A new variant of the differential gravimetry approach

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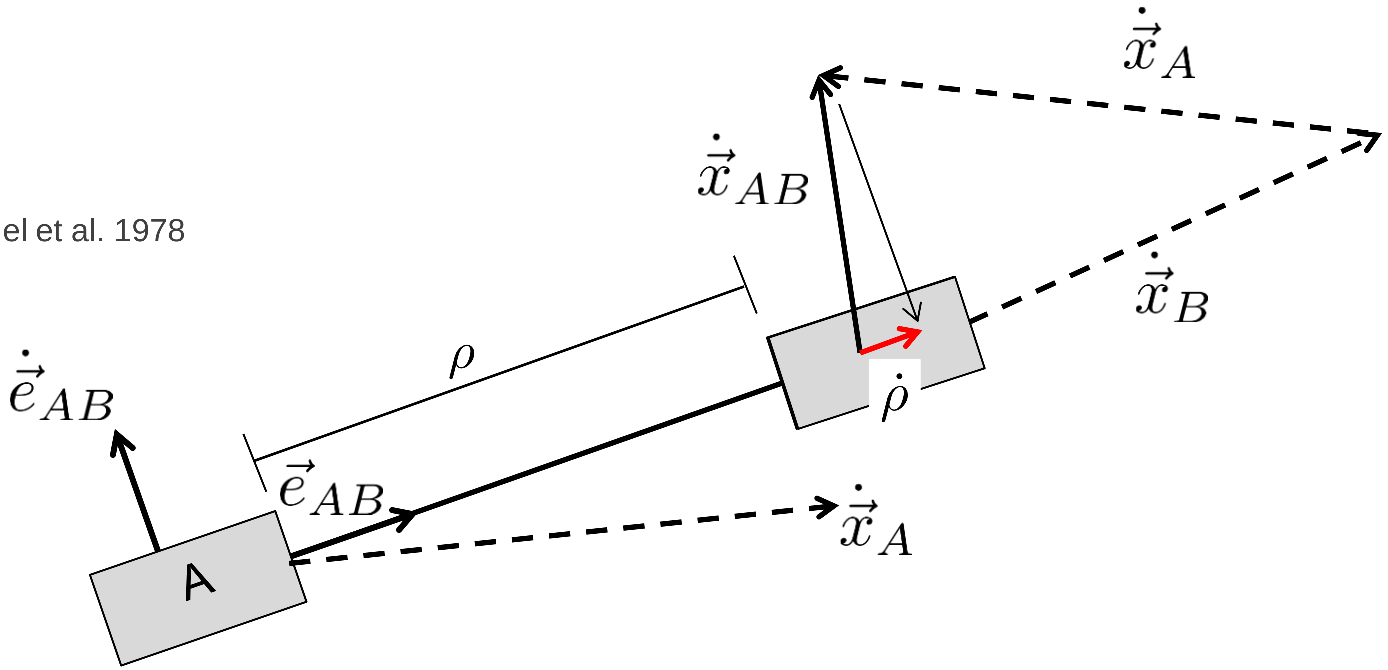
Geometry of the GRACE system

Rummel et al. 1978



Differentiation ↓

Integration ↑

$$\rho = \vec{x}_{AB} \cdot \vec{e}_{AB}$$

$$\dot{\rho} = \dot{\vec{x}}_{AB} \cdot \vec{e}_{AB}$$

$$\ddot{\rho} = \ddot{\vec{x}}_{AB} \cdot \vec{e}_{AB} + \dot{\vec{x}}_{AB} \cdot \dot{\vec{e}}_{AB}$$

$$= \nabla V_{AB}$$

Solution strategies

Variational equations

Classical
(Reigber 1989, Tapley 2004)
$\rho, \dot{\rho}$

Celestial Mechanics approach
(Beutler et al. 2010, Jäggi 2007)
$\rho, \dot{\rho}, \Delta\rho$

Short-arc method
(Mayer-Gürr 2006)
$\rho, \dot{\rho}$

...

In-situ observations

Energy Integral
(Han 2003, Ramillien et al. 2010)
$\dot{\rho}$

Differential gravimetry
(Liu 2010)
$\ddot{\rho}$

LoS Gradiometry
(Keller and Sharifi 2005)
$\frac{\ddot{\rho}}{\rho}$

...

DIFFERENTIAL GRAVIMETRY
THE STANDARD APPROACH

Differential gravimetry

Range observables: $\vec{x}_{AB} = \rho\, \vec{e}^{\,a}_{AB}$

Differential gravimetry

Range observables:

$$\vec{x}_{AB} = \rho\, \vec{e}^{\,a}_{AB}$$

$$\dot{\vec{x}}_{AB} = \dot{\rho}\, \vec{e}^{\,a}_{AB} + \rho\, \dot{\vec{e}}^{\,a}_{AB}$$

Differential gravimetry

Range observables:

$$\vec{x}_{AB} = \rho\,\vec{e}^{\,a}_{AB}$$

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Differential gravimetry

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$$\vec{x}_{AB} = \rho\, \vec{e}^{\,a}_{AB}$$

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Multiplication with unit vectors in along-track, cross-track and radial direction:

$$\begin{aligned}
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} &= \ddot{\rho} \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} &= 0 \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,r}_{AB} &= 0 \quad + \quad 2\,\dot{\rho}\, \|\dot{\vec{e}}^{\,a}_{AB}\| \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}
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Differential gravimetry

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Multiplication with unit vectors in along-track, cross-track and radial direction:

GRACE

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

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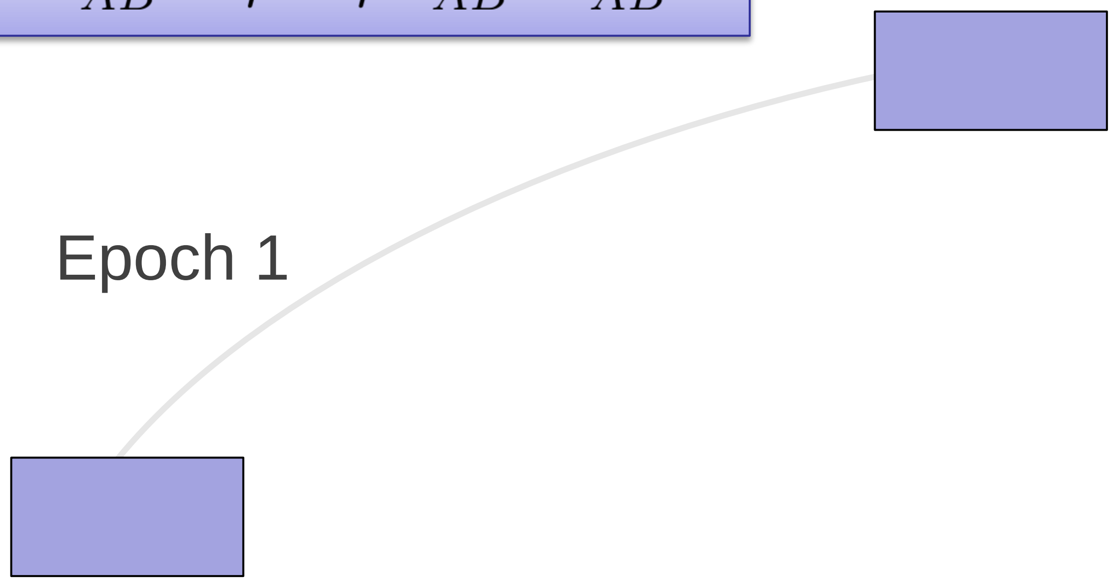
Relative motion

