi(p)}$ through p is the preimage of the point $\pi(p)$ by the subimmersion $\pi : E \to M$, it is an embedded submanifold of E and its tangent space $T_pE_{\pi(p)}$ at p is the preimage of 0 by $\pi_{*p} : T_pE \to T_{\pi(p)}M$, i.e. the kernel $\ker \pi_{*p}$. So $T_pE_{\pi(p)}$ is a subspace of T_pE. This space

$$\mathcal{V}_p = T_pE_{\pi(p)} = \ker \pi_{*p}$$

is called the vertical subspace of T_pE. Its vectors are the vertical vectors at p.

5. In order to define horizontal vectors at p, we need a connection or (better) a covariant derivative ∇ on E. First observe that any curve $\gamma : I \to E$ of E can be viewed as the lift of its projection. Indeed, if $c = \pi \circ \gamma : I \to M$, we have $\gamma(t) \in E_{c(t)}$ for any $t \in I$, so that $\gamma \in \mathrm{Sec}(c^*E)$. A curve $\gamma : I \to E$, i.e. a section $\gamma \in \mathrm{Sec}(c^*E)$ with $c = \pi \circ \gamma$, is said to be horizontal if c is an immersion and if $\nabla^c\gamma = 0$. A tangent vector in T_pE is called horizontal if it vanishes or is tangent to a horizontal curve of E through p. The space of horizontal vectors at p is denoted by $\mathcal{H}_p$. These spaces are actually supplementary to the corresponding vertical space $\mathcal{V}_p$. This distribution of subspaces $\mathcal{H}_p$ $(p \in E)$, such that

$$T_pE = \mathcal{V}_p \oplus \mathcal{H}_p,$$

induced by the covariant derivative ∇, is called an Ehresmann connection. In principal bundle theory, a connection on a principal bundle P will be defined as a smooth distribution

$$\mathcal{H} : P \ni p \to \mathcal{H}_p \subset T_pP,$$

such that subspace $\mathcal{H}_p$ is supplementary with subspace $\mathcal{V}_p$ and such that some invariance condition holds. Of course there is no canonical (Ehresmann) connection.

Exercise 15. We just understood that the 'integration' of a connection provides a parallel transport. In other words, a connection is the infinitesimal counterpart of a parallel transport. Explain how a parallel transport, i.e. the existence, for each curve c of M, of a family of maps $\tau(c)_t^{t'} : E_{c(t)} \to E_{c(t')}$ (that verify the usual compatibility conditions), allows to define a covariant derivative at $c(0)$ in the direction of the initial tangent vector of c of a section of E over c.

5 Principal bundles

5.1 Definition and examples

When working in Mechanics or Physics, say in the Euclidean space $\mathbb{R}^3$, we often encounter physical quantities that are characterized in any basis $u = (u_i)$ by a triplet

$f = (f^i)$ of real numbers. Such a quantity can be viewed as a vector if and only if the triplets f and f' obtained in different bases u and u' respectively, verify the vector law

$$f^i = s^i_j f'^j, \tag{20}$$

where $s \in \mathrm{GL}(3, \mathbb{R})$ is the transition matrix of the base transformation $u \to u'$. In other words, matrix s is defined by

$$u'_i = s^j_i u_j. \tag{21}$$

These equations (20) and (21) can be written in the compact form $u' = us$ and $f' = s^{-1}f$ respectively. Hence a vector can be viewed as the class

$$\langle u, f\rangle = \{(us, s^{-1}f) : s \in \mathrm{GL}(3, \mathbb{R})\},$$

where u is a basis and f a triplet of coordinates.

If we confine ourselves to orthonormal bases or positively oriented orthonormal bases, the emerging Lie group reduces to O(3) or SO(3). This structure group characterizes the considered geometry.

In principal bundle theory, we construct the vectors of the geometry defined by the chosen Lie group, in the just described way, over each point of the base manifold.

In the main, a principal bundle is a fiber bundle P, the typical fiber of which is a Lie group G that acts on P from the right.

The action of $s \in G$ on $u \in P$ will often be denoted $u.s$. We think of the fiber P_x over a point x of the base manifold, as the submanifold of bases over x and of G as the group of transition matrices. The vectors over each point x will be constructed later.

Definition 10. *Let P and M be two manifolds, $\pi : P \to M$ a smooth surjective map, and G a Lie group. The manifold P is a principal bundle over the base manifold M, with projection π and structure group G, if and only if P is endowed with a right G-action, such that*

1. *at each point $u \in P$, the G-orbit of u coincides with the fiber of P trough u, i.e.*

$$u.G = \pi^{-1}(\pi(u)), \tag{22}$$

2. *manifold P has G-compatible local trivializations, i.e. for any $x \in M$, there is a neighborhood U in M, and a diffeomorphism*

$$\varphi : \pi^{-1}(U) \to U \times G, \tag{23}$$

such that for any $u \in \pi^{-1}(U)$ and any $s \in G$,

$$\varphi(u) = (\pi(u), \phi(u)) \tag{24}$$

and

$$\phi(u.s) = (\phi(u))s. \tag{25}$$

Remarks:

1. The above defined principal bundle will be denoted by (P, M, G, π) or simply by $P(M, G)$.
2. It is understood that the right G-action is a differentiable action of Lie group G on the manifold P.
3. Condition (22) states that the fiber over any point is made up by "the accepted bases over this point" and compatibility condition (25) requires that ϕ intertwines the G-action and the G-multiplication.
4. It is easily checked that condition (25) can be rewritten in the form

$$\varphi^{-1}(x, ss') = \left(\varphi^{-1}(x, s)\right).s',$$

for all $x \in U$ and all $s, s' \in G$.

5. The preceding definition entails that the G-action is free and regular. Indeed, if $u.s = u.s'$, $u \in P$, $s, s' \in G$, it follows from the compatibility that $s = s'$. As for regularity, simply remark that there is a canonical 1-to-1 correspondence $\Pi : P/G \to M$ between the orbit-space and the base manifold. Now endow P/G with the differential structure that makes Π a diffeomorphism. It is clear that the fractorization map $\Pi^{-1} \circ \pi : P \to P/G$ is then a submersion.

The following useful proposition is the inverse of the preceding remark.

Proposition 8. *If a Lie group G acts freely and regularly on a manifold P, the manifold can be endowed with a principal bundle structure over the base manifold P/G, with structure group G, and canonical projection $\pi : P \to P/G$.*

Corollary 1. *A closed subgroup H of a Lie group G turns G into a principal bundle $(G, G/H, H, \pi)$, where π is the canonical projection.*

The right action of H on G is of course nothing but the action $j : (s, t) \ni G \times H \to st \in G$, which is free and regular.

We now depict the "frame bundle", the prototype of principal bundles.

Let M be an n-dimensional manifold. For any $x \in M$, we denote by $F_x(M)$ or simply by F_x, the set of frames (bases) of T_xM, or better, the set of ordered pairs (x, u), where u is a frame of T_xM. The "frame bundle" is the disjoint union $F(M)$, or F for short, of the F_x: $F = \cup_{x \in M} F_x$.

Proposition 9. *The set $F(M)$ can be endowed with a principal bundle structure over M, with structure group $G = \mathrm{GL}(n, \mathbb{R})$, $n = \dim M$, and canonical projection $\pi : F(M) \to M$.*

The reader might have observed that up till now the set F has no differential structure. This remark leads to the strategy of the proof. We will define canonical local bijections $\varphi : \pi^{-1}(U) \to U \times G$, transport the product manifold structure of $U \times G$ to $\pi^{-1}(U)$, thus making φ a diffeomorphism, then we glue these local differential structures together and get a global structure on F.

Proof. Let $(U, x^1, \ldots, x^n)$ be a chart of M and denote by $(\partial_{x^1}, \ldots, \partial_{x^n})$ the corresponding basis of T_xM, $x \in U$. It is obvious that the map

$$\varphi_U : \pi^{-1}(U) \ni (x, u) \to (x, s) \in U \times G,$$

where s is the transition matrix of the base transformation $\partial_{x^j} \to u_j$, is bijective. If we transport the differential structure of $U \times G$ as explained, this correspondence becomes a diffeomorphism. Moreover, it is clear that condition (24) is satisfied.

We must now check that the differential structures of $\pi^{-1}(U)$ and $\pi^{-1}(V)$ constructed by means of two bijections φ_U and φ_V, associated with two coordinate systems $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ in two overlapping open subsets U and V of M, coincide over $U \cap V$. More precisely, if φ_{VU} is the restriction of φ_U to the open subset $\pi^{-1}(U \cap V)$ of $\pi^{-1}(U)$, we have to prove that transition bijection

$$\psi_{VU} = \varphi_{UV} \circ \varphi_{VU}^{-1} : (U \cap V) \times G \to (U \cap V) \times G$$

is a diffeomorphism. This is not obvious. Indeed,

$$\varphi_{VU} \in \mathrm{Diff}\left(\left(\pi^{-1}(U \cap V)\right)_U, (U \cap V) \times G\right)$$

and

$$\varphi_{UV} \in \mathrm{Diff}\left(\left(\pi^{-1}(U \cap V)\right)_V, (U \cap V) \times G\right),$$

where we denote by $(\pi^{-1}(U \cap V))_U$ the manifold structure induced by $\pi^{-1}(U)$. However, if ψ_{VU} is a diffeomorphism, then

$$\mathrm{id} \in \mathrm{Diff}\left(\left(\pi^{-1}(U \cap V)\right)_U, \left(\pi^{-1}(U \cap V)\right)_V\right),$$

so that the differential structures of $\pi^{-1}(U)$ and $\pi^{-1}(V)$ actually coincide over $U \cap V$.

Let us at present focus on smoothness of, for instance, ψ_{VU}. For any $(x, s) \in (U \cap V) \times G$, we have $\psi_{VU}(x, s) = \varphi_{UV}\left(\varphi_{VU}^{-1}(x, s)\right) = \varphi_{UV}(x, u) = (x, s')$, where u and s' are defined by $u_j = s^i{}_j \partial_{x^i}$ and $u_j = s'^k{}_j \partial_{y^k}$ respectively. Since $\partial_{x^i} = \partial_{x^i} y^k \partial_{y^k}$, we finally get $s'^k{}_j = \partial_{x^i} y^k s^i{}_j$, so that ψ_{VU} is smooth.

We still have to define the right G-action on F. Let $(x, u) \in F$ and $s \in G$. Choose a chart $(U, x^1, \ldots, x^n)$ of M around x, denote by φ the associated trivialization of F over U, and remember that $\phi_x \in \mathrm{Diff}(F_x, G)$. We set

$$(x, u).s = \phi_x^{-1}\left((\phi_x(x, u))\, s\right),$$

so that conditions (22) and (25) are satisfied. If $\phi_x(x, u) = s'$ and $\phi_x^{-1}\left((\phi_x(x, u))\, s\right) = u'$, one has

$$u'_j = (s's)^i{}_j \partial_{x^i} = s^k{}_j s'^i{}_k \partial_{x^i} = s^k{}_j u_k.$$

Hence the preceding definition is independent of the considered chart. Furthermore, the defined right action is really a Lie group action on F. ∎

If we substitute $G = \mathrm{SO}(n)$ to $G = \mathrm{GL}(n, \mathbb{R})$, we get the principal bundle of orthonormal frames over M, which is important in General Relativity.

