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Preface

The present volume of “Travaux Mathématiques" collects some of the contributions to the international conference “Harmonic Analysis - Luxembourg/Metz" which was jointly organized by the Séminaire de mathématique du Centre Universitaire de Luxembourg and the Laboratoire de Mathématiques de l'Université de Metz, 9-12 September 2002.

This was the first workshop of this kind. Others are certainly to follow in the future, as such a meeting represents an excellent opportunity to establish links between the mathematical communities in Luxembourg and Metz.

The conference met a large resonance: 63 participants coming from 12 countries attended the conference and 25 talks were given (see the programme on the next pages and the abstracts at the end of the volume). The themes treated during the workshop covered most of the spectrum of the so-called non-commutative harmonic analysis, that is, analysis on non-abelian groups and group-like structures: analysis on symmetric spaces, representation theory and harmonic analysis on specific locally compact groups (amenable groups, compact groups, nilpotent Lie groups, discrete groups, Kazhdan's groups, p-adic groups ...), infinite dimensional Lie groups, von Neumann algebras, hypergroups...

The contributions submitted to this volume are the following. They all have been individually refereed.

- **V. Heiermann**: Spectral decomposition on a p-adic group
- **K.-H. Neeb**: Root graded Lie groups
- **D. Poguntke**: Synthesis properties of orbits of compact groups
- **J.-F. Quint**: Property (T) and exponential growth of discrete subgroups
- **G. Robertson** : Singular masas of von Neumann algebras : examples from the geometry of spaces of nonpositive curvature

Acknowlegments

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- Fonds National de la Recherche du Luxembourg

- Gouvernement Luxembourgeois
- Université de Metz
- Laboratoire de mathématiques de l'Université de Metz
- Région Lorraine
- Ville de Metz

We express our gratitude to all of them. We are grateful for the hospitality and the technical support of the

- Centre Universitaire de Luxembourg

and the

- Université de Metz

where the conference took place. We would also like to thank the speakers and participants: they are responsible for the success of the conference. Last but not least, we thank all the members and co-workers of the Centre Universitaire de Luxembourg, of the Société mathématique du Luxembourg and of the Laboratoire de mathématiques de l'Université de Metz for their help and assistance.

The organizers : **B. Bekka, J. Ludwig, C. Molitor–Braun, N. Poncin**

PROGRAMME

Monday, September 9 in Luxembourg

9h00-9h45 Welcome of the participants

9h45-10h00 Opening session

10h00-10h50 D. Müller (University of Kiel, Germany)
Bochner-Riesz means on the Heisenberg group and fractional integration on the dual

11h10-12h00 H. Fujiwara (Kinki University, Japan)
Inducing and restricting unitary representations of nilpotent Lie groups

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14h00-14h50 C. Anantharaman-Delaroche (University of Orléans, France)
On spectral characterizations of amenability

15h10-16h00 V. Nekrashevysh (University of Kiev, Ukraine)
Hilbert bimodules associated to self-similar group actions

16h30-17h20 G. Robertson (University of Newcastle, Australia)
Singular masas of von Neumann algebras : examples from the geometry of spaces of nonpositive curvature

Parallel sessions

17h40-18h10
N. Prudhon (University Louis Pasteur, Strasbourg, France)
Théorie des représentations et K-théorie
and
F. J. Gonzalez (University of Lausanne, Switzerland)
Fourier inversion on rank one compact symmetric spaces

Tuesday, September 10 in Luxembourg

9h00-9h50 A. Lubotzky (University of Jerusalem, Israel)
Ramanujan complexes

10h20-11h10 A. Valette (University of Neuchâtel, Switzerland)
Property (T) *and harmonic maps*

11h30-12h00 G. Litvinov (International Sophus Lie Center, Moscow, Russia)
Integral Geometry and hypergroups

14h00-14h50 J.-P. Anker (University of Orléans, France)
The heat kernel on symmetric spaces, fifteen years later

15h10-16h00 P. Torasso (University of Poitiers, France)
The Plancherel formula for almost algebraic groups

16h30-17h20 E. Damek (University of Wroclaw, Poland)
Asymptotic behavior of the Poisson kernel on NA *groups*

Parallel sessions

17h40-18h10
H. Biller (University of Darmstadt, Germany)
Harish-Chandra decomposition of Banach-Lie groups
and
V. Heiermann (Humbold University, Berlin, Germany)
Spectral decomposition and discrete series on a p-adic group

Wednesday, September 11 in Metz

Bus tranfer from Luxembourg to Metz

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14h00-14h50 J.-Y. Benoist (ENS Paris, France)
Divisible convex sets and prehomogeneous vector spaces

15h10-16h00 Y. Shalom (University of Jerusalem, Israel)
Harmonic analysis and the geometry of amenable groups

16h30-17h20 T. Steger (University of Sassari, Italy)
Free group representations and their realizations on the boundary

Parallel sessions

17h40-18h10 J.-F. Quint (ENS Paris, France)
Property (T) and exponential growth of discrete subgroups
and
J. Galindo (University Jaume I de Castellón, Spain)
Unitary duality, weak topologies and thin sets in locally compact groups

Thursday, September 12 in Metz

9h00-9h50 A.M. Vershik (University of St. Petersbourg, Russia)
Harmonic analysis on the groups which are similar to infinite dimensional groups

10h20-11h10 K.-H. Neeb (University of Darmstadt, Germany)
Root graded Lie groups

11h30-12h20 Y. Neretin (University of Moscow)
Closures of quasiinvariant actions of infinite-dimensional groups, polymorphisms, and Poisson configurations

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14h30-15h20 E. Kaniuth (University of Paderborn, Germany)
Extending positive definite functions from subgroups of locally compact groups

15h40-16h30 D. Poguntke (University of Bielefeld, Germany)
Synthesis properties of orbits of compact groups

List of participants

Mrs. Claire Anantharaman-Delaroche	Orléans, France
Mr. Yann Angeli	Nancy, France
Mr. Jean-Philippe Anker	Orléans, France
Mr. Didier Arnal	Dijon, France
Mr. Bachir Bekka	Metz, France
Mr. Yves Benoist	Paris, France
Mr. Harald Biller	Darmstadt, Germany
Mr. Marek Bozejko	Wroclaw, Poland
Mr. Dariusz Buraczewski	Wroclaw, Poland
Mr. Jean-Louis Clerc	Nancy, France
Mrs. Ewa Damek	Wroclaw, Poland
Mr. Antoine Derighetti	Lausanne, Switzerland
Mr. Jacek Dziubanski	Wroclaw, Poland
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Mr. Jorge Galindo	Castellón, Spain
Mr. Helge Glöckner	Darmstadt, Germany
Mr. Raoul Gloden	Luxembourg, Luxembourg
Mr. Francisco J. Gonzalez	Lausanne, Switzerland
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Mrs. Junko Inoue	Tottori, Japan
Mr. Hideyuki Ishi	Yokohama, Japan
Mr. Eberhard Kaniuth	Paderborn, Germany
Mr. Guy Kass	Luxembourg, Luxembourg
Mr. Koufany Khalid	Nancy, France
Mr. Michael Leinert	Heidelberg, Germany
Mr. Grigori Litvinov	Moscow, Russia
Mr. Nicolas Louvet	Metz, France
Mr. Alex Lubotzki	Jerusalem, Israel
Mr. Jean Ludwig	Metz, France
Mr. Moshen Masmoudi	Metz, France
Mr. Christian Masse	Metz, France
Mrs. Carine Molitor-Braun	Luxembourg, Luxembourg
Mr. Raymond Mortini	Metz, France
Mr. Detlef Müller	Kiel, Germany

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Mr. Volodia Nekrashevych	Kiev, Ukraine
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Mrs. Monique Peiffer	Luxembourg, Luxembourg
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Mr. Norbert Poncin	Luxembourg, Luxembourg
Mr. Nicolas Prudhon	Strasbourg, France
Mr. Jean-François Quint	Paris, France
Mr. Guyan Robertson	Newcastle, Australia
Mr. Jang Schiltz	Luxembourg, Luxembourg
Mr. Günter Schlichting	München, Germany
Mr. Laurent Scuto	Metz, France
Mr. Yehuda Shalom	Jerusalem, Israel
Mr. Yves Stalder	Neuchâtel, Switzerland
Mr. Tim Steger	Sassari, Italy
Mr. Aleksander Strasburger	Warsaw, Poland
Mr. Pierre Torasso	Poitiers, France
Mr. Bartosz Trojan	Wroclaw, Poland
Mr. Oliver Ungermann	Bielefeld, Germany
Mr. Roman Urban	Wroclaw, Poland
Mr. Alain Valette	Neuchâtel, Switzerland
Mr. Anatoly Vershik	St. Petersbourg, Russia
Mr. Tilmann Wurzbacher	Metz, France
Mr. Jacek Zienkiewicz	Wroclaw, Poland

Spectral decomposition and discrete series representations on a p-adic group

by Volker Heiermann

Abstract

Let G be a p-adic group. After a short survey on the representation theory of G, I outline my proof of a conjecture of A. Silberger on the infinitesimal character of discrete series representations of G. The conjecture says the following: a cuspidal representation τ of a Levi subgroup L of G corresponds to the infinitesimal character of a discrete series representation of G, if and only if τ is a pole of Harish-Chandra's μ-function of order equal to the parabolic rank of L. The proof uses a spectral decomposition based on a Fourier inversion formula analog to the Plancherel formula. To illustrate the method, the case of the unramified principal series of a semi-simple split group of type B_2 is worked out at the end.

1. Notations: Let F be a non-Archimedean local field. This is a topological field equipped with a discrete valuation $|.|_F$. It will be supposed to be normalized such that the Haar measure on F satisfies the transformation formula $d_F(xy) = |y|_F d_F x$. The topology of F is then defined by the ultrametric distance $d_F(x,y) := |x-y|_F$ and F is complete with respect to this metric. There exists a unique generator of the image of $|.|_F$ which is > 1. It will be denoted by q.

Let $\underline{G}$ be a connected reductive group defined over F and G the group of its F-rational points. So G is a locally compact and totally disconnected group.

The set of equivalence classes of irreducible representations of G will be denoted $\mathcal{E}(G)$. As usual, a representation will often be identified with its equivalence class. The subset of classes of square-integrable representations (i.e. whose matrix coefficients are square integrable functions on G modulo its center) will be denoted $\mathcal{E}_2(G)$. To any representation π in $\mathcal{E}_2(G)$ one can associate the formal degree $\deg(\pi)$, which is defined up to the choice of a Haar measure on G. Sometimes it is necessary to consider the bigger set $\mathfrak{A}(G)$ of equivalence classes of admissible representations of G, which contains the above. (These are no more irreducible but the space of vectors invariant by an open compact subgroup is finite dimensional.)

A subgroup P of G will be called a parabolic subgroup, if it is the group of F-rational points of a parabolic subgroup $\underline{P}$ of $\underline{G}$ defined over F. I will fix a maximal split torus in $\underline{G}$ and let T be the group of its F-rational points. A parabolic subgroup of G will be called semi-standard, if it contains T. There is then a unique Levi factor $\underline{L}$ of $\underline{P}$ which is defined over F and such that the group

L of its F-rational points contains T. The expression "$P = LU$ is a semi-standard parabolic subgroup of G" will then mean that P is a semi-standard parabolic subgroup, U its unipotent radical and L the group of F-rational points of its unique Levi factor which contains T. The functor i_P^G of parabolic induction sends $\mathfrak{A}(L)$ to $\mathfrak{A}(G)$ and will be supposed to be normalized such that it takes unitary representations to unitary representations. (So a representation parabolically induced from an irreducible representation is admissible.)

Let $\mathrm{Rat}(G)$ be the group of rational characters of $\underline{G}$ defined over F and G^1 the intersection of the kernels of the characters of G of the form $|\chi|_F$, $\chi \in \mathrm{Rat}(G)$. An unramified character of G is a homomorphism from the group quotient G/G^1 to $\mathbb{C}^\times$. The group formed by these characters will be designed by $\mathfrak{X}^{\mathrm{ur}}(G)$ and the subgroup formed by the unitary characters by $\mathfrak{X}_0^{\mathrm{ur}}(G)$. The group $\mathfrak{X}^{\mathrm{ur}}(G)$ is an algebraic tori isomorphic to $(\mathbb{C}^\times)^d$ with d equal to the rank of G, $\mathfrak{X}_0^{\mathrm{ur}}(G)$ being isomorphic to $(S^1)^d$. The first group acts on the set $\mathcal{E}(G)$ and the second group on the subset $\mathcal{E}_2(G)$. An orbit with respect to this action will be denoted by $\mathcal{O}$ in the first and $\mathcal{O}_2$ in the second case. So $\mathcal{O}$ is an algebraic variety. The space $\mathcal{O}$ (resp. $\mathcal{O}_2$) is, through the choice of a Haar measure on $\mathfrak{X}^{\mathrm{ur}}(G)$, equipped with a unique measure, such that the action of $\mathfrak{X}^{\mathrm{ur}}(G)$ on $\mathcal{O}$ (resp. of $\mathfrak{X}_0^{\mathrm{ur}}(G)$ on $\mathcal{O}_2$) preserves locally the measures.