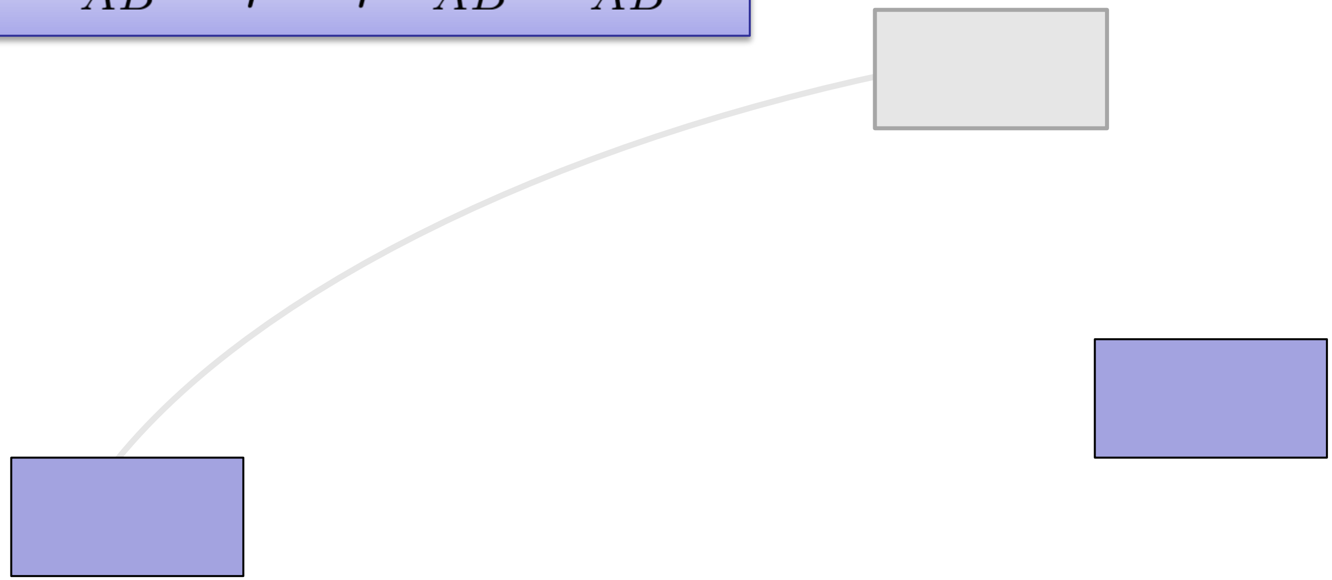
$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$



Relative motion

absolute motion neglected!

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

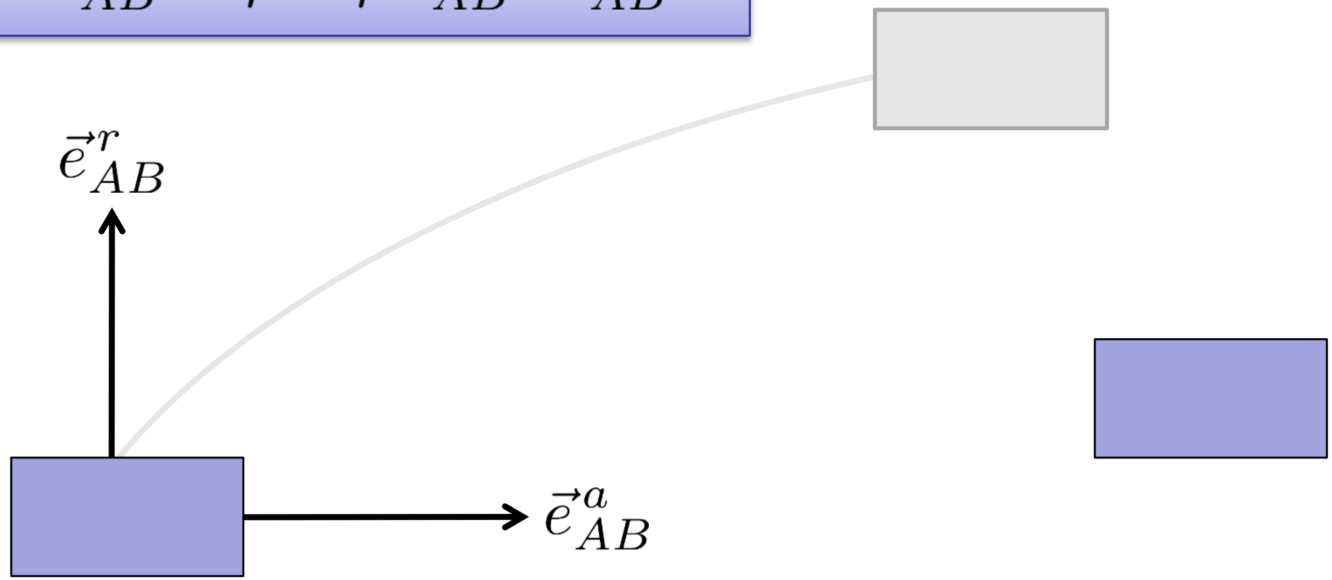


Epoch 2

Relative motion

absolute motion neglected!

