HODGE DUALITY OPERATORS ON LEFT COVARIANT EXTERIOR ALGEBRAS OVER TWO AND THREE DIMENSIONAL QUANTUM SPHERES

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ABSTRACT. Using non canonical braidings, we first introduce a notion of symmetric tensors and corresponding Hodge operators on a class of left-covariant 3d differential calculi over $\mathrm{SU}_q(2)$, then we induce Hodge operators on the left covariant 2d exterior algebra over the Podleś quantum sphere.

CONTENTS

1. INTRODUCTION

Following the formalism developed by Woronowicz [19], various aspects of the differential geometry induced on a large class of quantum groups $\mathcal{H}$ equipped with suitable bicovariant differential first order calculi (d, Γ) have been intensively studied.

Among those, the general problem of defining meaningful symmetric tensors and Hodge duality operators acting on the higher order differential calculi $\Gamma_{\sigma^{\pm}}$ constructed via the canonical braiding σ on $\Gamma^{\otimes 2} = \Gamma \otimes_{\mathcal{H}} \Gamma$ (and its inverse $\sigma^{-1} = \sigma^{-}$, the braiding being no longer the classical flip) for the calculus has been considered in detail in [7, 8, 10]. Given finite N-dimensional calculi such that $\dim \Gamma^{k}_{\sigma^{\pm}} = \dim \Gamma^{N-k}_{\sigma^{\pm}}$ for each space of exterior k- and $(N-k)$-forms, the properties of the canonical braiding and of its corresponding antisymmetriser operators $A^{(k)}_{\sigma^{\pm}}$ on $\Gamma^{k}_{\sigma^{\pm}}$ allow to define a rank two symmetric tensor over $\Gamma^{\otimes 2}$ and Hodge operators $\star_{\sigma^{\pm}} : \Gamma^{k}_{\sigma^{\pm}} \to \Gamma^{N-k}_{\sigma^{\pm}}$ satisfying $\star_{\sigma^{\pm}} \star_{\sigma^{\mp}} = 1$.

It is crucial in this formulation that the first order differential calculus is bicovariant, a condition which is sufficient to have a canonical braiding [11]. The properties of the canonical braiding also core the formalism developed in [12, 13] to introduce the concept of Riemannian quantum group

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and braided Killing form as a part of a more general formulation aimed to describe framings and coframings over quantum groups as a gauge theory of quantum differential forms.

A somehow reversed strategy for the specific example of the quantum SU(2) group equipped with the bicovariant $4D_+$ calculus gives in [23] results which largely agree to (and slightly generalize) those presented in the former approach. A Hodge operator is there meant as a bijection $\star_{\sigma^\pm} : \Gamma^k_{\sigma^\pm} \to \Gamma^{N-k}_{\sigma^\pm}$ whose square has, for a suitably defined class of symmetric tensors, the same degeneracy of the antisymmetrisers of the calculus.

The bicovariance of the calculus seems to play in this approach no explicit role, and this suggests that it is possible to study the problem of defining Hodge duality operators even on exterior algebras built over left-covariant calculi on quantum groups, provided they have a consistent – although not canonical – braiding. Presenting the first results obtained in this direction is one of the aim of the present paper.

Using the classification of [9], we equip the quantum group $\mathrm{SU}_q(2)$ with a set (that we call $\mathcal{K}$) of left covariant first order 3d calculi having a non canonical braiding σ and study the exterior algebras associated to the corresponding antisymmetriser operators. We introduce then Hodge operators acting on such exterior algebras: by *consistent* we mean that their squares present the same degeneracy of the antisymmetriser operators. Such Hodge operators exist for a class of properly defined symmetric tensors g acting on $\Gamma^{\otimes 2}$, with a notion of symmetry which does not necessarily coincide with the standard $\mathrm{g} \circ \sigma = \mathrm{g}$. The class $\mathcal{K}$ of calculi we consider contains the most famous Woronowicz' 3d calculus [18]: this paper then deepens the analysis on scalar products and duality operators presented in [22], and enlarges its results.

That the formalism we develop is consistent for any calculus in $\mathcal{K}$ poses the further question this paper aims to analyze. Is it possible to introduce a notion eventually selecting a proper and interesting subclass of elements in $\mathcal{K}$, i.e. of calculi on $\mathrm{SU}_q(2)$ among those that are being considered? We propose this condition to be the requirement that the previously introduced notion of symmetry for a tensor g does coincide with the standard one, namely that $\mathrm{g} \circ \sigma = \mathrm{g}$. This condition actually selects a subset $\tilde{\mathcal{K}} \subset \mathcal{K}$ of calculi on $\mathrm{SU}_q(2)$, a subset we can describe following a further interesting characterization.

It is well known that the Podleś standard sphere S^2_q is the quantum homogeneous space defined by a U(1)-coaction over $\mathrm{SU}_q(2)$; for a set $\mathcal{K}_\pi \subset \mathcal{K}$ of so called projectable calculi on $\mathrm{SU}_q(2)$ this topological Hopf fibration acquires compatible differential structures. The restriction to S^2_q of such projectable calculi gives in particular different isomorphic realizations of the unique 2d left covariant differential calculus introduced by Podleś himself [16]. Considering any of these realizations of the exterior algebra $\Gamma(\mathrm{S}^2_q)$ in terms of the frame bundle approach [13] allows to describe how the restriction of the above Hodge operators (on $\mathrm{SU}_q(2)$) meaningfully introduces a class (since they correspond to elements in $\mathcal{K}_\pi$) of different bijections as maps $\check{S} : \Gamma^k(\mathrm{S}^2_q) \to \Gamma^{2-k}(\mathrm{S}^2_q)$. It turns out that the operators $\check{S}$ have the expected degeneracy (i.e. the degeneracy of a Hodge duality on a classical 2 dimensional exterior algebra) only if they are induced by the Hodge operators on $\mathrm{SU}_q(2)$ equipped with the calculi $\tilde{\mathcal{K}} \subset \mathcal{K}$.

It means (this being the last result that this paper presents) that the formalism developed here allows to induce Hodge operators acting on the left invariant 2d exterior algebra $\Gamma(\mathrm{S}^2_q)$ starting from a suitable formulation of symmetric tensors and Hodge operators on the quantum group $\mathrm{SU}_q(2)$.

The paper is organised as follows. Section 2 describes the geometrical setting of the analysis, namely those aspects of differential calculi and exterior algebras over classical and quantum group we shall use, and present the class $\mathcal{K}$ of differential calculi over the quantum SU(2) we shall consider. Starting from a tensor whose coefficients give contraction map, section 3 present families of scalar products and corresponding dual Hodge operators. The example of the Woronowicz' calculus is used as a guide, the results are then extended to the whole class $\mathcal{K}$ of calculi. In section 4 we finally study Hodge operators for the quantum sphere S^2_q.

2. The geometrical setting

2.1. **A classical setting.** Consider a N-dimensional connected Lie group G given as the real form of a complex connected Lie group. Its group manifold is parallelizable: the space of 1-forms $\Omega^1(G)$ is a free bicovariant N-dimensional $\mathcal{A}(G)$-bimodule on the basis of left (right) invariant $\{\phi^a\}$ ($\{\eta^a\}$) 1-forms. The associated first order differential calculus is given by $(\mathrm{d}, \Omega^1(G))$ with the exterior differential given by $\mathrm{d}h = (L_a h)\omega^a = (R_a h)\eta^a$ in terms of the action of the dual left (right) invariant derivations L_a ($\{R_a\}$) on $h \in \mathcal{A}(G)$.

The standard flip given[1] on a basis by $\tau : \omega^a \otimes \omega^b \mapsto \omega^b \otimes \omega^a$ is a braiding on $\Omega^{\otimes 2}$; the corresponding antisymmetriser operators $A^{(k)} : \Omega^{\otimes k} \to \Omega^{\otimes k}$ ($k \in \mathbb{N}$) give the exterior algebra $\Omega^\wedge = (\oplus_{k=1}^N \Omega^k, \wedge)$ as $\Omega^{\otimes k} \supset \Omega^k = (\mathrm{Range}\, A^{(k)}) \simeq \Omega^{\otimes k}/\ker A^{(k)}$, with

$$\omega^{a_1} \wedge \ldots \wedge \omega^{a_k} \;=\; A^{(k)}(\omega^{a_1} \otimes \ldots \otimes \omega^{a_k}) \;=\; \sum_{\pi \in S_k} (-1)^\pi \omega^{\pi(a_1)} \otimes \ldots \otimes \omega^{\pi(a_k)} \tag{2.1}$$

where S_k is the set of permutations of k elements. The differential calculus $(\Omega^\wedge, \mathrm{d})$ is given by equipping the exterior algebra with the unique consistent graded derivative operator $\mathrm{d} : \Omega^k \to \Omega^{k+1}$ satisfying $\mathrm{d}^2 = 0$ and a graded Leibniz rule. Every Ω^k is a bicovariant free $\mathcal{A}(G)$-bimodule with $\dim \Omega^k = N!/(k!(N-k)!)$ and $\Omega^k = \emptyset$ for $k > N$. The antisymmetrisers have a completely degenerate spectral decomposition,

$$A^{(k)}(\omega^{a_1} \wedge \ldots \wedge \omega^{a_k}) = k!(\omega^{a_1} \wedge \ldots \wedge \omega^{a_k}). \tag{2.2}$$

We consider a non degenerate tensor $\mathrm{g} : \Omega^1 \times \Omega^1 \to \mathcal{A}(G)$, whose components we use to set an $\mathcal{A}(G)$-bimodule contraction $\mathrm{g} : \Omega^{\otimes k} \times \Omega^{\otimes(k+k')} \to \Omega^{k'}$ given on a basis by

$$\mathrm{g}(\omega^{a_1} \otimes \ldots \otimes \omega^{a_k}, \omega^{b_1} \otimes \ldots \otimes \omega^{b_{k+k'}}) \;=\; \left\{\Pi_{j=1,\ldots,k}\, \mathrm{g}(\omega^{a_j}, \omega^{b_j})\right\} \omega^{b_{k+1}} \otimes \ldots \otimes \omega^{b_{k'}}; \tag{2.3}$$

the properties of the antisymmetriser operators allow then to prove that the position (see (2.1))

$$\mathrm{g}(\omega^{a_1} \wedge \ldots \wedge \omega^{a_k}, \omega^{b_1} \wedge \ldots \wedge \omega^{b_{k+k'}}) \;=\; g(A^{(k)}(\omega^{a_1} \otimes \ldots \otimes \omega^{a_k}), A^{(k+k')}(\omega^{b_1} \otimes \ldots \otimes \omega^{b_{k+k'}})) \tag{2.4}$$

consistently generalizes the contraction map (2.3) to $\mathrm{g} : \Omega^k \times \Omega^{k+k'} \to \Omega^{k'}$. If $\mu = m\,\theta = \mu^*$ is a volume form, with $\theta = \omega^1 \wedge \ldots \wedge \omega^N$ the top form corresponding to an ordering of the basis elements $\{\omega^a\}$ and $m \in \mathbb{C}$), we define the operator $S : \Omega^k \to \Omega^{N-k}$ by

$$S(\phi) \;=\; \frac{1}{k!}\, \mathrm{g}(\phi, \mu), \tag{2.5}$$

on any k-form ϕ. The following equivalence holds ($\phi, \phi' \in \Omega^1$)

$$\mathrm{g}(\phi, \phi') \;=\; \mathrm{g}(\phi', \phi) \qquad \Leftrightarrow \qquad S^2(\phi) = (-1)^{N-1}\{S^2(1)\}\phi. \tag{2.6}$$

The tensor g is symmetric if and only if the action of the restiction of S^2 on Ω^1 is a constant depending on the volume[2]; given a symmetric g one has also that $S^2(\phi) = (-1)^{k(N-k)}\{S^2(1)\}\phi$ with $\phi \in \Omega^k$. But such an operator S is not (yet) an Hodge operator: it has to be real, and the reality condition comes as the equivalence

$$\mathrm{g}(\phi, \phi')^* \;=\; \mathrm{g}(\phi'^*, \phi^*) \qquad \Leftrightarrow \qquad S(\phi^*) \;=\; (S(\phi))^*. \tag{2.7}$$

The compatibility of the action of S with the hermitian conjugation on Ω^1 and Ω^{N-1} turns out to be sufficient to have $[S, ^*] = 0$ on the whole exterior algebra $\Omega^\wedge$. Such a symmetric and real operator

[1] We denote $\Omega^{\otimes k} = \Omega^1(G) \otimes_{\mathcal{A}(G)} \cdots \otimes_{\mathcal{A}(G)} \Omega^1(G)$ and drop the overall obvious dependence on G.

[2] It is true that the factor can be arbitrary, and that g is symmetric if and only if $S^2(\phi^a) = \zeta\,\phi^a$ with $0 \neq \zeta \in \mathbb{C}$. The choice in (2.6) will give the possibility of the usual overall normalization.

S is then recovered as the Hodge operator corresponding to the (inverse) of the (metric) tensor g on the group manifold. The choice $S^2(1) = \mathrm{sgn}(\mathrm{g})$ fixes the modulus of the scale parameter m of the volume so to have

$$S^2(\phi) = \mathrm{sgn}(\mathrm{g})(-1)^{k(N-k)}\phi$$

for any $\phi \in \Omega^k$.

