On the numerical stability in the derivation of Slepian base functions

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Outline

- Slepian representation
- Meissl scheme
- Dependency on the data distribution
- Combination GRACE + GPS

Spatial and spectral concentration problem

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- Solution is an eigenvalue problem:

$$\lambda \bar{g} = D \bar{g} \quad \text{with} \quad D = \int\limits_R \bar{Y}_{lm} \bar{Y}_{nk} d\Omega \quad \Rightarrow \quad D = G \Lambda G^{-1}$$

Slepian representation

- Spherical harmonic representation:

$$f = \sum_{l=0}^{L} \sum_{m=0}^{l} \left(\bar{C}_{lm} \cos m\lambda + \bar{S}_{lm} \sin m\lambda\right) \bar{P}_{lm} = \sum_{l=0}^{L} \sum_{m=-l}^{l} \bar{K}_{lm} \bar{Y}_{lm}$$

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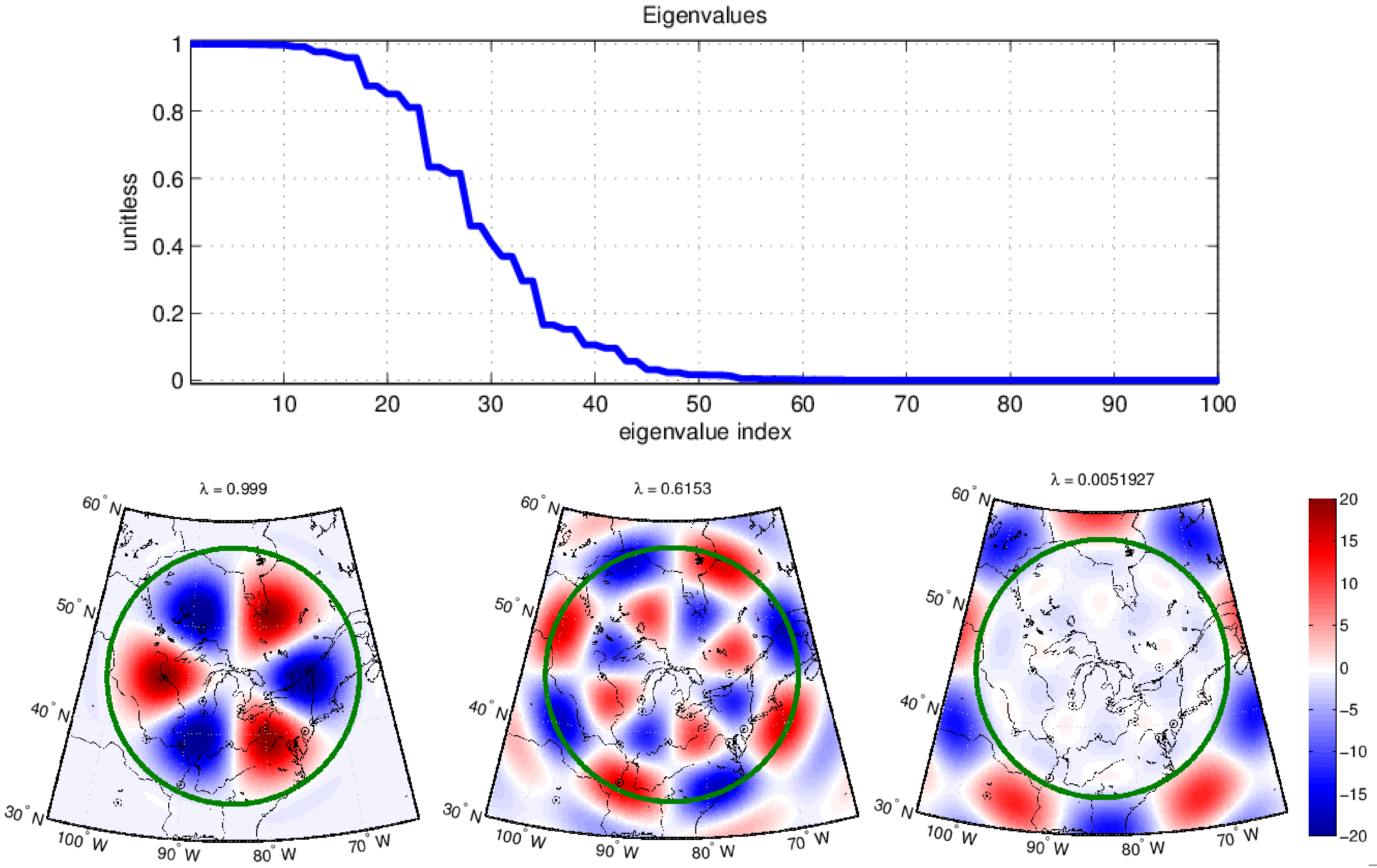
- Transformation:

$$\bar{\beta}_i = \sum_{l=0}^{L} \sum_{m=-l}^{l} \bar{K}_{lm} \, \bar{g}_{i,lm} \qquad \bar{K}_{lm} = \sum_{i=1}^{(L+1)^2} \bar{\beta}_i \, \bar{g}_{i,lm}$$