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

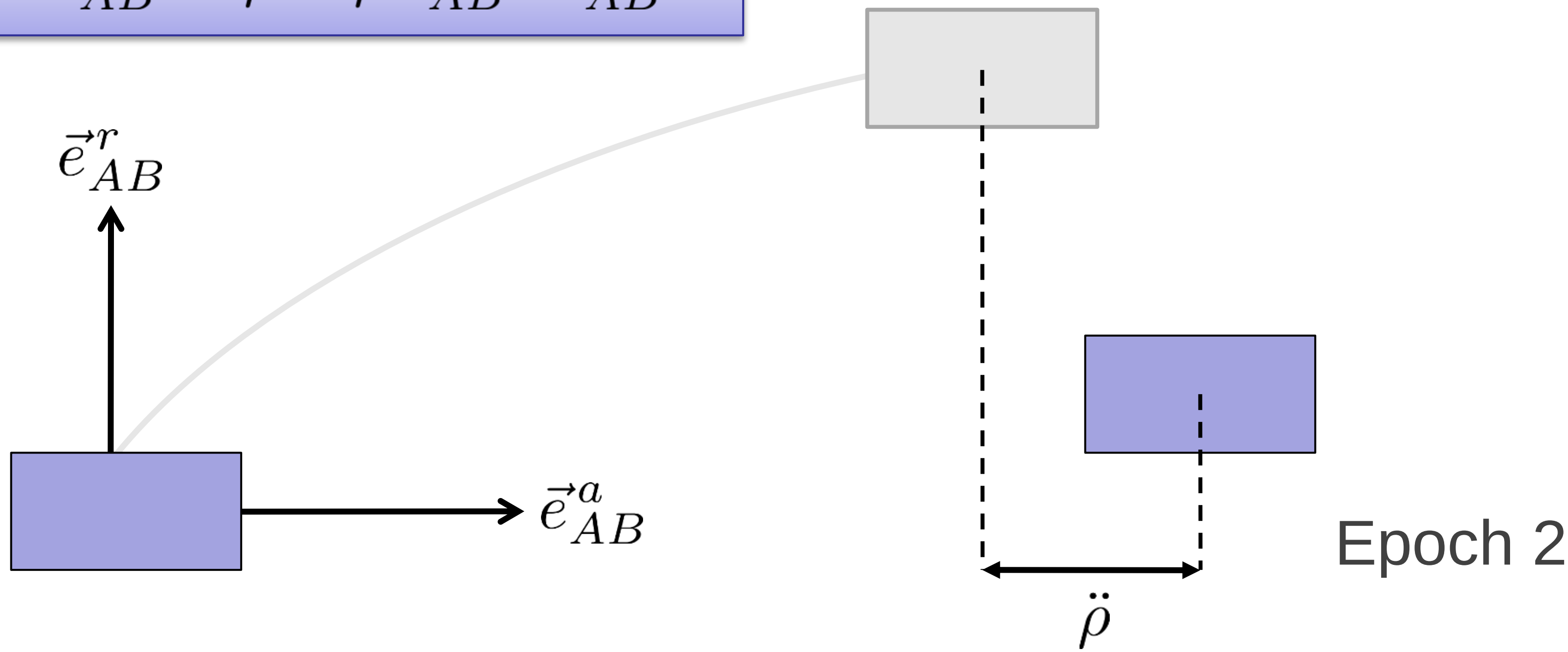


Epoch 2

Relative motion

absolute motion neglected!

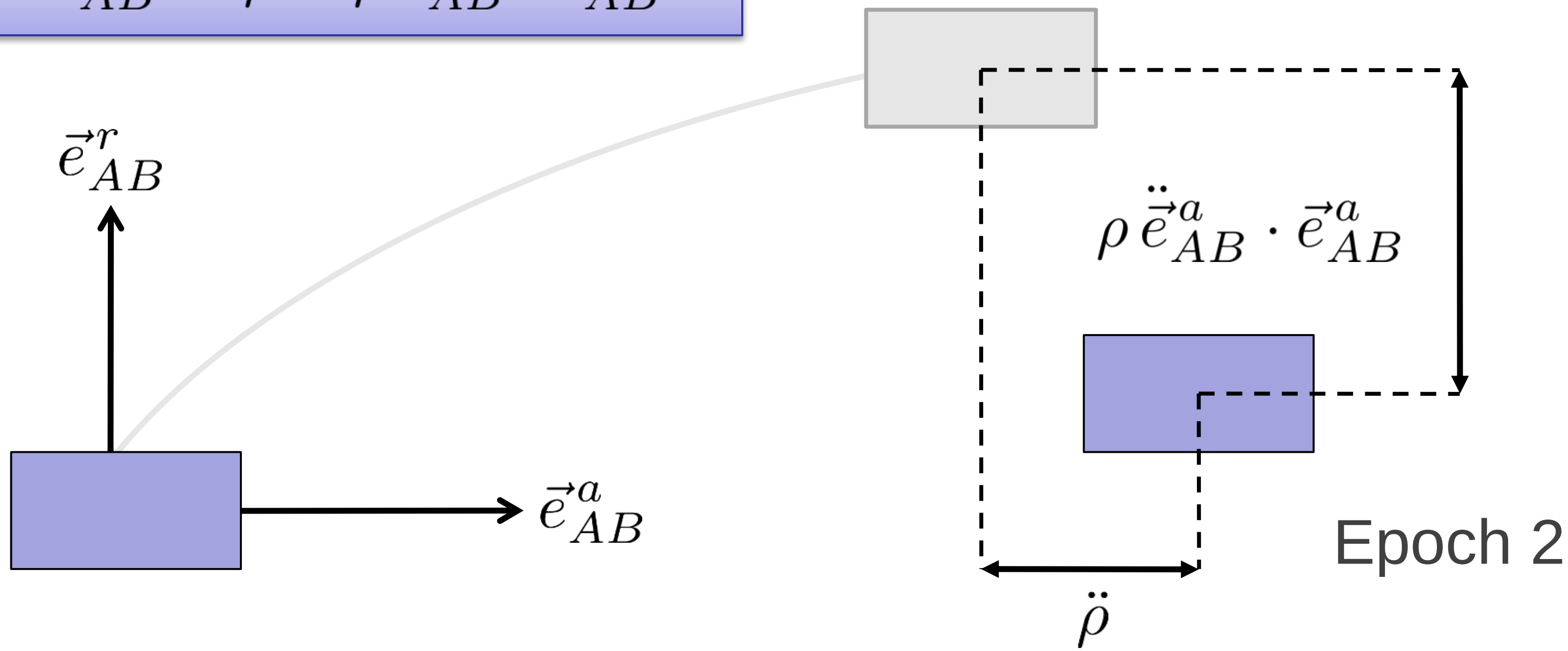
$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$



Relative motion

absolute motion neglected!

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$



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absolute motion neglected!