Hodge operators can be introduced also on homogeneous spaces. Let $K \subset G$ be a compact Lie subgroup of G. The quotient of its right action $\mathrm{r}_k(g) = g\,k$ for $k \in K$ and $g \in G$ gives a principal fibration $\pi : G \to G/K$. A homogeneous space is not necessarily parallelizable: the exterior algebra $\Omega(G/K) \subset \Omega(G)$ is given by horizontal and right K-invariant forms on G,

$$\Omega(G/K) = \{\psi \in \Omega(G) : i_{X_V}\psi = 0 ; \mathrm{r}_k^*(\psi) = \psi\}, \tag{2.8}$$

with X_V the vertical fields of the fibrations (i.e. the infinitesimal (left-invariant) generators of the right K action on G), and r_k^* the natural pull-back action to $\Omega(G)$. The $\Omega^s(G/K)$ sets (with $0 < s < N'$) are no longer free $\mathcal{A}(G/K)$-bimodules; the dimension of the exterior algebra $\Omega(G/K)$ is given as the highest integer N' so that $\Omega^{N'+1}(G/K) = \emptyset$, and coincides with $N' = \dim G - \dim K$. The set $\Omega^{N'}(G/K)$ is indeed a free 1-dimensional $\mathcal{A}(G/K)$ bimodule with a basis element given by $\check{\theta} = i_{X_{V^1}} \cdots i_{X_{V^{\dim K}}}\theta$ for a basis X_{V^a} of the Lie algebra of vertical vector fields of the fibration. We have then a consistent (up to scalars) left invariant volume form $\check{\mu} = \check{\mu}^*$ on the homogeneous space: if we consider right K-invariant metric tensors g on G whose restriction to the homogeneus space G/K is non degenerate, then the map $\check{S} : \Omega^j(G/K) \to \Omega^{N'-j}(G/K)$ given by

$$\check{S}(\psi) = \frac{1}{j!}\,\mathrm{g}(\psi, \check{\mu}) \tag{2.9}$$

is a well-defined bijection, satisfying the relation $\check{S}^2(\psi) = \mathrm{sgn}(\mathrm{g}(\check{\mu}, \check{\mu}))(-1)^{s(N'-s)}\psi$ for any $\psi \in \Omega^k(G/K)$ after a natural normalisation.

Both the Hodge operators above can be formulated following a different path. Starting from a non degenerate tensor g, a sesquilinear map $\langle\ ,\ \rangle_G : \Omega^k \times \Omega^k \to \mathcal{A}(G)$ can be defined by

$$\langle \phi, \phi' \rangle_G = \frac{1}{k!}\,\mathrm{g}(\phi^*, \phi'); \tag{2.10}$$

the equation

$$\phi^* \wedge T(\phi') = (\langle \phi, \phi' \rangle_G)\mu \tag{2.11}$$

uniquely defines a bijective $T : \Omega^k \to \Omega^{N-k}$ with $T(1) = \mu$, $T(\mu) = m$. It is immediate to check the equivalence $T(\phi) = S(\phi)$ on any $\phi \in \Omega^k$, which comes from

$$\mathrm{g}(\phi, \phi')\,\mu = \phi^* \wedge \mathrm{g}(\phi', \mu). \tag{2.12}$$

for any pair $\phi, \phi' \in \Omega^k$. It is analogously immediate to see that the restriction

$$\langle \psi, \psi' \rangle_{G/K} = \langle \psi, \psi' \rangle_G$$

of the sesquilinear allows to consistently set

$$\psi^* \wedge \check{T}(\psi') = (\langle \psi, \psi' \rangle_{G/K})\check{\mu} \tag{2.13}$$

as a definition for the operator $\check{T} : \Omega^j(G/K) \to \Omega^{N'-j}(G/K)$. One has clearly $\check{T} = \check{S}$ as the Hodge operators on $\Omega(G/K)$ corresponding to projecting the right K-invariant (inverse) metric tensor g onto the homogeneous space.

2.2. **A quantum setting: left covariant differential calculi over quantum groups.** Consider $\mathcal{H}$ to be the unital $*$-Hopf algebra $\mathcal{H} = (\mathcal{H}, \Delta, \varepsilon, S)$ over $\mathbb{C}$, with Γ an $\mathcal{H}$-bimodule. The pair (Γ, d) is a (first order) differential calculus over $\mathcal{H}$ provided the linear map $\mathrm{d} : \mathcal{H} \to \Gamma$ satisfies the Leibniz rule, $\mathrm{d}(h\,h') = (\mathrm{d}h)h' + h\,\mathrm{d}h'$ for $h, h' \in \mathcal{H}$, and Γ is generated by $\mathrm{d}(\mathcal{H})$ as a $\mathcal{H}$-bimodule. It is called a $*$-calculus provided there is an anti-linear involution $* : \Gamma \to \Gamma$ such that $(h_1(\mathrm{d}h)h_2)^* = h_2^*(\mathrm{d}(h^*))h_1^*$ for any $h, h_1, h_2 \in \mathcal{H}$.

A first order differential calculus is said left covariant provided a left coaction $\Delta_L^{(1)} : \Gamma \to \mathcal{H} \otimes \Gamma$ exists, such that $\Delta_L^{(1)}(\mathrm{d}h) = (1\otimes\mathrm{d})\Delta(h)$ and $\Delta_L^{(1)}(h_1\,\alpha\,h_2) = \Delta(h_1)\Delta_L^{(1)}(\alpha)\Delta(h_2)$ for any $h, h_1, h_2 \in \mathcal{H}$ and $\alpha \in \Gamma$. The set Γ turns out to be a free left covariant $\mathcal{H}$-bimodule, with a free basis Γ_{L} of left invariant one forms, namely the elements $\omega_a \in \Gamma$ such that $\Delta_L^{(1)}(\omega_a) = 1 \otimes \omega_a$. Its dimension is called the dimension of the first order calculus. The map $\mathfrak{R} : \mathcal{H} \to \Gamma_{\mathrm{L}}$ given by

$$\mathfrak{R}(h) \;=\; S(h_{(1)})\,\mathrm{d}h_{(2)} \tag{2.14}$$

allows to characterise left covariant first order differential calculi: they correspond to the choice of a right ideal $\mathcal{Q} \subset \ker\varepsilon$ with

$$\mathcal{Q} \;=\; \{h \,\in\, \ker\,\varepsilon \,:\, \mathfrak{R}(h) = 0\}; \tag{2.15}$$

there is a left $\mathcal{H}$-modules isomorphism given by $\Gamma \simeq \mathcal{H} \otimes (\ker\varepsilon/\mathcal{Q})$, and a complex vector space isomorphism $\Gamma_{\mathrm{L}} \simeq \ker\varepsilon/\mathcal{Q}$.

The tangent space of the calculus is the complex vector space of elements out of $\mathcal{H}'$ – the dual space $\mathcal{H}'$ of functionals on $\mathcal{H}$ – defined by $\mathcal{X}_{\mathcal{Q}} := \{X \in \mathcal{H}' \;:\; X(1) = 0,\, X(Q) = 0,\, \forall\, Q \in \mathcal{Q}\}$. One has that (Γ, d) is a $*$ calculus if and only if its quantum tangent space is $*$-invariant. There exists a unique bilinear form

$$\{\ ,\ \} : \mathcal{X}_{\mathcal{Q}} \times \Gamma, \qquad \{X, x\mathrm{d}y\} := \varepsilon(x)X(y), \tag{2.16}$$

giving a non-degenerate dual pairing between the vector spaces $\mathcal{X}_{\mathcal{Q}}$ and Γ_{L}. The dual space $\mathcal{H}'$ has natural left and right (mutually commuting) actions on $\mathcal{H}$:

$$X \triangleright h := h_{(1)}X(h_{(2)}), \qquad h \triangleleft X := X(h_{(1)})h_{(2)}. \tag{2.17}$$

If the vector space $\mathcal{X}_{\mathcal{Q}}$ is finite dimensional, its elements belong to the dual Hopf algebra $\mathcal{H}' \supset \mathcal{H}^o = (\mathcal{H}^o, \Delta_{\mathcal{H}^o}, \varepsilon_{\mathcal{H}^o}, S_{\mathcal{H}^o})$, defined as the largest Hopf $*$-subalgebra contained in $\mathcal{H}'$. In such a case the $*$-structures are compatible with both actions,

$$X\triangleright h^* = ((S(X))^*\triangleright h)^*, \qquad h^*\triangleleft X = (h\triangleleft(S(X))^*)^*,$$

for any $X \in \mathcal{H}^o,\; h \in \mathcal{H}$ and the exterior derivative can be written as:

$$\mathrm{d}h := \sum_a\; (X_a \triangleright h)\;\omega_a = \sum_a \omega_a(-S^{-1}(X_a))\triangleright h, \tag{2.18}$$

where $\{X_a, \omega_b\} = \delta_{ab}$. The twisted Leibniz rule of derivations of the basis elements X_a is dictated by their coproduct:

$$\Delta_{\mathcal{H}^o}(X_a) = 1 \otimes X_a + \sum\nolimits_b X_b \otimes f_{ba}, \tag{2.19}$$

where the $f_{ab} \in \mathcal{H}^o$ consitute an algebra representation of $\mathcal{H}$, also controlling the $\mathcal{H}$-bimodule structure of $\Omega^1(\mathcal{H})$:

$$\omega_a h = \sum\nolimits_b (f_{ab} \triangleright h)\omega_b, \qquad h\omega_a = \sum\nolimits_b \omega_b\left((S^{-1}(f_{ab})) \triangleright h\right), \qquad \text{for} \quad h \in \mathcal{H}. \tag{2.20}$$

In order to build an exterior algebra over the FODC (d, Γ), consider $\Gamma^{\otimes k}$ as the k-fold tensor product $\Gamma \otimes_{\mathcal{H}} \cdots \otimes_{\mathcal{H}} \Gamma$ (with $\Gamma^0 = \mathcal{H}$) and $\Gamma^{\otimes} = \oplus_{k=0}^{\infty}\Gamma^{\otimes k}$, which is an algebra with multiplication $\otimes_{\mathcal{H}}$. From the map

$$\mathcal{S} \,:\, \mathcal{H} \to \Gamma_{\mathrm{L}}^{\otimes 2}, \qquad\qquad x \,\mapsto\, \sum \mathfrak{R}(x_{(1)}) \otimes \mathfrak{R}(x_{(2)}), \tag{2.21}$$

let $\mathcal{S}_{\mathcal{Q}} \subset \Gamma^{\otimes 2}$ be the 2-sided ideal in $\Gamma^{\otimes}$ generated by the range of its restriction to $x \in \mathcal{Q}$; the quotient $\Gamma^k_u = \Gamma^{\otimes k}/(\mathcal{S}_{\mathcal{Q}} \cap \Gamma^{\otimes k})$ is a well defined $\mathcal{H}$-bimodule.

This exterior algebra turns out to be a differential calculus over $\mathcal{H}$ once the exterior derivative d is extended as a graded derivation with $\mathrm{d}^2 = 0$, satisfying a graded Leibniz rule (that is $\mathrm{d}(\omega \wedge \omega') = (\mathrm{d}\omega) \wedge \omega' + (-1)^m \omega \wedge \mathrm{d}\omega'$ for any $\omega \in \Gamma^m$). The quotient Γ_u also inherits the natural extension of the left coaction of $\mathcal{H}$, which is compatible with the action of the operator d, so to have a left covariant differential calculus (d, Γ_u) over the FODC which is universal: any other left covariant differential calculus (d, Γ) over $\mathcal{H}$ with $\Gamma^1 = \Gamma$ is a suitable quotient of the universal one.

Given the left covariant bimodule Γ over $\mathcal{H}$, an invertible linear mapping $\sigma : \Gamma \otimes_{\mathcal{H}} \Gamma \to \Gamma \otimes_{\mathcal{H}} \Gamma$ is called a braiding for Γ provided σ is a $\mathcal{H}$-bimodule homomorphism which commutes with the left coaction on Γ and satisfies the braid equation

$$(1 \otimes \sigma) \circ (\sigma \otimes 1) \circ (1 \otimes \sigma) = (\sigma \otimes 1) \circ (1 \otimes \sigma) \circ (\sigma \otimes 1) \tag{2.22}$$

on $\Gamma^{\otimes 3}$. The next natural requirement is that $(1 - \sigma)(\mathcal{S}_{\mathcal{Q}} \cap \Gamma^{\otimes 2}) = 0$. Such a braiding neither needs to exist nor it is unique for a given left covariant differential calculus over $\mathcal{H}$: this is the main difference with bicovariant differential calculi, which present a canonical braiding. If a braiding does exists, then the corresponding antisymmetriser operators $A^{(k)} : \Gamma^{\otimes k} \to \Gamma^{\otimes k}$ are well defined and their ranges give the differential calculus $(\mathrm{d}, \Gamma_\sigma)$ since $\ker A^{(k)} \supset \mathcal{S}_{\mathcal{Q}}$ is a 2-sided graded ideal in $\Gamma^{\otimes k}$.

2.3. **A class of left covariant differential calculi over the quantum SU(2).** As quantum group $\mathrm{SU}_q(2)$ we consider the compact real form of the quantum group $\mathrm{SL}_q(2)$ and, following [20], we formulate it as the polynomial unital $*$-algebra $\mathcal{A}(\mathrm{SU}_q(2)) = (\mathrm{SU}_q(2), \Delta, S, \varepsilon)$ generated by elements a and c which we write using the matrix notation

$$u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}. \tag{2.23}$$

The Hopf algebra structure can then be expressed as

$$uu^* = u^*u = 1, \quad \Delta\, u = u \otimes u, \quad S(u) = u^*, \quad \varepsilon(u) = 1$$

with the deformation parameter $q \in \mathbb{R}$.