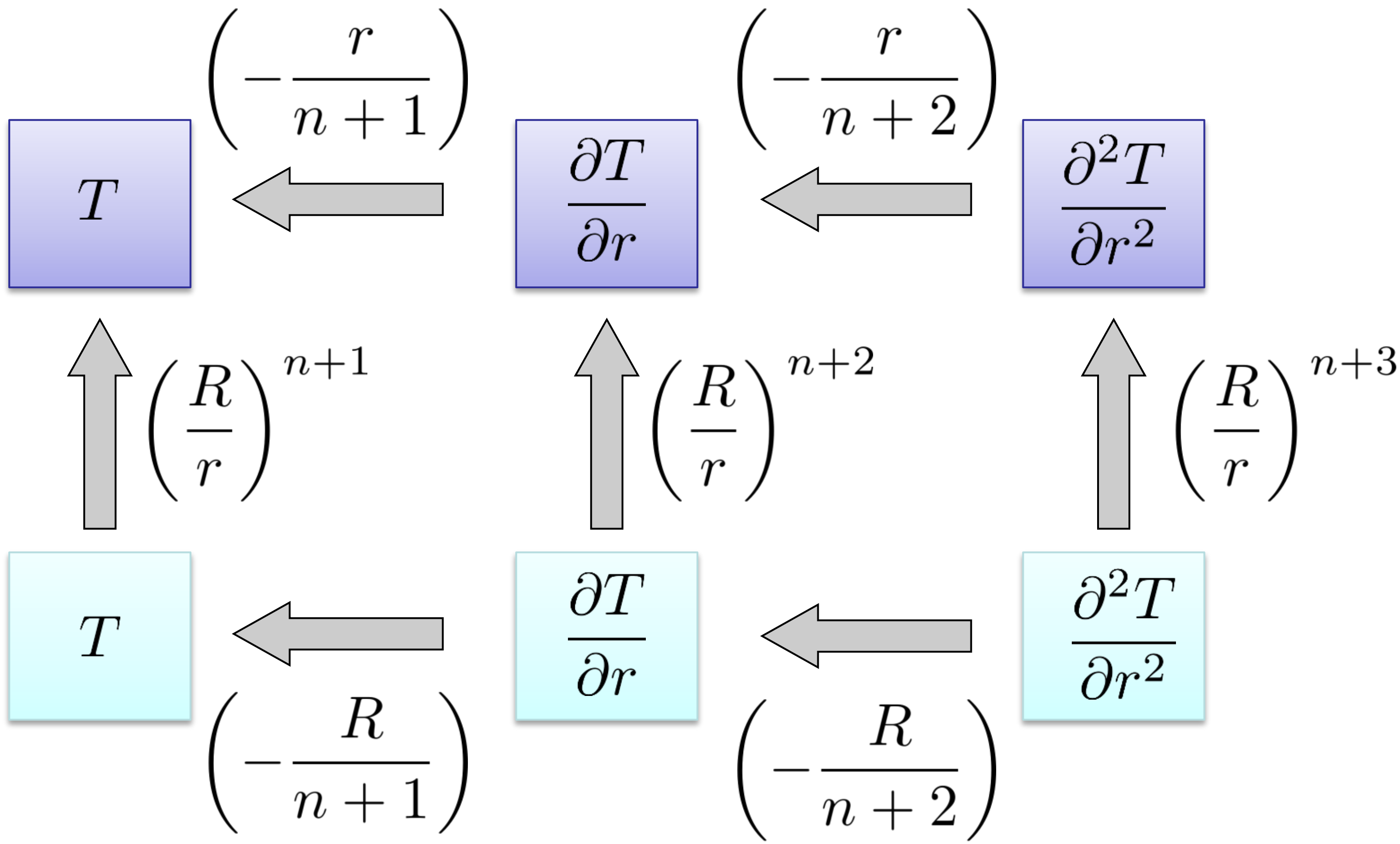
Localizing property



MEISSL SCHEME FOR SLEPIANS

Meissl scheme for spherical harmonics

From Rummel and van Gelderen (1995):



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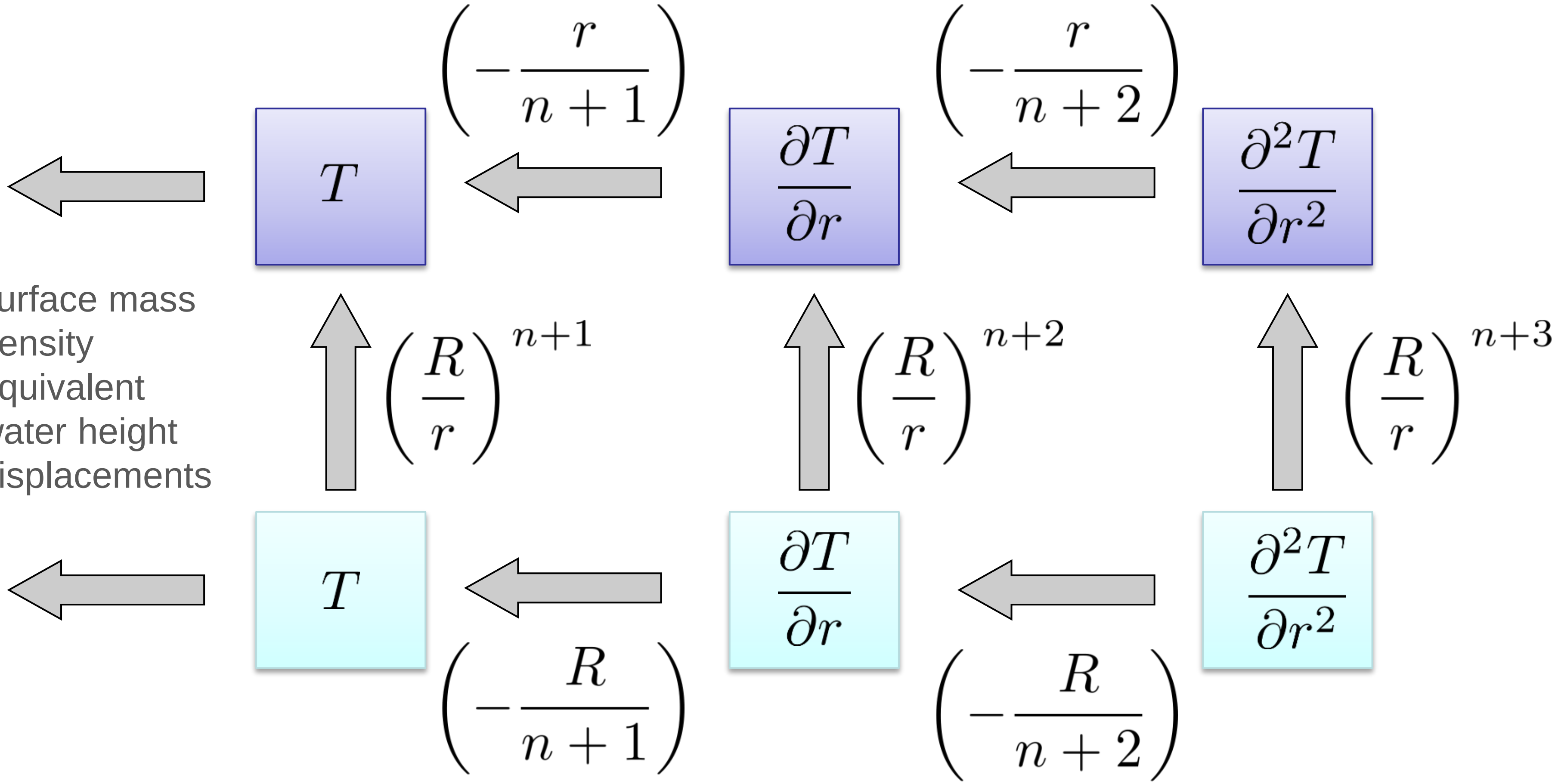

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- equivalent water height
- displacements