2. Plancherel formula: The Plancherel formula for p-adic groups (due to Harish-Chandra (cf. [21])) expresses a smooth, compactly supported and complex valued function f on G by its Fourier transforms. More precisely, for f in $C_c^\infty(G)$ and $\pi \in \mathcal{E}(G)$ one defines an endomorphism $\pi(f)$ of the representation space V_π of π by

$$\pi(f) := \int_G f(g)\pi(g)dg.$$

Let $\Theta_2(G)$ be the set of pairs $(P = LU, \mathcal{O}_2)$ with $P = LU$ a semi-standard Levi subgroup of G and $\mathcal{O}_2$ an orbit in $\mathcal{E}_2(L)$. Two paires are called equivalent $\sim$, if they are conjugated by an element of G.

Let ρ be the action of G on $C_c^\infty(G)$ by right translations. Harish-Chandra defined for every pair $(P = LU, \mathcal{O}_2) \in \Theta_2(G)$ a constant $\gamma(G/L)$ and a function μ on $\mathcal{O}_2$, which extends to a rational function on the $\mathfrak{X}^{\mathrm{ur}}(G)$-orbit $\mathcal{O}$ which contains $\mathcal{O}_2$. He showed that for f in $C_c^\infty(G)$ one has (with a suitable normalization of the measures)

$$f(g) = \sum_{(P=LU,\mathcal{O}_2)\in\Theta_2/\sim} \gamma(G/L)\ |W(L,\mathcal{O}_2)|^{-1} \int_{\mathcal{O}_2} \mathrm{tr}((i_P^G\pi)(\rho(g)f))\deg(\pi)\mu(\pi)d\pi.$$

(Here $W(L, \mathcal{O}_2)$ denotes a subset of the Weyl group of G relative to T formed by elements which stabilize $\mathcal{O}_2$ and L.)

3. Representation theory of G: (cf. [4]) The functor i_P^G admits a left adjoint functor r_P^G which is called the *Jacquet functor.* A representation $\pi \in \mathcal{E}(G)$ is then called *cuspidal,* when $r_P^G\pi = 0$ for all proper parabolic subgroups P of G. The subset of cuspidal representations will be denoted $\mathcal{E}_c(G)$. Note that to any cuspidal representation π a formal degree $\deg(\pi)$ can be attached.

The classification of cuspidal representations is a deep arithmetical problem which is entirely solved only for GL_N (including SL_N) and the multiplicative group of a central division algebra over F respectively by C. Bushnell and P. Kutzko ([2] and subsequent work treating SL_N) and E.-W. Zink [22]. A conjectural parametrization of this set by N-dimensional irreducible representations of the Weil group of F (which is some distinguished subgroup of the absolut Galois group of F) is the aim of the local Langlands conjectures. For GL_N, they have been proved recently by M. Harris and R. Taylor [5] and, by a more elementary approach, by G. Henniart [9].

Given $\pi \in \mathcal{E}(G)$ there exist a semi-standard parabolic subgroup $P = LU$ and a cuspidal representation $\sigma \in \mathcal{E}_c(L)$ such that π is a subquotient of $i_P^G\sigma$. The G-conjugation class of L and σ is uniquely determined by π. It is called the *cuspidal support* of π.

Remark that any *unitary* cuspidal representation is square integrable and that any orbit $\mathcal{O}$ of a cuspidal representation is formed by cuspidal representations and contains a cuspidal representation that is unitary. On the other hand there exist square integrable representations which are not cuspidal. These are called *special representations.*

Example: Identify $\chi = |.|_F$ with a character of the diagonal subgroup L of SL_2 by the embedding $x \mapsto (x, x^{-1})$. Let B be the Borel subgroup formed by upper triangular matrices. Then the induced representation $i_B^G\chi$ is of length 2 and has a unique subrepresentation which is called the *Steinberg representation.* It is square integrable, but not cuspidal. (Remark that the other subquotient of $i_B^G\chi$ is the unit representation of G.)

Classification scheme: The Langlands classification (cf. [17]) gives a description of the set $\mathcal{E}(G)$ up to the knowledge of the tempered representations of its Levi subgroups. Tempered representations can be constructed by parabolic induction from square integrable representations. For GL_N a representation parabolically induced from a square integrable representation is irreducible, but for other groups this may fail and it is not known yet how to describe the different components.

The next step below is to construct all square integrable representations from the cuspidal ones. This is known for GL_N by the work of Bernstein and Zelevinsky [22], for unramified principal series representations by Kazhdan and Lusztig [10] and for split classical groups (under some assumption on the reducibility points) by the results of C. Moeglin and Moeglin-Tadic ([12] and [13]). There are also several results of A. Silberger in [18] and [19].

4. A conjecture of Silberger: A. Silberger conjectured also the following result:

Theorem: (cf. [8] corollaire **8.7**) *Let $P = LU$ be a parabolic subgroup of G and τ an irreducible cuspidal representation of L. Then $i_P^G\tau$ has a subquotient in $\mathcal{E}_2(G)$ precisely when the following two conditions hold:*

i) the restriction of τ to the center of G is a unitary representation;

ii) τ is a pole of μ of order $rk_{ss}(G) - rk_{ss}(L)$. (Here μ denotes Harish-Chandra's μ-function as defined in **2.***).*

Let us make the second condition more precise. For this, I will first explain the notion of an affine rootal hyperplane. Fix a maximal split torus T_L in the center of L and let $\Sigma(P)$ be the set of roots of T_L in $Lie(U)$. Define $a_L = \mathrm{Hom}(\mathrm{Rat}(L), \mathbb{R})$ and let a_L^* be the dual space. It contains $\Sigma(P)$. There is a natural map $H_L : L \to a_L$. One defines a surjection from the complexified vector space $a_{L,\mathbb{C}}^*$ to $\mathfrak{X}^{\mathrm{ur}}(L)$, by sending λ to the character χ_λ with $\chi_\lambda(l) := q^{-\langle H_L(l),\lambda\rangle}$ (recall that q is the unique generator > 1 of the image of $|.|_F$). The restriction of this map to a_L^* is injective and so $\Re(\chi_\lambda) := \Re(\lambda)$ is well defined. An affine rootal hyperplane in $a_{L,\mathbb{C}}^*$ is then by definition an affine hyperplane defined by a coroot $\alpha^\vee$, $\alpha \in \Sigma(P)$.

Let $\mathcal{O}$ be the orbit of τ. An affine rootal hyperplane in $\mathcal{O}$ is then by definition the image of an affine rootal hyperplane in $a_{L,\mathbb{C}}^*$ by the composed map $a_{L,\mathbb{C}}^* \to \mathfrak{X}^{\mathrm{ur}}(L) \to \mathcal{O}$, the second arrow being given by the action of $\mathfrak{X}^{\mathrm{ur}}(L)$ on $\mathcal{O}$.

It is known since Harish-Chandra that the poles and zeroes of μ lie on finitely many affine rootal hyperplanes in $\mathcal{O}$. Let $\mathcal{S}_0$ be the set of affine zero hyperplanes of μ and $\mathcal{S}_1$ the set of affine polar hyperplanes. The affine zero hyperplanes are of order 2 and the polar ones are of order 1. So the order of the pole of μ in τ is $|\{S \in \mathcal{S}_1 \,|\tau \in S\}| - 2|\{S \in \mathcal{S}_0 \,|\tau \in S\}|$.

Remark: By a conjecture of Langlands [11], which has been proved by F. Shahidi [16] in the case of G quasi-split and τ generic, the function μ on $\mathcal{O}$ can be expressed as product and quotient of L-functions attached to τ.

5. Strategy of proof: Let $\mathcal{O}$ be the orbit of τ in $\mathcal{E}_c(L)$. All its elements can be realized as representations in a same vector space which will be denoted E. For σ in an open set of $\mathcal{O}$ and $P' = LU'$ a second parabolic subgroup with Levi factor L, one has an operator $J_{P|P'}(\sigma) : i_{P'}^G E \to i_P^G E$ which intertwines the representations $i_{P'}^G \tau$ and $i_P^G \tau$. In an open cone of $\mathcal{O}$ it is defined by the converging integral

$$(J_{P|P'}(\sigma)v)(g) := \int_{U \cap U' \backslash U} v(ug)du,$$

where v is considered as an element of the space $i_{P'}^G E$ equipped with the representation $i_{P'}^G \sigma$. It is a rational function in σ and the composed operator $J_{P|\overline{P}}(\sigma) J_{\overline{P}|P}(\sigma)$ is scalar and equals the inverse of the μ-function. For w in the Weyl group W of G with respect to T one defines an operator $\lambda(w)$ which induces an isomorphism between the representations $i_P^G \sigma$ and $i_{wP}^G w\sigma$.

The following lemma was crucial for the proof of a matrix Paley-Wiener theorem in [7]:

Lemma: (cf. [7] **0.2**) *Let f be in $C_c^\infty(G)$. Identify $(i_P^G \sigma)(f)$ to an element of $i_P^G E \otimes i_P^G E^\vee$. There exists a polynomial map $\xi_f : \mathcal{O} \to i_{\overline{P}}^G E \otimes i_P^G E^\vee$ with image in a finite dimensional space such that*

$$(i_P^G \sigma)(f) = \sum_{w \in W(L,\mathcal{O})} (J_{P|\overline{wP}}(\sigma)\lambda(w) \otimes J_{P|wP}(\sigma)\lambda(w))\xi_f(w^{-1}\sigma).$$

as rational funtions in σ. (Here $W(L,\mathcal{O})$ has the same meaning than in **2.***.)*

Remark that the poles of $J_{P|\overline{P}}$ are on the affine rootal hyperplanes in $\mathcal{S}_0$ and that the poles of $\mu J_{P|\overline{P}}$ are on the affine rootal hyperplanes in $\mathcal{S}_1$.

Let $C_c^\infty(G)_{\mathcal{O}}$ be the subspace of $C_c^\infty(G)$ formed by the functions f such that $(i_{P'}^G \sigma')(f) = 0$ for all $\sigma' \in \mathcal{O}'$ with $(P', \mathcal{O}') \not\sim (P, \mathcal{O})$.

Proposition: *Let f be in $C_c^\infty(G)_{\mathcal{O}}$. Identify $\xi_f(\sigma)$ with an element in* $\mathrm{Hom}(i_{\overline{P}}^G E, i_P^G E)$*. For $g \in G$ one has*

$$(*) \qquad f(g) = \int_{\Re(\sigma) = r \gg_P 0} \gamma(G/L) \deg(\sigma) \operatorname{tr}((i_P^G \sigma)(g^{-1}) J_{P|\overline{P}}(\sigma) \xi_f(\sigma)) \mu(\sigma) d\Im(\sigma).$$

(The symbol $\int_{\Re(\sigma)=r\gg_P 0}$ means that one fixes r in a_L^* such that $\langle r, \alpha^\vee \rangle \gg 0$ for all $\alpha \in \Sigma(P)$ and that one integrates on the compact set $\chi_r \mathcal{O}_2$. Here $\mathcal{O}_2$ is the subset formed by the unitary representations in $\mathcal{O}$ and the integral is taken with respect to the fixed measure on $\mathcal{O}_2$.)

The strategy of proof of the theorem in **4.** is then to compare the expression (*) in the above proposition with the one given by the Plancherel formula in **2.**. This is done after a contour shift to the unitary axis.

Example: Suppose G semi-simple and L maximal. Then $\mathfrak{X}^{\mathrm{ur}}(L) \simeq \mathbb{C}^\times$ and the theory of complex functions in one variable applies: the integral (*) is a sum of residues and an integral over the unitary orbit $\mathcal{O}_2$. The residues correspond by the remark after the above lemma to the poles of μ.

In the Plancherel formula for $f \in C_c^\infty(G)_\mathcal{O}$, there appear only terms corresponding to the equivalence class of pairs $(P' = L'U', \mathcal{O}_2)$ with either $P' = P$ or with $P' = G$ and $\mathcal{O}_2$ equal to the set formed by a single square integrable representation of G. The cuspidal support of this representations is necessarily contained in the G-orbit of $\mathcal{O}$. The first term is an integral over $\mathcal{O}_2$ and the other terms are discrete.

It is then rather easy to show that the two integrals and the discrete terms in both formulas correspond to each other, proving the theorem in this simple case. (Remark that the theorem **4.** was already known in this case.)