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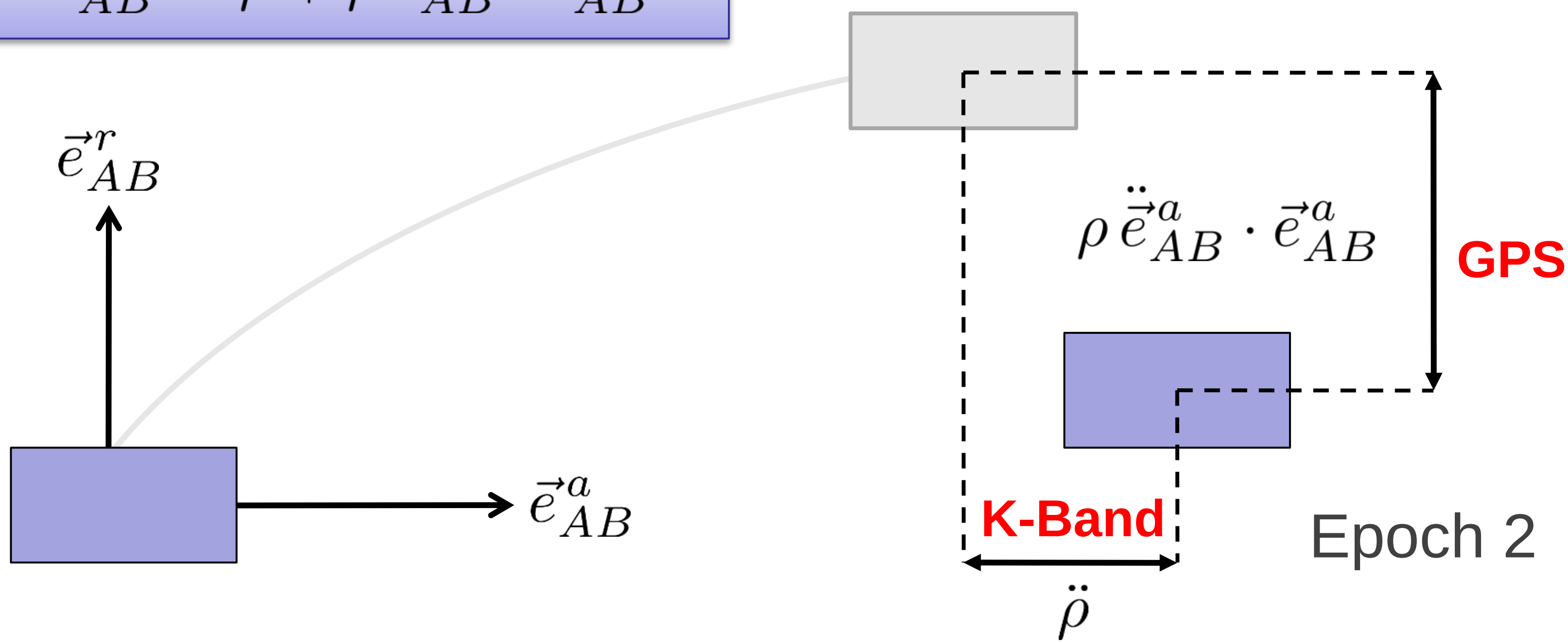

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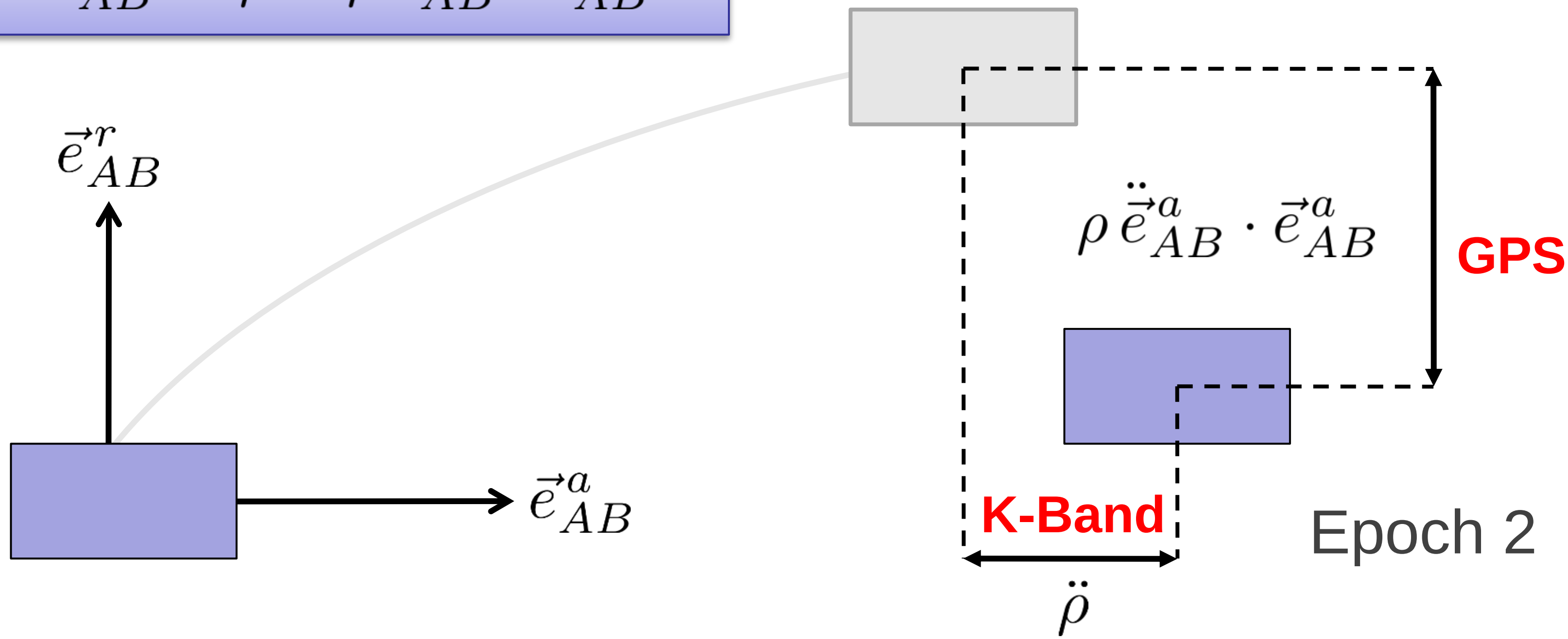
absolute motion neglected!

$$\ddot{\vec{x}}_{AB}\cdot\vec{e}^{\,a}_{AB} = \ddot{\rho} + \rho\,\ddot{\vec{e}}^{\,a}_{AB}\cdot\vec{e}^{\,a}_{AB}$$



$$\begin{aligned}\rho\,\ddot{\vec{e}}^{\,a}_{AB}\cdot\vec{e}_{AB} &= \vec{x}_{AB}\cdot\ddot{\vec{e}}^{\,a}_{AB} &&= -\dot{\vec{x}}_{AB}\cdot\dot{\vec{e}}^{\,a}_{AB}\\ &= -\rho\,\|\dot{\vec{e}}^{\,a}_{AB}\|^2 &&= -\frac{1}{\rho}\left(\dot{\vec{x}}_{AB}\cdot\dot{\vec{x}}_{AB}-\dot{\rho}^2\right)\end{aligned}$$

A VARIANT BASED ON ROTATIONS

Instantaneous relative reference frame

Remember:

$$\begin{aligned}
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} &= \ddot{\rho} \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} &= 0 \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,r}_{AB} &= 0 \quad + \quad 2\,\dot{\rho}\, \|\dot{\vec{e}}^{\,a}_{AB}\| \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}
\end{aligned}$$