In order to describe the quantum tangent spaces of the calculi that will be later introduced, we consider the set of functionals given by the unital Hopf $*$-algebra $\mathcal{A}(\widetilde{\mathrm{SU}_q(2)})$ over $\mathbb{C}$, satisfying the inclusions $\mathcal{A}(\mathrm{SU}_q(2))^o \supset \mathcal{A}(\widetilde{\mathrm{SU}_q(2)}) \supset \mathcal{U}_q(\mathfrak{su}(2))$ with $\mathcal{A}(\mathrm{SU}_q(2))^o$ the Hopf dual $*$-algebra and $\mathcal{U}_q(\mathfrak{su}(2))$ the universal envelopping algebra of $\mathcal{A}(\mathrm{SU}_q(2))$. As an algebra is $\mathcal{A}(\widetilde{\mathrm{SU}_q(2)})$ generated by the five elements $\{K^{\pm 1}, E, F, \varepsilon_-\}$, with $KK^{-1} = 1$ fullfilling the relations[3]:

$$\begin{aligned} \varepsilon_- K^{\pm} = K^{\pm}\varepsilon_-, &\qquad \varepsilon_-\varepsilon_- = 1, \\ \varepsilon_- E = E\varepsilon_-, &\qquad \varepsilon_- F = F\varepsilon_-, \\ K^{\pm}E = q^{\pm}EK^{\pm}, &\qquad K^{\pm}F = q^{\mp}FK^{\pm}, \\ [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}. & \end{aligned} \tag{2.24}$$

[3] We shall also denote $K^+ = K$, $K^- = K^{-1}$. It is clear that $\mathcal{A}(\widetilde{\mathrm{SU}_q(2)})$ is generated by the universal envelopping $\mathcal{U}_q(\mathfrak{su}(2))$ algebra together with the $\mathcal{A}(\mathrm{SU}_q(2))$-character ε_- acting as $\varepsilon_-(a) = \varepsilon_-(a^*) = -1$; $\varepsilon_-(c) = \varepsilon_-(c^*) = 0$.

The $*$-structure is $K^* = K$, $E^* = F$, $\varepsilon_-^* = \varepsilon_-$, while the Hopf algebra structures are

$$\Delta(K^{\pm}) = K^{\pm} \otimes K^{\pm}, \qquad \Delta(E) = E \otimes K + K^{-1} \otimes E,$$
$$\Delta(F) = F \otimes K + K^{-1} \otimes F, \qquad \Delta(\varepsilon_-) = \varepsilon_- \otimes \varepsilon_-,$$
$$S(K) = K^{-1}, \qquad S(E) = -qE, \qquad S(F) = -q^{-1}F, \qquad S(\varepsilon_-) = \varepsilon_-,$$
$$\varepsilon(K) = \varepsilon(\varepsilon_-) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$

The only non zero terms of its action on $\mathcal{A}(\mathrm{SU}_q(2))$ is given on the generators by

$$K^{\pm}(a) = q^{\mp 1/2}, \qquad K^{\pm}(a^*) = q^{\pm 1/2}, \qquad E(c) = 1, \qquad F(c^*) = -q^{-1},$$
$$\varepsilon_-(a) = \varepsilon_-(a^*) = -1. \tag{2.25}$$

Given the algebra $\mathcal{A}(\mathrm{U}(1)) := \mathbb{C}[z, z^*]/< zz^* - 1 >$, the map

$$\pi : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{U}(1)), \qquad \pi(a) = z, \quad \pi(a^*) = z^*, \quad \pi(c) = \pi(c^*) = 0 \tag{2.26}$$

is a surjective Hopf $*$-algebra homomorphism, so that $\mathrm{U}(1)$ is a quantum subgroup of $\mathrm{SU}_q(2)$ with right coaction:

$$\delta_R := (\mathrm{id} \otimes \pi) \circ \Delta : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1)). \tag{2.27}$$

This right coaction gives a decomposition

$$\mathcal{A}(\mathrm{SU}_q(2)) = \oplus_{n \in \mathbb{Z}} \mathcal{L}_n, \qquad \mathcal{L}_n := \{x \in \mathcal{A}(\mathrm{SU}_q(2)) \, : \, \delta_R(x) = x \otimes z^{-n}\}, \tag{2.28}$$

with $\mathcal{A}(\mathrm{S}^2_q) = \mathcal{L}_0$ the algebra of the standard Podleś sphere, each $\mathcal{A}(\mathrm{S}^2_q)$-bimodule $\mathcal{L}_n$ giving the set of (charge n) $\mathrm{U}(1)$-coequivariant maps for the topological quantum principal bundle $\mathcal{A}(\mathrm{S}^2_q) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$.

From [9] we know a classification of left covariant differential calculi over the quantum group $\mathrm{SL}_q(2)$. Among them we select those calculi which are compatible with the reality structure (the anti-hermitian involution) giving $\mathrm{SU}_q(2)$, and which present a consistent braiding. This means that we equip $\mathrm{SU}_q(2)$ with the left covariant differential calculi satisfying the following properties:

- Γ is a left covariant bimodule;
- a basis of Γ_{L} is given by $\mathfrak{R}(c), \mathfrak{R}(c^*), \mathfrak{R}(a - a^*)$;
- for the corresponding universal differential calculus it is $\dim \Gamma_u^{\wedge 2} \geq 3$;
- Γ is Hopf-invariant, i.e. for the corresponding right ideal $\mathcal{Q} \subset \ker \varepsilon$ the equality $\varphi(\mathcal{Q}) = \mathcal{Q}$ for any Hopf algebra automorphism φ on $\mathcal{A}(\mathrm{SU}_q(2))$ holds.
- On $\Gamma^{\otimes 2}$ a consistent braiding exists.

We are then left with seven (up to isomorphisms) such calculi, and we denote this class by $\mathcal{K}$. We present these calculi giving (2.15) the generators of the right ideals $\mathcal{Q}_j \subset \ker \varepsilon$ (with $j = 1, \ldots, 7$) they are characterised by, together with a basis of their quantum tangent spaces:

(1) $\mathcal{X}_{\mathcal{Q}_1} = \{X_z = \frac{K^{-2}-1}{q-1}, \quad X_+ = q^{-1/2} E K^{-1}, \quad X_- = q^{1/2} F K^{-1}\},$

$$\mathcal{Q}_1 = \{a + qa^* - (1+q), c^2, c^{*2}, cc^*, (a-q)c, (a-q)c^*\}; \tag{2.29}$$

(2) $\mathcal{X}_{\mathcal{Q}_2} = \{X_z \,=\, \frac{\varepsilon_- K^2 - 1}{q+1}, \quad X_+ \,=\, -q^{-1/2}\varepsilon_- EK^{-1}, \quad X_- \,=\, -q^{1/2}\varepsilon_- FK^{-1}\},$

$$\mathcal{Q}_2 \,=\{a - qa^* - (1-q), c^2, c^{*2}, cc^*, (a+q)c, (a+q)c^*\}; \tag{2.30}$$

(3) $\mathcal{X}_{\mathcal{Q}_3} = \{X_z \,=\, \frac{K^{-4}-1}{q^2-1}, \quad X_+ \,=\, q^{-3/2}EK^{-3}, \quad X_- \,=\, q^{3/2}FK^{-3}\},$

$$\mathcal{Q}_3 \,=\{a + q^2 a^* - (1+q^2), c^2, c^{*2}, cc^*, (a-q^2)c, (a-q^2)c^*\}; \tag{2.31}$$

(4) $\mathcal{X}_{\mathcal{Q}_4} = \{X_z \,=\, \frac{K^2-1}{q^{-1}-1}, \quad X_+ \,=\, q^{1/2}EK, \quad X_- \,=\, q^{-1/2}FK\},$

$$\mathcal{Q}_4 \,=\{a + q^{-1} a^* - (1+q^{-1}), c^2, c^{*2}, cc^*, (a-1)c, (a-1)c^*\}; \tag{2.32}$$

(5) $\mathcal{X}_{\mathcal{Q}_5} = \{X_z \,=\, \frac{\varepsilon_- K^2-1}{q^{-1}+1}, \quad X_+ \,=\, q^{1/2}EK, \quad X_- \,=\, q^{-1/2}FK\},$

$$\mathcal{Q}_5 \,=\{a - q^{-1} a^* - (1-q^{-1}), c^2, c^{*2}, cc^*, (a-1)c, (a-1)c^*\}; \tag{2.33}$$

(6) $\mathcal{X}_{\mathcal{Q}_6} = \{X_z \,=\, q(q^2-1)\{FEK^2 \,+\, \frac{q^3(K^4-1)}{(q^2-1)^2}\}, \quad X_+ \,=\, q^{1/2}EK, \quad X_- \,=\, q^{-1/2}FK\},$

$$\mathcal{Q}_6 \,=\{a + q^{-4} a^* - (1+q^{-4}), c^2, c^{*2}, cc^* + (q^3-q)(a-1), (a-1)c, (a-1)c^*\}; \tag{2.34}$$

(7) $\mathcal{X}_{\mathcal{Q}_7} = \{X_z = (1-q^{-2})^{-1}\frac{1-K^4}{1-q^{-2}}, \quad X_+ = q^{1/2}EK, \quad X_- = q^{-1/2}FK\},$

$$\mathcal{Q}_7 \,=\{a + q^{-2} a^* - (1+q^{-2}), c^2, c^{*2}, cc^*, (a-1)c, (a-1)c^*\}. \tag{2.35}$$

An immediate inspection shows that the first order differential calculus $\mathcal{Q}_7$ is the one introduced by Woronowicz [18]; calculi (2) and (5) are obtained by the calculi (1) and (4) after mapping $q \to -q$. The first order calculi associated to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ come as quotients of the four dimensional bicovariant $4D_\pm$ calculi introduced in [19].

Exterior algebras and differential calculi built over the first order calculi in $\mathcal{K}$ using the braidings from [9] present interesting common aspects which can be easily proved by straightforward although long computations that we prefer to omit. We remark that a general proof for these results does not exist, since the braidings we adopted are not canonical. By presenting them in the following lines, we also omit to explicitly write any dependence (of the bimodules, forms, braidings and antisymmetrisers) on index $j = (1, \dots, 7)$ labelling the calculi.

Given the basis of the quantum tangent space $\mathcal{X}_{\mathcal{Q}}$, exact one-forms can be written (2.18) as

$$\mathrm{d}x \,=\, \sum_a (X_a \triangleright x)\omega_a \qquad\qquad a = \{\pm, z\} \tag{2.36}$$

with $x \in \mathcal{A}(\mathrm{SU}_q(2))$ on the dual basis of left-invariant one forms. The antilinear hermitian conjugation on Γ_{L} is $\omega_-^* \,=\, -\omega_+$, $\omega_z^* \,=\, -\omega_z$. The $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule is right $\mathrm{U}(1)$-covariant with respect to the natural extension $\delta_R^{(1)} : \Gamma \to \Gamma \otimes \mathcal{A}(\mathrm{U}(1))$ to one forms of the coaction (2.27), set by

$$\delta_R^{(1)}(\omega_z) = \omega_z \otimes 1, \qquad\qquad \delta_R^{(1)}(\omega_\pm) = \omega_\pm \otimes z^{\pm 2}. \tag{2.37}$$

The braiding and its inverse σ, $\sigma^{-1}: \Gamma^{\otimes 2} \to \Gamma^{\otimes 2}$ are compatible with this U(1) grading, and have the following spectral decomposition,

$$\begin{aligned}(1-\sigma)(q^2+\sigma) &= 0, \\ (1-\sigma^{-1})(q^{-2}+\sigma^{-1}) &= 0\end{aligned} \tag{2.38}$$

with

$$\begin{aligned}\dim\ker(1-\sigma^{\pm}) &= 6, \\ \dim\ker(q^{\pm 2}+\sigma^{\pm}) &= 3.\end{aligned} \tag{2.39}$$

The antisymmetriser operators $A^{(k)}_{\sigma^{\pm}}: \Gamma^{\otimes k} \to \Gamma^{\otimes k}$ they give rise can be written as

$$\begin{aligned}A^{(2)}_{\sigma^{\pm}} &= 1-\sigma^{\pm}, && \text{on } \Gamma^{\otimes 2} \\ A^{(3)}_{\sigma^{\pm}} &= (1-\sigma^{\pm}_2)(1-\sigma^{\pm}_1+\sigma^{\pm}_1\sigma^{\pm}_2), && \text{on } \Gamma^{\otimes 3}\end{aligned} \tag{2.40}$$

with $\sigma^{\pm}_1 = (\sigma^{\pm}\otimes 1)$ and $\sigma^{\pm}_2 = (1\otimes\sigma^{\pm})$, while $A^{(k)}_{\sigma^{\pm}}$ are trivial for $k \geq 4$. They yield an isomorphic (as left-covariant $\mathcal{A}(\mathrm{SU}_q(2))$-bimodules) pair of exterior algebras $\mathrm{Range}(A^{(k)}_{\sigma}) = \Gamma^k_{\sigma} \sim \Gamma^k_{\sigma^-} = \mathrm{Range}(A^{(k)}_{\sigma^-})$ whose dimensions coincide to those in the classical setting, namely $\dim\Gamma^k_{\sigma^{\pm}} = 3!/(3-k)!$. A basis of left invariant two forms in Γ^2_{σ} is given by $\{\omega_-\wedge\omega_+, \omega_+\wedge\omega_z, \omega_z\wedge\omega_-\}$; it allows to write the isomorphism above as $\omega_a\wedge\omega_b = q^2\omega_a\vee\omega_b$ (with $a\neq b$) where the symbol $\vee$ clearly represents the wedge product in the exterior algebras Γ_{σ^-}. Given $\vartheta = \omega_-\otimes\omega_+\otimes\omega_z$, left invariant volume forms are then, up to complex numbers,

$$\theta_{\pm} = A^{(3)}_{\sigma^{\pm}}(\vartheta) \tag{2.41}$$

with $\theta_- = q^{-6}\theta_+$.

Equipped with the natural graded extension of the exterior differential in (2.36), these exterior algebras give isomorphic differential calculi $(\mathrm{d},\Gamma_{\sigma}) \sim (\mathrm{d},\Gamma_{\sigma^{-1}})$. Such differential calculi turn out to be isomorphic to the universal calculus (d,Γ_u), since the relation $\ker A^{(k)}_{\sigma^{\pm}} = \mathcal{S}_{\mathcal{Q}}$ among 2-sided ideals (see section 2.2) holds.