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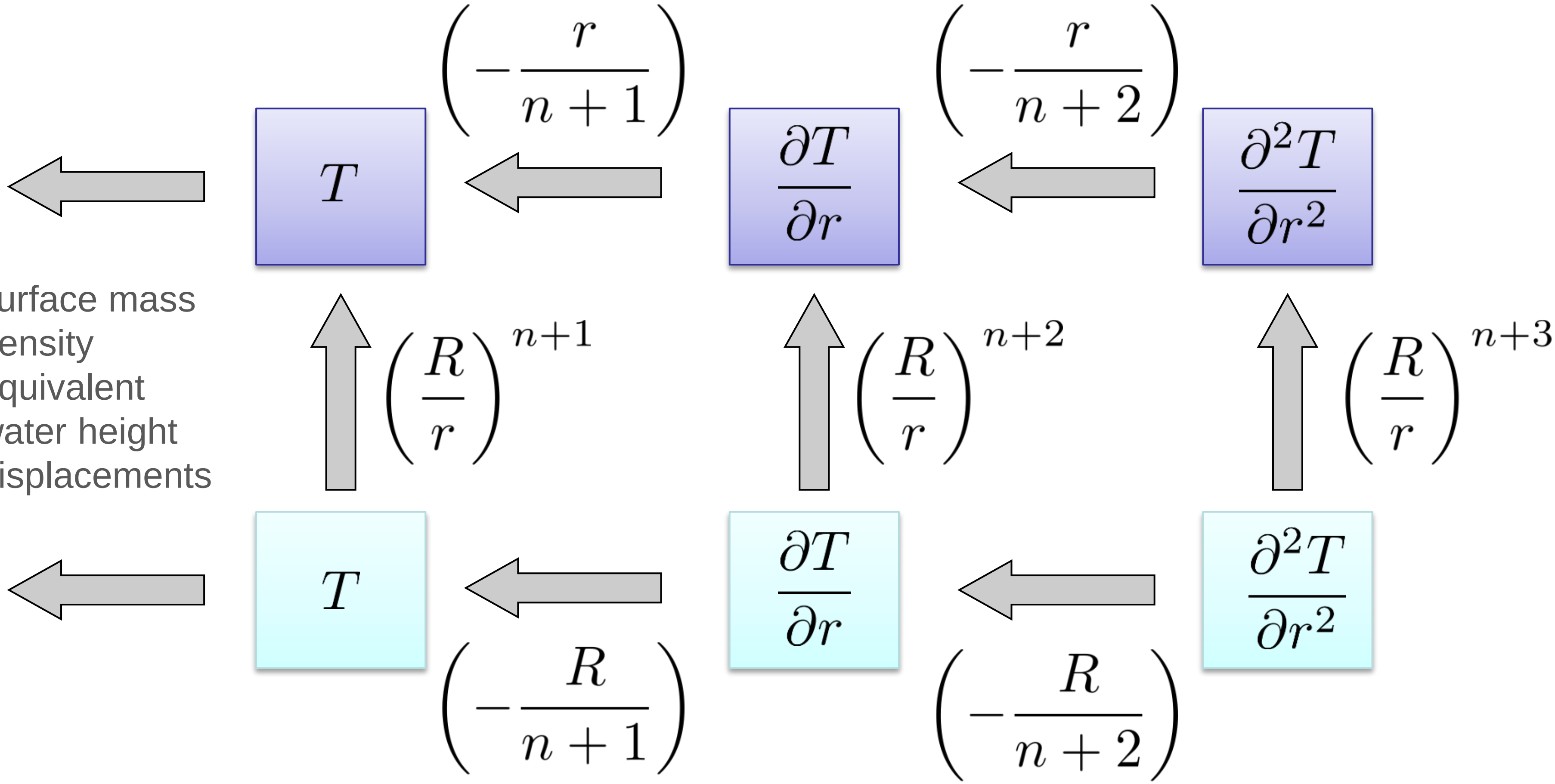

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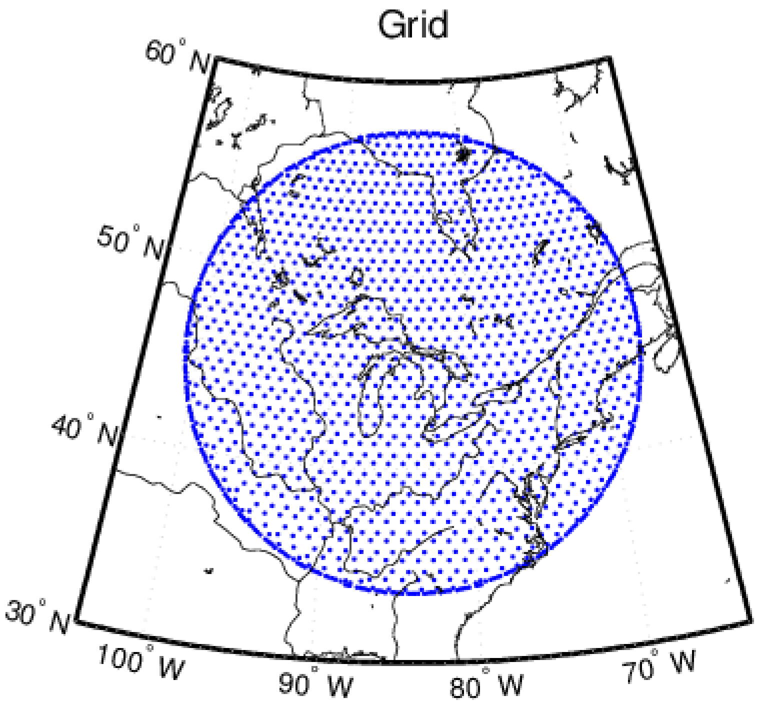
DEPENDENCY ON DATA DISTRIBUTION

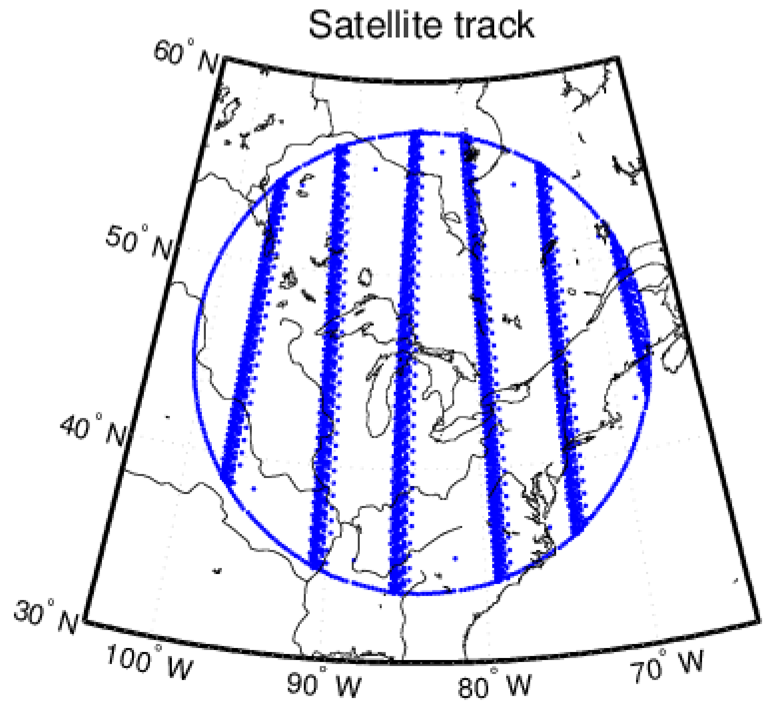
Motivation

Slepian base function depend implicitly on the data distribution:

$$D = G\Lambda G^{-1} = \int_R \bar{Y}_{lm}\bar{Y}_{nk}d\Omega \approx \sum_{i=1}^{I} \bar{Y}_{lm}\bar{Y}_{nk}\Delta\Omega_i$$

Examples:





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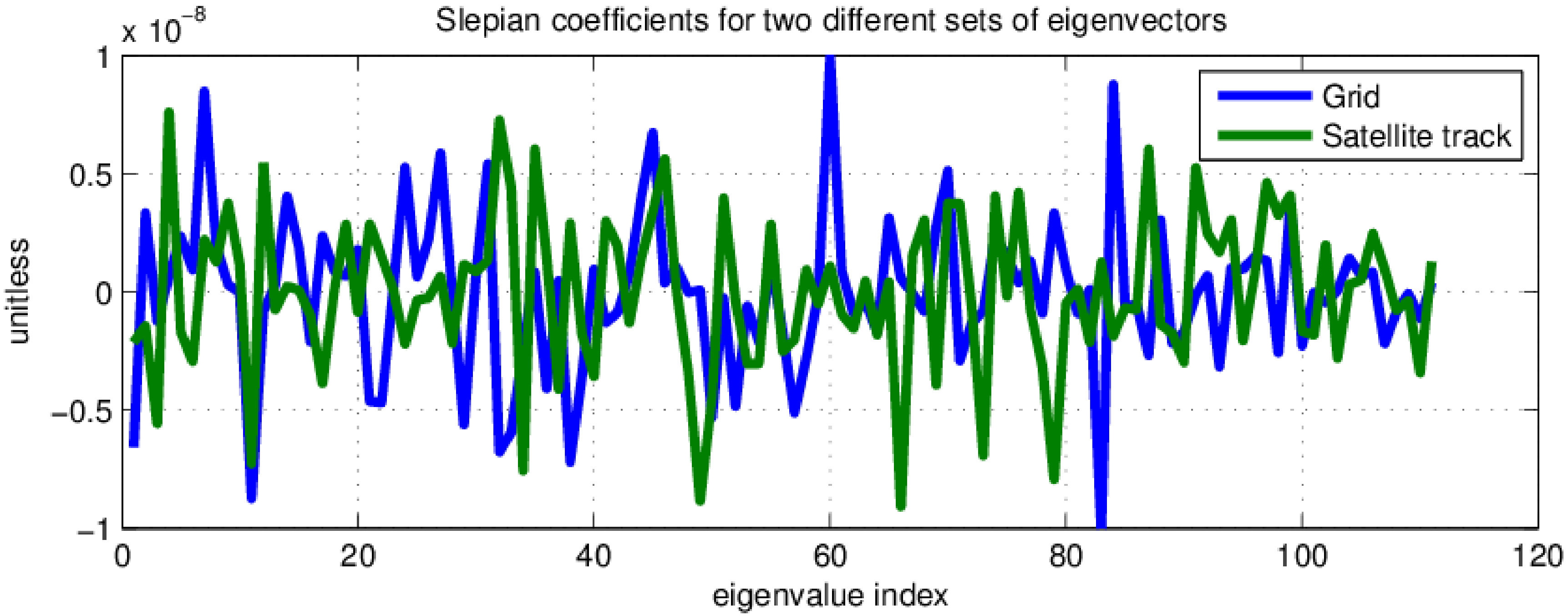

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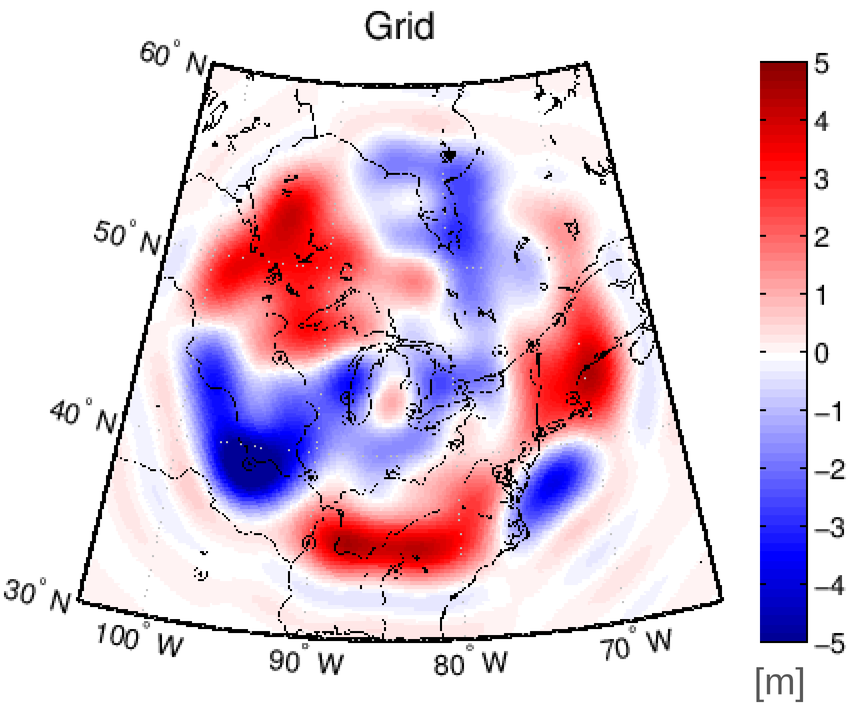

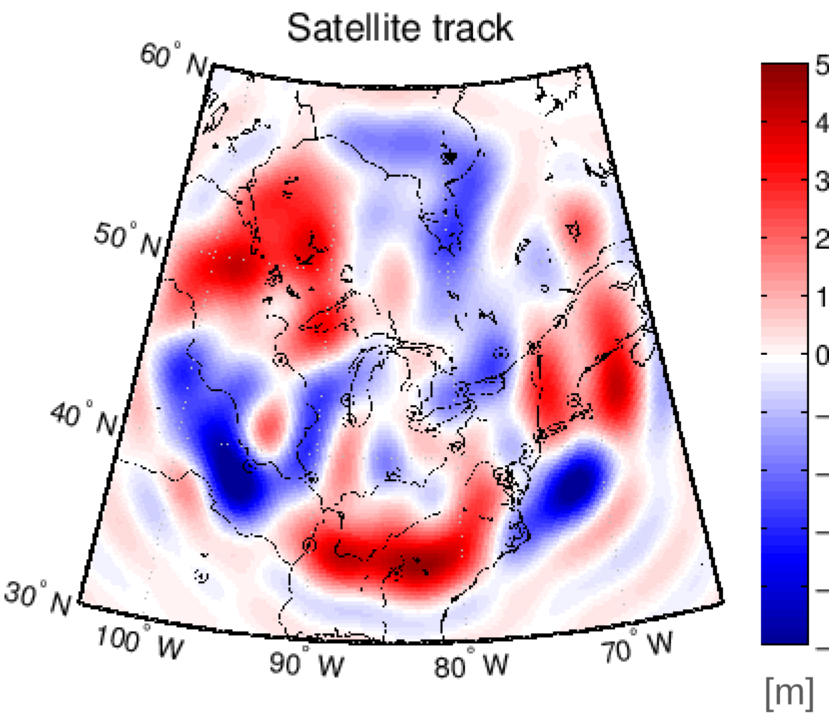

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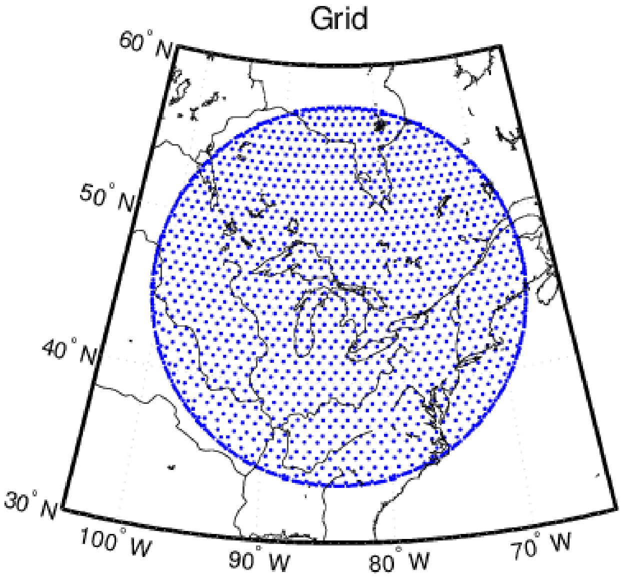