5.2 Triviality and Classification

We know that in a vector bundle of rank r, the choice of a trivialization is equivalent with the choice of a local frame, i.e. of r local sections that locally induce a frame in each fiber. There is a similar result for principal bundles. Here a trivialization implements a local section and vice versa. Indeed, if (P, M, G, π) is a principal bundle and if φ is a trivialization over U, we define a section σ over U by pulling the unit element e of G back to the fibers. In other words, we set, for any $x \in U$,

$$\sigma(x) = \phi_x^{-1}(e) = \varphi^{-1}(x, e).$$

Conversely, if a local section $\sigma \in \mathrm{Sec}(P_U)$ is given, we obtain a trivialization when setting, for any $x \in U$ and any $s \in G$,

$$\varphi^{-1}(x, s) = \varphi^{-1}(x, e).s = \sigma(x).s.$$

Alternatively, we may set, for $x \in U$ and $u \in P_x$,

$$\phi_x(u) = \phi_x(\sigma(x).\mathfrak{s}) = (\phi_x(\sigma(x)))\,\mathfrak{s} = \mathfrak{s},$$

where $\mathfrak{s}$ denotes the unique element in G such that $u = \sigma(x).\mathfrak{s}$. More generally, let $\mathcal{G}(G)$ be the graph

$$\mathcal{G}(G) = \{(u, u') \in P \times P : \exists s \in G : u' = u.s\}$$

of the equivalence relation induced by G, let $\mathfrak{s}(u, u')$ ($(u, u') \in \mathcal{G}(G)$) be the unique element of G that connects u with u' (i.e. $u' = u.\mathfrak{s}(u, u')$), and consider the map

$$\mathfrak{s} : \mathcal{G}(G) \ni (u, u') \to \mathfrak{s}(u, u') \in G.$$

In the following it will be interesting to remember that this map is actually smooth. Indeed, since G acts regularly, the graph $\mathcal{G}(G)$ is a closed embedded submanifold of $P \times P$ and if φ is a trivialization, the local form of $\mathfrak{s}$ is $(u, u') \to (\phi(u))^{-1}\,\phi(u')$.

Let us also mention that if σ and σ' are two local sections of P over U, i.e two trivializations or—in the terminology of Physics—two local gauges, we have

$$\sigma'(x) = \sigma(x).\mathfrak{t}(x), \forall x \in U,$$

where $\mathfrak{t} \in C^\infty(U, G)$ is nothing but the map defined by $\mathfrak{t}(x) = \mathfrak{s}(\sigma(x), \sigma'(x)), x \in U$. Conversely, if we consider a map $\mathfrak{t} \in C^\infty(U, G)$ and a local gauge $\sigma \in \mathrm{Sec}(P_U)$, then σ' defined by $\sigma'(x) = \sigma(x).\mathfrak{t}(x), \forall x \in U$ is another local gauge. The smooth mapping

$$\mathfrak{t} : U \to G$$

is the local gauge transformation.

The next proposition is evident.

Proposition 10. *A principal bundle is trivial if and only if it admits a global section.*

Let us now come back to transition functions, which encode—as we know—the entire information about the way to glue the local pieces of the bundle together.

Take a fiber bundle (E, M, π) with typical fiber F and a family $(\varphi_\alpha)_{\alpha \in \mathfrak{A}}$ of trivializations of E over an open cover $(U_\alpha)_{\alpha \in \mathfrak{A}}$ of M. The transition functions

$$\psi_{\alpha\beta} = \varphi_{\beta\alpha} \circ \varphi_{\alpha\beta}^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F,$$

where $U_{\alpha\beta}$ is a non-empty intersection $U_\alpha \cap U_\beta$ and where the other notations are those used above, are diffeomorphisms and verify the compatibility condition

$$\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}.$$

We have already mentioned that for a vector bundle of rank r over a field $\mathbb{K}$, we write

$$\psi_{\alpha\beta}(x, f) = (x, (\theta_{\alpha\beta}(x))(f)), \forall x \in U_{\alpha\beta}, \forall f \in \mathbb{K}^r$$

and think about transition functions as a family of mappings

$$\theta_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(r, \mathbb{K})$$

that are smooth and satisfy the compatibility condition

$$\theta_{\alpha\beta}(x)\theta_{\beta\gamma}(x) = \theta_{\alpha\gamma}(x), \forall x \in U_{\alpha\beta\gamma}.$$

In principal bundle theory the viewpoint is similar. Let M be a manifold, G a Lie group, and $(U_\alpha)_{\alpha \in \mathfrak{A}}$ an open cover of M. Consider a principal bundle (P, M, G, π) with base manifold M and structure group G, which is trivial over $(U_\alpha)_{\alpha \in \mathfrak{A}}$. Choose now a family $(\varphi_\alpha)_{\alpha \in \mathfrak{A}}$ of trivializations. If we write once more

$$\psi_{\alpha\beta}(x, s) = (x, (\theta_{\alpha\beta}(x))(s)), \forall x \in U_{\alpha\beta}, \forall s \in G,$$

we define diffeomorphisms $\theta_{\alpha\beta}(x)$ of the structure group G. We denote by $(\sigma_\alpha)_{\alpha \in \mathfrak{A}}$ the family of local sections $\sigma_\alpha(x) = \varphi_\alpha^{-1}(x, e)$, $x \in U_\alpha$ induced by the trivializations, and write $\mathfrak{s}_{\alpha\beta}(x)$ instead of $\mathfrak{s}(\sigma_\alpha(x), \sigma_\beta(x))$, $x \in U_{\alpha\beta}$. Then

$$\begin{aligned}(\theta_{\alpha\beta}(x))(s) = \; & \phi_\alpha\left(\varphi_\beta^{-1}(x, s)\right) = \phi_\alpha(\sigma_\beta(x).s) \\ & = (\phi_\alpha(\sigma_\beta(x)))\, s = (\phi_\alpha(\sigma_\alpha(x).\mathfrak{s}_{\alpha\beta}(x)))\, s = \mathfrak{s}_{\alpha\beta}(x)s,\end{aligned}$$

so that $\theta_{\alpha\beta}(x)$ nothing but left multiplication $\gamma_{\mathfrak{s}_{\alpha\beta}(x)}$ by $\mathfrak{s}_{\alpha\beta}(x)$. Hence we can view transition functions as a family of mappings

$$\mathfrak{s}_{\alpha\beta} : U_{\alpha\beta} \to G$$

that are smooth in view of smoothness of $\mathfrak{s} : \mathcal{G}(G) \to G$ and that verify once more the compatibility condition

$$\mathfrak{s}_{\alpha\beta}(x)\mathfrak{s}_{\beta\gamma}(x) = \mathfrak{s}_{\alpha\gamma}(x), \forall x \in U_{\alpha\beta\gamma}. \tag{26}$$

This family is called a cocycle of transition functions over $(U_\alpha)_{\alpha\in\mathfrak{A}}$.

We now look for the transformation law of this cocycle induced by a change of the chosen trivializations over the family $(U_\alpha)_{\alpha\in\mathfrak{A}}$, which is fixed. So we consider a new family of local gauges $(\sigma'_\alpha)_{\alpha\in\mathfrak{A}}$ and get a new cocycle $\mathfrak{s}'_{\alpha\beta} : U_{\alpha\beta} \to G$. Denoting the local gauge transformations $\mathfrak{t}_\alpha$, we have

$$\mathfrak{s}'_{\alpha\beta}(x) = \mathfrak{s}(\sigma'_\alpha(x), \sigma'_\beta(x)) = \mathfrak{s}(\sigma_\alpha(x).\mathfrak{t}_\alpha(x), \sigma_\beta(x).\mathfrak{t}_\beta(x))$$

and

$$\mathfrak{s}_{\alpha\beta}(x) = \mathfrak{s}(\sigma_\alpha(x), \sigma_\beta(x)),$$

for all $x \in U_{\alpha\beta}$. So

$$\sigma_\alpha(x).\left(\mathfrak{s}_{\alpha\beta}(x)\mathfrak{t}_\beta(x)\right) = \sigma_\beta(x).\mathfrak{t}_\beta(x) = \sigma_\alpha(x).(\mathfrak{t}_\alpha(x)\mathfrak{s}'_{\alpha\beta}(x)),$$

and since the action is free, we get

$$\mathfrak{s}'_{\alpha\beta}(x) = (\mathfrak{t}_\alpha(x))^{-1}\,\mathfrak{s}_{\alpha\beta}(x)\mathfrak{t}_\beta(x), \forall x \in U_{\alpha\beta}.$$

Finally

$$\mathfrak{s}'_{\alpha\beta} = \mathfrak{t}_\alpha^{-1}\mathfrak{s}_{\alpha\beta}\mathfrak{t}_\beta. \tag{27}$$

We say that the cocycles $\mathfrak{s}'_{\alpha\beta}$ and $\mathfrak{s}_{\alpha\beta}$ differ by a coboundary and are thus cohomologous. The cohomology classes of cocycles over the open cover $(U_\alpha)_{\alpha\in\mathfrak{A}}$ form a set $\check{H}^1((U_\alpha), G)$, the first Čech cohomology set of the cover $(U_\alpha)_{\alpha\in\mathfrak{A}}$ with coefficients in the sheaf $C^\infty(\cdot, G)$. This cohomological terminology is due to the similarity of Eq. (26) and Eq. (27) with the "skew-symmetric version" of the definition of the first Čech cohomology group of M with values in the sheaf $C^\infty(\cdot, G)$, where G is Abelian.

Note that to any principal bundle (P, M, G, π) with fixed base manifold M and fixed structure group G, which is trivial over a fixed open cover $(U_\alpha)_{\alpha\in\mathfrak{A}}$ of M, we have associated a unique class in $\check{H}^1((U_\alpha), G)$. The converse result also holds.

Proposition 11. *Let M be a manifold, G a Lie group, and $(U_\alpha)_{\alpha\in\mathfrak{A}}$ an open cover of M. For any cocycle over $(U_\alpha)_{\alpha\in\mathfrak{A}}$, i.e. any family $\mathfrak{s}_{\alpha\beta} : U_{\alpha\beta} \to G$ of smooth maps that satisfy Eq. (26), there is a unique principal bundle (P, M, G, π) over M with structure group G, which is trivial over $(U_\alpha)_{\alpha\in\mathfrak{A}}$ and such that the associated cocycle is the family $\mathfrak{s}_{\alpha\beta}$. Moreover, the principal bundle constructed from a cohomologous cocycle is isomorphic with (P, M, G, π).*

The proof of this proposition is not very stimulating and will not be given.

The preceding result states that the principal bundle associated with a given cohomology class is unique up to isomorphism. So let us give some information about homomorphisms and isomorphisms of principal bundles.