The action of the antisymmetrisers $A^{(k)}_{\sigma^{\pm}}$ on $\Gamma^k_{\sigma^{\pm}}$ is constant. Their spectral resolution $A^{(k)}_{\sigma^{\pm}}(\phi) = \lambda^{\pm}_{(k)}\phi$ with ϕ a k-form yields:

$$\lambda^{\pm}_{(2)} = (1+q^{\pm 2}) \qquad \lambda^{\pm}_{(3)} = (1+q^{\pm 2})(1+q^{\pm 2}+q^{\pm 4}). \tag{2.42}$$

The isomorphisms $\Gamma^k_{\sigma} \sim \Gamma^k_{\sigma^{-1}}$ can then be written in terms of these spectral resolutions:

$$\frac{\omega_a\wedge\omega_b}{\lambda^+_{(2)}} = \frac{\omega_a\vee\omega_b}{\lambda^-_{(2)}}, \qquad \frac{\theta_+}{\lambda^+_{(3)}} = \frac{\theta_-}{\lambda^-_{(3)}}. \tag{2.43}$$

Remark 2.1. *Considering the properties of the so called Drinfeld-Radford-Yetter modules, it is possible to define a braiding* Ψ *on* $\Gamma^{\otimes 2}$ *as a map which satisifies a braid equation* (2.22) *on* $\Gamma^{\otimes 3}$*, using the* U(1)*-right covariance of the calculi on* $\mathrm{SU}_q(2)$ *as explained in* [14]*. We notice that such a braiding has been considered in* [2, 3] *for the case of the Woronowicz' calculus, together with a further braiding* σ *(depending on* Ψ*) obtained in order to construct a meaningful Killing metric, and that they both do not coincide with the braiding we are using in this paper, coming from* [9]*.*

Explicit calculations show, this happens for any of the calculi we consider. The spectral decompositions of the braiding Ψ *corresponding to any of the calculi in* $\mathcal{K}$ *presents* $\dim\ker(1-\Psi) = 2$ *(see* (2.39)*). This shows moreover that* $\mathcal{S}_{\mathcal{Q}} \nsubseteq \ker(1-\Psi)$*: the exterior algebra* Γ_{Ψ} *(built using the braiding* Ψ*) is not a quotient of the universal exterior algebra* Γ_u *built over the first order differential calculus* Γ *as described in section 2.2. A direct inspection of section 6 in* [2] *moreover shows* *the differences between the braiding σ on the 3D Woronowicz' calculus used there and the braidings associated to $\mathcal{K}$ in our approach.*

3. Hodge operators and symmetric contractions over $\mathrm{SU}_q(2)$

The question is now to exploit how it is possible to suitably translate the classical path described in section 2.1 into a quantum path towards the introduction of a notion of Hodge duality operators and of symmetric contractions on the exterior algebras $\Gamma_{\sigma^\pm}$. We start by introducing an operator which parallels the classical one defined in (2.5).

3.1. Contractions and symmetry.

Since we are interested in Hodge duality operators whose corresponding Laplacians map line bundles elements $\mathcal{L}_n \subset \mathcal{A}(\mathrm{SU}_q(2))$ to themselves, we consider the class of non degenerate $\mathcal{A}(\mathrm{SU}_q(2))$-left invariant and $\mathrm{U}(1)$-right invariant contractions. We define it as the set of maps $\mathrm{g} : \Gamma_{\mathrm{L}} \times \Gamma_{\mathrm{L}} \to \mathbb{C}$, provided they fullfill the condition $\mathrm{g}(\omega_a, \omega_b) = 0$ if $n_a + n_b \neq 0$ with $\delta_R^{(1)} : \omega_j \mapsto \omega_j \otimes z^{n_j}$ via (2.37). The only non zero coefficients of the contraction are then (non degeneracy being equivalent to $\alpha\,\beta\,\gamma \neq 0$)

$$\mathrm{g}(\omega_-, \omega_+) = \alpha, \qquad \mathrm{g}(\omega_+, \omega_-) = \beta, \qquad \mathrm{g}(\omega_z, \omega_z) = \gamma. \tag{3.1}$$

This contraction is naturally extended to the left invariant part of $\Gamma_{\sigma^\pm}$; recalling the classical expressions (2.3), (2.4), via the actions of the quantum antisymmetrisers $A^{(k)}_{\sigma^\pm}$ (2.40) we set

$$\mathrm{g}(\omega_{a_1} \wedge \ldots \wedge \omega_{a_k}, \omega_{b_1} \wedge \ldots \wedge \omega_{b_s}) \;=\; \mathrm{g}(A^{(k)}_\sigma(\omega_{a_1} \otimes \ldots \otimes \omega_{a_k}), A^{(s)}_\sigma(\omega_{b_1} \otimes \ldots \otimes \omega_{b_s}))$$

together with the obvious analog definition on Γ_{σ^-}. From the ordered set of one forms given by $\vartheta \in \Gamma_{\mathrm{L}}^{\otimes 3}$, we define the quantum determinants of the contraction g – with respect to the braidings $\sigma^\pm$ – as

$$\det_{\sigma^\pm}\mathrm{g} \;=\; \frac{1}{\lambda^{\pm}_{(3)}}\,\mathrm{g}(\theta_\pm, \theta_\pm), \tag{3.2}$$

reading $\det_\sigma \mathrm{g} = q^6 \det_{\sigma^-}\mathrm{g}$. We set then the hermitian volume forms $\mu_\pm = m_\pm \theta_\pm = \mu_\pm^*$ from (2.41) with $m_\pm \in \mathbb{R}$, and generalising the classical (2.5) to the quantum setting, the linear $\mathcal{A}(\mathrm{SU}_q(2))$- linear operators $S_{\sigma^\pm} : \Gamma^k_{\sigma^\pm} \to \Gamma^{3-k}_{\sigma^\pm}$ as

$$S_{\sigma^\pm}(\omega) \;=\; \frac{1}{\lambda^{\mp}_{(k)}}\,\mathrm{g}(\omega, \mu_\pm) \tag{3.3}$$

on a left-invariant basis; the modulus of the scale factors of the volume are chosen by $S^2_{\sigma^\pm}(1) = \mathrm{sgn}(\det_{\sigma^\pm}\mathrm{g})$. Corresponding to these operators we introduce sesquilinear $\mathcal{A}(\mathrm{SU}_q(2))$-left invariant scalar products by

$$\begin{aligned}\{\omega, \omega'\}_\sigma \;&=\; \int_{\mu_+} \omega^* \wedge S_\sigma(\omega'),\\ \{\omega, \omega'\}_{\sigma^-} \;&=\; \int_{\mu_-} \omega^* \vee S_{\sigma^-}(\omega').\end{aligned} \tag{3.4}$$

The integral on $\Gamma^3_{\sigma^\pm}$ is defined in terms of the Haar functional h by $\int_{\mu_\pm} x\,\mu_\pm = h(x)$ for $x \in \mathcal{A}(\mathrm{SU}_q(2))$. The isomorphisms (2.43) allow to prove the following relations:

$$\begin{aligned}
S_{\sigma^-}(1) &= \left(\frac{m_-}{m_+}\right)\left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right) S_\sigma(1),\\
S_{\sigma^-}(\omega) &= \left(\frac{m_-}{m_+}\right)\left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right) S_\sigma(\omega),\\
S_{\sigma^-}(\phi) &= \left(\frac{m_-}{m_+}\right)\left(\frac{\lambda^-_{(2)}}{\lambda^+_{(2)}}\right)\left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right) S_\sigma(\phi),\\
S_{\sigma^-}(\theta) &= \left(\frac{m_-}{m_+}\right)\left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right)^2 S_\sigma(\theta),
\end{aligned} \tag{3.5}$$

for any 1-form ω, 2-form ϕ, 3-form θ. For the common scale factor one has

$$\left(\frac{m_-}{m_+}\right)^2 = \left(\frac{\lambda^+_{(3)}}{\lambda^-_{(3)}}\right)^3 \tag{3.6}$$

while, for the scalar products,

$$\begin{aligned}
\{\omega,\omega'\}_{\sigma^-} &= \left(\frac{\lambda^+_{(2)}}{\lambda^-_{(2)}}\right)\left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right)\{\omega,\omega'\}_\sigma,\\
\{\phi,\phi'\}_{\sigma^-} &= \left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right)\{\phi,\phi'\}_\sigma,\\
\{\theta,\theta'\}_{\sigma^-} &= \left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\right)\{\theta,\theta'\}_\sigma
\end{aligned} \tag{3.7}$$

for any pair ω, ω' of 1-forms, ϕ, ϕ' of 2-forms, θ, θ' of 3-forms.

3.2. **Scalar products and duality operators.** Before analysing the spectral properties of the operators above, we use the spectral resolution of the antisymmetrisers further and introduce via the contraction map another sesquilinear $\mathcal{A}(\mathrm{SU}_q(2))$-left invariant scalar product on the exterior algebras $\Gamma_{\sigma^\pm}$ by

$$\langle x\,\omega, x'\omega'\rangle_{\sigma^\pm} = h(x^*x')\,\frac{1}{\lambda^\pm_{(k)}}\,\mathrm{g}(\omega^*,\omega'), \tag{3.8}$$

where ω, ω' are left-invariant forms and $h(x^*x')$ is again the action of the Haar functional h with $x, x' \in \mathcal{A}(\mathrm{SU}_q(2))$. Recalling the classical definition (2.11), it is now natural to set the operators[4] $T_{\sigma^\pm} : \Gamma^k_{\sigma^\pm} \to \Gamma^{3-k}_{\sigma^\pm}$ by:

$$\begin{aligned}
\langle x\,\omega, x'\omega'\rangle_{\sigma} &= \int_{\mu_+} (x\,\omega)^* \wedge T_\sigma(x'\omega'),\\
\langle x\,\omega, x'\omega'\rangle_{\sigma^-} &= \int_{\mu_-} (x\,\omega)^* \vee T_{\sigma^-}(x'\omega').
\end{aligned} \tag{3.9}$$

[4]This operator is the one described in [22] for the Woronowicz calculus.

Provided the contraction g is non degenerate, such operators $T_{\sigma^\pm}$ are well-defined, bijective and left $\mathcal{A}(\mathrm{SU}_q(2))$-linear [6]. One immediately has

$$\begin{aligned} T_{\sigma^\pm}(1) &= \mu_\pm, \\ T_{\sigma^\pm}(\mu_\pm) &= \langle \mu_\pm, \mu_\pm \rangle_{\sigma^\pm} \; = \; m_\pm^2 \det{}_{\sigma^\pm} \mathrm{g}. \end{aligned} \tag{3.10}$$

From (3.1), the scalar product (3.8) reads on left-invariant 1-forms (we omit the subscripts $\sigma^\pm$ since they coincide)

$$\langle \omega_-, \omega_- \rangle = -\beta, \qquad \langle \omega_+, \omega_+ \rangle = -\alpha, \qquad \langle \omega_z, \omega_z \rangle = -\gamma. \tag{3.11}$$

If we assume the natural normalization condition $T^2_{\sigma^\pm}(1) = \mathrm{sgn}(\det_{\sigma^\pm}\mathrm{g})$, it is easy to prove that

$$\begin{aligned} \langle \phi, \phi' \rangle_{\sigma^-} &= \left(\frac{\lambda^-_{(2)}}{\lambda^+_{(2)}} \right)^3 \langle \phi, \phi' \rangle_\sigma \, , \\ \langle \theta, \theta' \rangle_{\sigma^-} &= \left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}} \right)^3 \langle \theta, \theta' \rangle_\sigma \end{aligned} \tag{3.12}$$

which are the counterparts of the (3.7) (with which they share the same notations) for the operators $T_{\sigma^\pm}$ while, from (3.12), we have

$$\begin{aligned} \lambda^-_{(3)} m_-^2 \; &= \lambda^+_{(3)} m_+^2 \\ T_{\sigma^-}(1) \; &= \left(\frac{m_-}{m_+} \right) \left(\frac{\lambda^-_{(3)}}{\lambda^+_{(3)}} \right) T_\sigma(1), \\ T_{\sigma^-}(\omega) \; &= \left(\frac{m_-}{m_+} \right) \left(\frac{\lambda^-_{(2)}}{\lambda^+_{(2)}} \right) T_\sigma(\omega), \\ T_{\sigma^-}(\phi) \; &= \left(\frac{m_-}{m_+} \right) T_\sigma(\phi), \\ T_{\sigma^-}(\theta) \; &= \left(\frac{m_-}{m_+} \right) T_\sigma(\theta), \end{aligned} \tag{3.13}$$

which are the counterparts of the previous (3.5), (3.6). It is now evident that the operators ($T_{\sigma^\pm}$, $S_{\sigma^\pm}$) differ, since the sesquilinear products (3.8), (3.4) differ and that they coincide only in the classical limit; one can easily for example check on 3-forms that

$$\{\theta_\pm, \theta_\pm\}_{\sigma^\pm} \; = \; \frac{1}{\lambda^\mp_{(3)}} \, \mathrm{g}(\theta_\pm, \theta_\pm) \; = \; \frac{\lambda^\pm_{(3)}}{\lambda^\mp_{(3)}} \, \langle \theta_\pm, \theta_\pm \rangle_{\sigma^\pm} \, . \tag{3.14}$$

In order to better understand the differences between the operators $S_{\sigma^\pm}, T_{\sigma^\pm}$ as well as their similarities, we explicitly present a deeper analysis in the case of the Woronowicz calculus $(\mathrm{d}, \Gamma^{\mathcal{W}}_{\sigma^\pm})$, that we use as an example.