Unfortunately, in the case of a Levi subgroup with corank bigger than one, poles of the intertwining operators do appear and it is not evident at all, that and how they cancel. The proof of the theorem **4.** follows then the following steps:

i) *Formulation of a convenient multi-dimensional residue theorem:* this is achieved by a generalization of the residue theory for root systems due to E. P. van den Ban and H. Schlichtkrull [3] to our situation (see also the paper [6] of G. Heckman and E. Opdam). Let $\mathcal{S}$ be the union of the sets $\mathcal{S}_0$ and $\mathcal{S}_1$. Define $\mathcal{A}(\mathcal{S})$ as the set of affine subspaces of $\mathcal{O}$ which are connected components of finite intersections of elements in $\mathcal{S}$. The subset of spaces in $\mathcal{A}(\mathcal{S})$, where μ can have a non trivial residue will be denoted $\mathcal{A}_\mu(\mathcal{S})$. I also fix a set $[\mathcal{A}(\mathcal{S})]$ of representatives of conjugation classes in $\mathcal{A}(\mathcal{S})$. To an affine hyperplane A in $\mathcal{A}(\mathcal{S})$ one attaches a semi-standard Levi subgroup L_A of G. The origine of A will be denoted $r(A)$ and ϵ_A will be an element in some positive Weyl chamber of $a_{L_A}^*$.

With $\Delta_\mathcal{O}$ some set of positive roots associated to $\mathcal{O}$, $W_{\Delta_\mathcal{O}}$, $W^{L_A}(L, \mathcal{O})$ and $W_{L_A}^+(L, \mathcal{O})$ some sets of Weyl group elements and $\mathcal{P}_\mathcal{S}(L_A)$ some set of generalized parabolic subgroups with Levi component L_A, the residue formula applied to the integral (*) gives then

$$\sum_{\Omega\subseteq\Delta_{\mathcal{O}}}\ \sum_{A\in[\mathcal{A}(\mathcal{S})],L_A=L_\Omega} |W_{\Delta_{\mathcal{O},L_A}}|^{-1}\,|\mathcal{P}_{\mathcal{S}}(L_A)|^{-1}\gamma(G/L)\int_{\Re(\sigma)=r(A)+\epsilon_A}\deg(\sigma)|\operatorname{Stab}(A)|^{-1}$$
$$\sum_{w'\in W^{L_A}(L,\mathcal{O})}\ \sum_{w\in W^+_{L_A}(L,\mathcal{O})}\operatorname{Res}^P_{w'A}(\operatorname{tr}((i_P^G\sigma)(g^{-1})J^{-1}_{\overline{P}|P}(ww'\sigma)\xi_f(ww'\sigma)))d_A\Im(\sigma).$$

Here Res^P_A is an operator from the space of rational functions on $\mathcal{O}$, which are regular outside the hyperplanes in $\mathcal{S}$, to some space of rational functions on A. It turns out to be uniquely determined by r and P. It is a sum of composed residue operators relative to affine hyperplanes in $\mathcal{S}$ containing A.

ii) *Identification of the continuous part:* This is done with help of an induction hypothesis. One sees then that one can replace ϵ_A by zero in the above formula.

iii) *Elimination of the undesirable poles with help of test functions.* These already appeared in [7] at a crucial step, although they played a different role there.

With this one gets the following result:

The induced representation $i_P^G\tau$ has a subquotient in $\mathcal{E}_2(G)$ if and only if the restriction of τ to the center of G is unitary, $A\in\mathcal{A}_\mu(\mathcal{S})$, $L_A=G$, and

(**) $$\sum_{w\in W(L,\mathcal{O})}(\operatorname{Res}_{wA}\mu)(w\sigma)\neq 0.$$

But a theorem of E. Opdam (cf. [15] theorem 3.29) shows, that the condition (**) is always satisfied, finishing the proof of the theorem in **4.**. Remark that Opdam proved in [15] a spectral decomposition for affine Hecke algebras, which applies in particular to Iwahori-Hecke algebras and through it for example to the unramified principal series of a p-adic group.

The identities with the terms in the Plancherel formula contain also informations on the formal degree and on the position of the discrete series representations in the induced representation. Opdam was for example able to deduce from his identities some invariance properties of the formal degree on L-packets of discrete series representations in his context.

The method employed here may be considered as a local analog of the spectral decomposition and the theory of the residual spectrum due to Langlands [11] in the field of automorphic forms.

Appendix: The case of B_2

Let now $\underline{G}$ be a semi-simple split group of type B_2 defined over F. Fix a minimal semi-standard parabolic subgroup $P = TU$ of G. Then $a_0^* := a_T^* \simeq \mathbb{R}^2$. The set $\Sigma(P)$ of roots of T in Lie(U) can be written in the form $\{\alpha, \beta, \alpha{+}2\beta, \alpha{+}\beta\}$, where β is the short root. Let $\Sigma^\vee(P)$ be the set of roots dual to the roots in $\Sigma(P)$. One has $\Sigma^\vee(P) = \{\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee, 2\alpha^\vee + \beta^\vee\}$. The set $\{\alpha^\vee, \beta^\vee\}$ is a base of a_0 and the dual bases of a_0^* will be denoted $\{\omega_\alpha, \omega_\beta\}$. Observe that $\langle\alpha^\vee, \beta\rangle = -1$ and $\langle\beta^\vee, \alpha\rangle = -2$.

Let τ be the trivial representation of T. The orbit $\mathcal{O}$ of τ with respect to $\mathfrak{X}^{\mathrm{ur}}(T)$ is isomorphic to $(\mathbb{C}^\times)^2$. Define $\tau_\lambda := \tau \otimes \chi_\lambda$. The μ-function on $\mathcal{O}$ is given by

$$\mu(\tau_{x\omega_\alpha + y\omega_\beta}) = C\frac{(1-q^x)(1-q^{-x})(1-q^y)(1-q^{-y})(1-q^{x+y})(1-q^{-x-y})}{(1-q^{1+x})(1-q^{1-x})(1-q^{1+y})(1-q^{1-y})(1-q^{1+x+y})(1-q^{1-x-y})} \times \frac{(1-q^{2x+y})(1-q^{-2x-y})}{(1-q^{1+2x+y})(1-q^{1-2x-y})},$$

where C is a constant > 0.

The affine hyperplanes of $\mathcal{O}$ which are polar for μ are the images of the affine hyperplanes in a_0^* of the form $\langle\gamma^\vee, \lambda\rangle = c$ with $c = -1$ or $c = 1$, $\gamma \in \Sigma(P)$. The zero affine hyperplanes are the images of the affine hyperplanes $\langle\gamma^\vee, \lambda\rangle = 0$ in a_0^*, $\gamma \in \Sigma(P)$. So they correspond to the lines generated respectively by the vectors $\overrightarrow{0\omega_\alpha}$, $\overrightarrow{0\omega_\beta}$, $\overrightarrow{0\alpha}$ and $\overrightarrow{0\beta}$.

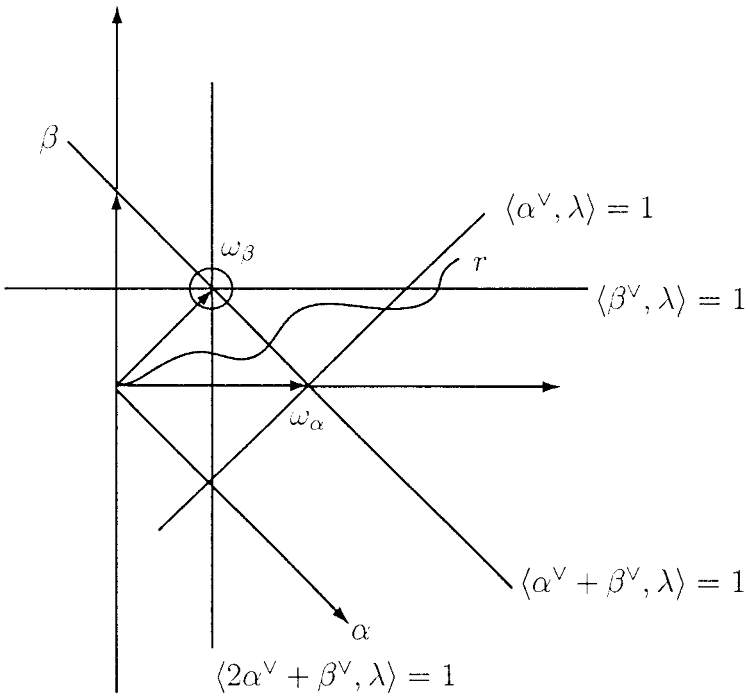
Fix $r = r_\alpha\omega_\alpha + r_\beta\omega_\beta \in a_0^*$ with $r_\alpha \gg 0$ and $r_\beta \gg 0$. To calculate for $f \in C_c^\infty(G)_\mathcal{O}$ the integral

$$(\#) \qquad \int_{\Re(\sigma)=r} \mathrm{tr}((i_P^G\sigma)(g^{-1})J_{P|\overline{P}}(\sigma)\xi_f(\sigma))\mu(\sigma)d\Im(\sigma),$$

one first moves r to 0 following the circuit below. Each intersection of this circuit with a polar hyperplane H gives a residue, which is a rational function in σ with $\Re(\sigma) \in H$. For each intersection point r_H, one has to move r_H on H near to the origine $r(H)$ of this affine hyperplane (which is the point with minimal distance to the origine of a_0^*). Each intersection point with an affine polar or zero hyperplane $H' \neq H$ of this segment on H can give rise to another non trivial residue. It turns out that there is only one case where a zero affine hyperplane gives a non trivial residue: this happens for the intersection of the polar affine hyperplane

$\langle\beta^\vee, \lambda\rangle = 1$ with the zero affine hyperplane $\langle\alpha^\vee, \lambda\rangle = 0$. The intersection point is ω_β.

Remark that the point r_H can be moved to the origine of the hyperplan H, if the origine is a regular point. (One verifies that only the affine polar hyperplane $\langle\alpha^\vee + \beta^\vee, \lambda\rangle = 1$ has an origine, which is not regular.)



To simplify the notations let $g = 1$. (For $g \neq 1$ one gets the analog result.) Then one sees that (#) is with $\epsilon > 0$ equal to

(1) $\quad \mathrm{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+\frac{3}{2}\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+\frac{3}{2}\omega_\alpha}))\,\mathrm{Res}_{x=\frac{3}{2}}\,\mathrm{Res}_{y=1}(\mu(\tau_{x\omega_\alpha+y\frac{\beta}{2}}))$

(2) $\quad +\,\mathrm{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+(1+\frac{\pi i}{\log q})\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+(1+\frac{\pi i}{\log q})\omega_\alpha}))\,\mathrm{Res}_{x=1+\frac{\pi i}{\log q}}\,\mathrm{Res}_{y=1}(\mu(\tau_{x\omega_\alpha+y\frac{\beta}{2}}))$

(3) $\quad +\,\mathrm{Res}_{x=\frac{1}{2}}(\mathrm{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+x\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+x\omega_\alpha}))\,\mathrm{Res}_{y=1}(\mu(\tau_{y\frac{\beta}{2}+x\omega_\alpha}))_{|x=\frac{1}{2}}$

(4) $\quad +\dfrac{\log q}{2\pi}\displaystyle\int_{x=0}^{\frac{2\pi}{\log q}} \mathrm{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+ix\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+ix\omega_\alpha}))\,\mathrm{Res}_{y=1}(\mu(\tau_{ix\omega_\alpha+y\frac{\beta}{2}}))dx$

(5) $\quad +\dfrac{\log q}{2\pi}\displaystyle\int_{y=0}^{\frac{2\pi}{\log q}} \mathrm{tr}(J_{P|\overline{P}}(\tau_{\frac{\alpha}{2}+iy\omega_\beta})\xi_f(\tau_{\frac{\alpha}{2}+iy\omega_\beta}))\,\mathrm{Res}_{x=1}(\mu(\tau_{x\frac{\alpha}{2}+iy\omega_\beta}))dy$

(6)ϵ $\quad +\dfrac{\log q}{2\pi}\displaystyle\int_{z=0}^{\frac{2\pi}{\log q}} \mathrm{tr}(J_{P|\overline{P}}(\tau_{(iz+\epsilon)\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{(iz+\epsilon)\frac{\alpha}{2}+\omega_\beta}))\,\mathrm{Res}_{x=1}(\mu(\tau_{(iz+\epsilon)\frac{\alpha}{2}+x\omega_\beta}))dz$

(7) $\quad +\dfrac{\log q}{2\pi}\displaystyle\int_{t=0}^{\frac{2\pi}{\log q}} \mathrm{tr}(J_{P|\overline{P}}(\tau_{it\beta+\frac{\omega_\alpha}{2}})\xi_f(\tau_{it\beta+\frac{\omega_\alpha}{2}}))\,\mathrm{Res}_{y=1}(\mu(\tau_{y\frac{\omega_\alpha}{2}+it\beta}))dt$

(8) $\quad +\displaystyle\int_{\Re(\sigma)=0} \mathrm{tr}(J_{P|\overline{P}}(\sigma)\xi_f(\sigma))\mu(\sigma)d\sigma.$

One observes that

$$(6)_\epsilon - (6)_{-\epsilon} = \mathrm{Res}_{z=0}(\mathrm{tr}(J_{P|\overline{P}}(\tau_{z\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{z\frac{\alpha}{2}+\omega_\beta}))) \,\mathrm{Res}_{x=1}(\mu(\tau_{z\frac{\alpha}{2}+x\omega_\beta}))_{|z=0}$$

and verfies that

$$0 = (3) + \frac{1}{2}\mathrm{Res}_{z=0}(\mathrm{tr}(J_{P|\overline{P}}(\tau_{z\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{z\frac{\alpha}{2}+\omega_\beta}))) \,\mathrm{Res}_{x=1}(\mu(\tau_{z\frac{\alpha}{2}+x\omega_\beta}))_{|z=0}.$$

So (3) cancels after replacing $(6)_\epsilon$ by $(6')_\epsilon = \frac{1}{2}((6)_\epsilon + (6)_{-\epsilon})$. According to our general results (cf. step ii) in **5.**), one verifies directly that the integrant in $((6)_\epsilon + (6)_{-\epsilon})$ is a regular function for $\epsilon = 0$, i.e. $(6')_\epsilon = (6')_0 =: (6')$.