Definition 11. *If (P, M, G, π) and (P', M', G', π') are two principal bundles, a morphism from (P, M, G, π) into (P', M', G', π') is a triplet (F, f, g) of smooth maps*

$$F \in C^\infty(P, P'),\ f \in C^\infty(M, M') \text{ and } h \in C^\infty(G, G'),$$

such that

$$\pi'(F(u)) = f(\pi(u)), \forall u \in P \tag{28}$$

and

$$F(u.s) = F(u).h(s), \forall u \in P, \forall s \in G. \tag{29}$$

The reader might miss the condition that h is a group homomorphism. This requirement is actually redundant, the property immediately follows from Eq. (29). It is even superfluous to assume existence of the map f. Indeed, if $F \in C^\infty(P, P')$ and $h \in C^\infty(G, G')$ verify $F(u.s) = F(u).h(s), \forall u \in P, \forall s \in G$, the image $F(u.G)$ of the orbit of u is a subset of the orbit of $F(u)$. So there is a (unique) map $f : M \to M'$ such that $\pi'(F(u)) = f(\pi(u))$, for all $u \in P$. It can be seen that this map is smooth.

Definition 12. *A principal bundle morphism (F, f, h) is a principal bundle isomorphism, if f and h are diffeomorphisms.*

Indeed, the assumptions entail that F is also a diffeomorphism. As $F(u.s) = F(u).h(s)$ for all $u \in P$ and all $s \in G$, we see that $F \in \text{Diff}(u.G, F(u).G')$ for any $u \in P$ and understand that $F \in \text{Diff}(P, P')$.

5.3 Associated bundle

Remember that the fibers of a principal bundle are viewed as the submanifolds of bases over the corresponding points and that the structure group can be interpreted as the group of transition matrices.

We are now ready to construct vectors over each base point. In order to achieve this goal, we need bases, i.e. a principal bundle (P, M, G, π), and components, i.e. a manifold F (e.g. $F = \mathbb{R}^n$). Since for $u \in P$ and $f \in F$, the vector with representative (u, f) will be defined—see above—as the class

$$\langle u, f\rangle = \{(u.s, s^{-1}.f) : s \in G\},$$

we also need a left G-action on F.

So consider a principal bundle (P, M, G, π) and a manifold F endowed with a differentiable left action of Lie group G (if necessary, we denote the left action on F by $\mathfrak{l} : G \times F \to F$, whereas the right action on P is denoted by $\mathfrak{r} : P \times G \to P$). Then

$$(P \times F) \times G \ni ((u, f), s) \to (u, f).s = (u.s, s^{-1}.f) \in P \times F$$

is a differentiable and a free right action of the Lie group G on the product manifold $P \times F$. We denote by E the orbit space $(P \times F)/G$, i.e.

$$E = (P \times F)/G = \{\langle u, f\rangle : u \in P, f \in F\}$$

is the "space" of "vectors", and we denote by π_F the canonical projection

$$\pi_F : P \times F \ni (u, f) \to \langle u, f\rangle \in E.$$

Note that the first elements, the "bases", of all the representatives $(u.s, s^{-1}.f)$, $s \in G$ of a vector $\langle u, f\rangle$, are members of the same fiber $P_{\pi(u)}$. So there is a (unique) map

$$\pi_E : E \ni \langle u, f\rangle \to \pi(u) \in M,$$

i.e. a (unique) map π_E such that the diagram

$$\begin{array}{ccc} P & \overset{\mathrm{pr}_1}{\longleftarrow} & P \times F \\ \downarrow \pi & & \downarrow \quad \pi_F \\ M & \overset{\pi_E}{\longleftarrow} & E \end{array}$$

is commutative.

We say that $E_x = \pi_E^{-1}(x)$, $x \in M$ is the space of vectors over x. Take now a basis $u \in P_x$ over x and any vector $\langle v, g\rangle \in E_x$ over x. Then there is a unique representative of $\langle v, g\rangle$ whose first element is u. This representative is $(v.s, s^{-1}.g)$, with $s = \mathfrak{s}(v, u)$. So, for any point x in the base manifold, any basis u over x induces a map

$$u^* : E_x \ni \langle v, g\rangle \to u^*\langle v, g\rangle = \mathfrak{s}(u, v).g \in F.$$

It is obvious that map u^* is a 1-to-1 correspondence. We think about this map as the "isomorphism" that, given a basis, associates to each vector its "components" in this basis.

We feel that E should be a vector bundle over M with typical fiber F and that, for any $u \in P$, the component-map u^* should be a vector space isomorphism. The following theorem gives an answer to this question.

Theorem 9. *In the situation summarized by the above diagram,*

1. *the set E can be endowed with a fiber bundle structure over M, with projection π_E and typical fiber F,*

2. *this bundle $E(P, M, G, F)$ is a vector bundle and the component-maps u^*, $u \in P$, are linear isomorphisms, if F is a vector space and the left G-action on F is a representation of the Lie group G on vector space F,*

3. *the manifold $P \times F$ can be endowed with a principal bundle structure over E, with structure group G and projection π_F.*

We say that bundle E is the fiber bundle associated with the principal bundle (P, M, G, π) (and the manifold F). It is often denoted $E(P, M, G, F)$. We also write $E = P \times_G F$ and think of bundle E as the manifold $P \times F$ twisted by G.

Proof.

1. Let U be an open subset of M over which P is trivial and denote by σ the corresponding section. In order to define a 1-to-1 correspondence between $\pi_E^{-1}(U)$ and $U \times F$, we set

$$\varphi_\sigma : \pi_E^{-1}(U) \ni \langle u, f\rangle \to (x, \sigma(x)^* \langle u, f\rangle) \in U \times F,$$

where $x = \pi(u)$. This map is really a bijection since it admits an inverse,

$$\varphi_\sigma^{-1} : U \times F \ni (x, f) \to \langle \sigma(x), f\rangle \in \pi_E^{-1}(U).$$

Let us for instance verify that $\varphi_\sigma^{-1} \circ \varphi_\sigma = \mathrm{id}$. We have

$$\varphi_\sigma^{-1} (\varphi_\sigma \langle u, f\rangle) = \varphi_\sigma^{-1} (x, \sigma(x)^* \langle u, f\rangle) = \langle \sigma(x), \sigma(x)^* \langle u, f\rangle\rangle = \langle u, f\rangle.$$

It is now possible proceed exactly as in the proof of Prop. 9 and to define a fiber bundle structure on E. We leave this construction to the reader.

2. Our objective is to define on any fiber E_x, $x \in M$ a vector space structure isomorphic to that of F. Choose as before a local section σ of P over a neighborhood U of x. Then

$$(\phi_\sigma)_x : E_x \ni \langle u, f\rangle \to \sigma(x)^* \langle u, f\rangle \in F \tag{30}$$

is a bijection. We endow E_x with the with the linear structure that turns $(\phi_\sigma)_x$ into an isomorphism. Once again we have to verify that the structures obtained from two different sections σ and σ' of P over U coincide. This means that we must check that

$$(\phi_{\sigma'})_x \circ (\phi_\sigma)_x^{-1} : F \to F$$

is an isomorphism. Since

$$(\phi_{\sigma'})_x \left((\phi_\sigma)_x^{-1}(f)\right) = \sigma'(x)^* \langle \sigma(x), f\rangle = \mathfrak{s}(\sigma'(x), \sigma(x)).f,$$

map $(\phi_{\sigma'})_x \circ (\phi_\sigma)_x^{-1}$ is the left action by $\mathfrak{s}(\sigma'(x), \sigma(x)) \in G$ and is thus an isomorphism of F, if the action is a representation of Lie group G on vector space F. Note that structure transport (30) entails that $\sigma^*(x)$ and even u^*, $u \in P$, are linear isomorphisms.

3. Since the Lie group G acts freely on the manifold $P \times F$, the conclusion follows, if we prove that this G-action is regular. Hence it suffices to check that $\pi_F : P \times F \to E$ is a submersion. This problem being a local one, we can simply examine the restriction $\pi_F = \varphi_\sigma^{-1} \circ (\varphi_\sigma \circ \pi_F)$, where σ is a section of P over an open subset U of M. For any $(u, f) \in \pi^{-1}(U) \times F$, we have

$$\pi_F(u, f) = \varphi_\sigma^{-1}(x, g),$$

where

$$x = \pi(u) \text{ and } g = \sigma(x)^* \langle u, f \rangle = \mathfrak{s}(\sigma(x), u).f = \mathfrak{l}_{\mathfrak{s}(\sigma(x),u)}(f).$$

So

$$\pi_{F*(u,f)} = \varphi^{-1}_{\sigma*(x,g)} \circ \begin{pmatrix} \pi_{*u} & 0 \\ \bullet & \mathfrak{l}_{\mathfrak{s}(\sigma(x),u)*f} \end{pmatrix}.$$

Since $\varphi^{-1}_{\sigma*(x,g)}$ and $\mathfrak{l}_{\mathfrak{s}(\sigma(x),u)*f}$ are bijective and π_{*u} is surjective, we conclude that $\pi_{F*(u,f)}$ is also surjective. ∎

Exercise. Complete the first point of the preceding proof.

Remarks:

1. All the fiber bundles commonly associated with a manifold M, e.g. TM, T^*M, $\otimes^p_q TM$, ... can be constructed as special cases of fiber bundles associated with a principal bundle. This remark allows to guess that principal bundle theory will turn out to be a powerful, global and unifying language. Let us briefly explain how the tangent bundle can be recovered in the principal bundle framework. Consider the frame bundle $(F(M), M, \mathrm{GL}(n, \mathbb{R}), \pi)$, $n = \dim M$. The Lie group $\mathrm{GL}(n, \mathbb{R})$ has a canonical linear representation on $\mathbb{R}^n$. It is clear that the vector bundle $E(F(M), M, \mathrm{GL}(n, \mathbb{R}), \mathbb{R}^n)$ associated with the preceding principal bundle, is diffeomorphic with the tangent bundle of M: $E = F(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \simeq TM$.

2. Consider a principal bundle (P, M, G, π) and a closed subgroup H of G.

   This embedded Lie subgroup has a canonical differentiable—in the following this property will be understood—right action (resp. left action) on G and the quotient space $G/_\delta H$ (resp. $G/_\gamma H$) is a quotient manifold. Moreover, G acts from the left (resp. right) on $G/_\delta H$ (resp. $G/_\gamma H$). Below we omit subscript δ, since only the first quotient manifold will be considered. We denote E the fiber bundle $E(P, M, G, G/H)$ associated with the principal bundle (P, M, G, π).

   Furthermore, the Lie subgroup H of G obviously acts freely on P from the right. We will explain that this action is also regular and that $E = P \times_G G/H$ is diffeomorphic to P/H, so that P can be endowed with a principal bundle structure over $E \simeq P/H$, with structure group H.

Here the sketch of the proof. The map

$$\mathcal{I} : E = P \times_G G/H \ni \langle u, sH \rangle \to (u.s).H \in P/H$$

is well-defined and bijective. We endow P/H with the differential structure that turns $\mathcal{I}$ into a diffeomorphism. It then suffices to show that $\pi' : P \to P/H$ is a submersion.

5.4 Reduced bundle

Let us first define the appropriate notion of subbundle of a principal bundle. It is natural to say that a principal bundle $P'(M', G')$ is a subbundle of another principal bundle $P(M, G)$, if P' and M' are submanifolds of P and M respectively, if G' is a Lie subgroup of G and if the triplet (i_1, i_2, i_3), where $i_1 : P' \to P$, $i_2 : M' \to M$, and $i_3 : G' \to G$ are the canonical injections, is a principal bundle morphism.

The results of this section require a more general definition.