3.3. **The guiding example.** On the Woronowicz first order differential calculus the braiding reads

$$\begin{aligned} \sigma(\omega_a \otimes \omega_a) &= \omega_a \otimes \omega_a, & a &= \pm, z \\ \sigma(\omega_- \otimes \omega_+) &= (1-q^2)\omega_- \otimes \omega_+ + q^{-2}\omega_+ \otimes \omega_-, & \sigma(\omega_+ \otimes \omega_-) &= q^4 \omega_- \otimes \omega_+, \\ \sigma(\omega_- \otimes \omega_z) &= (1-q^2)\omega_- \otimes \omega_z + q^{-4}\omega_z \otimes \omega_-, & \sigma(\omega_z \otimes \omega_-) &= q^6 \omega_- \otimes \omega_z, \\ \sigma(\omega_z \otimes \omega_+) &= (1-q^2)\omega_z \otimes \omega_+ + q^{-4}\omega_+ \otimes \omega_z, & \sigma(\omega_+ \otimes \omega_z) &= q^6 \omega_z \otimes \omega_+, \end{aligned} \tag{3.15}$$

so that the wedge product on the exterior algebra satisfies $\omega_a \wedge \omega_a = 0, \quad (a = \pm, z)$ with

$$\omega_- \wedge \omega_+ + q^{-2}\omega_+ \wedge \omega_- = 0, \qquad\qquad \omega_z \wedge \omega_\mp + q^{\pm 4}\omega_- \wedge \omega_z = 0. \tag{3.16}$$

From (2.41) and (2.43) it is

$$\theta \,=\, q^4\,\omega_- \otimes (\omega_+ \otimes \omega_z \,-\, q^6\,\omega_z \otimes \omega_+) \,+\, q^{-6}\,\omega_+ \otimes (\omega_z \otimes \omega_- \,-\, q^6\omega_- \otimes \omega_z) \\ +\, q^4\,\omega_z \otimes (\omega_- \otimes \omega_+ \,-\, q^{-4}\omega_+ \otimes \omega_-) \tag{3.17}$$

and

$$\mathrm{g}(\theta_+, \theta_+) \,=\, -6\,q^4\,\alpha\,\beta\,\gamma. \tag{3.18}$$

Together with (3.10), one has

$$\begin{aligned} T_\sigma(\omega_-) &= q^{-2}m_+ \,\langle\omega_-, \omega_-\rangle\; \omega_- \wedge \omega_z, & T_\sigma(\omega_- \wedge \omega_z) &= q^{-2}m_+ \,\langle\omega_- \wedge \omega_z, \omega_- \wedge \omega_z\rangle_\sigma\; \omega_- \\ T_\sigma(\omega_+) &= -m_+ \,\langle\omega_+, \omega_+\rangle\; \omega_+ \wedge \omega_z, & T_\sigma(\omega_+ \wedge \omega_z) &= -m_+ \,\langle\omega_+ \wedge \omega_z, \omega_+ \wedge \omega_z\rangle_\sigma\; \omega_+ \\ T_\sigma(\omega_z) &= -m_+ \,\langle\omega_z, \omega_z\rangle\; \omega_- \wedge \omega_+, & T_\sigma(\omega_- \wedge \omega_+) &= -m_+ \,\langle\omega_- \wedge \omega_+, \omega_- \wedge \omega_+\rangle_\sigma\; \omega_z \end{aligned} \tag{3.19}$$

while the relations (3.13) give the action of T_{σ^-}. The expressions above depend only on the wedge products relations (3.16); they are valid for any choice of a non degenerate scalar product on the space of left-invariant 1-, 2- and 3-forms. Since from the expression (3.4) we see that the scalar product $\{\ ,\ \}_{\sigma^\pm}$ characterises the operators $S_{\sigma^\pm}$ in the same way the scalar product $\langle\ ,\ \rangle_{\sigma^\pm}$ characterises the operators $T_{\sigma^\pm}$ (see (3.9)), it is immediate to recover that the action of the operators $S_{\sigma^\pm}$ can be written from the action of the operators $T_{\sigma^\pm}$ after the mapping $\langle\ ,\ \rangle_{\sigma^\pm} \mapsto \{\ ,\ \}_{\sigma^\pm}$ on each space $\Gamma^k_{\sigma^\pm}$.

But we want to show that a deeper analogy exists. It is clear from the structure of the braiding that the scalar product (3.8) on left-invariant k-forms ($k = 2, 3$) is a k-order polynomial in the first order terms (3.11). This means that the definition (3.8) amounts to set a specific choice for the extension to higher order forms of the scalar product $\langle\ ,\ \rangle_\sigma$ on 1-forms. For the example we are considering, we calculate

$$\begin{aligned} \langle\omega_- \wedge \omega_+, \omega_- \wedge \omega_+\rangle_\sigma &= 2\,\langle\omega_-, \omega_-\rangle\,\langle\omega_+, \omega_+\rangle\,/\lambda^+_{(2)}, \\ \langle\omega_- \wedge \omega_z, \omega_- \wedge \omega_z\rangle_\sigma &= 2\,q^{-2}\,\langle\omega_-, \omega_-\rangle\,\langle\omega_z, \omega_z\rangle\,/\lambda^+_{(2)}, \\ \langle\omega_+ \wedge \omega_z, \omega_+ \wedge \omega_z\rangle_\sigma &= 2\,q^6\,\langle\omega_+, \omega_+\rangle\,\langle\omega_z, \omega_z\rangle\,/\lambda^+_{(2)}, \end{aligned} \tag{3.20}$$

on 2-forms, and (see (3.18))

$$\langle\theta_+, \theta_+\rangle_\sigma \,=\, \frac{6\,q^4}{\lambda^+_{(3)}}\,\langle\omega_-, \omega_-\rangle\,\langle\omega_+, \omega_+\rangle\,\langle\omega_z, \omega_z\rangle\,. \tag{3.21}$$

on the volume form. Concerning the scalar product (3.4), we start by computing that we have

$$\begin{aligned} &\{\omega_-, \omega_-\}_\sigma = -\alpha, && \{\omega_+, \omega_+\}_\sigma = -q^4\beta, && \{\omega_z, \omega_z\}_\sigma = -q^2\gamma \\ &\{\omega_-, \omega_-\}_{\sigma^-} = -q^{-4}\alpha, && \{\omega_+, \omega_+\}_{\sigma^-} = -\beta, && \{\omega_z, \omega_z\}_{\sigma^-} = -q^{-2}\gamma \end{aligned} \tag{3.22}$$

on 1-forms. The next step is to understand how this scalar product on higher order forms can be written in terms of the scalar products among 1-forms. It turns then out that we can write:

$$\begin{aligned} \{\omega_- \wedge \omega_+, \omega_- \wedge \omega_+\}_\sigma &= 2\,\{\omega_-, \omega_-\}_\sigma\,\{\omega_+, \omega_+\}_\sigma/\lambda^+_{(2)}, \\ \{\omega_- \wedge \omega_z, \omega_- \wedge \omega_z\}_\sigma &= 2\,q^{-2}\{\omega_-, \omega_-\}_\sigma\,\{\omega_z, \omega_z\}_\sigma/\lambda^+_{(2)}, \\ \{\omega_+ \wedge \omega_z, \omega_+ \wedge \omega_z\}_\sigma &= 2\,q^6\{\omega_+, \omega_+\}_\sigma\,\{\omega_z, \omega_z\}_\sigma/\lambda^+_{(2)}, \end{aligned} \tag{3.23}$$

and (see (3.18))

$$\{\theta_+,\theta_+\}_\sigma \;=\; \frac{6\,q^4}{\lambda^+_{(3)}}\,\{\omega_-,\omega_-\}_\sigma\,\{\omega_+,\omega_+\}_\sigma\,\{\omega_z,\omega_z\}_\sigma. \tag{3.24}$$

on the volume form. The comparison is immediate: with respect to first order coefficients, the equations (3.23) have the same structure of the equations (3.20), the equation (3.24) analogously has the same structure of the equation (3.21). This equivalence holds also if we consider the scalar product defined by the operator S_{σ^-}, as one may easily infers from

$$\begin{aligned}\frac{\{\omega_a\vee\omega_b,\omega_a\vee\omega_b\}_{\sigma^-}}{\{\omega_a,\omega_a\}_{\sigma^-}\,\{\omega_b,\omega_b\}_{\sigma^-}} \;&=\; \frac{\lambda^-_{(2)}}{\lambda^+_{(2)}}\,\frac{\{\omega_a\wedge\omega_b,\omega_a\wedge\omega_b\}_\sigma}{\{\omega_a,\omega_a\}_\sigma\,\{\omega_b,\omega_b\}_\sigma}\\ \frac{\{\theta_-,\theta_-\}_{\sigma^-}}{\{\omega_-,\omega_-\}_{\sigma^-}\,\{\omega_+,\omega_+\}_{\sigma^-}\,\{\omega_z,\omega_z\}_{\sigma^-}} \;&=\; \frac{\lambda^-_{(3)}}{\lambda^+_{(3)}}\,\frac{\{\theta_+,\theta_+\}_\sigma}{\{\omega_-,\omega_-\}_\sigma\,\{\omega_+,\omega_+\}_\sigma\,\{\omega_z,\omega_z\}_\sigma},\end{aligned} \tag{3.25}$$

which parallel, for the specific example of the Woronowicz calculus, the relations (3.12).

The collection of these results prove that the scalar products $\{\ ,\ \}_{\sigma^\pm}$ among higher order (left-invariant) forms can be obtained from $\langle\ ,\ \rangle_{\sigma^\pm}$ via the replacements of the 1-forms terms, as (restoring only for this expression the index $\sigma^\pm$ to the $\langle\ ,\ \rangle$ scalar product between 1-forms)

$$\begin{aligned}\langle\omega_a,\omega_b\rangle_\sigma \quad&\leftrightarrow\quad \{\omega_a,\omega_b\}_\sigma \;=\; \frac{1}{\lambda^-_{(k)}}\int_{\mu_+}\omega_a^*\wedge \mathrm{g}(\omega_b,\mu_+),\\ \langle\omega_a,\omega_b\rangle_{\sigma^{-1}} \quad&\leftrightarrow\quad \{\omega_a,\omega_b\}_{\sigma^-} \;=\; \frac{1}{\lambda^+_{(k)}}\int_{\mu_-}\omega_a^*\vee \mathrm{g}(\omega_b,\mu_-);\end{aligned} \tag{3.26}$$

the comparison between (3.4) and (3.9) convinces that the arrows in (3.26) give the action of $T_{\sigma^\pm}$ from the action of $S_{\sigma^\pm}$ and viceversa.

3.4. **A shared pattern.** We come now to the crucial point of our analysis. Does the equivalence – described above for our guiding example – between the scalar products defined by (3.8) and (3.4) hold also for the other calculi in $\mathcal{K}$ on $\mathrm{SU}_q(2)$? Do – for any fixed calculus in $\mathcal{K}$ – these two families share the same pattern, once we look at the scalar products among higher order forms as polynomials over their first order coefficients? The answer is positive, and can be proved by straightforward but long explicit computations, since we miss a general theory for the set of non canonical braidings [9] we consider. Relations (3.25) hold for any of the calculi in $\mathcal{K}$; the mappings in (3.26) allow to obtain the action of the operators $T_{\sigma^\pm}$ from that of the operators $S_{\sigma^\pm}$ and viceversa.

3.5. **Symmetric and real contractions.** We now use these operators to introduce a notion of symmetry and reality for the contraction g. Let us define the contraction g:

- (i) $S_{\sigma^\pm}$-*symmetric* (resp. $T_{\sigma^\pm}$-*symmetric)*, provided the operators $S^2_{\sigma^\pm}$ (resp. $T^2_{\sigma^\pm}$) have the same degeneracy of the antisymmetrisers, namely is their action on one forms constant;
- (ii) $S_{\sigma^\pm}$-*real* (resp. $T_{\sigma^\pm}$-*real*), provided the relations $S_{\sigma^\pm}(\omega_a^*) = (S_{\sigma^\pm}(\omega_a))^*$ (resp. $T_{\sigma^\pm}(\omega_a^*) = (T_{\sigma^\pm}(\omega_a))^*$) on any left-invariant one form hold.

Denote by $G^\pm_S$ (resp. $G^\pm_T$) the set of real and symmetric contractions: we consider then the corresponding dualities $S_{\sigma^\pm}$ (resp. $T_{\sigma^\pm}$) as Hodge operators. The relations (3.13) enable to prove that $G^+_S = G^-_S$, while the relations (3.25) give also $G^+_T = G^-_T$ (from now on we shall then denote these sets by G_S, G_T). The requirements of reality and symmetry clearly amount to constraint the parameters α, β,γ; such sets are not void, and do not coincide, i.e. $G_T \neq G_S$.