With this one sees easily, that (8) corresponds to the term in the Plancherel formula coming from the unitary orbit of the unit representation of $L = T$, that $(6') + (5)$ corresponds to the term coming from the orbit of the Steinberg representation of the Levi subgroup L_α and that $(4) + (7)$ corresponds to the term coming from the Steinberg representation of the Levi subgroup L_β. The term (1) corresponds to the one in the Plancherel formula coming from the square-integrable representation of G which is the unique subrepresentation of $i_P^G\tau_{\omega_\alpha+\omega_\beta}$. The term (2) comes from the unique square-integrable representation of G, which is a subrepresentation of $i_P^G\tau_{\omega_\beta+(\frac{1}{2}+\frac{\pi i}{\log q})\omega_\alpha}$.

Remark that these results on the discrete series of G of type B_2 were already known to P. Sally and M. Tadic [20] (see also [1] for a complete Plancherel formula in this setting). However, in general it is much more difficult to find explicitly the subquotients of an induced representation which are square-integrable (see for example the case of a group of type G_2 studied in the appendix **A.** to [8] which is the local analog to the case studied in the appendix III to [14]).

The material of this article together with all the proofs will appear in the Journal de l'Institut de Mathématiques de Jussieu.

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Locally convex root graded Lie algebras

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Abstract

In the present paper we start to build a bridge from the algebraic theory of root graded Lie algebras to the global Lie theory of infinite-dimensional Lie groups by showing how root graded Lie algebras can be defined and analyzed in the context of locally convex Lie algebras. Our main results concern the description of locally convex root graded Lie algebras in terms of a locally convex coordinate algebra and its universal covering algebra, which has to be defined appropriately in the topological context. Although the structure of the isogeny classes is much more complicated in the topological context, we give an explicit description of the universal covering Lie algebra which implies in particular that in most cases (called regular) it depends only on the root system and the coordinate algebra. Not every root graded locally convex Lie algebra is integrable in the sense that it is the Lie algebra of a Lie group. In a forthcoming paper we will discuss criteria for the integrability of root graded Lie algebras.

Introduction

Let $\mathbb{K}$ be a field of characteristic zero and Δ a finite reduced irreducible root system. We write $\mathfrak{g}_\Delta$ for the corresponding finite-dimensional split simple $\mathbb{K}$-Lie algebra and fix a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Delta$. In the algebraic context, a Lie algebra $\mathfrak{g}$ is said to be Δ-graded if it contains $\mathfrak{g}_\Delta$ and $\mathfrak{g}$ decomposes as follows as a direct sum of simultaneous $\operatorname{ad}\mathfrak{h}$-eigenspaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{g}_0 = \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

It is easy to see that the latter requirement is equivalent to $\mathfrak{g}$ being generated by the *root spaces* $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, and that it implies in particular that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e., that $\mathfrak{g}$ is a perfect Lie algebra. Recall that two perfect Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are called *(centrally) isogenous* if $\mathfrak{g}_1/\mathfrak{z}(\mathfrak{g}_1) \cong \mathfrak{g}_2/\mathfrak{z}(\mathfrak{g}_2)$. A perfect Lie algebra $\mathfrak{g}$ has a unique universal central extension $\widetilde{\mathfrak{g}}$, called its universal covering algebra ([We95, Th. 7.9.2]). Two isogenous perfect Lie algebras have isomorphic universal central extensions, so that the *isogeny class of* $\mathfrak{g}$ consists of all quotients of $\widetilde{\mathfrak{g}}$ by central subspaces.

The systematic study of root graded Lie algebras was initiated by S. Berman and R. Moody in [BM92], where they studied Lie algebras graded by simply laced root systems, i.e., types A, D and E. The classification of Δ-graded Lie algebras proceeds in two steps. First one considers isogeny classes of Δ-graded Lie algebras and then describes the elements of a fixed isogeny class as quotients of the corresponding universal covering Lie algebra. Berman and Moody show that for a fixed simply laced root system of type Δ the isogeny classes are in one-to-one correspondence with certain classes of unital coordinate algebras which are

(1) commutative associate algebras for types D_r, $r \geq 4$, E_6, E_7 and E_8,
(2) associative algebras for type A_r, $r \geq 3$, and
(3) alternative algebras for type A_2.

The corresponding result for type A_1 is that the coordinate algebra is a Jordan algebra, which goes back to results of J. Tits ([Ti62]).

Corresponding results for non-simply laced root systems have been obtained by G. Benkart and E. Zelmanov in [BZ96], where they also deal with the A_1-case. In these cases the isogeny classes are determined by a class of coordinate algebras, which mostly is endowed with an involution, where the decomposition of the algebra into eigenspaces of the involution corresponds to the division of roots into short and long ones. Based on the observation that all root systems except E_8, F_4, and G_2 are 3-graded, E. Neher obtains in [Neh96] a uniform description of the coordinate algebras of 3-graded Lie algebras by Jordan theoretic methods. Neher's approach is based on the observation that if Δ is 3-graded, then each Δ-graded Lie algebra can also be considered as an A_1-graded Lie algebra, which leads to a unital Jordan algebra as coordinate algebra. Then one has to identify the types of Jordan algebras corresponding to the different root systems.

The classification of root graded Lie algebras was completed by B. Allison, G. Benkart and Y. Gao in [ABG00]. They give a uniform description of the isogeny classes as quotients of a unique Lie algebra $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$, depending only on the root system Δ and the coordinate algebra $\mathcal{A}$, by central subspaces. Their construction implies in particular the existence of a functor $\mathcal{A} \mapsto \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ from the category of coordinate algebras associated to Δ to centrally closed Δ-graded Lie algebras.

Apart from split simple Lie algebras, there are two prominent classes of root graded Lie algebras, which have been studied in the literature from a different point of view. The first class are the affine Kac–Moody algebras which can be described as root graded Lie algebras ([Ka90, Ch. 6] and Examples I.4 and I.11 below). The other large class are the isotropic finite-dimensional simple Lie algebras $\mathfrak{g}$ over fields of characteristic zero. These Lie algebras have a restricted root decomposition with respect to a maximal toral subalgebra $\mathfrak{h}$[1]. The corresponding root system Δ is irreducible, but it may also be non-reduced, i.e., of type BC_r ([Se76]). If it is reduced, then $\mathfrak{g}$ is Δ-graded in the sense defined above. In the general case, one needs the notion of BC_r-graded Lie algebras which has been developed by B. Allison, G. Benkart and Y. Gao in [ABG02]. Since three different root lengths occur in BC_r, we call the shortest ones the *short roots*, the longest ones the *extra-long roots*, and the other roots *long*. The main difference to

[1]We call a subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$ *toral* if $\operatorname{ad} \mathfrak{t} \subseteq \operatorname{der}(\mathfrak{g})$ consists of diagonalizable endomorphisms.

the reduced case is that there cannot be any grading subalgebra of type BC_r, so that one has to distinguish between different types, where the grading subalgebra is either of type B_r (the short and the long roots), type C_r (the long and the extra-long roots), or of type D_r (the long roots).

The theory of root graded Lie algebras has a very geometric flavor because the coordinatization theorems for the different types of root systems are very similar to certain coordinatization results in synthetic geometry. That the Lie algebra $\mathfrak{g}$ under consideration is simple implies that the coordinate algebra is simple, too. In geometric contexts, in addition, the coordinate algebras are mostly division algebras or forms of division algebras. For a nice account on the geometry of groups corresponding to the root systems A_2, $B_2 \cong C_2$ and G_2 we refer to the memoir [Fa77] of J. R. Faulkner. Here type A_2 corresponds to generalized triangles, type B_2 to generalized quadrangles and G_2 to generalized hexagons.

An important motivation for the algebraic theory of root graded Lie algebras was to find a class of Lie algebras containing affine Kac–Moody algebras ([Ka90]), isotropic finite-dimensional simple Lie algebras ([Se76]), certain ones of Slodowy's intersection matrix algebras ([Sl86]), and extended affine Lie algebras (EALAs) ([AABGP97]), which can roughly be considered as those root graded Lie algebras with a root decomposition. Since a general structure theory of infinite-dimensional Lie algebras does not exist, it is important to single out large classes with a uniform structure theory. The class of root graded Lie algebras satisfies all these requirements in a very natural fashion. It is the main point of the present paper to show that root graded Lie algebras can also be dealt with in a natural fashion in a topological context, where it covers many important classes of Lie algebras, arising in such diverse contexts as mathematical physics, operator theory and geometry.

With the present paper we start a project which connects the rich theory of root graded Lie algebras, which has been developed so far on a purely algebraic level, to the theory of infinite-dimensional Lie groups. A *Lie group* G is a manifold modeled on a locally convex space $\mathfrak{g}$ which carries a group structure for which the multiplication and the inversion map are smooth ([Mi83], [Gl01a], [Ne02b]). Identifying elements of the tangent space $\mathfrak{g} := T_{\mathbf{1}}(G)$ of G in the identity $\mathbf{1}$ with left invariant vector fields, we obtain on $\mathfrak{g}$ the structure of a *locally convex Lie algebra*, i.e., a Lie algebra which is a locally convex space and whose Lie bracket is continuous. Therefore the category of locally convex Lie algebras is the natural setup for the "infinitesimal part" of infinite-dimensional Lie theory. In addition, it is an important structural feature of locally convex spaces that they have natural tensor products.

In Section I we explain how the concept of a root graded Lie algebra can be adapted to the class of locally convex Lie algebras. The main difference to the algebraic concept is that one replaces the condition that $\sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ coincides with $\mathfrak{g}_0$ by the requirement that it is a dense subspace of $\mathfrak{g}_0$. This turns out to make the theory of locally convex root graded Lie algebras somewhat harder than the algebraic theory, but it is natural, as a closer inspection of the topological versions of the Lie algebras $\mathfrak{sl}_n(A)$ for locally convex associative algebras A shows. In Section I we also discuss some natural classes of "classical" locally convex root graded Lie algebras such as symplectic and orthogonal Lie algebras and the Tits–Kantor–Koecher–Lie algebras associated to Jordan algebras.

In Section II we undertake a detailed analysis of locally convex root graded Lie algebras. Here the main point is that the action of the grading subalgebra $\mathfrak{g}_\Delta$ on $\mathfrak{g}$ is semisimple with at most three isotypical components, into which $\mathfrak{g}$ decomposes topologically. The corresponding simple modules are the trivial module $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the adjoint module $\mathfrak{g}_\Delta$ and the simple module V_s whose highest weight is the maximal short root with respect to a positive system $\Delta^+ \subseteq \Delta$. In the algebraic context, the decomposition of $\mathfrak{g}$ is a direct consequence of Weyl's Theorem, but here we need that the isotypical projections are continuous operators, a result which can be derived from the fact that they come from the center of the enveloping algebra $U(\mathfrak{g}_\Delta)$. The underlying algebraic arguments are provided in Appendix A. If A, B, resp., D, are the multiplicity spaces with respect to $\mathfrak{g}_\Delta$, V_s, resp., $\mathbb{K}$, then $\mathfrak{g}$ decomposes topologically as

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D.$$

A central point in our structural analysis is that the direct sum $\mathcal{A} := A \oplus B$ carries a natural (not necessarily associative) unital locally convex algebra structure, that D acts by derivations on $\mathcal{A}$, and that we have a continuous alternating map $\delta^D\colon \mathcal{A} \times \mathcal{A} \to D$ satisfying a certain cocycle condition. Here the type of the root system Δ dictates certain identities for the multiplication on $\mathcal{A}$, which leads to the coordinatization results mentioned above ([BM92], [BZ96] and [Neh96]). The main new point here is that $\mathcal{A}$ inherits a natural locally convex structure, that the multiplication is continuous and that all the related maps such as δ^D are continuous. We call the triple $(\mathcal{A}, D, \delta^D)$ the *coordinate structure of* $\mathfrak{g}$.