Definition 13. *Let $P'(M', G')$ and $P(M, G)$ be two principal bundles. If (F, f, h) is a principal bundle homomorphism from $P'(M', G')$ into $P(M, G)$ and if the smooth maps F, f, g are injective immersions, we identify $P' \simeq F(P')$, $M' \simeq f(M')$, and $G' \simeq h(G')$ and say that $P'(M', G')$ is a subbundle of $P(M, G)$. If $M' = M$ and $f = \mathrm{id}$, the subbundle $P'(M, G')$ is called a reduced bundle. Given a principal bundle $P(M, G)$ and a Lie subgroup G' of G, we say that the structure group G is reducible to G', if there is a reduced bundle $P'(M, G')$.*

In order to understand this definition, note that if we endow $f(M')$ with the manifold structure that makes $\tilde{f} : M' \ni x \to f(x) \in f(M')$ a diffeomorphism, the injection $i = f \circ \tilde{f}^{-1} : f(M') \to M$ is an immersion. Hence, if we identify $M' \simeq f(M')$, then M' becomes a submanifold of M. Similarly, $P' \simeq F(P')$ can be viewed as a submanifold of P and $G' \simeq h(G')$ as a Lie subgroup of G. Moreover, $(i_1, i_2, i_3) = (F, f, h)$ is a principal bundle morphism.

Proposition 12. *The structure group G of a principal bundle $P(M, G)$ is reducible to a Lie subgroup G' of G, if and only if there is an open cover $(U_\alpha)_{\alpha \in \mathfrak{A}}$ of M such that $P(M, G)$ is trivial and the corresponding cocycle takes its values in G'.*

In this section, only sketches of proofs will be given.

Sketch of the proof. Let $P'(M, G')$ be a reduced bundle and let $(\sigma'_\alpha)_{\alpha \in \mathfrak{A}}$ be a family of local sections of P' over an open cover $(U_\alpha)_{\alpha \in \mathfrak{A}}$ of M. The corresponding cocycle $\mathfrak{s}_{\alpha\beta}$ is of course valued in G'. It is visible that these sections are also local sections of $P(M, G)$ and that the associated cocycle is nothing but $\mathfrak{s}_{\alpha\beta}$.

Conversely, if there is a G'-valued cocycle of $P(M, G)$ over a covering of M, the cocycle mappings are smooth as maps valued in G, but also as maps whose target manifold is G'. So we can construct a principal fiber bundle $P'(M, G')$. This bundle is a reduced bundle of $P(M,G)$. ∎

In the next result we consider the situation depicted in Remark 2 of the last section.

Proposition 13. *Let H be a closed subgroup of the structure group G of a principal bundle $P(M,G)$. The group G is reducible to H if and only if the associated fiber bundle $E(P,M,G,G/H)$ admits a global section $\sigma : M \to E \simeq P/H$.*

Sketch of the proof. If G is reducible to H, there is an open covering $(U_\alpha)_{\alpha\in\mathfrak{A}}$ of M, matching local sections $(\sigma_\alpha)_{\alpha\in\mathfrak{A}}$ of P, and a corresponding cocycle $\mathfrak{s}_{\alpha\beta}$ valued in H. So, for any $x \in U_{\alpha\beta}$, we have $\sigma_\beta(x) = \sigma_\alpha(x).\mathfrak{s}_{\alpha\beta}(x)$, with $\mathfrak{s}_{\alpha\beta}(x) \in H$. Hence

$$\sigma_\beta(x).H = \sigma_\alpha(x).H$$

and σ defined for $x \in U_\alpha$ by $\sigma(x) = \sigma_\alpha(x).H \in P/H$ is well-defined on M. Since in view of Remark 2 of the preceding section, $\sigma_\alpha(x).H \simeq \langle\sigma_\alpha(x), H\rangle$, we have $\pi_E(\sigma(x)) = x$, so that σ is a global section of $E \simeq P/H$.

Conversely, let

$$\sigma : M \to E \simeq P/H$$

be a global section. The objective is to construct a cover of M such that P is trivial and the corresponding cocycle is valued in H. Hence we consider the principal bundle $(P, P/H, H, \pi')$, a covering $(U'_\alpha)_{\alpha\in\mathfrak{A}}$ of P/H, and matching local sections

$$\sigma'_\alpha : U'_\alpha \to P$$

of P. Then we set $U_\alpha = \sigma^{-1}(U'_\alpha) \subset M$ and

$$\sigma_\alpha : U_\alpha \ni x \to \sigma'_\alpha(\sigma(x)) \in P.$$

It is clear that $\sigma'_\alpha(\sigma(x))$ and $\sigma'_\beta(\sigma(x))$, $x \in U_{\alpha\beta}$, "differ" by an element of H. ∎

Proposition 14. *Let $F(M)$ be the frame bundle over an n-dimensional manifold M. Any reduction of the structure group* $\mathrm{GL}(n,\mathbb{R})$ *of $F(M)$ to its closed subgroup $O(n)$ of orthogonal matrices, provides a Riemannian metric on M and vice versa.*

If g is a Riemannian structure on M, the corresponding reduced bundle with structure group $O(n)$ is the bundle of orthonormal frames over M. We observe that a reduction of the structure group means an enrichment of the considered geometric structure.

Sketch of the proof. Existence of a reduced bundle entails existence of a global section

$$\sigma : M \ni x \to \{(x, u.s) : s \in O(n)\} \in F(M)/O(n),$$

where u is a basis of T_xM. The image $\sigma(x)$ can be identified with the unique inner product g_x on T_xM such that the bases $u.s$, $s \in O(n)$, are orthonormal. Hence a Riemannian metric g on M. ∎

6 Connections on principal bundles

6.1 Vertical distribution, horizontal distribution

Consider a right action $\mathfrak{r}$ of a Lie group G on a manifold P. If $s \in G$ and $u \in P$, we set $u.s = \mathfrak{r}(u, s)$, $\mathfrak{r}_u = \mathfrak{r}(u, \cdot)$ and $\mathfrak{r}_s = \mathfrak{r}(\cdot, s)$. Remember that the definition of such an action is actually a substitute for the requirement "$\mathfrak{r} : G \to \mathrm{Diff}(P)$ is a Lie group homomorphism". It is well-known that this Lie group action of G on P induces a Lie algebra action of $\mathfrak{g} = \mathrm{Lie}(G)$ on P, i.e. a Lie algebra homomorphism

$$\mathfrak{r}_* : \mathfrak{g} \ni h \to X^h \in \mathcal{X}(P).$$

Remember here that the algebra $\mathcal{X}(M)$ of vector fields of M is the Lie algebra of the infinite-dimensional Lie group $\mathrm{Diff}(M)$. Let us also recall that the fundamental vector field X^h of P is defined for any $u \in P$ by

$$(\mathfrak{r}_* h)_u = X^h_u = \mathrm{d}_t \left(u.\exp(th)\right)|_{t=0} = \mathfrak{r}_{u*} h.$$

Observe that $\exp(th)$ induces a 1-parameter group $\mathfrak{r}_{\exp(th)}$ of diffeomorphisms of P, and that X^h is nothing but the complete vector field of P whose flow is

$$\phi^h_t(u) = u.\exp(th).$$

The following property of X^h will turn out to be of importance.

Proposition 15. *If $\mathfrak{r} : P \times G \to P$ is a right action of a Lie group G on a manifold M, we have, for any $s \in G$, $u \in P$, and $h \in \mathfrak{g}$,*

$$\mathfrak{r}_{s*u} X^h_u = X^{\mathrm{Ad}(s^{-1})h}_{u.s},$$

where Ad *is the adjoint action of G.*

We often simply write

$$\mathfrak{r}_{s*} X^h = X^{\mathrm{Ad}(s^{-1})h}$$

and say that the "exterior action by s coincides with the interior action by s^{-1}".

Proof. Let c_s, $s \in G$, be the conjugation by s. For any $s, S \in G$ and $u \in P$, we have

$$u.(s^{-1}Ss) = \left(\mathfrak{r}_u \circ c_{s^{-1}}\right)(S) \text{ and } u.(s^{-1}Ss) = ((u.s^{-1}).S).s = \left(\mathfrak{r}_s \circ \mathfrak{r}_{u.s^{-1}}\right)(S).$$

It suffices now to compute the derivative of this equality at $S = e$. ∎

Proposition 16. *Let $P(M, G)$ be a principal bundle with right action $\mathfrak{r}$. For any $u \in P$, the linear map $\mathfrak{r}_{u*} : \mathfrak{g} \to T_uP$ is an isomorphism from $\mathfrak{g}$ onto $T_u(u.G)$, i.e.*

$$\mathfrak{r}_{u*} \in \mathrm{Isom}(\mathfrak{g}, T_u(u.G)).$$

Proof. Since the orbit $u.G$ is an embedded submanifold of P, the map $\mathfrak{r}_u : G \to u.G$ is smooth and $\mathfrak{r}_{u*} : \mathfrak{g} \to T_u(u.G)$ is linear. Assume now that $\mathfrak{r}_{u*}h = 0$, for some $h \in \mathfrak{g}$. Then $X^h_u = 0$ and $\gamma : \mathbb{R} \ni t \to u \in P$ is an integral curve of X^h with initial value u. So we have $u = u.\exp(th)$, for all $t \in \mathbb{R}$, and $\exp(th) = e$, for all $t \in \mathbb{R}$, since the action is free. This entails that $h = 0$ and that $\mathfrak{r}_{u*}$ is injective. Hence it suffices to show that the dimensions of $\mathfrak{g}$ and $T_u(u.G)$ coincide. This however is clear, since in any fiber bundle any fiber is diffeomorphic with the typical fiber. ∎

Take now a principal bundle $P(M, G)$ and let $u \in P$. We know that $u.G = \pi^{-1}(\pi(u))$ is a closed embedded submanifold of P and that the tangent space at u to this fiber,

$$V_u := \{X^h_u : h \in \mathfrak{g}\} = \mathfrak{r}_{u*}(\mathfrak{g}) = T_u(u.G) = T_u(\pi^{-1}(\pi(u))) = \ker \pi_{*u},$$

is a linear subspace of T_uP, which is called the vertical subspace at u. A vector field $X \in \mathcal{X}(P)$ is said to be vertical, if $X_u \in V_u$, for any $u \in P$. The space of vertical vector fields of P will be denoted by $\mathcal{V}(P)$.

Remark 2. *The just defined distribution,*

$$V : P \ni u \to V_u \subset T_uP,$$

is smooth, i.e. for any $u \in P$, the space V_u is spanned by $\{X_u : X \in \mathcal{V}(P)\}$, and it is G-invariant, i.e. for any $s \in G$, $u \in P$,

$$\mathfrak{r}_{s*u}V_u = V_{u.s}.$$

Exercise. Prove the statements of the preceding remark.

This distribution is called the vertical distribution and the isomorphism

$$\mathfrak{r}^{-1}_{u*} : V_u \ni \nu \to \tilde{\nu} \in \mathfrak{g},$$

see Prop. 16, will often be called the vertical isomorphism. We already mentioned that there is no canonical supplementary distribution. Hence the following definition.