Instantaneous relative reference frame

Remember:

$$
\begin{aligned}
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} &= \ddot{\rho} &+& \quad 0 &+& \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} &= 0 &+& \quad 0 &+& \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,r}_{AB} &= 0 &+& \quad 2\,\dot{\rho}\, \|\dot{\vec{e}}^{\,a}_{AB}\| &+& \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}
\end{aligned}
$$

Instantaneous relative reference frame

Remember:

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + 0 + \rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} = 0 + 0 + \rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB}$$

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,r}_{AB} = 0 + 2\,\dot{\rho}\,\|\dot{\vec{e}}^{\,a}_{AB}\| + \rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}$$

Definition:

$$R^I_F = \begin{pmatrix} (\vec{e}^{\,a}_{AB})^T \\ (\vec{e}^{\,c}_{AB})^T \\ (\vec{e}^{\,r}_{AB})^T \end{pmatrix}$$

Instantaneous relative reference frame

Remember:

$$
\begin{aligned}
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} &= \ddot{\rho} \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} &= 0 \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB} \\
\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,r}_{AB} &= 0 \quad + \quad 2\,\dot{\rho}\, \|\dot{\vec{e}}^{\,a}_{AB}\| \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}
\end{aligned}
$$

Definition:

$$
R^I_F = \begin{pmatrix} (\vec{e}^{\,a}_{AB})^T \\ (\vec{e}^{\,c}_{AB})^T \\ (\vec{e}^{\,r}_{AB})^T \end{pmatrix}
$$

$$
\vec{e}^{\,a}_{AB,F} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{e}^{\,c}_{AB,F} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{e}^{\,r}_{AB,F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Instantaneous relative reference frame

Remember:

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} = 0 \quad + \quad 0 \quad + \quad \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB}$$

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Definition:

$$R^I_F = \begin{pmatrix} (\vec{e}^{\,a}_{AB})^T \\ (\vec{e}^{\,c}_{AB})^T \\ (\vec{e}^{\,r}_{AB})^T \end{pmatrix}$$

$$\Rightarrow \quad \vec{e}^{\,a}_{AB,F} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}^{\,c}_{AB,F} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{e}^{\,r}_{AB,F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \dot{\vec{e}}^{\,a,c,r}_{AB,F} = \ddot{\vec{e}}^{\,a,c,r}_{AB,F} = \vec{0}$$

Instantaneous relative reference frame

Remember:

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,a}_{AB} = \ddot{\rho} + 0 + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB}$$

$$\ddot{\vec{x}}_{AB} \cdot \vec{e}^{\,c}_{AB} = 0 + 0 + \rho\, \ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB}$$

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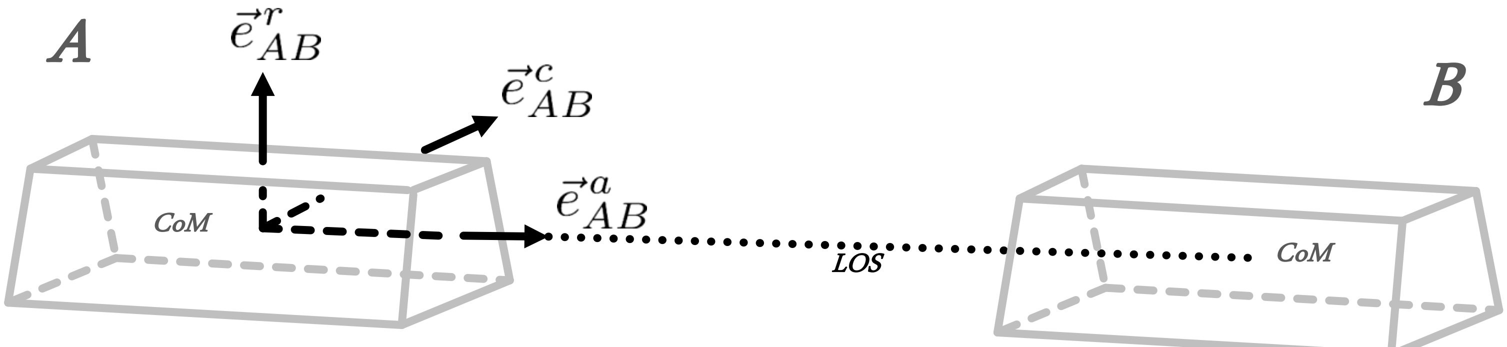

Instantaneous relative reference frame

Need for moving frame quantities:

$$R_F^I \vec{e}_{AB}^{\,a,c,r} = \vec{e}_{AB,F}^{\,a,c,r}$$

Instantaneous relative reference frame

Need for moving frame quantities:

$$\dot{\vec{e}}^{\,a,c,r}_{AB,F} = \ddot{\vec{e}}^{\,a,c,r}_{AB,F} = \vec{0}$$

$$R^I_F \vec{e}^{\,a,c,r}_{AB} = \vec{e}^{\,a,c,r}_{AB,F}$$

$$R^I_F \dot{\vec{e}}^{\,a}_{AB} = \dot{\vec{e}}^{\,a}_{AB,F} + \vec{\omega} \times \vec{e}^{\,a}_{AB,F} = \vec{\omega} \times \vec{e}^{\,a}_{AB,F}$$

Instantaneous relative reference frame

Need for moving frame quantities:

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$$R^I_F \ddot{\vec{e}}^{\,a}_{AB} = \ddot{\vec{e}}^{\,a}_{AB,F} + 2\vec{\omega} \times \dot{\vec{e}}^{\,a}_{AB,F} + \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F}$$

$$= \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F}$$

Instantaneous relative reference frame

Need for moving frame quantities:

$$\dot{\vec{e}}^{\,a,c,r}_{AB,F} = \ddot{\vec{e}}^{\,a,c,r}_{AB,F} = \vec{0}$$

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$$R^I_F \dot{\vec{e}}^{\,a}_{AB} = \dot{\vec{e}}^{\,a}_{AB,F} + \vec{\omega} \times \vec{e}^{\,a}_{AB,F} = \vec{\omega} \times \vec{e}^{\,a}_{AB,F}$$

$$\begin{aligned} R^I_F \ddot{\vec{e}}^{\,a}_{AB} &= \ddot{\vec{e}}^{\,a}_{AB,F} + 2\vec{\omega} \times \dot{\vec{e}}^{\,a}_{AB,F} + \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F} \\ &= \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F} \end{aligned}$$

What about $\vec{\omega}$?