In the algebraic context, the coordinate algebra $\mathcal{A}$ and the root system Δ classify the isogeny classes. The isogeny class of $\mathfrak{g}$ contains a unique centrally closed Lie algebra $\widetilde{\mathfrak{g}}$ and a unique center-free Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. In the locally convex context, the situation is more subtle because we have to work with generalized central extensions instead of ordinary central extensions: a morphism $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of locally convex Lie algebras is called a *generalized central extension* if it has dense range and there exists a continuous bilinear map $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ for which $b \circ (q \times q)$ is the Lie bracket on $\widehat{\mathfrak{g}}$. This condition implies that $\ker q$ is central, but the requirement that $\ker q$ is central would be too weak for most of our purposes. The subtlety of generalized central extensions is that q need not be surjective and if it is surjective, it does not need to be a quotient map. Fortunately these difficulties are compensated by the nice fact that each *topologically perfect Lie algebra* $\mathfrak{g}$, meaning that the commutator algebra is dense, has a universal generalized central extension $q_\mathfrak{g}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$, called the *universal covering Lie algebra of* $\mathfrak{g}$.

We call two topologically perfect locally convex Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ *(centrally) isogenous* if $\widetilde{\mathfrak{g}}_1 \cong \widetilde{\mathfrak{g}}_2$. We thus obtain a locally convex version of isogeny classes of locally convex Lie algebras. The basic results on generalized central extensions are developed in Section III.

In Section IV we apply this concept to locally convex root graded Lie algebras and give a description of the universal covering Lie algebras of root graded Lie algebras. It turns out that in the locally convex context, this description is more complicated than in the algebraic context ([ABG00]). Here a central point is that for any generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ the Lie algebra $\widehat{\mathfrak{g}}$ is Δ-graded if and only if $\mathfrak{g}$ is Δ-graded. An *isogeny class* contains a Δ-graded element if and only if it consists entirely of Δ-graded Lie algebras. The universal covering algebra $\widetilde{\mathfrak{g}}$ of a root graded Lie algebra $\mathfrak{g}$ has a coordinate structure $(\mathcal{A}, \widetilde{D}, \delta^{\widetilde{D}})$, where $q_{\mathfrak{g}} \mid_{\widetilde{D}}\colon \widetilde{D} \to D$ is a generalized central extension, but since D need not be topologically perfect, the Lie algebra $\widetilde{D}$ cannot always be interpreted as the universal covering algebra of D. Moreover, we construct for each root system Δ and a corresponding coordinate algebra $\mathcal{A}$ a Δ-graded Lie algebra $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ which is functorial in $\mathcal{A}$, and which has the property that for each Δ-graded Lie algebra $\mathfrak{g}$ with coordinate algebra $\mathcal{A}$ we have a natural morphism $q^\sharp\colon \widetilde{\mathfrak{g}}(\Delta, \mathcal{A}) \to \mathfrak{g}$ with dense range and central kernel, but this map is not always a generalized central extension. The universal covering Lie algebra $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ also depends, in addition, on the Lie algebra D, and we characterize those Lie algebras for which $\widetilde{\mathfrak{g}} \cong \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$. They are called *regular* and many naturally occuring Δ-graded Lie algebras have this property.

We also show that there are non-isomorphic center-free root graded Lie algebras with the same universal covering and describe an example where $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ is not the universal covering Lie algebra of $\mathfrak{g}$ (Example IV.24). All these problems are due to the fact that the Lie algebras $\mathfrak{g}$ with coordinate algebra $\mathcal{A}$ are obtained from the centrally closed Lie algebra $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ by a morphism $q^\sharp\colon \widetilde{\mathfrak{g}}(\Delta, \mathcal{A}) \to \mathfrak{g}$ with dense range and central kernel. As $q^\sharp$ is not necessarily a quotient map or a generalized central extension, the topology on $\mathfrak{g}$ is *not* determined by the topology on $\mathcal{A}$, resp., $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ (Proposition III.19, Examples IV.23/24).

A Lie group G is said to be Δ-*graded* if its Lie algebra $\mathbf{L}(G)$ is Δ-graded. It is a natural question which root graded locally convex Lie algebras $\mathfrak{g}$ are *integrable* in the sense that they are the Lie algebra of a Lie group G. Although this question always has an affirmative answer if $\mathfrak{g}$ is finite-dimensional, it turns out to be a difficult problem to decide integrability for infinite-dimensional Lie algebras. These global questions will be pursued in another paper ([Ne03b], see also [Ne03a]). In Section V we give an outline of the global side of the theory and explain how it is related to K-theory and non-commutative geometry. One of the main points is that, in view of the results of Section IV, it mainly boils down to showing that at least one member $\mathfrak{g}$ of an isogeny class is integrable and then analyze the situation for its universal covering Lie algebra $\widetilde{\mathfrak{g}}$.

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0.1 Preliminaries and notation

The theory of root graded Lie algebras is a subject with great aesthetic appeal and rich connections to many other fields of mathematics. We therefore tried to keep the exposition of the present paper as self-contained as possible to make it accessible to readers from different mathematical subcultures. In particular we include proofs for those results on the structure of the coordinate algebras which can be obtained by short elementary arguments; for the more refined structure theory related to the exceptional and the low rank algebras we refer to the literature. On the algebraic level we essentially build on the representation theory of finite-dimensional semisimple split Lie algebras (cf. [Dix74] or [Hum72]); the required Jordan theoretic results are elementary and provided in Appendices B and C. On the functional analytic level we do not need much more than some elementary facts on locally convex spaces such as the existence of the projective tensor product.

All locally convex spaces in this paper are vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If X and Y are locally convex spaces, then we write $\mathrm{Lin}(X, Y)$ for the space of continuous linear maps $X \to Y$.

A *locally convex algebra* $\mathcal{A}$ is a locally convex topological vector space together with a continuous bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. In particular a *locally convex Lie algebra* $\mathfrak{g}$ is a Lie algebra which is a locally convex space for which the Lie bracket is a continuous bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

The assumption that the topological Lie algebras we consider are locally convex spaces is motivated by the fact that such Lie algebras arise naturally as Lie algebras of Lie groups and by the existence of tensor products, which will be used in Section III to construct the universal covering Lie algebra. Tensor products of locally convex spaces are defined as follows.

Let E and F be locally convex spaces. On the tensor product $E \otimes F$ there exists a natural locally convex topology, called the *projective topology*. It is defined by the seminorms

$$(p \otimes q)(x) = \inf \left\{ \sum_{j=1}^{n} p(y_j) q(z_j) \colon x = \sum_j y_j \otimes z_j \right\},$$

where p, resp., q are continuous seminorms on E, resp., F (cf. [Tr67, Prop. 43.4]). We write $E \otimes_\pi F$ for the locally convex space obtained by endowing $E \otimes F$ with the locally convex topology defined by this family of seminorms. It is called the *projective tensor product of E and F.* It has the universal property that for a locally convex space G the continuous bilinear maps $E \times F \to G$ are in one-to-one correspondence with the continuous linear maps $E \otimes_\pi F \to G$. We write $E \widehat{\otimes}_\pi F$ for the completion of the projective tensor product of E and F. If E and F are Fréchet spaces, their topology is defined by a countable family of seminorms, and this property is inherited by $E \widehat{\otimes}_\pi F$. Hence this space is also Fréchet.

If E and F are Fréchet spaces, then every element θ of the completion $E \widehat{\otimes}_\pi F$ can be written as $\theta = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$, where $\lambda \in \ell^1(\mathbb{N}, \mathbb{K})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ ([Tr67, Th. 45.1]). If, in addition, E and F are Banach spaces, then the tensor product of the two norms is a norm defining the topology on $E \otimes F$ and $E \widehat{\otimes}_\pi F$ also is a Banach space. For $\|\theta\| < 1$ we then obtain a representation with $\|\lambda\|_1 < 1$ and $\|x_n\|, \|y_n\| < 1$ for all $n \in \mathbb{N}$ ([Tr67, p.465]).

I Root graded Lie algebras

In this section we introduce locally convex root graded Lie algebras. In the algebraic setting it is natural to require that root graded Lie algebras are generated by their root spaces, but in the topological context this condition would be unnaturally strong. Therefore it is weakened to the requirement that the root spaces generate the Lie algebra topologically. As we will see below, this weaker condition causes several difficulties which are not present in the algebraic setting, but this defect is compensated by the well behaved theory of generalized central extensions (see Section IV).

I.1 Basic definitions

Definition I.1. Let Δ be a finite irreducible reduced root system and $\mathfrak{g}_\Delta$ the corresponding finite-dimensional complex simple Lie algebra.

A locally convex Lie algebra $\mathfrak{g}$ is said to be Δ-*graded* if the following conditions are satisfied:

(R1) $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$.

(R2) There exist elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \neq 0$, and a subspace $\mathfrak{h} \subseteq \mathfrak{g}_0$ with $\mathfrak{g}_\Delta \cong \mathfrak{h} + \sum_{\alpha\in\Delta} \mathbb{K} x_\alpha$.

(R3) For $\alpha \in \Delta \cup \{0\}$ we have $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \colon (\forall h \in \mathfrak{h})\, [h, x] = \alpha(h)x\}$, where we identify Δ with a subset of $\mathfrak{h}^*$.

(R4) $\sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is dense in $\mathfrak{g}_0$.

The subalgebra $\mathfrak{g}_\Delta$ of $\mathfrak{g}$ is called a *grading subalgebra*. We say that $\mathfrak{g}$ is *root graded* if $\mathfrak{g}$ is Δ-graded for some Δ.

A slight variation of the concept of a Δ-graded Lie algebra is obtained by replacing (R2) by

(R2') There exist a sub-root system $\Delta_0 \subseteq \Delta$ and elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_0$, and a subspace $\mathfrak{h} \subseteq \mathfrak{g}_0$ with $\mathfrak{g}_{\Delta_0} \cong \mathfrak{h} + \sum_{\alpha\in\Delta_0} \mathbb{K}x_\alpha$.

A Lie algebra satisfying (R1), (R2'), (R3) and (R4) is called (Δ, Δ_0)-*graded*.■

Remark I.2. (a) Suppose that a locally convex Lie algebra $\mathfrak{g}$ satisfies (R1)-(R3). Then the subspace

$$\sum_{\alpha\in\Delta} \mathfrak{g}_\alpha + \sum_{\alpha\in\Delta} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

is invariant under each root space $\mathfrak{g}_\alpha$ and also under $\mathfrak{g}_0$, hence an ideal. Therefore its closure satisfies (R1)-(R4), hence is a Δ-graded Lie algebra.
(b) Sometimes one starts with the subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and the corresponding weight space decomposition, so that we have (R1) and (R3). Let Π be a basis of the root system $\Delta \subseteq \mathfrak{h}^*$ and $\check{\alpha}$, $\alpha \in \Delta$, the coroots. If there exist elements $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ for $\alpha \in \Pi$ such that $[x_\alpha, x_{-\alpha}] = \check{\alpha}$, then we consider the subalgebra $\mathfrak{g}_\Delta \subseteq \mathfrak{g}$ generated by $\{x_{\pm\alpha} \colon \alpha \in \Pi\}$. Then the weight decomposition of $\mathfrak{g}$ with weight set $\Delta \cup \{0\}$ easily implies that the generators $x_{\pm\alpha}$, $\alpha \in \Pi$, satisfy the Serre relations, and therefore that $\mathfrak{g}_\Delta$ is a split simple Lie algebra with root system Δ satisfying (R2). ■