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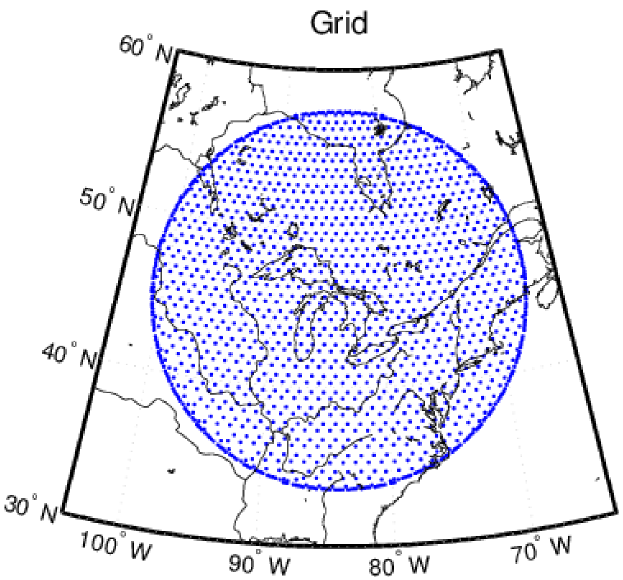

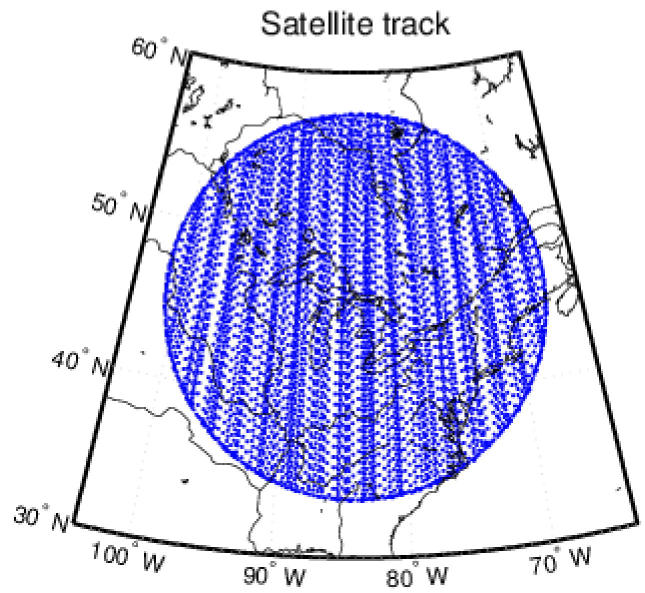

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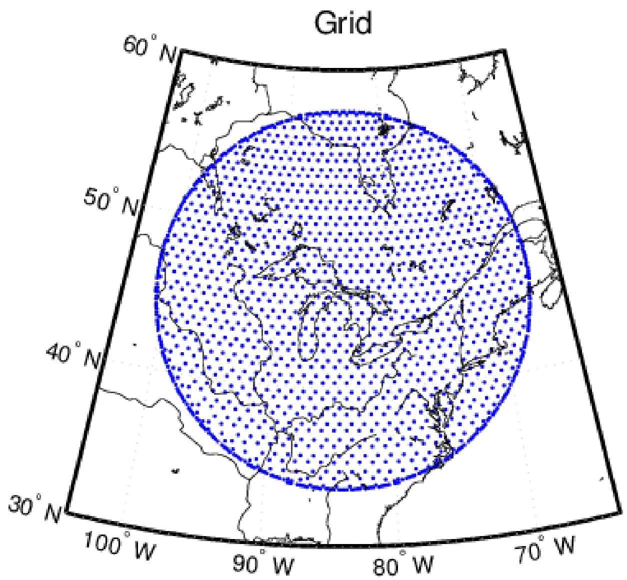