It is easy to compute, for the Woronowicz calculus,

$$\begin{array}{lll} \mathrm{g} \in G_T & \Leftrightarrow & \{\beta = q^6\alpha \in \mathbb{R},\ \gamma \in \mathbb{R}\} \\ \mathrm{g} \in G_S & \Leftrightarrow & \{\beta = q^{10}\alpha \in \mathbb{R},\ \gamma \in \mathbb{R}\} \end{array} \tag{3.27}$$

Following the long analysis of the two scalar products and their corresponding Hodge duality operators, comes naturally that these two sets of constraints are equivalent, if they are written in terms of the corresponding scalar products on 1-forms, namely

$$\begin{array}{lll} \mathrm{g} \in G_T & \Leftrightarrow & \{\langle\omega_-,\omega_-\rangle = q^6\,\langle\omega_+,\omega_+\rangle \in \mathbb{R},\ \langle\omega_z,\omega_z\rangle \in \mathbb{R}\} \\ \mathrm{g} \in G_S & \Leftrightarrow & \{\{\omega_-,\omega_-\}_{\sigma^\pm} = q^6\{\omega_+,\omega_+\}_{\sigma^\pm} \in \mathbb{R},\ \{\omega_z,\omega_z\}_{\sigma^\pm} \in \mathbb{R}\}. \end{array} \tag{3.28}$$

Such an equivalence is again given by (3.26). We close this part by saying that the equivalences given in (3.28) and (3.30) hold for any of the other six calculi (2.29)-(2.34) in $\mathcal{K}$. This means, that what we have actually introduced is (for any of the calculi in $\mathcal{K}$ on $\mathrm{SU}_q(2)$) a notion of real and symmetric tensor over the vector space $\mathcal{X}^{\otimes 2} = \mathcal{X} \otimes_{\mathbb{C}} \mathcal{X}$, which we write as

$$\mathrm{g} = \sum_{a,b=\pm,z} \mathrm{g}_{ab}\, X_a \otimes X_b \tag{3.29}$$

with $\mathrm{g}_{ab} = \mathrm{g}(\omega_a,\omega_b)$ belonging either to G_S or G_T. We notice that no compelling reason at this level of mathematical analysis allows to select the notion of $T_{\sigma^\pm}$-symmetry with respect to the one of $S_{\sigma^\pm}$-symmetry, nor to select one of the calculi we have considered.

What differs (only for $q \neq 1$) are the spectra of the Laplacians associated to the Hodge operators $T_{\sigma^\pm}$, $S_{\sigma^\pm}$ and the symmetric tensor (3.29), which turn out to be

$$\begin{aligned} \square^{(T)}x &= \mathrm{sgn}(\det{}_{\sigma^\pm}\mathrm{g}) \sum_{a,b=\pm,z} [\langle\omega_a^*,\omega_b\rangle\ X_aX_b \triangleright x] \\ \square^{(S)}x &= \mathrm{sgn}(\det{}_{\sigma^\pm}\mathrm{g}) \sum_{a,b=\pm,z} [\{\omega_a^*,\omega_b\}_{\sigma^\pm}\ X_aX_b \triangleright x]\,. \end{aligned} \tag{3.30}$$

It is interesting to notice that such Laplacians depend on the scalar products associated to the symmetric and real tensor (3.29).

In the next part we present, for each of these calculi, the sufficient ingredients to build the isomorphic exterior algebras and the Hodge operator S_σ together with its corresponding class of real and symmetric contractions (those which are $\mathcal{A}(\mathrm{SU}_q(2))$-left invariant and U(1)-right coinvariant). As we saw, this is enough to construct the operators S_{σ^-}, $T_{\sigma^\pm}$.

3.6. **Hodge operators.** We follow the numbering in section 2.3; we assume the non degeneracy of the contraction g in (3.1), that is $\alpha\,\beta\,\gamma \neq 0$.

(1) Given the quantum tangent space $\mathcal{X}_{\mathcal{Q}_1}$ the exact one forms are

$$\begin{aligned} \mathrm{d}a &= -q\,c^*\omega_+ + a\,\omega_z, & \mathrm{d}c &= a^*\omega_+ + c\,\omega_z, \\ \mathrm{d}a^* &= c\,\omega_- - q^{-1}a^*\omega_z, & \mathrm{d}c^* &= -q^{-1}a\,\omega_- - q^{-1}c^*\omega_z; \end{aligned}$$

and the braiding reads

$$\sigma(\omega_\pm\otimes\omega_\pm)=\omega_\pm\otimes\omega_\pm,$$

$$\sigma(\omega_z\otimes\omega_z)=\omega_z\otimes\omega_z\,+\,\frac{q(1-q)}{1+q^{-1}}(\omega_+\otimes\omega_-\,-\,\omega_-\otimes\omega_+),$$

$$\sigma(\omega_-\otimes\omega_+)=q^2\omega_+\otimes\omega_-\,+\,(1-q^2)\omega_-\otimes\omega_+,\qquad \sigma(\omega_+\otimes\omega_-)=\omega_-\otimes\omega_+$$

$$\sigma(\omega_-\otimes\omega_z)=q^2\omega_z\otimes\omega_-\,+\,(1-q^2)\omega_-\otimes\omega_z,\qquad \sigma(\omega_z\otimes\omega_-)=\omega_-\otimes\omega_z$$

$$\sigma(\omega_z\otimes\omega_+)=q^2\omega_+\otimes\omega_z\,+\,(1-q^2)\omega_z\otimes\omega_+,\qquad \sigma(\omega_+\otimes\omega_z)=\omega_z\otimes\omega_+.$$

The hermitian structure over left-invariant two forms and the wedge product antisymmetry are

$$(\omega_-\wedge\omega_+)^*=-\omega_-\wedge\omega_+=q^2\omega_+\wedge\omega_-,$$
$$(\omega_-\wedge\omega_z)^*=-\omega_z\wedge\omega_+=q^2\omega_+\wedge\omega_z,$$
$$(\omega_+\wedge\omega_z)^*=-\omega_z\wedge\omega_-=q^{-2}\omega_-\wedge\omega_z;$$

the volume form θ_+ turns out to be a multiple of the classical one, namely the one we would obtain if the braiding were the classical flip,

$$\theta_+=q^4(\omega_-\otimes(\omega_+\otimes\omega_z-\omega_z\otimes\omega_+)+\omega_+\otimes(\omega_z\otimes\omega_--\omega_-\otimes\omega_z)+\omega_z\otimes(\omega_-\otimes\omega_+-\omega_+\otimes\omega_-)),\quad (3.31)$$

while for the normalisations of the Hodge operators one needs $\mathrm{g}(\theta_+,\theta_+)=-6q^8(\alpha\beta\gamma)$. It is

$$\{\omega_-,\omega_-\}_\sigma\,=\,-\alpha,\qquad \{\omega_+,\omega_+\}_\sigma\,=\,-q^4\,\beta,\qquad \{\omega_z,\omega_z\}_\sigma\,=\,-q^2\,\gamma.$$

A contraction (3.1) is $S_{\sigma^\pm}$-real and symmetric

$$\mathrm{g}\,\in\,G_S\qquad\Leftrightarrow\qquad\{\alpha\,=\,\beta\,\in\,\mathbb{R},\,\gamma\,\in\,\mathbb{R}\},\qquad (3.32)$$

while $\mathrm{g}\,\in\,G_T\Leftrightarrow\{\alpha\,=\,q^4\,\beta\,\in\,\mathbb{R},\,\gamma\,\in\,\mathbb{R}\}$. As corresponding Hodge operator we have:

$$\begin{aligned}
S_\sigma(1)&=\mu_+, & S_\sigma(\mu_+)&=-\mathrm{sgn}(\gamma),\\
S_\sigma(\omega_-)&=m_+\,q^2\{\omega_-,\omega_-\}_\sigma\,\omega_-\wedge\omega_z, & S_\sigma(\omega_-\wedge\omega_z)&=2m(q^6/\lambda^+_{(2)})\{\omega_-,\omega_-\}_\sigma\,\{\omega_z,\omega_z\}_\sigma\,\omega_-\\
S_\sigma(\omega_+)&=-m_+\,\{\omega_+,\omega_+\}_\sigma\,\omega_+\wedge\omega_z, & S_\sigma(\omega_+\wedge\omega_z)&=-2m_+(1/\lambda^+_{(2)})\{\omega_+,\omega_+\}_\sigma\,\{\omega_z,\omega_z\}_\sigma\,\omega_+\\
S_\sigma(\omega_z)&=-m_+\,\{\omega_z,\omega_z\}_\sigma\,\omega_-\wedge\omega_+, & S_\sigma(\omega_-\wedge\omega_+)&=-2m(q^4/\lambda^+_{(2)})\{\omega_-,\omega_-\}_\sigma\,\{\omega_+,\omega_+\}_\sigma\,\omega_z
\end{aligned}\qquad (3.33)$$

with a normalisation condition $m_+^2\,\det_\sigma\mathrm{g}\,=\,-\mathrm{sgn}\gamma$.

(2) As we have already noticed, the structure of the exterior algebra corresponding to this calculus is obtained by that corresponding to the previous one by mapping $q\,\to\,-q$. Since the relations (3.33) (and then the (3.32)) are invariant by this reflection, the Hodge duality we obtain is exactly the previous one.

(3) Given the quantum tangent space $\mathcal{X}_{\mathcal{Q}_3}$ the exact one forms are

$$\mathrm{d}a=-q\,c^*\omega_+\,+\,a\,\omega_z,\qquad \mathrm{d}c=a^*\omega_+\,+\,c\,\omega_z,$$
$$\mathrm{d}a^*=c\,\omega_-\,-\,q^{-2}a^*\omega_z,\qquad \mathrm{d}c^*=-q^{-1}a\,\omega_-\,-\,q^{-2}c^*\omega_z;$$

with a braiding:

$$\sigma(\omega_\pm\otimes\omega_\pm)=\omega_\pm\otimes\omega_\pm,$$
$$\sigma(\omega_z\otimes\omega_z)=\omega_z\otimes\omega_z\,+\,(q^2-1)(\omega_-\otimes\omega_+\,-\,q^4\omega_+\otimes\omega_-),$$
$$\sigma(\omega_-\otimes\omega_+)=q^6\omega_+\otimes\omega_-\,+\,(1-q^2)\omega_-\otimes\omega_+,\qquad \sigma(\omega_+\otimes\omega_-)=q^{-4}\omega_-\otimes\omega_+$$
$$\sigma(\omega_-\otimes\omega_z)=q^4\omega_z\otimes\omega_-\,+\,(1-q^2)\omega_-\otimes\omega_z,\qquad \sigma(\omega_z\otimes\omega_-)=q^{-2}\omega_-\otimes\omega_z$$
$$\sigma(\omega_z\otimes\omega_+)=q^4\omega_+\otimes\omega_z\,+\,(1-q^2)\omega_z\otimes\omega_+,\qquad \sigma(\omega_+\otimes\omega_z)=q^{-2}\omega_z\otimes\omega_+.$$

The hermitian structure over left-invariant two forms is

$$(\omega_-\wedge\omega_+)^*=-\omega_-\wedge\omega_+=q^6\omega_+\wedge\omega_-,$$
$$(\omega_-\wedge\omega_z)^*=-\omega_z\wedge\omega_+=q^4\omega_+\wedge\omega_z,$$
$$(\omega_+\wedge\omega_z)^*=-\omega_z\wedge\omega_-=q^{-4}\omega_-\wedge\omega_z;$$

the volume form θ_+ is

$$\theta_+=q^4\omega_-\otimes(\omega_+\otimes\omega_z-q^{-2}\omega_z\otimes\omega_+)-q^8\omega_+\otimes(\omega_-\otimes\omega_z-q^2\omega_z\otimes\omega_-)+q^4\omega_z\otimes(\omega_-\otimes\omega_+-q^4\omega_+\otimes\omega_-),\tag{3.34}$$

so to have $\mathrm{g}(\theta_+,\theta_+)\,=\,-\,6q^{12}(\alpha\beta\gamma)$. It is

$$\{\omega_-,\omega_-\}_\sigma\,=\,-\alpha,\qquad \{\omega_+,\omega_+\}_\sigma\,=\,-q^4\,\beta,\qquad \{\omega_z,\omega_z\}_\sigma\,=\,-q^2\,\gamma.$$

A contraction (3.1) is $S_{\sigma^\pm}$-real and symmetric

$$\mathrm{g}\,\in\,G_S\qquad\Leftrightarrow\qquad\{q^6\,\alpha\,=\,\beta\,\in\,\mathbb{R},\,\gamma\,\in\,\mathbb{R}\};\tag{3.35}$$

while the conditions of T_σ-reality and symmetry for the same contraction are $\mathrm{g}\,\in\,G_T\Leftrightarrow\{\alpha\,=\,q^{10}\beta\,\in\,\mathbb{R},\,\gamma\,\in\,\mathbb{R}\}$. The Hodge operator is:

$$S_\sigma(1)=\mu_+,\qquad S_\sigma(\mu_+)=-\mathrm{sgn}(\gamma),$$
$$S_\sigma(\omega_-)=m_+\,q^6\{\omega_-,\omega_-\}_\sigma\,\omega_-\wedge\omega_z,\qquad S_\sigma(\omega_-\wedge\omega_z)=2m_+(q^{12}/\lambda^+_{(2)})\{\omega_-,\omega_-\}_\sigma\,\{\omega_z,\omega_z\}_\sigma\,\omega_-$$
$$S_\sigma(\omega_+)=-m_+\,\{\omega_+,\omega_+\}_\sigma\,\omega_+\wedge\omega_z,\qquad S_\sigma(\omega_+\wedge\omega_z)=-2m_+(q^{-2}/\lambda^+_{(2)})\{\omega_+,\omega_+\}_\sigma\,\{\omega_z,\omega_z\}_\sigma\,\omega_+$$
$$S_\sigma(\omega_z)=m_+\,\{\omega_z,\omega_z\}_\sigma\,\omega_-\wedge\omega_+,\qquad S_\sigma(\omega_-\wedge\omega_+)=-2m_+(q^8/\lambda^+_{(2)})\{\omega_-,\omega_-\}_\sigma\,\{\omega_+,\omega_+\}_\sigma\,\omega_z\tag{3.36}$$

with a normalisation condition $m_+^2\,\det_\sigma\mathrm{g}\,=\,-\mathrm{sgn}\gamma$.