Remark I.3. (a) In the algebraic context one replaces (R4) by the requirement that $\mathfrak{g}_0 = \sum_{\alpha\in\Delta}[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This is equivalent to $\mathfrak{g}$ being generated by the spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta$.
(b) The concept of a Δ-graded Lie algebra can be defined over any field of characteristic 0. Here it already occurs in the classification theory of simple Lie algebras as follows. Let $\mathfrak{g}$ be a simple Lie algebra which is *isotropic* in the sense that it contains non-zero elements x for which $\operatorname{ad} x$ is diagonalizable. The latter condition is equivalent to the existence of a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a maximal toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then $\mathfrak{g}$ has an $\mathfrak{h}$-weight decomposition, and the corresponding set of weights $\Delta \subseteq \mathfrak{h}^*$ is a not necessarily reduced irreducible root system (cf. [Se76, pp.10/11]). If this root system is reduced, then one can use the method from Remark I.2(b) to show that $\mathfrak{g}$ is Δ-graded in the sense defined above. For restricted root systems of type BC_r this argument produces grading subalgebras of type C_r, hence (BC_r, C_r)-graded Lie algebras ([Se76]).
(c) (R4) implies in particular that $\mathfrak{g}$ is topologically perfect, i.e., that $\mathfrak{g}' := \overline{[\mathfrak{g}, \mathfrak{g}]} = \mathfrak{g}$.
(d) Suppose that $\mathfrak{g}$ is Δ-graded and

$$\mathfrak{d} \subseteq \operatorname{der}_\Delta(\mathfrak{g}) := \{D \in \operatorname{der}(\mathfrak{g}) \colon (\forall \alpha \in \Delta) D.\mathfrak{g}_\alpha \subseteq \mathfrak{g}_\alpha\}$$

is a Lie subalgebra with a locally convex structure for which the action $\mathfrak{d} \times \mathfrak{g} \to \mathfrak{g}$ is continuous. Then $\mathfrak{g} \rtimes \mathfrak{d}$ satisfies (R1)–(R3) with $(\mathfrak{g} \rtimes \mathfrak{d})_0 = \mathfrak{g}_0 \rtimes \mathfrak{d}$. ■

I.2 Examples of root graded Lie algebras

Example I.4. Let Δ be an irreducible reduced finite root system and $\mathfrak{g}_\Delta$ be the corresponding simple split $\mathbb{K}$-Lie algebra. If A is a locally convex associative commutative algebra with unit $\mathbf{1}$, then $\mathfrak{g} := A \otimes \mathfrak{g}_\Delta$ is a locally convex Δ-graded Lie algebra with respect to the bracket

$$[a \otimes x, a' \otimes x'] := aa' \otimes [x, x'].$$

The embedding $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$ is given by $x \mapsto \mathbf{1} \otimes x$. ■

Example I.5. Now let A be an associative unital locally convex algebra. Then the $(n \times n)$-matrix algebra $M_n(A) \cong A \otimes M_n(\mathbb{K})$ also is a locally convex associative algebra. We write $\mathfrak{gl}_n(A)$ for this algebra, endowed with the commutator bracket and

$$\mathfrak{g} := \overline{[\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]}$$

for the closure of the commutator algebra of $\mathfrak{gl}_n(A)$. We claim that this is an A_{n-1}-graded Lie algebra with grading subalgebra $\mathfrak{g}_\Delta = \mathbf{1} \otimes \mathfrak{sl}_n(\mathbb{K})$. It is clear that $\mathfrak{g}_\Delta$ is a subalgebra of $\mathfrak{g}$. Let

$$\mathfrak{h} := \Big\{ \operatorname{diag}(x_1, \ldots, x_n) \colon x_1, \ldots, x_n \in \mathbb{K}, \sum_j x_j = 0 \Big\} \subseteq \mathfrak{g}_\Delta$$

denote the canonical Cartan subalgebra and define linear functionals ε_j on $\mathfrak{h}$ by

$$\varepsilon_j(\operatorname{diag}(x_1, \ldots, x_n)) = x_j.$$

Then the weight space decomposition of $\mathfrak{g}$ satisfies

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = A \otimes E_{ij}, \quad i \neq j,$$

where E_{ij} is the matrix with one non-zero entry 1 in position (i, j). From

$$[aE_{ij}, bE_{kl}] = ab\delta_{jk}E_{il} - ba\delta_{li}E_{kj}$$

we derive that

$$[aE_{ij}, bE_{ji}] = abE_{ii} - baE_{jj} \in [a, b] \otimes E_{ii} + A \otimes \mathfrak{sl}_n(\mathbb{K}) = \frac{1}{n}[a, b] \otimes \mathbf{1} + A \otimes \mathfrak{sl}_n(\mathbb{K}).$$

In view of $A \otimes \mathfrak{sl}_n(\mathbb{K}) = [\mathfrak{g}_\Delta, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$, it is now easy to see that

$$\mathfrak{g}_0 = \Big\{ \operatorname{diag}(a_1, \ldots, a_n) \colon \sum_j a_j \in \overline{[A, A]} \Big\} = (A \otimes \mathfrak{h}) \oplus (\overline{[A, A]} \otimes \mathbf{1}).$$

From the formulas above, we also see that (R4) is satisfied, so that $\mathfrak{g}$ is an A_{n-1}-graded locally convex Lie algebra.

We have a natural non-commutative trace map

$$\mathrm{Tr}\colon \mathfrak{gl}_n(A) \to A/\overline{[A,A]}, \quad x \mapsto \Big[\sum_{j=1}^n x_{jj}\Big],$$

where $[a]$ denotes the class of $a \in A$ in $A/\overline{[A,A]}$. Then the discussion above implies that

$$\mathfrak{sl}_n(A) := \ker \mathrm{Tr} = \mathfrak{g} = (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus (\overline{[A,A]} \otimes \mathbf{1}).$$

To prepare the discussion in Example I.9(b) and in Section II below, we describe the Lie bracket in $\mathfrak{sl}_n(A)$ in terms of the above direct sum decomposition. First we note that in $\mathfrak{gl}_n(A)$ we have

$$[a \otimes x, a' \otimes x'] = aa' \otimes xx' - a'a \otimes x'x = \frac{aa' + a'a}{2} \otimes [x,x'] + \frac{1}{2}[a,a'] \otimes (xx' + x'x).$$

For $x, x' \in \mathfrak{sl}_n(\mathbb{K})$ we have

$$x * x' := xx' + x'x - 2\frac{\mathrm{tr}(xx')}{n}\mathbf{1} \in \mathfrak{sl}_n(\mathbb{K}),$$

so that for $a, a' \in A$ and $x, x' \in \mathfrak{sl}_n(\mathbb{K})$ we have

$$(1.1)\quad [a \otimes x, a' \otimes x'] = \Big(\frac{aa' + a'a}{2} \otimes [x,x'] + \frac{1}{2}[a,a'] \otimes x * x'\Big) + [a,a'] \otimes \frac{\mathrm{tr}(xx')}{n}\mathbf{1},$$

according to the direct sum decomposition $\mathfrak{sl}_n(A) = (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus ([A,A] \otimes \mathbf{1})$, and

$$[d \otimes \mathbf{1}, a \otimes x] = [d,a] \otimes x, \quad a, d \in A, x \in \mathfrak{sl}_n(\mathbb{K}).$$ ■

Remark I.6. A Lie algebra $\mathfrak{g}$ can be root graded in several different ways. Let $\mathfrak{s} \subseteq \mathfrak{g}$ be a subalgebra with $\mathfrak{s} = \mathrm{span}\{h,e,f\} \cong \mathfrak{sl}_2(\mathbb{K})$ and the relations

$$[h,e] = 2e, \quad [h,f] = -2f \quad \text{and} \quad [e,f] = h.$$

If $\mathrm{ad}_{\mathfrak{g}} h$ is diagonalizable with $\mathrm{Spec}(\mathrm{ad}_{\mathfrak{g}} h) = \{2,0,-2\}$, then the eigenspaces of $\mathrm{ad}_{\mathfrak{g}} h$ yield on $\mathfrak{g}$ the structure of an A_1-grading with $\mathfrak{g}_\Delta := \mathfrak{s}$. This shows in particular that for any associative algebra A the Lie algebra $\mathfrak{sl}_n(A)$, $n \geq 3$, has many different A_1-gradings in addition to its natural A_{n-1}-grading. ■

Example I.7. Let $\mathcal{A}$ be a locally convex unital associative algebra with a continuous *involution* $\sigma\colon a \mapsto a^\sigma$, i.e., σ is a continuous involutive linear antiautomorphism:

$$(ab)^\sigma = b^\sigma a^\sigma \quad \text{and} \quad (a^\sigma)^\sigma = a, \quad a, b \in \mathcal{A}.$$

If $\sigma = \mathrm{id}_{\mathcal{A}}$, then $\mathcal{A}$ is commutative. We write

$$\mathcal{A}^{\pm\sigma} := \{a \in \mathcal{A} \colon a^\sigma = \pm a\}$$

and observe that $\mathcal{A} = \mathcal{A}^\sigma \oplus \mathcal{A}^{-\sigma}$.

The involution σ extends in a natural way to an involution of the locally convex algebra $M_n(\mathcal{A})$ of $n \times n$-matrices with entries in $\mathcal{A}$ by $(x_{ij})^\sigma := (x_{ji}^\sigma)$. If $\sigma = \mathrm{id}_{\mathcal{A}}$, then $x^\sigma = x^\top$ is just the transposed matrix.

(a) Let $\mathbf{1} \in M_n(\mathcal{A})$ be the identity matrix and define

$$J := \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \in M_{2n}(\mathcal{A}).$$

Then $J^2 = -\mathbf{1}$, and

$$\mathfrak{sp}_{2n}(\mathcal{A}, \sigma) := \{x \in \mathfrak{gl}_{2n}(\mathcal{A}) \colon J x^\sigma J^{-1} = -x\}$$

is a closed Lie subalgebra of $\mathfrak{gl}_{2n}(\mathcal{A})$. Writing x as a (2×2)-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(M_n(\mathcal{A}))$, this means that

$$\mathfrak{sp}_{2n}(\mathcal{A}, \sigma) = \left\{ \begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathcal{A}) \colon b^\sigma = b, c^\sigma = c \right\}.$$

For $\mathcal{A} = \mathbb{K}$ we have $\sigma = \mathrm{id}$, and we obtain $\mathfrak{sp}_{2n}(\mathbb{K}, \mathrm{id}_{\mathbb{K}}) = \mathfrak{sp}_{2n}(\mathbb{K})$. With the identity element $\mathbf{1} \in \mathcal{A}$ we obtain an embedding $\mathbb{K} \cong \mathbb{K}\mathbf{1} \hookrightarrow \mathcal{A}$, and hence an embedding

$$\mathfrak{sp}_{2n}(\mathbb{K}) \hookrightarrow \mathfrak{sp}_{2n}(\mathcal{A}, \sigma).$$

Let

$$\mathfrak{h} := \{\mathrm{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n) \colon x_1, \ldots, x_n \in \mathbb{K}\}$$

denote the canonical Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{K})$. Then the $\mathfrak{h}$-weights with respect to the adjoint action of $\mathfrak{h}$ on $\mathfrak{sp}_{2n}(\mathcal{A}, \sigma)$ coincide with the set

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \colon i, j = 1, \ldots, n\}$$

of roots of $\mathfrak{sp}_{2n}(\mathbb{K})$, where $\varepsilon_j(\mathrm{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n)) = x_j$ for $j = 1, \ldots, n$. Typical root spaces are

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \{aE_{ij} - a^\sigma E_{j+n,i+n} \colon a \in \mathcal{A}\}, \quad \mathfrak{g}_{\varepsilon_i + \varepsilon_j} = \{aE_{i,j+n} + a^\sigma E_{j,i+n} \colon a \in \mathcal{A}\},\ i \neq j,$$

$$\mathfrak{g}_{2\varepsilon_j} = \mathcal{A}^\sigma E_{j,j+n}, \quad \text{and} \quad \mathfrak{g}_0 = \{\mathrm{diag}(a_1, \ldots, a_n, -a_1^\sigma, \ldots, -a_n^\sigma) \colon a_1, \ldots, a_n \in \mathcal{A}\}.$$

As $\mathfrak{sp}_{2n}(\mathcal{A}, \sigma)$ is a semisimple module of $\mathfrak{sp}_{2n}(\mathbb{K})$ (it is a submodule of $\mathfrak{gl}_{2n}(\mathcal{A}) = \mathcal{A} \otimes \mathfrak{gl}_{2n}(\mathbb{K})$), the centralizer of the subalgebra $\mathfrak{sp}_{2n}(\mathbb{K})$ is

$$\mathfrak{z}_{\mathfrak{sp}_{2n}(\mathcal{A},\sigma)}(\mathfrak{sp}_{2n}(\mathbb{K})) = \mathcal{A}^{-\sigma}\mathbf{1},$$

and therefore
$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = [\mathfrak{sp}_{2n}(\mathbb{K}), \mathfrak{sp}_{2n}(\mathcal{A},\sigma)] \oplus \mathcal{A}^{-\sigma}\mathbf{1}.$$
From Example I.5 we know that a necessary condition for an element $a\mathbf{1}$ to be contained in the closure of the commutator algebra of $\mathfrak{gl}_{2n}(A)$ is $a \in \overline{[\mathcal{A},\mathcal{A}]}$. On the other hand, the embedding

$$\mathfrak{sl}_n(\mathcal{A}) \hookrightarrow \mathfrak{sp}_{2n}(\mathcal{A},\sigma), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & -x^\sigma \end{pmatrix}$$

implies that the elements

$$\begin{pmatrix} a\mathbf{1} & 0 \\ 0 & -a^\sigma\mathbf{1} \end{pmatrix}, \quad a \in \overline{[\mathcal{A},\mathcal{A}]},$$

are contained in the closure $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ of the commutator algebra of $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$. This proves that

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma)' = [\mathfrak{sp}_{2n}(\mathbb{K}), \mathfrak{sp}_{2n}(\mathcal{A},\sigma)] \oplus (\overline{[\mathcal{A},\mathcal{A}]}^{-\sigma} \otimes \mathbf{1}).$$

Using Example I.5 again, we now obtain (R4), and therefore that $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ is a C_n-graded Lie algebra with grading subalgebra $\mathfrak{sp}_{2n}(\mathbb{K})$. We refer to Example II.9 and Definitions II.7 and II.8 for a description of the bracket in $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$ in the spirit of (1.1) in Example I.5.