Definition 14. *A connection on a principal bundle $P(M, G)$ is a smooth distribution*

$$H : P \ni u \to H_u \subset T_uP,$$

such that

$$T_uP = H_u \oplus V_u, \forall u \in P$$

and

$$\mathfrak{r}_{s*u}H_u = H_{u.s}, \forall s \in G, \forall u \in P.$$

Remarks:

1. A connection in the sense of the preceding definition is an Ehresmann connection.

2. The linear subspace H_u of T_uP, $u \in P$, is called the horizontal subspace at u and the distribution H is the horizontal distribution. So a connection H on a principal bundle $P(M,G)$ provides a unique decomposition of any tangent vector $\tau \in T_uP$ into a horizontal component and a vertical component,

$$\tau = \mathrm{p_H}\,\tau + \mathrm{p_V}\,\tau = \eta + \nu,$$

where $\mathrm{p_H}\,\tau = \eta \in H_u$ and $\mathrm{p_V}\,\tau = \nu \in V_u$. Moreover, any vector field $X \in \mathcal{X}(P)$ that satisfies $X_u \in H_u$, for all $u \in P$, is a horizontal vector field. The space of horizontal vector fields of P is denoted by $\mathcal{H}(P)$. Any vector field $X \in \mathcal{X}(P)$ can be uniquely decomposed into a sum

$$X = \mathrm{p_H}\,X + \mathrm{p_V}\,X$$

of a horizontal vector field $\mathrm{p_H}\,X \in \mathcal{H}(P)$ and a vertical vector field $\mathrm{p_V}\,X \in \mathcal{V}(P)$. Smoothness of the projections $\mathrm{p_H}\,X$ and $\mathrm{p_V}\,X$ is a consequence of smoothness of the horizontal distribution. It can even be proven that distribution H is smooth if and only if the projections $\mathrm{p_H}\,X$ and $\mathrm{p_V}\,X$ of any (smooth) vector field $X \in \mathcal{X}(P)$ are smooth (vector fields).

3. The last requirement in the definition, the G-invariance, means that all the horizontal subspaces of a given fiber can be constructed out of one of them. This condition is of importance also in connection with parallel transport.

4. Since $\pi_{*u} : T_uP \to T_xM$, $u \in P$, $x = \pi(u)$, is a surjective linear map, and as $T_uP = H_u \oplus V_u$, the isomorphism $\tilde{\pi}_{*u} : T_uP/\ker \pi_{*u} \to \mathrm{im}\,\pi_{*u}$ can be viewed as an isomorphism

$$\tilde{\pi}_{*u} : H_u \to T_xM$$

from H_u onto T_xM. This isomorphism will be called the horizontal isomorphism.

6.2 Connection 1-form

We know that a connection on a vector bundle of rank r is characterized, in any trivialization over some open subset U, by a connection 1-form

$$\mathcal{A} \in \mathrm{Sec}(T^*U \otimes (U \times \mathrm{gl}(r, \mathbb{R}))).$$

Similarly an Ehresmann connection on a principal bundle $P(M,G)$ is characterized by a connection 1-form

$$\omega \in \mathrm{Sec}(T^*P \otimes (P \times \mathfrak{g})),$$

and more precisely by a $\mathfrak{g}$-valued differential 1-form on P that verifies some requirements.

Indeed, let $P(M,G)$ be a principal bundle and choose a connection H on P. In order to show that H defines such a 1-form, we have to identify a smooth assignment

$$\omega : P \ni u \to \omega_u \in T_u^*P \otimes \mathfrak{g} \simeq \mathcal{L}(T_uP, \mathfrak{g}).$$

In view of the vertical isomorphism $\mathfrak{r}_{u*}^{-1} : V_u \ni \nu \to \tilde{\nu} \in \mathfrak{g}$, $u \in P$, it suffices to set

$$\omega_u(\tau) = \tilde{\nu} = \mathfrak{r}_{u*}^{-1}\left(\mathrm{p_V}\,\tau\right),$$

for any $\tau \in T_uP$. In other words, $\omega_u(\tau)$ is the unique element in $\mathfrak{g}$, such that $\mathfrak{r}_{u*}(\omega_u(\tau)) = X_u^{\omega_u(\tau)}$ coincides with $\nu = \mathrm{p_V}\,\tau$. It can be shown that ω is actually smooth. The differential 1-form ω is called the connection 1-form of H. Note that the dependence of ω upon H is via $\mathrm{p_V}$. The tight link between ω and H is obvious, since for any $u \in P$ and any $\tau \in T_uP$, we have $\omega_u(\tau) = 0 \iff \tau \in H_u$, i.e.

$$H_u = \ker \omega_u.$$

Proposition 17. *The connection* 1*-form* ω *of a connection on a principal bundle* $P(M,G)$ *has the following properties:*

1. *For any* $h \in \mathfrak{g}$*, the function* $\omega(X^h) \in C^\infty(P,\mathfrak{g})$ *has constantly the value* h*, i.e.*

$$\left(\omega(X^h)\right)_u = h, \forall u \in P.$$

2. *For any* $s \in G$*,*

$$\mathfrak{r}_s^*\omega = \mathrm{Ad}(s^{-1})\omega.$$

Conversely, given a $\mathfrak{g}$*-valued differential* 1*-form* ω *on* P*, which satisfies these conditions 1 and 2, there is a unique connection on* $P(M,G)$ *whose connection* 1*-form is* ω*.*

Let us first explain Condition 2. Note that the pullback by $\mathfrak{r}_s : P \to P$ acts on the "form part" of ω, whereas $\mathrm{Ad}(s^{-1}) \in \mathrm{GL}(\mathfrak{g})$ acts on the "Lie algebra part" in ω. More precisely, we define $\mathfrak{r}_s^*\omega$ by

$$(\mathfrak{r}_s^*\omega)(X) = \omega(\mathfrak{r}_{s*}X) \circ \mathfrak{r}_s,$$

which is the usual pullback definition, and we define $\mathrm{Ad}(s^{-1})\,\omega$ by

$$(\mathrm{Ad}(s^{-1})\,\omega)(X) = \mathrm{Ad}(s^{-1})\,(\omega(X)),$$

where $X \in \mathcal{X}(P)$. For $\omega = \alpha \otimes h$, $\alpha \in \Omega^1(P)$ and $h \in \mathfrak{g}$, we then actually find

$$\mathfrak{r}_s^*(\alpha \otimes h) = (\mathfrak{r}_s^*\alpha) \otimes h$$

and

$$\mathrm{Ad}(s^{-1})(\alpha \otimes h) = \alpha \otimes \mathrm{Ad}(s^{-1})\,h.$$

Condition 2 of the preceding proposition now reads

$$(\mathfrak{r}^* \otimes \mathrm{Ad})(s)(\alpha \otimes h) := (\mathfrak{r}_s^* \alpha) \otimes (\mathrm{Ad}(s)\, h) = \alpha \otimes h, \forall s \in G,$$

so it means that $\omega = \alpha \otimes h$ is invariant under the product action $\mathfrak{r}^* \otimes \mathrm{Ad}$ of G.

Proof of Proposition 17. Property 1 is obvious. Indeed, since $X^h \in \mathcal{V}(P)$, for every $h \in \mathfrak{g}$, we have $\omega_u(X_u^h) = \mathfrak{r}_{u*}^{-1} X_u^h = h$. Property 2 is a consequence of the G-invariance of fundamental vector fields,

$$\mathfrak{r}_{s*} X^h = X^{\mathrm{Ad}(s^{-1})\, h},$$

where $s \in G$. Property 2,

$$\mathfrak{r}_s^* \omega = \mathrm{Ad}(s^{-1})\, \omega,$$

reads

$$\omega_{u.s}(\mathfrak{r}_{s*u} \tau) = \mathrm{Ad}(s^{-1})\, \omega_u(\tau),$$

where $u \in P$ and $\tau \in T_u P$, or equivalently,

$$\mathrm{p_V}(\mathfrak{r}_{s*u} \tau) = X_{u.s}^{\mathrm{Ad}(s^{-1})\, \omega_u(\tau)} = \mathfrak{r}_{s*u} X_u^{\omega_u(\tau)}.$$

Since $\mathfrak{r}_{s*}$ commutes—as easily checked—with the projectors $\mathrm{p_H}$ and $\mathrm{p_V}$, this last condition is satisfied.

Conversely, let ω be a $\mathfrak{g}$-valued differential 1-form on P that verifies Conditions 1 and 2. If there is a connection H whose form is ω, we necessarily have $H_u = \ker \omega_u$, $u \in P$. It is easily seen that $T_u P = H_u \oplus V_u$. Moreover, G-invariance of the horizontal distribution is a consequence of the G-invariance of ω and smoothness a consequence of smoothness of ω. ∎

Exercise. Complete the preceding proof.

6.3 Maurer-Cartan forms, local connection forms

Let us first recall that a left-invariant form on a Lie group G is a differential form ϖ on G that verifies $\gamma_s^* \varpi = \varpi$, for all $s \in G$. Of course γ_s is nothing but left multiplication by s. We know that the Lie algebra $\mathcal{X}_{\mathrm{inv}}(G)$ of left-invariant vector fields on G is isomorphic with the Lie algebra $\mathfrak{g}$ of G. The isomorphism is

$$\mathfrak{g} \ni h \to H \in \mathcal{X}_{\mathrm{inv}}(G), H_s = \gamma_{s*} h, h = H_e.$$

Similarly the space $\Omega^1_{\mathrm{inv}}(G)$ of left-invariant 1-forms on G is isomorphic with the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Here the isomorphism is given by

$$\mathfrak{g}^* \ni \theta \to \Theta \in \Omega^1_{\mathrm{inv}}(G), \Theta_s = {}^t\gamma_{s^{-1}*} \theta, \theta = \Theta_e.$$

If (e_i) denotes a basis of $\mathfrak{g}$ and (ϵ^i) is the dual basis in $\mathfrak{g}^*$, then (E_i) and $(\mathcal{E}^i)$, defined by

$$E_{i,s} = \gamma_{s*} e_i \in T_s G$$

and

$$\mathcal{E}^i_s = {}^t\gamma_{s^{-1}*}\epsilon^i \in T^*_s G,$$

are bases of $\mathcal{X}_{\mathrm{inv}}(G)$ and $\Omega^1_{\mathrm{inv}}(G)$ respectively. They are also globally defined bases of the $C^\infty(G)$-modules $\mathcal{X}(G)$ and $\Omega^1(G)$.

A Maurer-Cartan 1-form on a Lie group G is a $\mathfrak{g}$-valued left-invariant differential 1-form on G. The space $\Omega^1_{\mathrm{inv}}(G) \otimes \mathfrak{g}$ of these forms is of course isomorphic with the space $\mathfrak{g}^* \otimes \mathfrak{g} \simeq \mathrm{End}(\mathfrak{g})$ of endomorphisms of $\mathfrak{g}$. The canonical Maurer-Cartan 1-form on G is the Maurer-Cartan 1-form

$$\Theta_0 = \sum_i \mathcal{E}^i \otimes e_i$$

associated with the identity map $\sum_i \epsilon^i \otimes e_i = \mathrm{id}_{\mathfrak{g}}$. So for any $X \in \mathcal{X}(G)$ and any $s \in G$, we have

$$\Theta_0(X)(s) = \sum_i \epsilon^i(\gamma_{s^{-1}*}X_s)e_i = \gamma_{s^{-1}*}X_s. \tag{31}$$

Exercise. Prove that the canonical Maurer-Cartan 1-form satisfies the equation

$$d\Theta_0 + \frac{1}{2}[\![\Theta_0, \Theta_0]\!] = 0.$$

Consider now a principal bundle $P(M,G)$. In Physics, the base manifold M is often a model for a physical space or space-time or for a phase space or phase space-time of a general mechanical system. So the base manifold is frequently the physically relevant object. It is therefore crucial to examine how abstract geometric objects defined on the total space P can be projected onto M and to understand the link of these projections with concrete physical situations.