(4) Given the quantum tangent space $\mathcal{X}_{\mathcal{Q}_4}$, exact one forms are

$$\mathrm{d}a=-q\,c^*\omega_+\,+\,a\,\omega_z,\qquad \mathrm{d}c=a^*\omega_+\,+\,c\,\omega_z,$$
$$\mathrm{d}a^*=c\,\omega_-\,-\,q\,a^*\omega_z,\qquad \mathrm{d}c^*=-q^{-1}a\,\omega_-\,-\,q\,c^*\omega_z;$$

with a braiding:

$$\sigma(\omega_\pm\otimes\omega_\pm)=\omega_\pm\otimes\omega_\pm,$$
$$\sigma(\omega_z\otimes\omega_z)=\omega_z\otimes\omega_z\,+\,\frac{1-q}{1+q}\,(\omega_-\otimes\omega_+\,-\,\omega_+\otimes\omega_-),$$
$$\sigma(\omega_+\otimes\omega_-)=q^2\omega_-\otimes\omega_+\,+\,(1-q^2)\omega_+\otimes\omega_-,\qquad \sigma(\omega_-\otimes\omega_+)=\omega_+\otimes\omega_-$$
$$\sigma(\omega_z\otimes\omega_-)=q^2\omega_-\otimes\omega_z\,+\,(1-q^2)\omega_z\otimes\omega_-,\qquad \sigma(\omega_-\otimes\omega_z)=\omega_z\otimes\omega_-$$
$$\sigma(\omega_+\otimes\omega_z)=q^2\omega_z\otimes\omega_+\,+\,(1-q^2)\omega_+\otimes\omega_z,\qquad \sigma(\omega_z\otimes\omega_+)=\omega_+\otimes\omega_z.$$

The hermitian structure and wedge products read over left-invariant two forms:

$$(\omega_- \wedge \omega_+)^* = -\omega_- \wedge \omega_+ = q^{-2}\omega_+ \wedge \omega_-,$$
$$(\omega_- \wedge \omega_z)^* = -\omega_z \wedge \omega_+ = q^{-2}\omega_+ \wedge \omega_z,$$
$$(\omega_+ \wedge \omega_z)^* = -\omega_z \wedge \omega_- = q^{2}\omega_- \wedge \omega_z;$$

so that the volume form θ_+ is again a quantum multiple of the classical one:

$$\theta_+ = q^2(\omega_-\otimes(\omega_+\otimes\omega_z - \omega_z\otimes\omega_+) + \omega_+\otimes(\omega_z\otimes\omega_- - \omega_-\otimes\omega_z) + \omega_z\otimes(\omega_-\otimes\omega_+ - \omega_+\otimes\omega_-)), \quad (3.37)$$

giving the following expression $\mathrm{g}(\theta_+, \theta_+) = -6q^4(\alpha\beta\gamma)$. It is

$$\{\omega_-, \omega_-\}_\sigma = -q^4\,\alpha, \qquad \{\omega_+, \omega_+\}_\sigma = -\beta, \qquad \{\omega_z, \omega_z\}_\sigma = -q^2\,\gamma.$$

The set of $S_{\sigma^\pm}$-real and symmetric contraction is

$$\mathrm{g} \in G_S \qquad \Leftrightarrow \qquad \{\alpha = \beta \in \mathbb{R},\, \gamma \in \mathbb{R}\}; \qquad (3.38)$$

while the conditions of T_σ-reality and symmetry for the same contraction are $\mathrm{g} \in G_T \Leftrightarrow \{\alpha = q^{-4}\beta \in \mathbb{R},\, \gamma \in \mathbb{R}\}$. The Hodge operator is:

$$\begin{aligned}
S_\sigma(1) &= \mu_+, & S_\sigma(\mu_+) &= -\mathrm{sgn}(\gamma),\\
S_\sigma(\omega_-) &= m_+\,q^{-2}\{\omega_-, \omega_-\}_\sigma\,\omega_- \wedge \omega_z, & S_\sigma(\omega_- \wedge \omega_z) &= 2m_+(q^{-2}/\lambda^+_{(2)})\{\omega_-, \omega_-\}_\sigma\,\{\omega_z, \omega_z\}_\sigma\,\omega_-\\
S_\sigma(\omega_+) &= -m_+\,\{\omega_+, \omega_+\}_\sigma\,\omega_+ \wedge \omega_z, & S_\sigma(\omega_+ \wedge \omega_z) &= -2m_+(q^{4}/\lambda^+_{(2)})\{\omega_+, \omega_+\}_\sigma\,\{\omega_z, \omega_z\}_\sigma\,\omega_+\\
S_\sigma(\omega_z) &= -m_+\,\{\omega_z, \omega_z\}_\sigma\,\omega_- \wedge \omega_+, & S_\sigma(\omega_- \wedge \omega_+) &= -2m_+(1/\lambda^+_{(2)})\{\omega_-, \omega_-\}_\sigma\,\{\omega_+, \omega_+\}_\sigma\,\omega_z
\end{aligned} \qquad (3.39)$$

with a normalisation condition $m_+^2\,\det_\sigma \mathrm{g} = -\mathrm{sgn}\gamma$.

(5) Again we refer to what already noticed, and do not explicitly report the results concerning the calculus generated by $\mathcal{Q}_5$ since they can be obtained by the those of the previous calculus by mapping $q \to -q$. Once more, the relations (3.39) (and then the (3.38)) being invariant by this reflection, the Hodge duality we obtain is exactly the previous one.

(6) Given the quantum tangent space $\mathcal{X}_{\mathcal{Q}_6}$ the exact one forms are

$$\begin{aligned}
\mathrm{d}a &= -q\,c^*\omega_+ - a\,\omega_z, & \mathrm{d}c &= a^*\omega_+ - c\,\omega_z,\\
\mathrm{d}a^* &= c\,\omega_- + q^4a^*\omega_z, & \mathrm{d}c^* &= -q^{-1}a\,\omega_- + q^4c^*\omega_z;
\end{aligned}$$

and the braiding reads

$$\begin{aligned}
\sigma(\omega_a \otimes \omega_a) &= \omega_a \otimes \omega_a, & &\text{for } a = \pm, z\\
\sigma(\omega_- \otimes \omega_+) &= q^{-4}\omega_+ \otimes \omega_- + (1-q^2)\omega_- \otimes \omega_+ + q^2(q^2-1)\omega_z \otimes \omega_z,\\
\sigma(\omega_+ \otimes \omega_-) &= q^{6}\omega_- \otimes \omega_+ - q^6(q^2-1)\omega_z \otimes \omega_z\\
\sigma(\omega_- \otimes \omega_z) &= q^{-2}\omega_z \otimes \omega_- + (1-q^2)\omega_- \otimes \omega_z, & \sigma(\omega_z \otimes \omega_-) &= q^4\omega_- \otimes \omega_z\\
\sigma(\omega_z \otimes \omega_+) &= q^{-2}\omega_+ \otimes \omega_z + (1-q^2)\omega_z \otimes \omega_+, & \sigma(\omega_+ \otimes \omega_z) &= q^4\omega_z \otimes \omega_+.
\end{aligned}$$

The hermitian structure over left-invariant two forms and the wedge product antisymmetry are

$$(\omega_- \wedge \omega_+)^* = -\omega_- \wedge \omega_+ = q^{-4}\omega_+ \wedge \omega_-,$$
$$(\omega_- \wedge \omega_z)^* = -\omega_z \wedge \omega_+ = q^{-2}\omega_+ \wedge \omega_z,$$
$$(\omega_+ \wedge \omega_z)^* = -\omega_z \wedge \omega_- = q^{2}\omega_- \wedge \omega_z;$$

and the volume form

$$\theta_{+} = q^{2}\omega_{-}\otimes(q^{2}\omega_{+}\otimes\omega_{z} - q^{6}\omega_{z}\otimes\omega_{+}) + q^{-6}\omega_{+}\otimes(\omega_{z}\otimes\omega_{-} - q^{4}\omega_{-}\otimes\omega_{z}) + q^{4}\omega_{z}\otimes(\omega_{-}\otimes\omega_{+} - q^{-6}\omega_{+}\otimes\omega_{-} + (1-q^{2})\omega_{z}\otimes\omega_{z}), \tag{3.40}$$

with $\mathrm{g}(\theta_{+},\theta_{+}) = -q^{2}\gamma(6\,\alpha\,\beta-(1-q^{2})^{2}\gamma^{2})$. For the scalar product (3.4) one has

$$\{\omega_{-},\omega_{-}\}_{\sigma} = -\alpha, \qquad \{\omega_{+},\omega_{+}\}_{\sigma} = -q^{4}\,\beta, \qquad \{\omega_{z},\omega_{z}\}_{\sigma} = -q^{2}\,\gamma.$$

The conditions of reality and symmetry of the contraction (3.1) with respect to S_{σ} can be expressed by

$$\mathrm{g}\in G_{S} \qquad \Leftrightarrow \qquad \{\alpha = -\mathrm{i}\,q^{6}\,\xi,\ \beta = \mathrm{i}\,q^{4}\rho,\ (q^{2}-1)\gamma = \pm2\,q^{-2}\xi;\ 0\neq\xi\in\mathbb{R}\}, \tag{3.41}$$

while $\mathrm{g}\in G_{T}\Leftrightarrow\{\alpha = \mathrm{i}\,\rho,\ \beta = -\mathrm{i}\,q^{6}\rho,\ (q^{2}-1)\gamma = \pm2\,\rho;\ 0\neq\rho\in\mathbb{R}\}$. Since these conditions appear counterintuitive, we report the expression that the determinant of the contraction for symmetric and real contractions acquires, namely

$$\det_{\sigma^{\pm}}\mathrm{g} = -\frac{2q^{8}}{\lambda^{\pm}_{(3)}}\,\gamma\,\rho^{2}. \tag{3.42}$$

The corresponding Hodge operator is:

$$\begin{aligned}
S_{\sigma}(1) &= \mu_{+}, & S_{\sigma}(\mu_{+}) &= -\mathrm{sgn}(\gamma),\\
S_{\sigma}(\omega_{-}) &= m_{+}\,q^{-4}\{\omega_{-},\omega_{-}\}_{\sigma}\,\omega_{-}\wedge\omega_{z}, & S_{\sigma}(\omega_{-}\wedge\omega_{z}) &= 2m_{+}(q^{-4}/\lambda^{+}_{(2)})\{\omega_{-},\omega_{-}\}_{\sigma}\,\{\omega_{z},\omega_{z}\}_{\sigma}\,\omega_{-}\\
S_{\sigma}(\omega_{+}) &= -m_{+}\,\{\omega_{+},\omega_{+}\}_{\sigma}\,\omega_{+}\wedge\omega_{z}, & S_{\sigma}(\omega_{+}\wedge\omega_{z}) &= -2m_{+}(q^{4}/\lambda^{+}_{(2)})\{\omega_{+},\omega_{+}\}_{\sigma}\,\{\omega_{z},\omega_{z}\}_{\sigma}\,\omega_{+}\\
S_{\sigma}(\omega_{z}) &= -m_{+}\,\{\omega_{z},\omega_{z}\}_{\sigma}\,\omega_{-}\wedge\omega_{+}, & &
\end{aligned}$$

$$S_{\sigma}(\omega_{-}\wedge\omega_{+}) = -(m_{+}/\lambda^{+}_{(2)})(2q^{-2}\{\omega_{-},\omega_{-}\}_{\sigma}\{\omega_{+},\omega_{+}\}_{\sigma} - (q(q^{2}-1)(\{\omega_{z},\omega_{z}\}_{\sigma})^{2})\omega_{z} \tag{3.43}$$

with a normalisation condition $m_{+}^{2}\det_{\sigma}\mathrm{g} = -\mathrm{sgn}\,\gamma$. Due to the symmetry and reality conditions (3.41), the Laplacians (3.30) have for this calculus a spectrum which is not real. This is a condition that characterises this calculus and the corresponding symmetric and real tensors g as in (3.29) with respect to the others.

Once we have the explicit families of real and symmetric tensors g in (3.29) for any of the calculi in $\mathcal{K}$, we further address a question arose in the previous pages, namely we wonder whether it is possible to set a condition eventually selecting between the notions of $T_{\sigma^{\pm}}$ and $S_{\sigma^{\pm}}$ symmetry on one side, and even among the calculi considered above on the other.

Our approach is straightforward. We consider the set $\mathfrak{G}$ of rank 2 tensors (3.29) satisfying the symmetry condition $\mathrm{g} = \mathrm{g}\circ\sigma$ (the meaning of this condition has been extensively analysed when the braiding is the canonical one associated to a bicovariant calculus), and we compare it with G_{S}, G_{T}. A direct inspection shows that $\mathfrak{G}\neq G_{T}$ for any calculus in $\mathcal{K}$, while $\mathfrak{G} = G_{S}$ *only for* the calculi (1,2) and (4,5) following the list above (i.e. those defined in (2.29), (2.30), (2.32), (2.33)). We call this subset $\tilde{K}\subset\mathcal{K}$. It is interesting to notice that the Woronowicz calculus does not fulfill this condition; the only calculi satisfying this condition are those, whose volume form is a multiple of the classical one, as it can be immediately seen from the explicit expressions in (3.17), (3.31), (3.34), (3.37), (3.40).

We shall explore these calculi from a possible different perspective by studying an extension to the homogeneous space S^{2}_{q} of the formalism giving Hodge dualities on $\mathrm{SU}_{q}(2)$.

4. Hodge operators over the standard Podleś sphere

In section 2.3 we introduced the standard Podleś sphere $\mathcal{A}(\mathrm{S}^2_q)$ as the subalgebra of U(1)-coinvariant elements in $\mathcal{A}(\mathrm{SU}_q(2))$ by the coaction given in (2.27). As a set of generators for the algebra $\mathcal{A}(\mathrm{S}^2_q)$ we consider

$$B_- = -ac^*, \qquad B_+ = qca^*, \qquad B_0 = \frac{q^2}{1+q^2} - q^2cc^*, \tag{4.1}$$

with $B_0^* = B_0$, $B_+^* = -qB_-$ [5].