The preceding description of the commutator algebra shows that each element $x = \begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} \in \mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ satisfies

$$\mathrm{tr}(x) = \mathrm{tr}(a - a^\sigma) = \mathrm{tr}(a) - \mathrm{tr}(a)^\sigma \in \overline{[\mathcal{A},\mathcal{A}]}.$$

That the latter condition is sufficient for x being contained in $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)'$ follows from
$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = [\mathfrak{sp}_{2n}(\mathbb{K}), \mathfrak{sp}_{2n}(\mathcal{A},\sigma)] \oplus \mathcal{A}^{-\sigma} \otimes \mathbf{1}.$$

The Lie algebra $\mathfrak{sp}_{2n}(\mathcal{A},\sigma)$ also has a natural 3-grading

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma) = \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_+ \oplus \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_0 \oplus \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_-$$

with

$$\mathfrak{sp}_{2n}(\mathcal{A},\sigma)_\pm \cong \mathrm{Herm}_n(\mathcal{A},\sigma) := \{x \in M_n(\mathcal{A})\colon x^\sigma = x\} \quad \text{and} \quad \mathfrak{sp}_{2n}(\mathcal{A},\sigma)_0 \cong \mathfrak{gl}_n(\mathcal{A}),$$

obtained from the (2×2)-matrix structure.
(b) Now we consider the symmetric matrix

$$I := \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \in M_{2n}(\mathcal{A}),$$

which satisfies $I^2 = \mathbf{1}$. We define the associated closed Lie subalgebra of $\mathfrak{gl}_{2n}(\mathcal{A})$ by

$$\begin{aligned}\mathfrak{o}_{n,n}(\mathcal{A},\sigma) &:= \{x \in \mathfrak{gl}_{2n}(\mathcal{A})\colon Ix^\sigma I^{-1} = -x\} \\ &= \left\{\begin{pmatrix} a & b \\ c & -a^\sigma \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathcal{A})\colon b^\sigma = -b, c^\sigma = -c\right\}.\end{aligned}$$

For $\mathcal{A} = \mathbb{K}$ we have $\sigma = \mathrm{id}$, and we obtain $\mathfrak{o}_{n,n}(\mathbb{K}, \mathrm{id}_\mathbb{K}) = \mathfrak{o}_{n,n}(\mathbb{K})$. With the identity element $\mathbf{1} \in \mathcal{A}$ we obtain an embedding $\mathbb{K} \cong \mathbb{K}\mathbf{1} \hookrightarrow \mathcal{A}$, and hence an embedding

$$\mathfrak{o}_{n,n}(\mathbb{K}) \hookrightarrow \mathfrak{o}_{n,n}(\mathcal{A},\sigma).$$

Again,

$$\mathfrak{h} := \{\mathrm{diag}(x_1,\ldots,x_n,-x_1,\ldots,-x_n)\colon x_1,\ldots,x_n \in \mathbb{K}\}$$

is the canonical Cartan subalgebra of $\mathfrak{o}_{n,n}(\mathbb{K})$. The $\mathfrak{h}$-weights with respect to the adjoint action of $\mathfrak{h}$ on $\mathfrak{o}_{n,n}(\mathcal{A},\sigma)$ coincide with the set

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j\colon i,j = 1,\ldots,n\}.$$

Typical root spaces are

$$\mathfrak{g}_{\varepsilon_i-\varepsilon_j} = \{aE_{ij} - a^\sigma E_{j+n,i+n}\colon a \in \mathcal{A}\}, \quad \mathfrak{g}_{\varepsilon_i+\varepsilon_j} = \{aE_{i,j+n} - a^\sigma E_{j,i+n}\colon a \in \mathcal{A}\},\ i \neq j,$$

$$\mathfrak{g}_{2\varepsilon_j} = \mathcal{A}^{-\sigma}E_{j,j+n}, \quad \text{and} \quad \mathfrak{g}_0 = \{\mathrm{diag}(a_1,\ldots,a_n,-a_1^\sigma,\ldots,-a_n^\sigma)\colon a_1,\ldots,a_n \in \mathcal{A}\}.$$

The root spaces $\mathfrak{g}_{2\varepsilon_j}$ are non-zero if and only if $\mathcal{A}^{-\sigma} \neq \{0\}$, which is equivalent to $\sigma \neq \mathrm{id}_\mathcal{A}$.

As in (a), we obtain

$$\mathfrak{z}_{\mathfrak{o}_{n,n}(\mathcal{A})}(\mathfrak{o}_{n,n}(\mathbb{K})) = \mathcal{A}^{-\sigma} \otimes \mathbf{1}, \quad \mathfrak{o}_{n,n}(\mathcal{A}) = [\mathfrak{o}_{n,n}(\mathbb{K}), \mathfrak{o}_{n,n}(\mathcal{A})] \oplus (\mathcal{A}^{-\sigma} \otimes \mathbf{1}),$$

and

$$\mathfrak{o}_{n,n}(\mathcal{A})' = [\mathfrak{o}_{n,n}(\mathbb{K}), \mathfrak{o}_{n,n}(\mathcal{A})] \oplus (\overline{[\mathcal{A},\mathcal{A}]}^{-\sigma} \otimes \mathbf{1}).$$

If $\sigma_\mathcal{A} = \mathrm{id}_\mathcal{A}$, then Δ is of type D_n, the root system of $\mathfrak{o}_{n,n}(\mathbb{K})$, and $\mathfrak{o}_{n,n}(\mathcal{A}) := \mathfrak{o}_{n,n}(\mathcal{A}, \mathrm{id}_\mathcal{A})$ is a D_n-graded Lie algebra. In this case $\mathcal{A} = \mathcal{A}^\sigma$ is commutative, and

$$\mathfrak{o}_{n,n}(\mathcal{A}) \cong \mathcal{A} \otimes \mathfrak{o}_{n,n}(\mathbb{K}),$$

so that this case is also covered by Example I.4.

If $\sigma_\mathcal{A} \neq \mathrm{id}_\mathcal{A}$, then we obtain a (C_n, D_n)-graded Lie algebra with grading subalgebra $\mathfrak{o}_{n,n}(\mathbb{K})$ of type D_n. ■

Lemma I.8. *Let $\mathbb{K}$ be a field with $2 \in \mathbb{K}^\times$. For $x, y, z \in \mathfrak{sl}_2(\mathbb{K})$ we have the relations*

$$xy + yx = \operatorname{tr}(xy)\mathbf{1}, \tag{1.2}$$

and

$$[x, [y, z]] = 2\operatorname{tr}(xy)z - 2\operatorname{tr}(xz)y. \tag{1.3}$$

Proof. For $x \in \mathfrak{sl}_2(\mathbb{K})$ let

$$p(t) = \det(t\mathbf{1} - x) = t^2 - \operatorname{tr} x \cdot t + \det x = t^2 + \det x$$

denote the characteristic polynomial of x. Then the Cayley–Hamilton Theorem implies

$$0 = p(x) = x^2 + (\det x)\mathbf{1}.$$

On the other hand $-2\det x = \operatorname{tr} x^2$ follows by consideration of eigenvalues $\pm\lambda$ of x in a quadratic extension of $\mathbb{K}$. We therefore obtain $2x^2 - \operatorname{tr}(x^2)\mathbf{1} = 2x^2 + 2(\det x)\mathbf{1} = 0$. By polarization (taking derivatives in direction y), we obtain from $2x^2 = \operatorname{tr}(x^2)\mathbf{1}$ the relation $2xy + 2yx = \operatorname{tr}(xy + yx)\mathbf{1} = 2\operatorname{tr}(xy)\mathbf{1}$, which leads to

$$xy + yx = \operatorname{tr}(xy)\mathbf{1}.$$

We further get

$$\begin{aligned} \operatorname{tr}(xy)z - \operatorname{tr}(xz)y &= (xy + yx)z - y(xz + zx) = xyz - yzx = [x, yz] \\ &= \frac{1}{2}[x, [y, z] + (yz + zy)] \\ &= \frac{1}{2}[x, [y, z] + \operatorname{tr}(yz)\mathbf{1}] = \frac{1}{2}[x, [y, z]]. \end{aligned}$$

■

Example I.9. (a) Let J be a locally convex Jordan algebra with identity $\mathbf{1}$ (cf. Appendix B). We endow the space $J \otimes J$ with the projective tensor product topology and define

$$\langle J, J\rangle := (J \otimes J)/I,$$

where $I \subseteq J \otimes J$ is the closed subspace generated by the elements of the form $a \otimes a$ and

$$ab \otimes c + bc \otimes a + ca \otimes b, \quad a, b, c \in J.$$

We write $\langle a, b\rangle$ for the image of $a \otimes b$ in $\langle J, J\rangle$. Then

$$\langle a, b\rangle = -\langle b, a\rangle \quad \text{and} \quad \langle ab, c\rangle + \langle bc, a\rangle + \langle ca, b\rangle = 0, \quad a, b, c \in J.$$

It follows in particular that $\langle \mathbf{1}, c\rangle + 2\langle c, \mathbf{1}\rangle = 0$, which implies $\langle \mathbf{1}, c\rangle = 0$ for each $c \in J$.

Let $L(a)b := ab$ denote the left multiplication in J. From the identity

$$[L(a), L(bc)] + [L(b), L(ca)] + [L(c), L(ab)] = 0$$

(Proposition B.2(1)) and the continuity of the maps $(a, b, x) \mapsto [L(a), L(b)].x$ we derive that the map

$$\delta_J \colon J \otimes J \to \mathrm{der}(J), \quad (a, b) \mapsto 2[L(a), L(b)]$$

(cf. Corollary B.3 for the fact that it maps into $\mathrm{der}(J)$) factors through a map

$$\delta_J \colon \langle J, J\rangle \to \mathrm{der}(J).$$

It therefore makes sense to define

$$\langle a, b\rangle.x := 2[L(a), L(b)].x, \quad a, b, x \in J. \tag{1.4}$$

We now define a bilinear continuous bracket on

$$\widetilde{\mathrm{TKK}}(J) := (J \otimes \mathfrak{sl}_2(\mathbb{K})) \oplus \langle J, J\rangle$$

by

$$\begin{aligned} [a \otimes x, a' \otimes x'] &:= aa' \otimes [x, x'] + \langle a, a'\rangle \operatorname{tr}(xx'), \quad [\langle a, b\rangle, c \otimes x] := \langle a, b\rangle.c \otimes x \\ [\langle a, b\rangle, \langle c, d\rangle] &:= \langle \langle a, b\rangle.c, d\rangle + \langle c, \langle a, b\rangle.d\rangle. \end{aligned}$$

The label TKK refers to Tits, Kantor and Koecher who studied the relation between Jordan algebras and Lie algebras from various viewpoints (see Appendices B and C). It is clear from the definitions that if we endow $\widetilde{\mathrm{TKK}}(J)$ with the natural locally convex topology turning it into a topological direct sum of $J \otimes \mathfrak{sl}_2(\mathbb{K})$ and $\langle J, J\rangle$, then $\widetilde{\mathrm{TKK}}(J)$ is a locally convex space with a continuous bracket. That the bracket is alternating follows for the $\langle J, J\rangle$-term from the calculation in Example III.10(3) below. To see that $\widetilde{\mathrm{TKK}}(J)$ is a Lie algebra, it remains to verify the Jacobi identity. The trilinear map

$$J(\alpha, \beta, \gamma) := [[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] =: \sum_{\text{cycl.}} [[\alpha, \beta], \gamma]$$

is alternating. Therefore we only have to show that it vanishes for entries in $J \otimes \mathfrak{sl}_2(\mathbb{K})$ and $\langle J, J\rangle$. The essential case is where all elements are in $J \otimes \mathfrak{sl}_2(\mathbb{K})$. In the last step of the following calculation we use Lemma I.8:

$$\begin{aligned} [[a \otimes x, b \otimes y], c \otimes z] &= [ab \otimes [x, y] + \operatorname{tr}(xy)\langle a, b\rangle, c \otimes z] \\ &= (ab)c \otimes [[x, y], z] + \operatorname{tr}([x, y]z)\langle ab, c\rangle + \langle a, b\rangle.c \otimes \operatorname{tr}(xy)z \\ &= 2(ab)c \otimes (\operatorname{tr}(zy)x - \operatorname{tr}(zx)y) + \langle a, b\rangle.c \otimes \operatorname{tr}(xy)z \\ &\quad + \operatorname{tr}([x, y]z)\langle ab, c\rangle. \end{aligned}$$

Now the vanishing of $J(a \otimes x, b \otimes y, c \otimes z)$ follows from

$$\sum_{\text{cycl.}} \operatorname{tr}([x,y]z)\langle ab, c\rangle = \operatorname{tr}([x,y]z) \sum_{\text{cycl.}} \langle ab, c\rangle = 0$$

and

$$(\langle a,b\rangle.c - 2(bc)a + 2(ca)b) \otimes \operatorname{tr}(xy)z = 0.$$

Note that this also explains the factor 2 in (1.4).