Instantaneous relative reference frame

Need for moving frame quantities:

$$\dot{\vec{e}}^{\,a,c,r}_{AB,F} = \ddot{\vec{e}}^{\,a,c,r}_{AB,F} = \vec{0}$$

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$$\begin{aligned} R^I_F \ddot{\vec{e}}^{\,a}_{AB} &= \ddot{\vec{e}}^{\,a}_{AB,F} + 2\vec{\omega} \times \dot{\vec{e}}^{\,a}_{AB,F} + \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F} \\ &= \vec{\omega} \times \left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) + \dot{\vec{\omega}} \times \vec{e}^{\,a}_{AB,F} \end{aligned}$$

What about $\vec{\omega}$?

defined by the Cartan-Matrix:

$$\Omega = R^I_F \left(\dot{R}^I_F\right)^T = \begin{pmatrix} 0 & -\omega^r & \omega^c \\ \omega^r & 0 & -\omega^a \\ -\omega^c & \omega^a & 0 \end{pmatrix} \qquad \dot{R}^I_F = \begin{pmatrix} \left(\dot{\vec{e}}^{\,a}_{AB}\right)^T \\ \left(\dot{\vec{e}}^{\,c}_{AB}\right)^T \\ \left(\dot{\vec{e}}^{\,r}_{AB}\right)^T \end{pmatrix}$$

Differential gravimetry in the IRRF

Inertial: $$\ddot{\vec{x}}_{AB} \quad = \quad \ddot{\rho}\,\vec{e}^{\,a}_{AB} + 2\,\dot{\rho}\,\dot{\vec{e}}^{\,a}_{AB} + \rho\,\ddot{\vec{e}}^{\,a}_{AB}$$

Differential gravimetry in the IRRF

Inertial:

$$\ddot{\vec{x}}_{AB} = \ddot{\rho}\,\vec{e}^{\,a}_{AB} + 2\,\dot{\rho}\,\dot{\vec{e}}^{\,a}_{AB} + \rho\,\ddot{\vec{e}}^{\,a}_{AB}$$

IRRF:

$$\begin{aligned}\ddot{\vec{x}}_{AB,F} &= \ddot{\rho}\,\vec{e}^{\,a}_{AB,F} + 2\,\dot{\rho}\left(\vec{\omega} \times \vec{e}^{\,a}_{AB,F}\right) \\ &\quad + \rho\left(\vec{\omega} \times \left(\vec{\omega} \times \vec{e}_{AB,F}\right)\right) + \rho\left(\dot{\vec{\omega}} \times \vec{e}_{AB,F}\right)\end{aligned}$$

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Inertial: $$\ddot{\vec{x}}_{AB} = \ddot{\rho}\,\vec{e}^{\,a}_{AB} + 2\,\dot{\rho}\,\dot{\vec{e}}^{\,a}_{AB} + \rho\,\ddot{\vec{e}}^{\,a}_{AB}$$

IRRF: $$\begin{aligned}\ddot{\vec{x}}_{AB,F} &= \ddot{\rho}\,\vec{e}^{\,a}_{AB,F} + 2\,\dot{\rho}\left(\vec{\omega}\times\vec{e}^{\,a}_{AB,F}\right)\\ &\quad +\rho\left(\vec{\omega}\times\left(\vec{\omega}\times\vec{e}_{AB,F}\right)\right) + \rho\left(\dot{\vec{\omega}}\times\vec{e}_{AB,F}\right)\end{aligned}$$

with $\vec{\omega} = (\omega^a, \omega^c, \omega^r)^T$ and introducing $\nabla V_{AB,F}$:

$$\begin{aligned}\nabla V_{AB,F}\cdot\vec{e}^{\,a}_{AB,F} &= \ddot{\rho} - \rho\left((\omega^c)^2 + (\omega^r)^2\right)\\ \nabla V_{AB,F}\cdot\vec{e}^{\,c}_{AB,F} &= 2\,\dot{\rho}\,\omega^r - \rho\,\omega^a\omega^c + \rho\,\dot{\omega}^r\\ \nabla V_{AB,F}\cdot\vec{e}^{\,r}_{AB,F} &= -2\,\dot{\rho}\,\omega^c + \rho\,\omega^a\omega^r - \rho\,\dot{\omega}^c\end{aligned}$$

Differential gravimetry in the IRRF

Inertial: $$\ddot{\vec{x}}_{AB} = \ddot{\rho}\,\vec{e}^{\,a}_{AB} + 2\,\dot{\rho}\,\dot{\vec{e}}^{\,a}_{AB} + \rho\,\ddot{\vec{e}}^{\,a}_{AB}$$

IRRF: $$\begin{aligned}\ddot{\vec{x}}_{AB,F} &= \ddot{\rho}\,\vec{e}^{\,a}_{AB,F} + 2\,\dot{\rho}\left(\vec{\omega}\times\vec{e}^{\,a}_{AB,F}\right)\\ &\quad +\rho\left(\vec{\omega}\times\left(\vec{\omega}\times\vec{e}_{AB,F}\right)\right) + \rho\left(\dot{\vec{\omega}}\times\vec{e}_{AB,F}\right)\end{aligned}$$

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INSTANTANEOUS RELATIVE REFERENCE FRAME

What is the optimal choice for the IRRF?

Obviously given: $\vec{e}^{\,a}_{AB}$

"Ambiguity" for $\vec{e}^{\,r}_{AB}$:

- radial direction of GRACE A
- radial direction of GRACE B
- radial direction of midpoint
- …