Let H be a connection on P characterized by its connection 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$. The best way to "project ω onto M" is to pull it back by sections of P. Since—in general—there is no global section, we choose an open covering of M, say $(U_\alpha)_{\alpha \in \mathfrak{A}}$, such that P is trivial over each U_α, and use a family $(\sigma_\alpha)_{\alpha \in \mathfrak{A}}$ of corresponding sections. In order to simplify the vocabulary, such a cover, together with matching trivializations of P, will be called a bundle atlas of P. Set now

$$\mathcal{A}_\alpha = \sigma_\alpha^* \omega \in \Omega^1(U_\alpha) \otimes \mathfrak{g}.$$

The form $\mathcal{A}_\alpha$ is the local form of the connection 1-form ω in the considered trivialization. Since the objects that appear in Physics should be the local connection 1-forms $\mathcal{A}_\alpha$, we look for a compatibility condition on which a family of such forms is induced by a unique connection 1-form on P.

Proposition 18. *Let $P(M,G)$ be a principal bundle, ω a connection 1-form on P, and $(U_\alpha, \sigma_\alpha)_{\alpha \in \mathfrak{A}}$ a bundle atlas. The associated local connection 1-forms $\mathcal{A}_\alpha$ verify*

$$\mathcal{A}_\beta = \mathrm{Ad}(\mathfrak{s}^{-1}_{\alpha\beta})\mathcal{A}_\alpha + \Theta_{\alpha\beta}, \tag{32}$$

where $\mathfrak{s}_{\alpha\beta}$ is the corresponding cocycle and where $\Theta_{\alpha\beta} = \mathfrak{s}^*_{\alpha\beta}\Theta_0$ is the pullback of the canonical Maurer-Cartan 1-form. Conversely, any family $\mathcal{A}_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$, $\alpha \in \mathfrak{A}$, which verifies compatibility condition (32), is implemented by a unique connection 1-form on P, i.e. there is a unique $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, such that $\mathcal{A}_\alpha = \sigma^*_\alpha \omega$, for any $\alpha \in \mathfrak{A}$.

Proof. Since we look for the relationship between $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, where $\mathcal{A}_{\alpha,x}(v) = \omega_{\sigma_\alpha(x)}(\sigma_{\alpha*x}v)$, $x \in U_\alpha$, $v \in T_xM$, it is clear that we differentiate the equation

$$\sigma_\beta(x) = \sigma_\alpha(x).\,\mathfrak{s}_{\alpha\beta}(x) = \mathfrak{r}(\sigma_\alpha(x), \mathfrak{s}_{\alpha\beta}(x)),$$

$x \in U_{\alpha\beta}$. Using the differentiation theorem of compound mappings, we get

$$\sigma_{\beta*x}v = \mathfrak{r}_{\mathfrak{s}_{\alpha\beta}(x)*\sigma_\alpha(x)}(\sigma_{\alpha*x}v) + \mathfrak{r}_{\sigma_\alpha(x)*\mathfrak{s}_{\alpha\beta}(x)}(\mathfrak{s}_{\alpha\beta*x}v),$$

where $v \in T_xM$. So the value of $\omega_{\sigma_\beta(x)}$ on $\sigma_{\beta*}v$ is

$$\mathcal{A}_{\beta,x}(v) = \omega_{\sigma_\beta(x)}(\sigma_{\beta*}v) = \omega_{\sigma_\beta(x)}(\mathfrak{r}_{\mathfrak{s}_{\alpha\beta}(x)*}(\sigma_{\alpha*}v)) + \omega_{\sigma_\beta(x)}(\mathfrak{r}_{\sigma_\alpha(x)*}(\mathfrak{s}_{\alpha\beta*}v)).$$

If we apply the G-invariance $\omega(\mathfrak{r}_{s*}X) = \mathrm{Ad}(s^{-1})(\omega(X))$ (we have omitted composition of the LHS with $\mathfrak{r}_s$), we see that the first term of the RHS reads

$$\mathrm{Ad}(s^{-1}_{\alpha\beta}(x))(\mathcal{A}_{\alpha,x}(v)),$$

which is part of the announced result. As for the second term, we need some preparation.

Let us first mention that, in view of Eq. (31),

$$\Theta_0(H) = h,$$

and remind that

$$\omega(X^h) = h,$$

for any $h \in \mathfrak{g}$. But, as $\mathfrak{r}_u \circ \gamma_s = \mathfrak{r}_{u.s}$, $u \in P, s \in G$, the left-invariant vector field H and the fundamental field X^h are $\mathfrak{r}_u$-related, i.e.

$$\mathfrak{r}_{u*}H_s = X^h_{u.s}.$$

In order to write the second term

$$\omega_{\sigma_\beta(x)}(\mathfrak{r}_{\sigma_\alpha(x)*\mathfrak{s}_{\alpha\beta}(x)}(\mathfrak{s}_{\alpha\beta*}v))$$

of the RHS of the above equation, in the form "$\omega(\mathfrak{r}_{u*}H_s)$", we consider the left-invariant vector field H that verifies

$$H_{\mathfrak{s}_{\alpha\beta}(x)} = \gamma_{\mathfrak{s}_{\alpha\beta}(x)*}h = \mathfrak{s}_{\alpha\beta*}v \in T_{\mathfrak{s}_{\alpha\beta}(x)}G.$$

The second term now reads

$$\omega_{\sigma_\beta(x)}\left(\mathfrak{r}_{\sigma_\alpha(x)*}\left(H_{\mathfrak{s}_{\alpha\beta}(x)}\right)\right) = \omega_{\sigma_\beta(x)}\left(X^h_{\sigma_\beta(x)}\right) = h$$

$$= \Theta_{0,\mathfrak{s}_{\alpha\beta}(x)}\left(H_{\mathfrak{s}_{\alpha\beta}(x)}\right) = (\mathfrak{s}^*_{\alpha\beta}\Theta_0)_x(v) = \Theta_{\alpha\beta,x}(v).$$

Hence the first part of the preceding proposition. We do not give the proof of the second part. ∎

Equation (32) can be written in a more user-friendly form, if G is a matrix group, say $G = \mathrm{GL}(n,\mathbb{R})$. It is well-known that the adjoint representation of G, i.e.

$$\mathrm{Ad}(s)h = \text{“}shs^{-1}\text{”} = \gamma_{s*}\delta_{s^{-1}*}h = c_{s*}h$$

(where $\delta_{s^{-1}}$ is the right multiplication by s^{-1} and c_s the inner automorphism $c_s : G \ni S \to sSs^{-1} \in G$), then actually reads

$$\mathrm{Ad}(s)h = shs^{-1},$$

where $s \in G = \mathrm{GL}(n,\mathbb{R})$, $h \in \mathfrak{g} = \mathrm{gl}(n,\mathbb{R})$, and where the multiplication in the RHS is the matrix multiplication. This is easily checked. Note that $\mathfrak{c} : \mathbb{R} \ni t \to \exp(th) \in G$ is a curve in G, such that $\mathfrak{c}(0) = e$ and $d_t\mathfrak{c}|_{t=0} = \mathfrak{c}_{*0}(1) = h$. Hence,

$$\mathrm{Ad}(s)h = c_{s*e}h = (c_s \circ \mathfrak{c})_{*0}(1) = d_t(c_s \circ \mathfrak{c})|_{t=0} = shs^{-1},$$

since

$$c_s \circ \mathfrak{c} : \mathbb{R} \ni t \to s\exp(th)s^{-1} \in G.$$

Furthermore, using the just recalled usual technique, we see that for $G = \mathrm{GL}(n,\mathbb{R})$,

$$\gamma_{s*S} : \mathfrak{g} \ni h \to sh \in \mathfrak{g},$$

for all $s, S \in G$. So, if $x \in U_{\alpha\beta}$ and $v \in T_xM$, we get

$$\Theta_{\alpha\beta,x}(v) = \Theta_{0,\mathfrak{s}_{\alpha\beta}(x)}\left(\mathfrak{s}_{\alpha\beta*x}v\right) = \gamma_{\mathfrak{s}^{-1}_{\alpha\beta}(x)*}\left((\mathrm{d}\,\mathfrak{s}_{\alpha\beta})_x\, v\right) = \mathfrak{s}^{-1}_{\alpha\beta}(x)\left((\mathrm{d}\,\mathfrak{s}_{\alpha\beta})_x\, v\right).$$

Hence the following result:

Remark 3. *If G is a matrix group, the compatibility condition (32) reads*

$$\mathcal{A}_\beta = \mathfrak{s}^{-1}_{\alpha\beta}\mathcal{A}_\alpha\mathfrak{s}_{\alpha\beta} + \mathfrak{s}^{-1}_{\alpha\beta}\,\mathrm{d}\,\mathfrak{s}_{\alpha\beta}.$$

In the terminology of Physics, each local gauge $\sigma \in \mathrm{Sec}(P_U)$ gives rise to a local connection 1-form $\mathcal{A} = \sigma^*\omega$ and a local gauge transformation $\sigma' = \sigma.\mathfrak{t}$, $\mathfrak{t} : U \to G$ induces a change of the local connection 1-form given by

$$\mathcal{A}' = \mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}. \tag{33}$$

6.4 Gauge theory (first part)

6.4.1 Electromagnetic tensor

Consider the electromagnetic field in the vacuum. With respect to any inertial observer $\mathcal{R}$, Maxwell's equations read

$$\operatorname{div}\vec{B} = 0, \ \ \operatorname{curl}\vec{E} = -\partial_t\vec{B}, \tag{34}$$

$$\operatorname{div}\vec{E} = 0, \ \ \operatorname{curl}\vec{B} = \frac{1}{c^2}\partial_t\vec{E}, \tag{35}$$

where c is the celerity of light and where the electric and magnetic fields $\vec{E}$ and $\vec{B}$ depend on the coordinates (x, y, z) and the time t of the observer. The first two equations are equivalent with

$$\vec{B} = \operatorname{curl}\vec{A}, \ \ \vec{E} = -\operatorname{grad}\phi - \partial_t\vec{A},$$

where $\vec{A}$ and ϕ are the vector and scalar potentials. It is well-known that the chosen inertial observer can be viewed as a frame in Minkowski's space. Set

$$\mathcal{A}_1 = A_1, \mathcal{A}_2 = A_2, \mathcal{A}_3 = A_3, \mathcal{A}_4 = \frac{i}{c}\phi,$$

where A_1, A_2, A_3 are the components of $\vec{A}$ and where $i = \sqrt{-1}$ appears, since we use the coordinates $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$, which allow to view Minkowski's space as a Euclidean space. Physicists assume that the quadruplets $(\mathcal{A}_\lambda)$ and $(\mathcal{A}'_\lambda)$, associated with different inertial observers $\mathcal{R}$ and $\mathcal{R}'$, verify the "vector law" $\mathcal{A}_\lambda = \Lambda_{\lambda\mu}\mathcal{A}'_\mu$ in Minkowski's space, where Λ is the Lorentz matrix. Hence, these quadruplets define a unique four-vector field, the electromagnetic potential. The curl of this four-vector

$$\mathcal{F}_{\lambda\mu} = \partial_\lambda\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\lambda,$$

where $\partial_\lambda = \partial_{x_\lambda}$, is of course assumed to be a skew-symmetric four-tensor field of order 2, called the electromagnetic tensor. It is easily checked that

$$\mathcal{F}_{\lambda\mu} = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix},$$

where E_1, E_2, E_3 (resp. B_1, B_2, B_3) are the components of $\vec{E}$ (resp. $\vec{B}$). Furthermore, one immediately sees that Equations (34) and (35) read

$$\partial_\lambda\mathcal{F}_{\mu\nu} + \partial_\mu\mathcal{F}_{\nu\lambda} + \partial_\nu\mathcal{F}_{\lambda\mu} = 0 \tag{36}$$

and

$$\partial_\lambda\mathcal{F}_{\lambda\mu} = 0 \tag{37}$$

respectively.