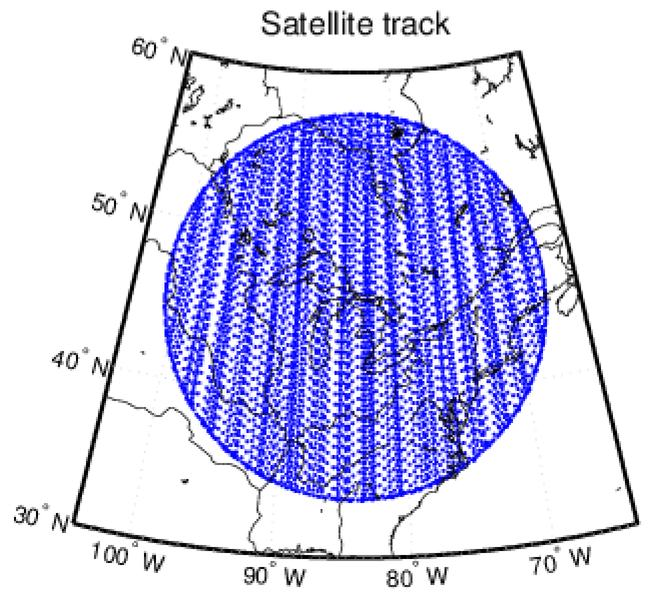

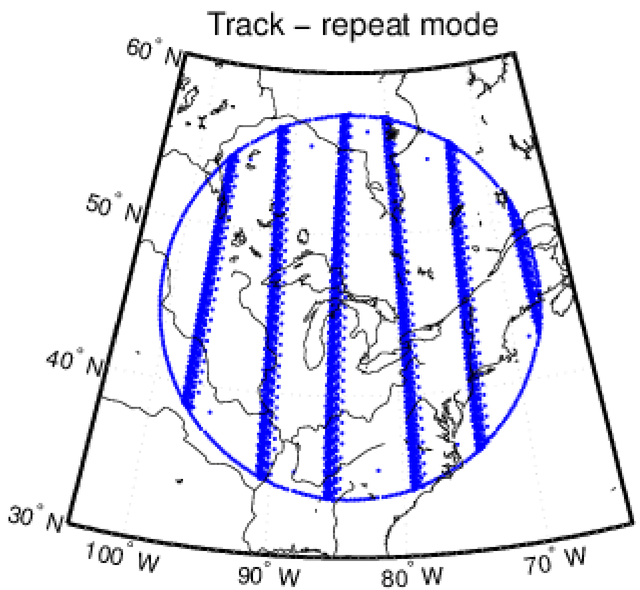

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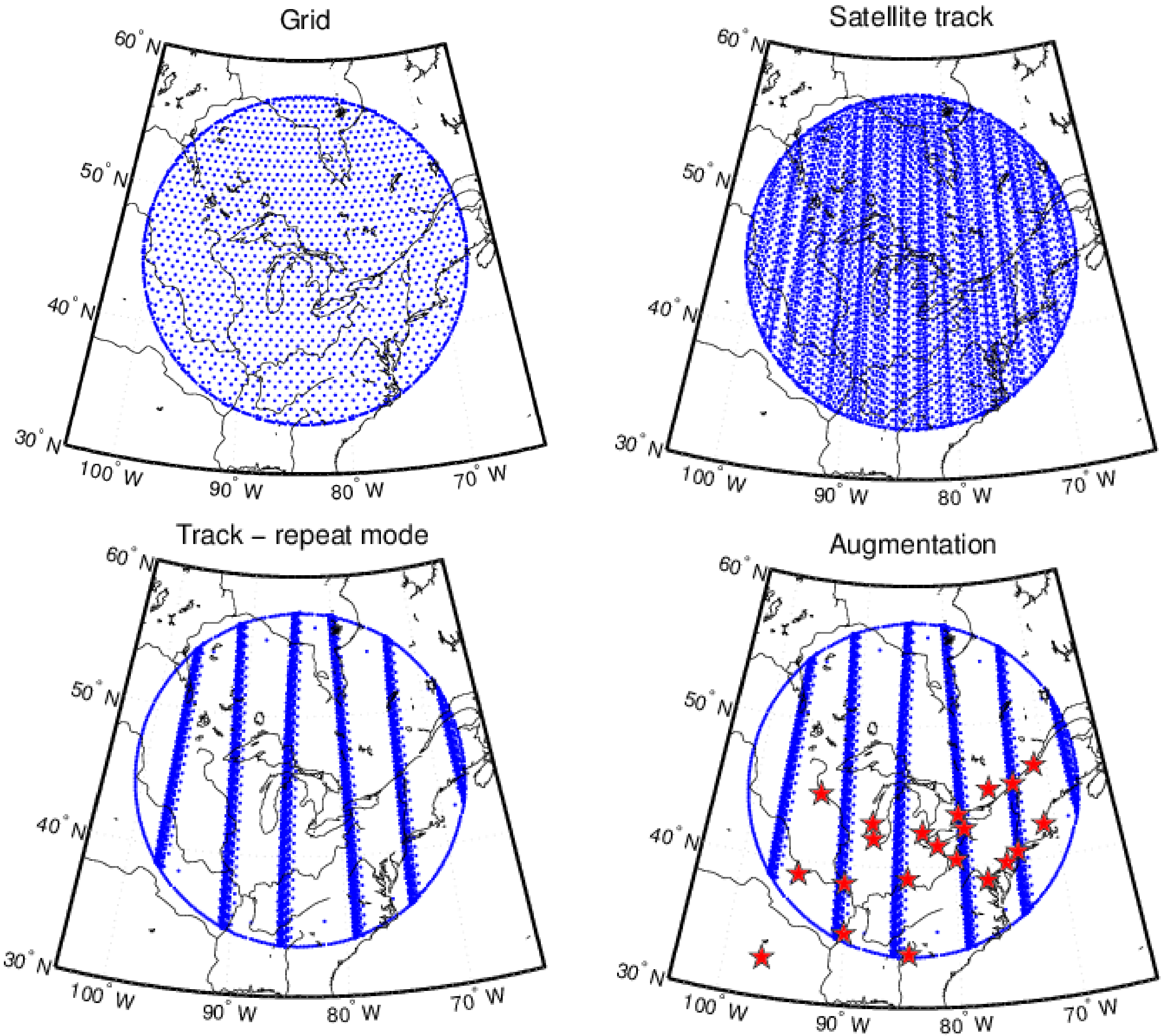

How does the estimate change?

Given: Slepian base functions based on a grid

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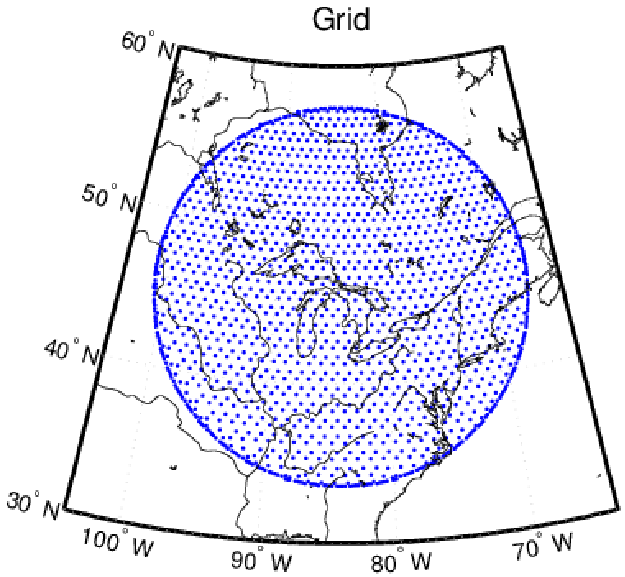
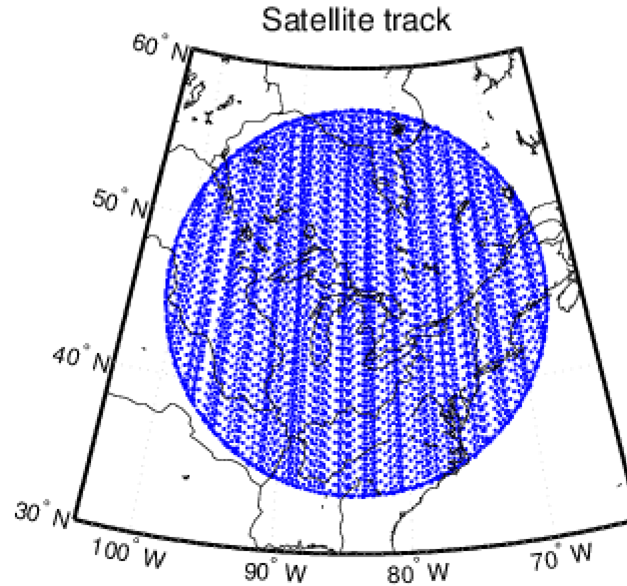
Case I: observations on the same grid



Case II: observations on a (dense) satellite track



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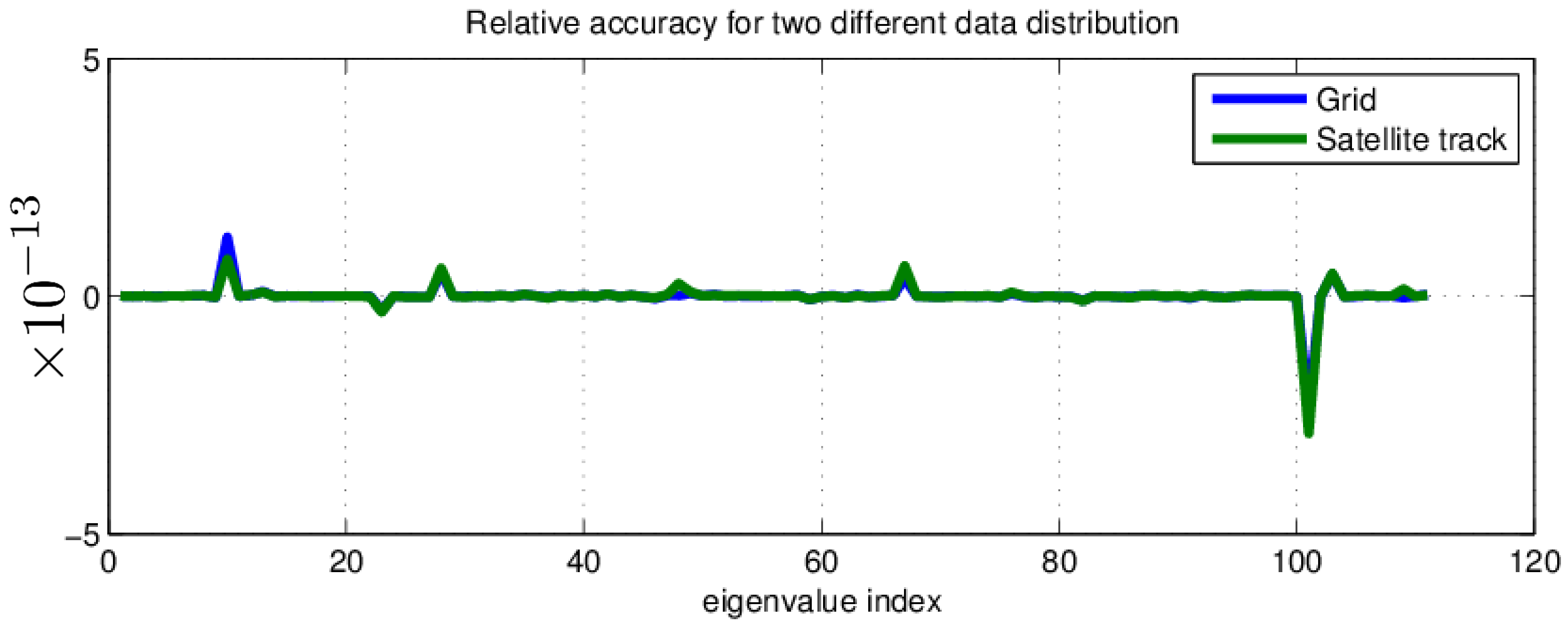