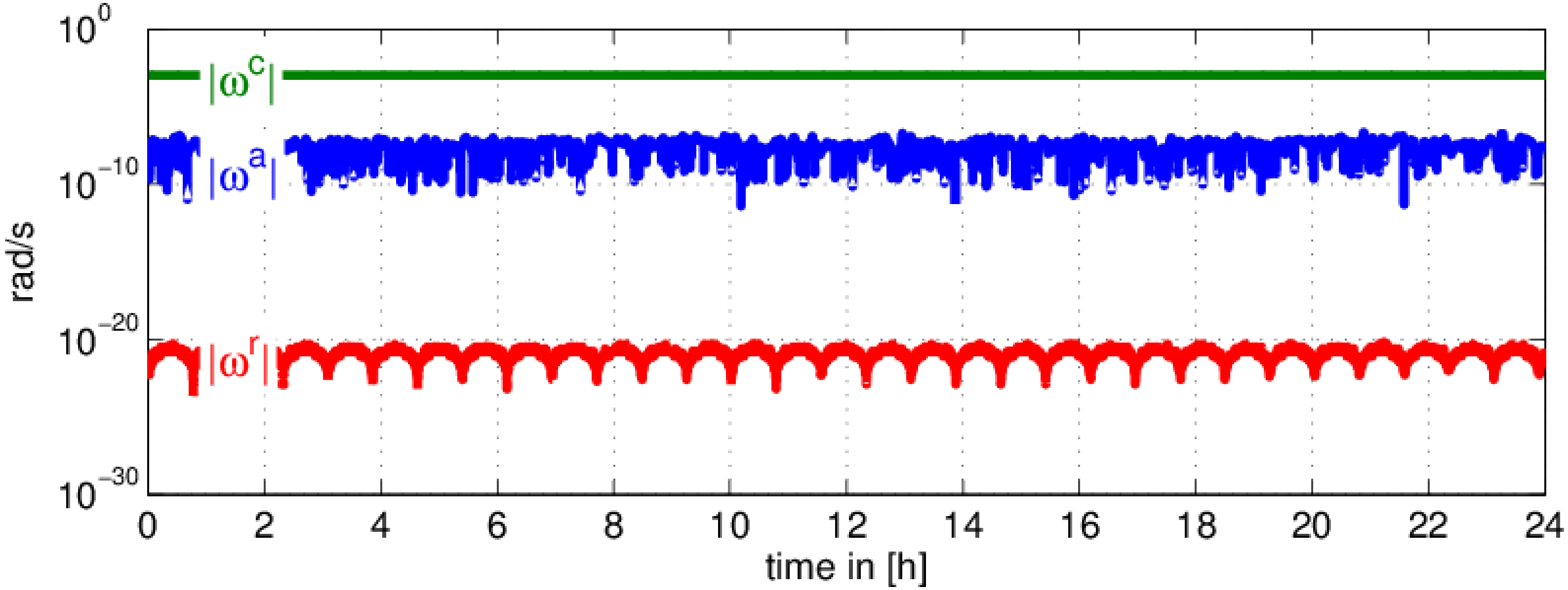
Other considerations:

- accessibility
- accuracy
- simplification
- physical meaning

Instantaneous relative reference frame

Most useful implementation at the current stage:

$$\vec{e}^{\,v}_{AB} = \frac{\dot{\vec{x}}_{AB}}{\|\dot{\vec{x}}_{AB}\|} \qquad \vec{e}^{\,c}_{AB} = \frac{\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}}{\|\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}\|} \qquad \vec{e}^{\,r}_{AB} = \frac{\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}}{\|\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}\|}$$



Instantaneous relative reference frame

Best approximation at the current stage:

$$\vec{e}^{\,v}_{AB} = \frac{\dot{\vec{x}}_{AB}}{\|\dot{\vec{x}}_{AB}\|} \qquad \vec{e}^{\,c}_{AB} = \frac{\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}}{\|\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}\|} \qquad \vec{e}^{\,r}_{AB} = \frac{\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}}{\|\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}\|}$$

Instantaneous relative reference frame

Best approximation at the current stage:

$$\vec{e}^{\,v}_{AB} = \frac{\dot{\vec{x}}_{AB}}{\|\dot{\vec{x}}_{AB}\|} \qquad \vec{e}^{\,c}_{AB} = \frac{\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}}{\|\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}\|} \qquad \vec{e}^{\,r}_{AB} = \frac{\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}}{\|\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}\|}$$

$$\begin{aligned}
\nabla V_{AB,F} \cdot \vec{e}^{\,a}_{AB,F} &= \ddot{\rho} && -\rho\,(\omega^c)^2 \\
\nabla V_{AB,F} \cdot \vec{e}^{\,c}_{AB,F} &= && -\rho\,\omega^a \omega^c \\
\nabla V_{AB,F} \cdot \vec{e}^{\,r}_{AB,F} &= -2\,\dot{\rho}\,\omega^c && -\rho\,\dot{\omega}^c
\end{aligned}$$

Instantaneous relative reference frame

Best approximation at the current stage:

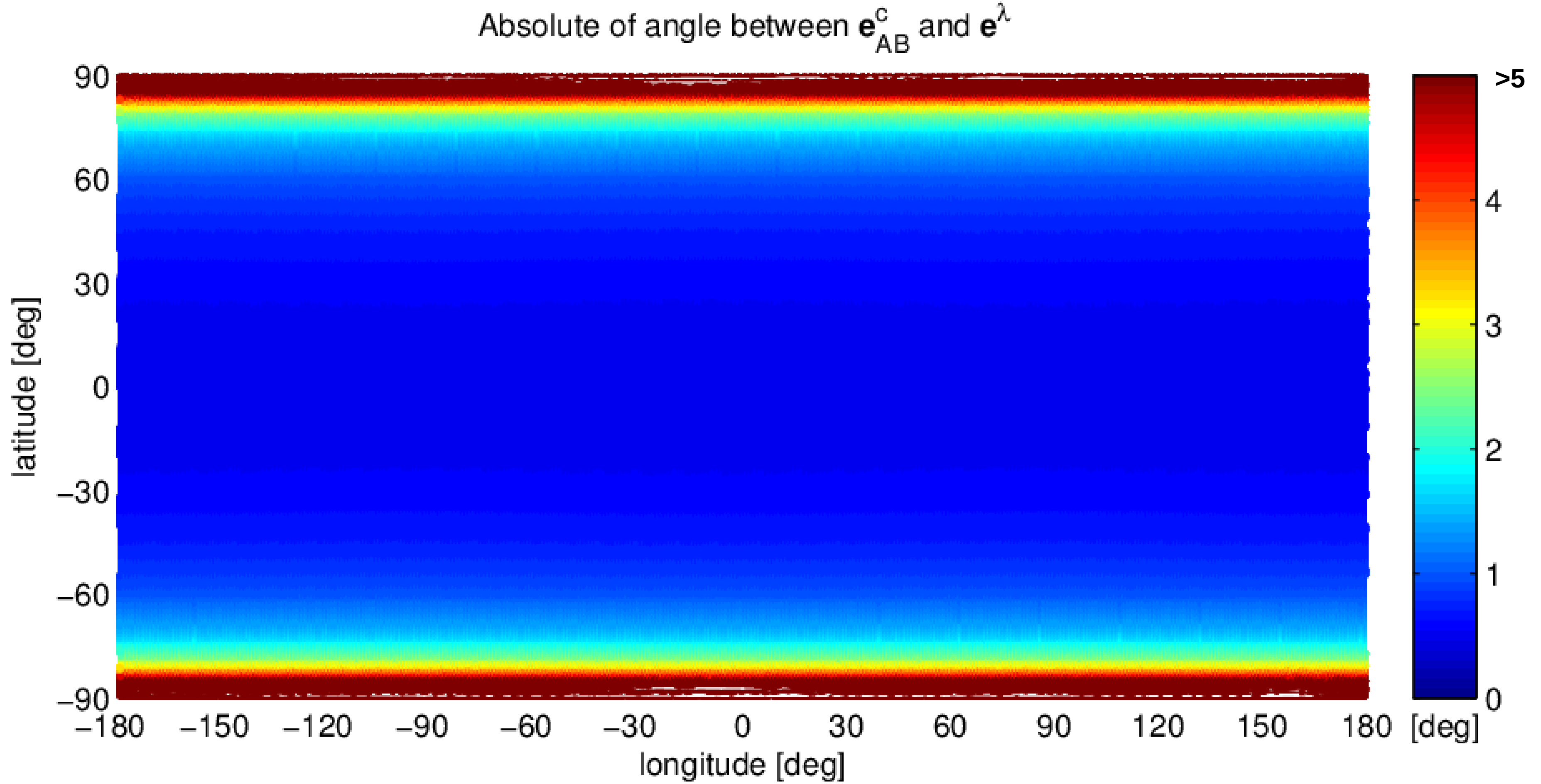
$$\vec{e}^{\,v}_{AB} = \frac{\dot{\vec{x}}_{AB}}{\|\dot{\vec{x}}_{AB}\|} \qquad \vec{e}^{\,c}_{AB} = \frac{\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}}{\|\vec{e}^{\,v}_{AB} \times \vec{e}^{\,a}_{AB}\|} \qquad \vec{e}^{\,r}_{AB} = \frac{\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}}{\|\vec{e}^{\,a}_{AB} \times \vec{e}^{\,c}_{AB}\|}$$