The assumptions above, regarding the nature of the electromagnetic potential and the electromagnetic tensor, are in accordance with the principle of Special Relativity, since they entail that Maxwell's equations are tensorial in Minkowski's space, so have the same form for each inertial observer. Moreover, these hypotheses imply specific transformation laws for $\vec{A}$, ϕ, $\vec{E}$, and $\vec{B}$, which are in conformity with experience.

Observe finally that Maxwell's equations are invariant under a gauge transformation

$$\mathcal{A}'_\lambda = \mathcal{A}_\lambda + \partial_\lambda \psi, \tag{38}$$

where ψ is a four-scalar field, since the electromagnetic tensor is invariant.

6.4.2 Geometric framework

Let us now look at Electrodynamics from the geometric viewpoint. Consider a physical space-time, i.e. a pseudo-Riemannian manifold M of dimension 4. Let $(x^1, \ldots, x^4)$ be a system of local coordinates in an open subset U of M. Remark that the assumed pseudo-Riemannian character entails that the fourth coordinate is real here. So the components $\mathcal{A}_\lambda$ of the electromagnetic potential define a 1-form

$$\mathcal{A} = i\mathcal{A}_\lambda dx^\lambda \in \Omega^1(U) \otimes \mathfrak{g},$$

where $\mathfrak{g} = u(1) = i\mathbb{R}$ is the Lie algebra of the unitary group $G = U(1) = S^1$ of dimension 1. Choose now a gauge transformation in the geometric sense, i.e. a map $\mathfrak{t} : U \ni x \to \mathfrak{t}(x) = e^{i\psi(x)} \in G$, where $\psi : U \to \mathbb{R}$. As the involved group is Abelian, compatibility condition (33) reduces here to

$$\mathcal{A}' = \mathcal{A} + i\,\mathrm{d}\,\psi,$$

i.e. to

$$\mathcal{A}'_\lambda = \mathcal{A}_\lambda + \partial_\lambda \psi.$$

Comparing with Eq. (38), we see that the electromagnetic potentials give rise to a connection 1-form ω on a $U(1)$-bundle.

Note that gauge transformation $\mathfrak{t}$ is valued in an Abelian group. In the fifties Yang and Mills studied non-Abelian gauge transformations that are of basic importance in Elementary Particle Physics. Their gauge transformation is

$$\mathcal{A}' = \mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t},$$

i.e. nothing but Eq. (33).

We thus understand that the appropriate geometric framework for gauge theories is the theory of connections on principle bundles. We will define the curvature of a connection on a principle bundle and see that the electromagnetic tensor can be viewed as the curvature of the above-mentioned connection ω. A similar remark holds for the Yang-Mills field strength. Moreover, the geometric part (34) of Maxwell's equations can be recovered from general results on connections on principal bundles. The dynamical part (35) can be obtained from Maxwell's action.

6.5 Horizontal lift

We consider a principal bundle $P(M,G)$ endowed with a connection H.

A horizontal lift of a vector field $X \in \mathcal{X}(M)$ of the base manifold, is a horizontal vector field $X^* \in \mathcal{H}(P)$ of the total space, which projects onto X, i.e. satisfies

$$\pi_{*u}X^*_u = X_{\pi(u)}, \forall u \in P.$$

This condition of course means that the fields X^* and X are π-related. Since $\pi_{*u} \in \mathrm{Isom}(H_u, T_{\pi(u)}M)$, it is clear that the horizontal lift exists and is unique. Smoothness is a consequence of the smoothness of X. Moreover, this horizontal lift is G-invariant, i.e. verifies

$$\mathfrak{r}_{s*u}X^*_u = X^*_{u.s}, \forall s \in G, \forall u \in P.$$

This is of course equivalent with saying that X^* is $\mathfrak{r}_s$-related to itself for all $s \in G$. In order to prove this G-invariance, note that $X^*_{u.s}$ is the unique horizontal vector that projects onto $X_{\pi(u.s)}$. However, as H is G-invariant, i.e. as $\mathfrak{r}_{s*u}H_u = H_{u.s}$, vector $\mathfrak{r}_{s*u}X^*_u$ is horizontal. Furthermore, since $\pi \circ \mathfrak{r}_s = \pi$, this vector projects properly, i.e.

$$\pi_{*u.s}\left(\mathfrak{r}_{s*u}X^*_u\right) = \pi_{*u}X^*_u = X_{\pi(u)} = X_{\pi(u.s)}.$$

It is quite easy to see that—conversely—any G-invariant horizontal vector field of P is the lift of a vector field of M.

We claim that

$$* : \mathcal{X}(M) \ni X \to X^* \in \mathcal{X}(P)$$

is not a Lie algebra isomorphism. Indeed,

Proposition 19. *The map $*$ has the following properties:*

1. $$(X+Y)^* = X^* + Y^*, \forall X, Y \in \mathcal{X}(M),$$

2. $$(fX)^* = (f \circ \pi)X^*, \forall f \in C^\infty(M), \forall X \in \mathcal{X}(M),$$

3. $$[X,Y]^* = \mathrm{p_H}\,[X^*,Y^*], \forall X, Y \in \mathcal{X}(M).$$

Proof. The first two results are obvious. Just check that the RHS is horizontal and projects in the appropriate way. For the last property, we have to show that

$$\pi_{*u}\left(\mathrm{p_H}\,[X^*,Y^*]_u\right) = [X,Y]_{\pi(u)}.$$

First remember that $[X^*,Y^*]$ is π-related with $[X,Y]$, since X^* is π-related to X and Y^* to Y. As $V_u = \ker \pi_{*u}$, we then obtain

$$\pi_{*u}\left(\mathrm{p_H}\,[X^*,Y^*]_u\right) = \pi_{*u}\left([X^*,Y^*]_u\right) = [X,Y]_{\pi(u)}. \blacksquare$$

Below we define the curvature of a connection H on a principle bundle. We will see that the curvature measures the homomorphism deficiency of map $*$ and is tightly connected with integrability of the distribution H. Hence the importance of the next proposition.

Proposition 20. *Distribution H is involutive if and only if the map $*$ is a Lie algebra homomorphism.*

Proof. Necessity of this condition is a direct consequence of the third part of Prop. (19). For the converse result, note that if (X_i) is a frame of TM over an open subset U of M, then (X_i^*) is a frame of H over $\pi^{-1}(U)$. Indeed, $X_{i,u}^* = \pi_{*u}^{-1} X_{i,\pi(u)}$, $u \in \pi^{-1}(U)$ and $\pi_{*u} \in \mathrm{Isom}(H_u, T_{\pi(u)}M)$. Take now two horizontal vector fields $X, Y \in \mathcal{H}(P)$. In order to see that $[X,Y]_u$ is horizontal, we assume that $u \in \pi^{-1}(U)$ and write X and Y locally in the form $X = \sum_i X^i X_i^*$ and $Y = \sum_j Y^j X_j^*$, where X^i and Y^j are functions on $\pi^{-1}(U)$. It is then easily seen that $[X,Y]_u \in H_u$, as by assumption $[X_i^*, X_j^*] = [X_i, X_j]^*$. ∎

6.6 Exterior covariant derivative, curvature of a connection

Let us recall that a connection 1-form on a principal bundle $P(M,G)$ is a G-invariant and $\mathfrak{g}$-valued 1-form ω on P that satisfies $\omega(X^h) = h$, for each $h \in \mathfrak{g}$. Observe here that ω is not horizontal, i.e. does not vanish if evaluated on a vertical vector (ω is in fact vertical, since its value on any horizontal vector vanishes). The presence of the second term in the RHS of compatibility condition (32) for instance, is a consequence of the fact that ω is not horizontal.

We now extend the notion of G-invariant, $\mathfrak{g}$-valued differential form on the total space P of a principal bundle $P(M,G)$. Let (V, ϱ) be a linear representation of G. A V-valued differential k-form ϖ on P that is G-invariant in the sense that

$$\mathfrak{r}_s^* \varpi = \varrho(s^{-1})\varpi, \forall s \in G,$$

is called a pseudo-tensorial k-form of type (V, ϱ). If in addition ϖ is horizontal, i.e. $\varpi(X_1, \ldots, X_k) = 0$ if one at least of the vector fields $X_i \in \mathcal{X}(P)$ is vertical, we say that ϖ is tensorial.

Hence, in principal bundle theory, a connection is characterized by a pseudo-tensorial 1-form $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ of type $(\mathfrak{g}, \mathrm{Ad})$ (vector bundle framework: $\mathcal{A} \in \Omega^1(U) \otimes \mathrm{gl}(r, \mathbb{R})$). The curvature form of a connection will be a tensorial 2-form $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ of same type (vector bundle framework: $\mathcal{F} \in \Omega^2(U) \otimes \mathrm{gl}(r, \mathbb{R})$). To define this curvature form Ω, we need an exterior covariant derivative D that maps pseudo-tensorial k-forms into tensorial $(k+1)$-forms of the same type. We then set $\Omega = \mathrm{D}\,\omega$ (vector bundle framework: $R^\nabla = \mathrm{d}^\nabla \circ \nabla$).

Proposition 21. *Let $P(M,G)$ be a principal bundle endowed with a connection. If ϖ is a pseudo-tensorial k-form on P of type (V,ϱ), then $\mathrm{D}\,\varpi$ defined, for all $X_1,\ldots,X_{k+1}\in\mathcal{X}(P)$, by*

$$(\mathrm{D}\,\varpi)(X_1,\ldots,X_{k+1})=(\mathrm{d}\,\varpi)(\mathrm{p_H}\,X_1,\ldots,\mathrm{p_H}\,X_{k+1}),$$

is a tensorial $(k+1)$-form on P of type (V,ϱ).

The just defined operator D is called exterior covariant derivative.

Exercise. Prove the preceding statement.