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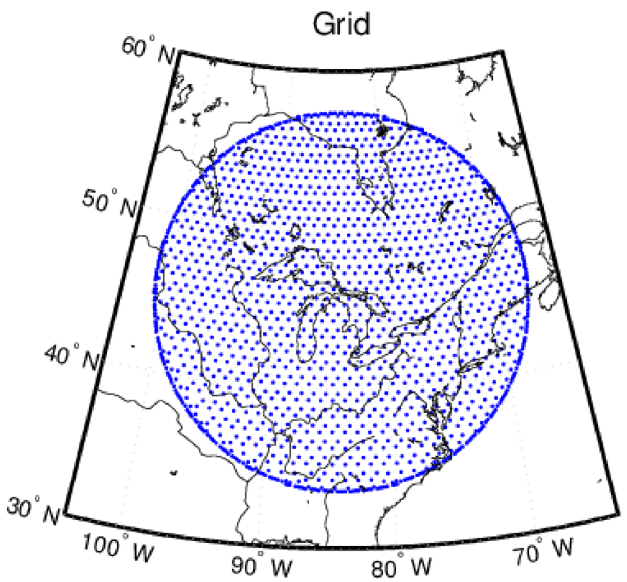
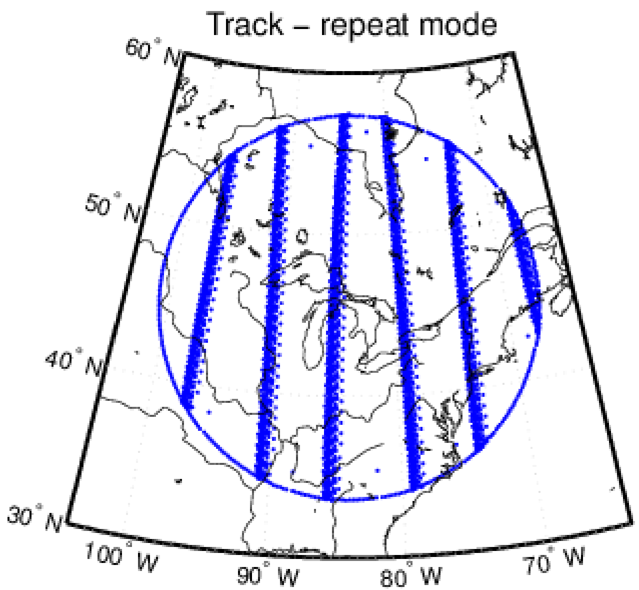


Case III: observations on a (sparse) satellite track



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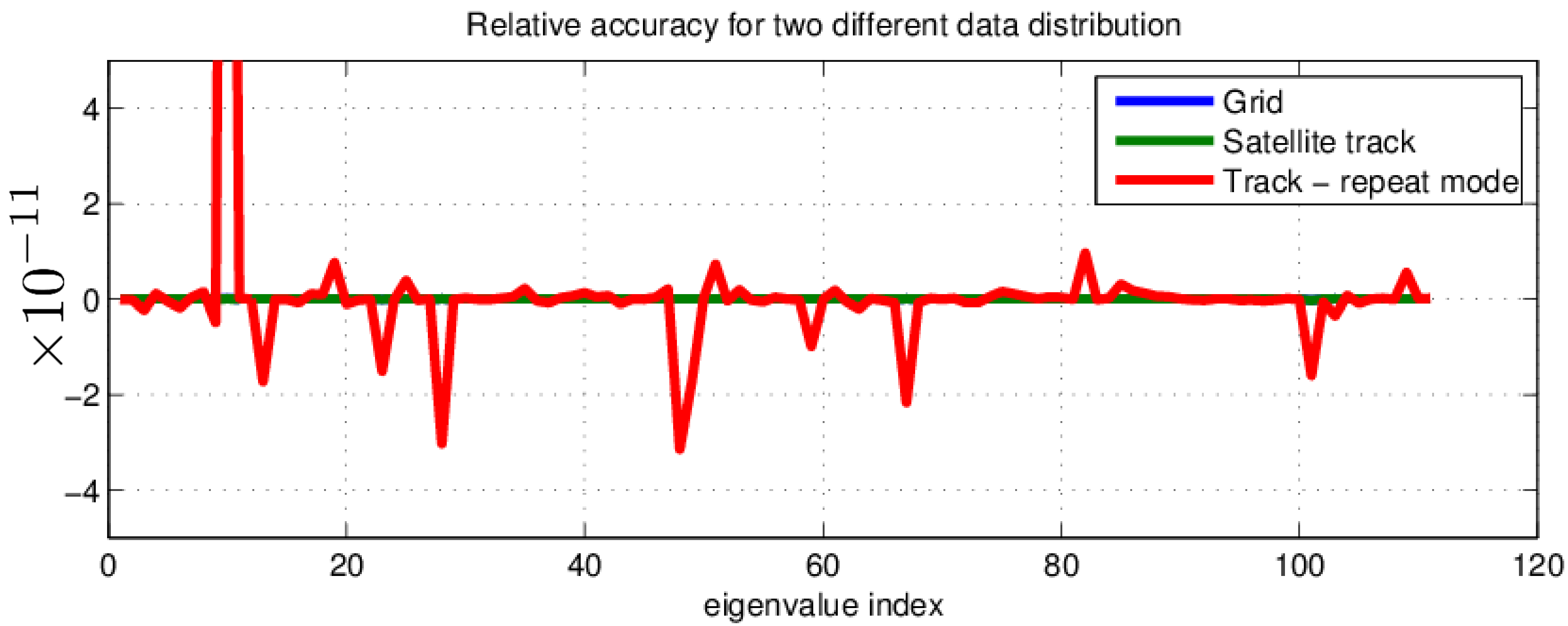