Apply now the formalism developed in [5]: given the 3d calculi in $\mathcal{K}$ characterised by $\mathcal{Q}_a \subset \ker \varepsilon_{\mathrm{SU}_q(2)}$, the position $\pi(\mathcal{Q}_a) \subset \ker \varepsilon_{\mathrm{U}(1)}$ defines – from (2.26) – a calculus on U(1). For $a \neq 6$ this calculus on U(1) is bicovariant and 1 dimensional – we call *projectable* such 3d calculi on $\mathrm{SU}_q(2)$, and denote them by $\mathcal{K}_\pi \subset \mathcal{K}$ – and notice that the position $\pi(\mathcal{Q}_6)$ would induce on U(1) a trivial (0-dimensional) calculus. The restriction to S^2_q of the projectable calculi gives left-covariant calculi: in terms of the generators (4.1) the first order part of such calculi (that is, for any of the realizations of the operator d) is characterized as the left covariant bimodule given by the quotient $\Gamma(\mathrm{S}^2_q) = \mathcal{A}(\mathrm{S}^2_q)\,\{\mathrm{d}B_\pm,\, \mathrm{d}B_0\}/\omega_0$ where one has defined

$$\omega_0 \,=\, q^{-1}B_-\mathrm{d}B_+ \,+\, qB_+\mathrm{d}B_- \,-\,(1+q^{-2})B_0\mathrm{d}B_0, \tag{4.5}$$

while their higher order part is given as the quotient of the tensor products $\Gamma^{\otimes}(\mathrm{S}^2_q)$ by the differential ideal with generators $\{\omega_0,\, \mathrm{d}\omega_0\}$. Such a characterization allows to understand that all these calculi on S^2_q are isomorphic to the well known 2d left covariant calculus described by Podleś in [16] (the proof of this equivalence is straightforward, mimicking the one in [17, §3.4] which holds for all the projectable calculi over $\mathrm{SU}_q(2)$).

It is moreover immediate to prove that this setting describes the geometry of U(1) Hopf fibrations over the quantum sphere S^2_q with compatible calculi. This compatibility allows to meaningful recover that the quantum tangent space $\mathcal{X}_{\pi(\mathcal{Q}_a)}$ associated to the 1 dimensional calculs over U(1) is vertical for the fibrations, while the 2d exterior algebra over S^2_q is given by horizontal and U(1)-coequivariant exterior forms on $\mathrm{SU}_q(2)$ (more details can be found for example in [4]). Adopting the so called frame bundle approach [15] we write the exterior algebras over the Podleś sphere (corresponding to any of the projectable calculi over $\mathrm{SU}_q(2)$) as

$$\Gamma_\sigma(\mathrm{S}^2_q) = \mathcal{A}(\mathrm{S}^2_q) \oplus (\mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+) \oplus \mathcal{A}(\mathrm{S}^2_q)\omega_- \wedge \omega_+ \tag{4.6}$$

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[5] They satisfy the algebraic relations:

$$\begin{aligned}
(1+q^{-2})(B_-B_+ + q^2B_+B_-) &= q((1+q^{-2})^2B_0^2 - 1),\\
q(B_-B_+ - B_+B_-) + (q^{-2}-q^2)B_0^2 &= (1-q^2)B_0,\\
(1+q^{-2})(B_-B_0 - q^2B_0B_-) &= (1-q^2)B_-,\\
(1+q^{-2})(B_0B_+ - q^2B_+B_0) &= (1-q^2)B_+
\end{aligned} \tag{4.2}$$

The isomorphism (compatible with the $*$ anti-hermitian conjugation) to the algebra generated by $\{e_{\pm1}, e_0\}$ with relations given in (1)-(4) from [1] (for the real form $\mathrm{SU}_q(2)$ of $\mathrm{SL}_q(2)$ it is $e_{+1}^* = e_{-1}$) is given by:

$$\begin{aligned}
(1+q^{-2})B_0 &\mapsto e_0\\
B_- &\mapsto \pm i\, e_{+1}\\
B_+ &\mapsto \pm iq\, e_{-1}
\end{aligned} \tag{4.3}$$

with the identification

$$\lambda = (1-q^2), \qquad \rho = 1. \tag{4.4}$$

(i.e. in a more complete – and heavier – notation one should write $\Gamma^{(a)}_{\sigma}(\mathrm{S}^2_q)$ and $\omega^a_{\pm}$ with $(1,\ldots,7)\ni a\neq 6$. We also remark that one has an isomorphism $\Gamma_{\sigma^-}(\mathrm{S}^2_q)\sim\Gamma_{\sigma}(\mathrm{S}^2_q)$ for any projectable calculus). This expression is the counterpart in the quantum setting of the classical (2.8): the sets $\mathcal{L}_{\pm2}\omega_{\pm}$ are not free $\mathcal{A}(\mathrm{S}^2_q)$-bimodules, the top form bimodule *does* indeed have a 1 dimensional free $\mathcal{A}(\mathrm{S}^2_q)$ left invariant basis. Given the left invariant volume 2-forms

$$\check{\mu}_{+}\,=\,\mathrm{i}\,\check{m}_{+}\omega_{-}\wedge\omega_{+}\qquad\qquad \check{\mu}_{-}\,=\,\mathrm{i}\,\check{m}_{-}\omega_{-}\vee\omega_{+} \tag{4.7}$$

with $\check{m}_{\pm}\in\mathbb{R}$ so to have $\check{\mu}^*_{\pm}\,=\,\check{\mu}_{\pm}$, we define the operators $\check{S}_{\pm}\,:\,\Gamma^k_{\sigma^{\pm}}(\mathrm{S}^2_q)\,\to\,\Gamma^{2-k}_{\sigma^{\pm}}(\mathrm{S}^2_q)$, $(k=0,1,2)$ via a tensor $\mathrm{g}\in G_S$ (i.e. g is S-symmetric and real on $\mathrm{SU}_q(2)$) in terms of the decomposition (4.6),

$$\check{S}_{\sigma^{\pm}}(\omega)\,=\,\frac{1}{\lambda^{\mp}_{(k)}}\,\mathrm{g}(\omega,\check{\mu}_{\pm}). \tag{4.8}$$

This contraction operator clearly retains the left $\mathcal{A}(\mathrm{SU}_q(2))$-linearity in the l.h.s. factor. The scale is fixed by the natural normalization condition:

$$\check{S}^2_{\sigma^{\pm}}(1)\,=\,\mathrm{sgn}\,\{\,\mathrm{g}(\mathrm{i}\,\omega_{-}\wedge\omega_{+},\mathrm{i}\,\omega_{-}\wedge\omega_{+})\}. \tag{4.9}$$

Notice that only for the projectable calculi the braiding over $\mathrm{SU}_q(2)$ consistently restricts to a braiding among 1-forms on S^2_q as in the classical setting, so that $\check{S}_{\sigma^{\pm}}$ is well defined; for those calculi the spectral decomposition of the antisymmetrisers on S^2_q is given by the restriction of the one on $\mathrm{SU}_q(2)$ with the same eigenvalues $\lambda^{\pm}_{(k)}$ [6].

Since for any choice of a projectable calculus on $\mathrm{SU}_q(2)$ we have a specific realisation of the unique 2d left covariant $\Gamma(\mathrm{S}^2_q)$, what we introduce in (4.8) are different (because they come from different calculi on $\mathrm{SU}_q(2)$ and corresponding different G_S sets of symmetric and real tensors g) contraction operators acting on the same exterior algebra.

Which is their degeneracy? The only non trivial behaviour to analyze is how they act on 1-forms: a direct proof shows that $(\check{S}_{\sigma^{\pm}})^2$ is a multiple of the identity operator on 1-forms on S^2_q *only* for those calculi on $\mathrm{SU}_q(2)$ for which one has that $G_S\,=\,\mathfrak{G}$, namely the calculi in $\tilde{K}\subset\mathcal{K}$. Following the same approach we used for $\mathrm{SU}_q(2)$, we shall then define those contraction operators (4.8) corresponding to the calculi in $\tilde{\mathcal{K}}$ on $\mathrm{SU}_q(2)$ as the Hodge operators on the unique left covariant 2d calculus on the Podleś sphere.

We explicitly write these two pairs of Hodge operators.

- We realize the 2d differential calculus (4.6) on S^2_q as a restriction of the calculus (2.29) on $\mathrm{SU}_q(2)$, characterized by $\mathcal{Q}_1\subset\ker\varepsilon_{\mathrm{SU}_q(2)}$. The action of the Hodge operator turns out to be (we know from (3.32) that it depends on the real parameter $\alpha\neq 0$):

$$\begin{aligned}\check{S}_{\sigma}(1)&=\,\mathrm{i}\check{m}_{+}\,\omega_{-}\wedge\omega_{+},\\ \check{S}_{\sigma}(x_{\pm}\,\omega_{\pm})&=\,\pm\mathrm{i}\check{m}_{+}\,q^2\,\alpha\,x_{\pm}\,\omega_{\pm},\\ \check{S}_{\sigma}(\omega_{-}\wedge\omega_{+})&=\,-\mathrm{i}\check{m}_{+}\,2\,q^4\,\alpha^2/\lambda^{-}_{(2)},\end{aligned} \tag{4.10}$$

[6] It is moreover possible to prove that the element

$$\Gamma^{\otimes2}(\mathrm{S}^2_q)\,\ni\,\phi\,=\,q^{-1}\mathrm{d}B_{-}\otimes\mathrm{d}B_{+}\,+\,q\mathrm{d}B_{+}\otimes\mathrm{d}B_{-}\,-(1+q^{-2})\mathrm{d}B_0\otimes\mathrm{d}B_0$$

is the only (up to scalars) to be left invariant and symmetric for any of the resulting braidings: this result extends then the result proved in the proposition 4.2 in [13].

where $x_\pm \in \mathcal{L}_{\pm 2}$, while the normalization reads $\check{m}_+^2\, 2\, q^4\, \alpha^2/\lambda^-_{(2)} = 1$. Given the isomorphism $\Gamma_\sigma(\mathrm{S}^2_q) \sim \Gamma_{\sigma^-}(\mathrm{S}^2_q)$, from (2.43) it is immediate to see that

$$\begin{aligned} \check{S}_{\sigma^-}(1) &= \mathrm{i} q^{-2} \check{m}_-\, \omega_- \wedge \omega_+, \\ \check{S}_\sigma(x_\pm\, \omega_\pm) &= \pm \mathrm{i} \check{m}_-\, \alpha\, x_\pm\, \omega_\pm, \\ \check{S}_\sigma(\omega_- \wedge \omega_+) &= -\mathrm{i} \check{m}_-\, 2\, q^2\, \alpha^2/\lambda^+_{(2)}, \end{aligned} \tag{4.11}$$

with $\check{m}_-^2\, 2\, \alpha^2/\lambda^+_{(2)} = 1$. We recall that the Hodge operators on S^2_q corresponding to those defined on $\mathrm{SU}_q(2)$ via the calculus (2) (that is in (2.30)) coincide with those above, since it is obtained by mapping $q \to -q$.

- In the same way, if we realize the 2d differential calculus (4.6) on S^2_q as a restriction of the calculus (2.32) on $\mathrm{SU}_q(2)$, characterized by $\mathcal{Q}_4 \subset \ker \varepsilon_{\mathrm{SU}_q(2)}$, then the Hodge operator we obtain is (recall from (3.38) that it also depends on the real parameter $\alpha \neq 0$):

$$\begin{aligned} \check{S}_\sigma(1) &= \mathrm{i} \check{m}_+\, \omega_- \wedge \omega_+, \\ \check{S}_\sigma(x_\pm\, \omega_\pm) &= \pm \mathrm{i} \check{m}_+\, \alpha\, x_\pm\, \omega_\pm, \\ \check{S}_\sigma(\omega_- \wedge \omega_+) &= -\mathrm{i} \check{m}_+\, 2\, \alpha^2/\lambda^-_{(2)}, \end{aligned} \tag{4.12}$$

where $x_\pm \in \mathcal{L}_{\pm 2}$, while the normalization reads $\check{m}_+^2\, 2\, \alpha^2/\lambda^-_{(2)} = 1$. Given the isomorphism $\Gamma_\sigma(\mathrm{S}^2_q) \sim \Gamma_{\sigma^-}(\mathrm{S}^2_q)$, from (2.43) it is immediate to see that

$$\begin{aligned} \check{S}_{\sigma^-}(1) &= \mathrm{i} q^{-2} \check{m}_-\, \omega_- \wedge \omega_+, \\ \check{S}_\sigma(x_\pm\, \omega_\pm) &= \pm \mathrm{i} q^{-2} \check{m}_-\, \alpha\, x_\pm\, \omega_\pm, \\ \check{S}_\sigma(\omega_- \wedge \omega_+) &= -\mathrm{i} \check{m}_-\, 2\, q^{-2}\, \alpha^2/\lambda^+_{(2)}, \end{aligned} \tag{4.13}$$

with $\check{m}_-^2\, 2q^{-4}\, \alpha^2/\lambda^+_{(2)} = 1$. Once more we have that the Hodge operators on S^2_q corresponding to those defined on $\mathrm{SU}_q(2)$ via the calculus (5) (i.e. (2.33)) coincide with the ones above.

We close this analysis by describing the Laplacian operators introduced on S^2_q by the Hodge operators above. A direct calculation shows that one has (due to the normalization condition given in (4.9))

$$\Box_{\mathrm{S}^2_q} f = q\,(EF + FE) \triangleright f \tag{4.14}$$

(on $f \in \mathcal{A}(\mathrm{S}^2_q)$) for any of the calculi in $\tilde{\mathcal{K}}$ and for any realization of the exterior algebra in terms of the braidings $\sigma^\pm$. The action of this operator moreover coincides with the action of the Laplacian operator on S^2_q obtained in [21] following a different formulation.

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