Definition 15. *Consider a connection 1-form ω on a principal bundle $P(M,G)$. The curvature form of ω is the tensorial 2-form $\Omega\in\Omega^2(P)\otimes\mathfrak{g}$ of type $(\mathfrak{g},\mathrm{Ad})$, defined by $\Omega=\mathrm{D}\,\omega$.*

6.7 Cartan's structure equation

Cartan's structure equation known from vector bundle theory,

$$\mathcal{F}=\mathrm{d}\,\mathcal{A}+\frac{1}{2}[\![\mathcal{A},\mathcal{A}]\!]=\mathrm{d}\,\mathcal{A}+[\mathcal{A},\mathcal{A}]=\mathrm{d}\,\mathcal{A}+\mathcal{A}\wedge\mathcal{A},$$

is still valid in the principal bundle setting.

Theorem 10. *If Ω is the curvature 2-form of a connection 1-form on a principal bundle $P(M,G)$, we have*

$$\Omega=\mathrm{D}\,\omega=\mathrm{d}\,\omega+\frac{1}{2}[\![\omega,\omega]\!]=\mathrm{d}\,\omega+[\omega,\omega]. \tag{39}$$

Proof. We have to show that

$$(\mathrm{D}\,\omega)(\tau,\tau')=(\mathrm{d}\,\omega)(\tau,\tau')+[\omega(\tau),\omega(\tau')],$$

for any $\tau,\tau'\in T_uP$ and any $u\in P$. Since the LHS and the RHS of this equation are bilinear and skew-symmetric in τ,τ', and as $\tau=\mathrm{p_H}\,\tau+\mathrm{p_V}\,\tau$ and $\tau'=\mathrm{p_H}\,\tau'+\mathrm{p_V}\,\tau'$, it is sufficient to prove the equality in the following three cases.

- The vectors τ and τ' are horizontal. In this case the statement is obvious.
- The vectors τ and τ' are vertical. Then $\tau=X_u^h$ and $\tau'=X_u^{h'}$, for some $h,h'\in\mathfrak{g}$. It is clear that the LHS vanishes. The RHS is the value at u of

  $$X^h(\omega(X^{h'}))-X^{h'}(\omega(X^h))-\omega[X^h,X^{h'}]+[\omega(X^h),\omega(X^{h'})].$$

  The first two terms vanish and, since $[X^h,X^{h'}]=X^{[h,h']}$, the last two cancel out.

- Vector τ is vertical and vector τ' is horizontal. Here $\tau = X^h_u$, $h \in \mathfrak{g}$ and $\tau' = Y^*_u$, where Y is a vector field of M whose value at $\pi(u)$ is $\pi_*\tau'$. Hence the LHS and the second term of the RHS vanish. As for the first term of the RHS, it is easily seen that it vanishes, if the bracket $[X^h, Y^*]$ of a fundamental vector field and a horizontal vector field is horizontal. As the flow of X^h is $\phi_t = \mathfrak{r}_{\exp(th)}$, the value of $[X^h, Y]$, even for an arbitrary $Y \in \mathcal{H}(P)$, at any point $u \in P$ is given by

$$[X^h, Y]_u = \lim_0 \frac{1}{t}\left(\phi_{-t*}Y_{\phi_t(u)} - Y_u\right) = \lim_0 \frac{1}{t}\left(\mathfrak{r}_{\exp(-th)*}Y_{u.\exp(th)} - Y_u\right).$$

  The G-invariance of the horizontal distribution allows to conclude that this vector is a member of H_u. ∎

Proposition 22. *Let Ω be the curvature form of a connection form ω on a principal bundle $P(M, G)$. For any $X, Y \in \mathcal{H}(P)$, we have*

$$\Omega(X, Y) = -\omega[X, Y].$$

Proof. Consequence of the structure equation and the fact that ω is vertical. ∎

Observe that the last proposition entails that, for any vector fields $X, Y \in \mathcal{X}(M)$ and any $u \in P$, we have

$$\Omega(X^*, Y^*)|_u = -\mathfrak{r}_{u*}^{-1}\, \mathrm{p_V}[X^*, Y^*]_u = \mathfrak{r}_{u*}^{-1}\left([X, Y]^* - [X^*, Y^*]\right)_u,$$

where we have used the last statement of Prop. (19). Hence the curvature actually measures—as claimed above—the homomorphism deficiency of map $*$. In other words, the value $\Omega(X^*, Y^*)$ is at each point, up to an isomorphism, the vertical component of $[X^*, Y^*]$ at this point.

The next proposition clarifies the relationship between the curvature and integrability of the connection.

Proposition 23. *Let $P(M, G)$ be a principal bundle endowed with a connection H. We denote by Ω the corresponding curvature form. The connection H is flat, i.e. the curvature form Ω vanishes, if and only if the distribution H is involutive.*

Proof. Let ω be the connection 1-form of H. Remember first that for any $X \in \mathcal{X}(P)$, we have $\omega(X) = 0$ if and only if $X \in \mathcal{H}(P)$. Hence,

$$\Omega(X, Y) = 0 \Longleftrightarrow \omega[X, Y] = 0 \Longleftrightarrow [X, Y] \in \mathcal{H}(P),$$

for any $X, Y \in \mathcal{H}(P)$. As Ω vanishes on any vector fields if and only if it vanishes on horizontal vector fields, the conclusion follows. ∎

Exercise. Let (h_i) be a basis of $\mathfrak{g}$ and denote the structure constants of this Lie algebra by c^i_{jk}. Set $\omega = \sum_i \omega^i \otimes h_i$ and $\Omega = \sum_j \Omega^j \otimes h_j$. Show that the structure equation reads

$$\Omega^i = \mathrm{d}\,\omega^i + \frac{1}{2}\sum_{jk} c^i_{jk}\omega^j \wedge \omega^k$$

and prove Bianchi's identity

$$\mathrm{D}\,\Omega = 0.$$

Observe that $\mathrm{DD}\,\omega = 0$, but that $\mathrm{DD} \neq 0$.

6.8 Gauge theory (second part)

Take a principal bundle $P(M, G)$ endowed with a connection and denote the connection 1-form and the curvature 2-form by ω and Ω respectively. We know that each local gauge $\sigma \in \mathrm{Sec}(P_U)$, $U \subset M$, gives rise to a local connection 1-form $\mathcal{A} = \sigma^*\omega$ and that a local gauge transformation $\sigma' = \sigma.\mathfrak{t}$, $\mathfrak{t} : U \to G$, induces a change of the local connection 1-form given—if G is a matrix group—by

$$\mathcal{A}' = \mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}. \tag{40}$$

Similarly, the local form $\mathcal{F}$ in the gauge $\sigma \in \mathrm{Sec}(P_U)$ of the curvature 2-form is defined by

$$\mathcal{F} = \sigma^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}.$$

Since $\mathfrak{g}$ is here a Lie subalgebra of $\mathrm{gl}(n, \mathbb{R})$, the structure equation reads $\Omega = \mathrm{D}\,\omega = \mathrm{d}\,\omega + \omega \wedge \omega$. Hence,

$$\mathcal{F} = \mathrm{d}\,\sigma^*\omega + \sigma^*\omega \wedge \sigma^*\omega = \mathrm{d}\,\mathcal{A} + \mathcal{A} \wedge \mathcal{A}. \tag{41}$$

Remark 4. *If G is a matrix group, a local gauge transformation $\sigma' = \sigma.\mathfrak{t}$, $\mathfrak{t} : U \to G$ induces a transformation of the local curvature 2-form:*

$$\mathcal{F}' = \mathfrak{t}^{-1}\mathcal{F}\mathfrak{t},$$

where $\mathcal{F}$ and $\mathcal{F}'$ are the local curvature forms in the gauges σ and σ' respectively.

Proof. First observe that $0 = \mathrm{d}\,(\mathfrak{t}^{-1}\mathfrak{t}) = (\mathrm{d}\,\mathfrak{t}^{-1})\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}$, so that

$$\mathrm{d}\,\mathfrak{t}^{-1} = -\mathfrak{t}^{-1}\,(\mathrm{d}\,\mathfrak{t})\,\mathfrak{t}^{-1}.$$

Using Eq. (41) and Eq. (40), as well as the preceding result, we easily find

$$\mathcal{F}' = \mathrm{d}\left(\mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}\right) + \left(\mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}\right) \wedge \left(\mathfrak{t}^{-1}\mathcal{A}\,\mathfrak{t} + \mathfrak{t}^{-1}\,\mathrm{d}\,\mathfrak{t}\right)$$

$$= \mathfrak{t}^{-1}\,(\mathrm{d}\,\mathcal{A} + \mathcal{A} \wedge \mathcal{A})\,\mathfrak{t} = \mathfrak{t}^{-1}\mathcal{F}\mathfrak{t}. \blacksquare$$

Let us now come back to the above depicted geometric framework for Electrodynamics. We already understood that the electromagnetic potentials

$$\mathcal{A} = i\mathcal{A}_\lambda \, \mathrm{d}\, x^\lambda \in \Omega^1(U) \otimes \mathfrak{g}$$

(where $(x^1, \ldots, x^4)$ are local coordinates in $U \subset M$, where $\mathcal{A}_\lambda \in C^\infty(U)$, and $\mathfrak{g} = u(1) = i\mathbb{R}$), can be viewed as the local forms of a connection 1-form ω on a $U(1)$-principal bundle. We now show that well-known properties of the electromagnetic tensor are then nothing but special cases of some properties of the curvature form Ω of ω. Indeed, it follows from Eq. (41) that the local forms $\mathcal{F}$ of Ω and $\mathcal{A}$ of ω are related by

$$\mathcal{F} = \mathrm{d}\,\mathcal{A} + \frac{1}{2}[\![\mathcal{A}, \mathcal{A}]\!] = \mathrm{d}\,\mathcal{A},$$

as $\mathfrak{g} = u(1) = i\mathbb{R}$ is Abelian. Hence,

$$\mathcal{F} = \frac{i}{2}\left(\partial_\lambda \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\lambda\right) \mathrm{d}\, x^\lambda \wedge \mathrm{d}\, x^\mu = \frac{i}{2}\mathcal{F}_{\lambda\mu} \, \mathrm{d}\, x^\lambda \wedge \mathrm{d}\, x^\mu.$$

In other words, the local curvature coincides with the electromagnetic tensor. Due to commutativity, the transformation of the local curvature induced by a gauge transformation $\mathfrak{t}$ is just $\mathcal{F}' = \mathcal{F}$. So we recover that the electromagnetic tensor is invariant under a gauge transformation. Moreover, one immediately obtains

$$0 = \mathrm{d}^2\,\mathcal{A} = \mathrm{d}\,\mathcal{F} = \frac{i}{6}\left(\partial_\lambda \mathcal{F}_{\mu\nu} + \partial_\mu \mathcal{F}_{\nu\lambda} + \partial_\nu \mathcal{F}_{\lambda\mu}\right) \mathrm{d}\, x^\lambda \wedge \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu,$$

so that we geometrically recover the first part of Maxwell's equations

$$\partial_\lambda \mathcal{F}_{\mu\nu} + \partial_\mu \mathcal{F}_{\nu\lambda} + \partial_\nu \mathcal{F}_{\lambda\mu} = 0.$$

7 Exercises

Exercises will be suggested in separate files that will be made available in the UL Learning Management System MOODLE.

8 Individual work

Here some suggestions for individual work.

Topic 1. Induced connections on associated vector bundles.

Topic 2. Holonomy.

Topic 3. Connections in Riemannian Geometry.

Topic 4. Characteristic classes.

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