OUTLOOK:
COMBINATION OF GRACE AND GPS

Approach

Given: SH from GRACE and terrestrial GPS in local area

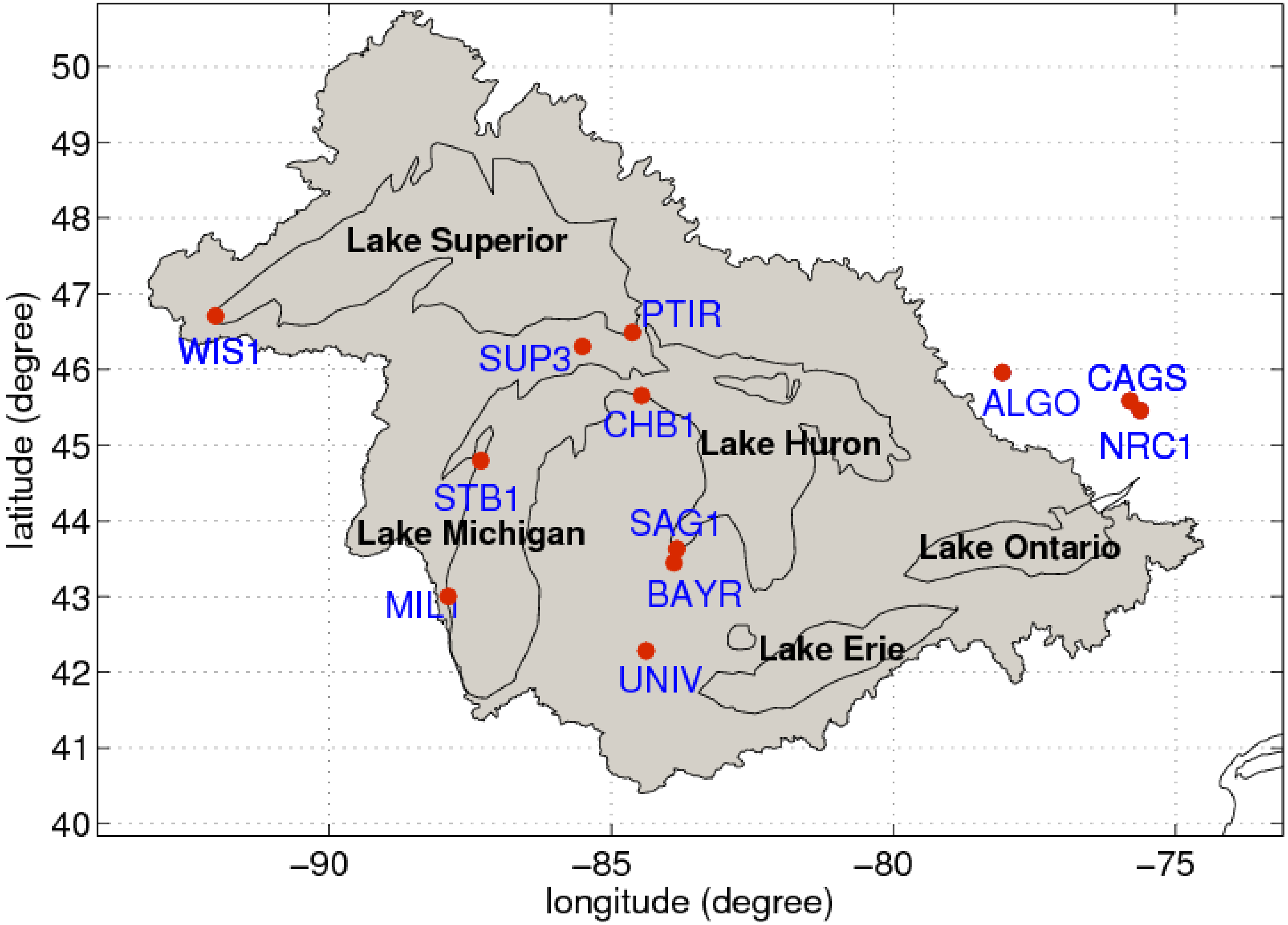
Methodology:

- Convert SH to Slepian coefficients
- Update Slepian coefficients by the terrestrial GPS observations with the help of a Kalman filter
- Kalman filter is implemented as continuous Wiener process acceleration (CWPA) model

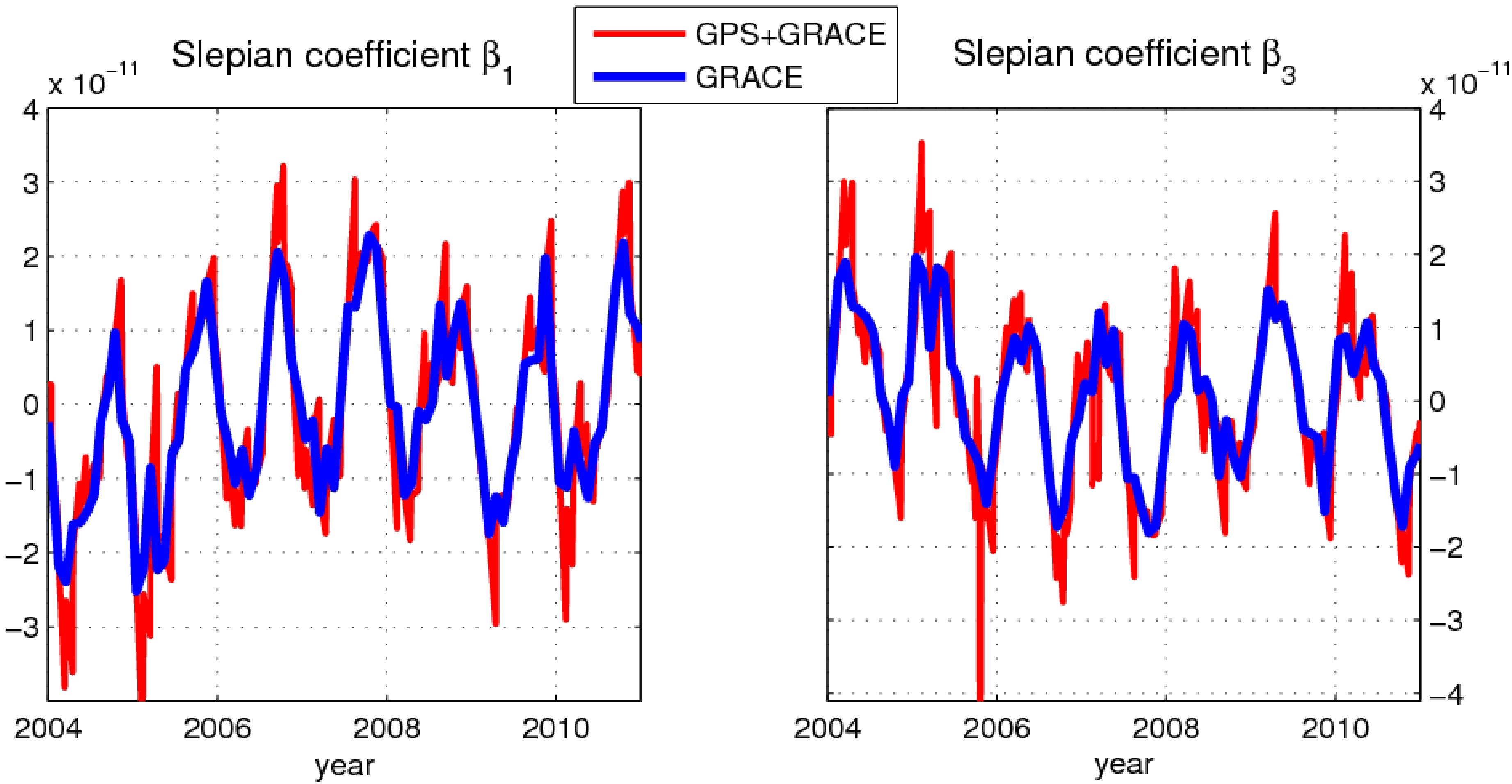
Data:

- GRACE GFZ Rel.05 with GAC restored
- Height observations of GPS time series of 12 stations

GPS site distribution



Preliminary results



Conclusion

- The direct conversion between specific and non-specific Slepian coefficients is possible.
  - → Meissl scheme for Slepians (analogue to SH)
  - → trivial approach numerically more efficient
- Data distribution is “not” critical for the estimation of the Slepian coefficients but for the derivation of the Slepian base function