$$\begin{aligned}
\nabla V_{AB,F} \cdot \vec{e}^{\,a}_{AB,F} &= \ddot{\rho} && -\rho\,(\omega^c)^2 \\
\nabla V_{AB,F} \cdot \vec{e}^{\,c}_{AB,F} &= && -\rho\,\omega^a\omega^c \\
\nabla V_{AB,F} \cdot \vec{e}^{\,r}_{AB,F} &= -2\,\dot{\rho}\,\omega^c && -\rho\,\dot{\omega}^c
\end{aligned}$$

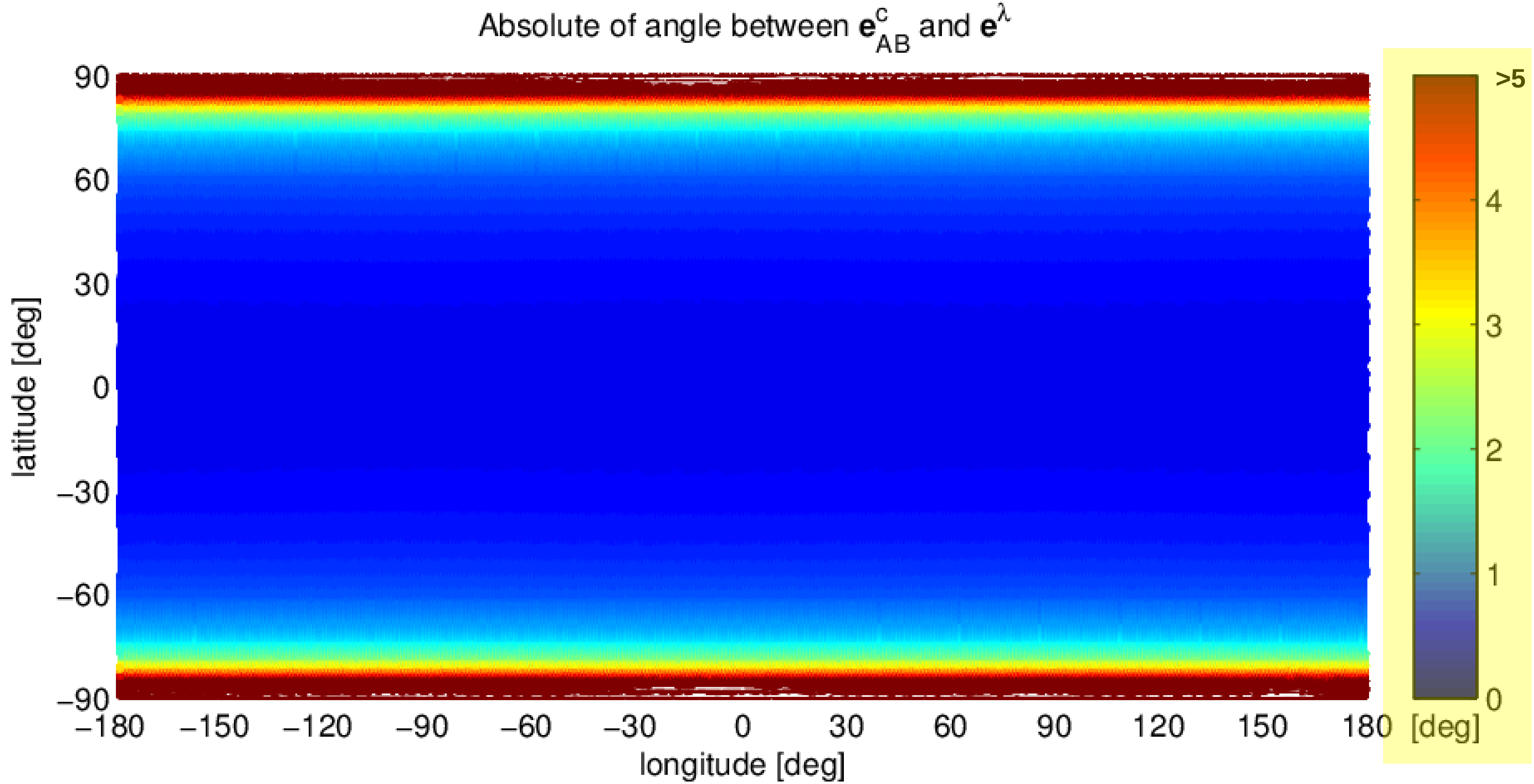
Note the similarity:

$$\begin{aligned}
\nabla V_{AB} \cdot \vec{e}^{\,a}_{AB} &= \ddot{\rho} && +\rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,a}_{AB} \\
\nabla V_{AB} \cdot \vec{e}^{\,c}_{AB} &= && +\rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,c}_{AB} \\
\nabla V_{AB} \cdot \vec{e}^{\,r}_{AB} &= 2\,\dot{\rho}\,\|\dot{\vec{e}}^{\,a}_{AB}\| && +\rho\,\ddot{\vec{e}}^{\,a}_{AB} \cdot \vec{e}^{\,r}_{AB}
\end{aligned}$$

Relation cross-track to East-West direction



Relation cross-track to East-West direction



Summary

- Derivation of the acceleration approach based on rotations
  → creating the chance of using star cameras instead of GPS
- (Free) choice of the moving frame allows for optimisation
- Determination of ω^c and $\dot{\omega}^c$ allows for a second (observed) component.
- Explanation of the low East-West sensitivity
- Necessity for GPS observations (determination of the frame)
- Open questions:
  - Can ω^c be determined by the star camera (accuracy) ?
  - Is it possible to find a frame which also fulfils: $\omega^a = 0$?

A new variant of the differential gravimetry approach

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