That the expression $J(\alpha,\beta,\gamma)$ vanishes if one entry is in $\langle J,J\rangle$ follows easily from the fact that $\delta_J(a,b) = 2[L(a),L(b)] \in \operatorname{der}(J)$. The case where two entries are in $\langle J,J\rangle$ corresponds to the relation

$$[\delta(a,b),\delta(c,d)] = \delta(\langle a,b\rangle.c,d) + \delta(c,\langle a,b\rangle.d)$$

in $\operatorname{der}(J)$, which in turn follows from the fact that for any $D \in \operatorname{der}(J)$ we have

$$\begin{aligned}[D,\delta(c,d)] &= 2[D,[L(c),L(d)]] = 2[[D,L(c)],L(d)] + 2[L(c),[D,L(d)]] \\ &= 2[L(D.c),L(d)] + 2[L(c),L(D.d)] = \delta(D.c,d) + \delta(c,D.d).\end{aligned}$$

The case where all entries of $J(\alpha,\beta,\gamma)$ are in $\langle J,J\rangle$ follows easily from the fact that the representation of $\operatorname{der}(J)$ on $J \otimes J$ factors through a Lie algebra representation on $\langle J,J\rangle$ given by $D.\langle a,b\rangle = \langle D.a,b\rangle + \langle a,D.b\rangle$. In this sense the latter three cases are direct consequences of the derivation property of the $\delta(a,b)$'s.

This proves that the bracket defined above is a Lie bracket on $\widetilde{\mathrm{TKK}}(J)$. The assignment $J \mapsto \widetilde{\mathrm{TKK}}(J)$ is functorial. It is clear that each derivation of J induces a natural derivation on $\widetilde{\mathrm{TKK}}(J)$ and that each morphism of unital locally convex Jordan algebras $\varphi\colon J_1 \to J_2$ defines a morphism $\widetilde{\mathrm{TKK}}(J_1) \to \widetilde{\mathrm{TKK}}(J_2)$ of locally convex Lie algebras.

It is interesting to observe that in general tensor products $A \otimes \mathfrak{k}$ of an algebra A and a Lie algebra $\mathfrak{k}$ carry only a natural Lie algebra structure if A is commutative and associative (Example I.4). For more general algebras one has to add an extra space such as $\langle J,J\rangle$ for a Jordan algebra J and $\mathfrak{k} = \mathfrak{sl}_2(\mathbb{K})$. The Jacobi identity for $\widetilde{\mathrm{TKK}}(J)$ very much relies on the identity for triple brackets in $\mathfrak{sl}_2(\mathbb{K})$ from Lemma I.8 and the definition of the action of $\langle a,b\rangle$ as $2[L(a),L(b)]$.

We have a natural embedding of $\mathfrak{sl}_2(\mathbb{K})$ into $\mathfrak{g} = \widetilde{\mathrm{TKK}}(J)$ as $\mathfrak{g}_\Delta := \mathbf{1} \otimes \mathfrak{sl}_2(\mathbb{K})$. Let $h,e,f \in \mathfrak{sl}_2(\mathbb{K})$ be a basis with

$$[h,e] = 2e, \quad [h,f] = -2f \quad \text{and} \quad [e,f] = h.$$

Then $\mathfrak{h} = \mathbb{K}h$ is a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{K})$, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g}_2 = J \otimes e, \quad \mathfrak{g}_{-2} = J \otimes f \quad \text{and} \quad \mathfrak{g}_0 = J \otimes h \oplus \langle J,J\rangle.$$

In view of $[\mathfrak{g}_\Delta, \mathfrak{g}] = J \otimes \mathfrak{sl}_2(\mathbb{K})$, the formula for the bracket implies that $\langle J, J\rangle \subseteq [\mathfrak{g}, \mathfrak{g}]$, and hence that $\mathfrak{g}$ is an A_1-graded locally convex Lie algebra.
(b) If A is a locally convex unital associative algebra, then A also carries the structure of a locally convex unital Jordan algebra A_J with respect to the product

$$a \circ b := \frac{1}{2}(ab + ba)$$

(Lemma B.7). It is interesting to compare $\widetilde{\mathrm{TKK}}(A_J)$ with the locally convex Lie algebra $\mathfrak{sl}_2(A)$ discussed in Example I.5, where we have seen that with respect to the decomposition

$$\mathfrak{sl}_2(A) = \Big(A \otimes \mathfrak{sl}_2(\mathbb{K})\Big) \oplus \Big(\overline{[A, A]} \otimes \mathbf{1}\Big),$$

the Lie bracket is given by

$$[a \otimes x, b \otimes y] = \frac{ab + ba}{2} \otimes [x, y] + \frac{1}{2}[a, b] \otimes x * y + [a, b] \otimes \frac{\mathrm{tr}(xy)}{2}\mathbf{1}.$$

In view of (1.2), we have $x * y = 0$, so that we obtain the simpler formula

$$[a \otimes x, b \otimes y] = (a \circ b) \otimes [x, y] + \frac{1}{2}[a, b] \otimes \mathrm{tr}(xy)\mathbf{1}.$$

Let $L_a(b) := ab$ and $R_a(b) := ba$. Then the left multiplication in the Jordan algebra is $L(a) = \frac{1}{2}(L_a + R_a)$, and therefore $\langle a, b\rangle$ acts on A_J as

$$\begin{aligned} 2[L(a), L(b)] &= \frac{1}{2}[L_a + R_a, L_b + R_b] = \frac{1}{2}([L_a, L_b] + [R_a, R_b]) \\ &= \frac{1}{2}(L_{[a,b]} - R_{[a,b]}) = \frac{1}{2}\,\mathrm{ad}([a, b]). \end{aligned}$$

From this it easily follows that

$$\varphi\colon \widetilde{\mathrm{TKK}}(A_J) \to \mathfrak{sl}_2(A), \quad a \otimes x \mapsto a \otimes x, \quad \langle a, b\rangle \mapsto \frac{1}{2}[a, b] \otimes \mathbf{1}$$

defines a morphism of locally convex Lie algebras.

From the discussion of the examples in Section IV below, we will see that this homomorphism is in general neither injective nor surjective.
(c) From the continuity of the map

$$\langle J, J\rangle \times J \to J, \quad (\langle a, b\rangle, x) \mapsto \delta_J(a, b).x = \langle a, b\rangle.x$$

it follows that $\ker \delta_J$ is a closed subspace of $\langle J, J\rangle$. Hence the space $\mathrm{ider}(J) := \mathrm{im}(\delta_J) \cong \langle J, J\rangle / \ker(\delta_J)$ carries a natural locally convex topology as the quotient space $\langle J, J\rangle / \ker(\delta_J)$.

The closed subspace $\ker(\delta_J) \subseteq \langle J, J\rangle$ also is a closed ideal of $\widetilde{\mathrm{TKK}}(J)$. The quotient Lie algebra

$$\mathrm{TKK}(J) := \widetilde{\mathrm{TKK}}(J)/\ker(\delta_J) = (J \otimes \mathfrak{sl}_2(\mathbb{K})) \oplus \mathrm{ider}(J)$$

is called the *topological Tits–Kantor–Koecher–Lie algebra* associated to the locally convex unital Jordan algebra J. The bracket of this Lie algebra is given by

$$\begin{aligned} [a \otimes x, a' \otimes x'] &:= aa' \otimes [x, x'] + 2\,\mathrm{tr}(x, x')[L(a), L(a')], \quad [d, c \otimes c] := d.c \otimes x \\ [d, d'] &:= dd' - d'd. \end{aligned}$$

Mostly $\mathrm{TKK}(J)$ is written in a different form, as $J \times \mathfrak{istr}(J) \times J$, where $\mathfrak{istr}(J) := L(J) + \mathrm{ider}(J)$ is the *inner structure Lie algebra of* J. The correspondence between the two pictures is given by the map

$$\Phi\colon \mathrm{TKK}(J) \to J \times \mathfrak{istr}(J) \times J, \quad a \otimes e + b \otimes h + c \otimes f + d \mapsto (a, 2L(b) + d, c).$$

To understand the bracket in the product picture, we observe that

$$(L(a) + [L(b), L(c)]).\mathbf{1} = a + b(c\mathbf{1}) - c(b\mathbf{1}) = a$$

implies

$$\mathfrak{istr}(J) = L(J) \oplus [L(J), L(J)] \cong J \oplus [L(J), L(J)].$$

For each derivation d of J we have $[d, L(a)] = L(d.a)$, which implies that

$$\sigma(L(x) + [L(y), L(z)]) = -L(x) + [L(y), L(z)]$$

defines an involutive Lie algebra automorphism on $\mathfrak{istr}(J)$. Now the bracket on $J \times \mathfrak{istr}(J) \times J$ can be described as

$$\begin{aligned} [(a, d, c), (a', d', c')] = {} & (d.a' - d'.a, 2L(ac') + 2[L(a), L(c')] - 2L(a'c) \\ & -2[L(a'), L(c)] + [d, d'], \sigma(d).c' - \sigma(d').c). \end{aligned}$$

From this formula it is clear that the map $\tau(a, d, c) := (c, \sigma(d), a)$ defines an involutive automorphism of $\mathrm{TKK}(J)$.
(d) Let A be a commutative algebra and

$$\mathfrak{o}_{n,n}(A) := \mathfrak{o}_{n,n}(A, \mathrm{id}) \cong A \otimes \mathfrak{o}_{n,n}(\mathbb{K})$$

(Example I.7(b)).

For the quadratic module $(M_n, q_n) := (A^{2n}, (q_A \oplus -q_A)^n)$ with

$$q(a_1, \ldots, a_{2n}) = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$$

the n-fold direct sum of the hyperbolic A-plane, we consider the associated Jordan algebra $J(M_n)$ (Lemma B.4). As $M_n \cong A \otimes \mathbb{K}^{2n}$ as quadratic modules, it is easy to see that

$$\mathrm{TKK}(J(M_n)) \cong A \otimes \mathrm{TKK}(J(\mathbb{K}^{2n})) \cong A \otimes \mathfrak{o}_{n,n+1}(\mathbb{K}),$$

which is a Lie algebra graded by the root system B_n.

If M is an orthogonal direct sum $M = M_0 \oplus M_n$, we have an inclusion $\mathrm{TKK}(J(M_n)) \hookrightarrow \mathrm{TKK}(J(M))$ which leads to an embedding

$$\mathfrak{o}_{n,n+1}(\mathbb{K}) \hookrightarrow \mathrm{TKK}(J(M)),$$

and further to a B_n-grading of $\mathrm{TKK}(J(M))$. ■

I.3 Twisted loop algebras

There are also so-called twisted versions of the Lie algebras $A \otimes \mathfrak{g}_\Delta$ from Example I.4. The construction is based on the following observation.

Let $\mathfrak{k}$ be a split simple $\mathbb{K}$-Lie algebra, $\mathfrak{h}_\mathfrak{k} \subseteq \mathfrak{k}$ a splitting Cartan subalgebra, and Γ a group of automorphisms of $\mathfrak{k}$ fixing a regular element of $\mathfrak{k}$ in $\mathfrak{h}_\mathfrak{k}$. Typical groups of this type arise from the outer automorphisms of $\mathfrak{k}$, which can be realised by automorphisms of $\mathfrak{k}$ preserving the root decomposition and a positive system of roots (see Example I.10 below). Let $\mathfrak{k}^\Gamma$ denote the subalgebra of all elements of $\mathfrak{k}$ fixed by Γ. Then $\mathfrak{k}^\Gamma$ contains a regular element x_0 of $\mathfrak{h}_\mathfrak{k}$, and therefore Γ preserves $\mathfrak{z}_\mathfrak{k}(x_0) = \mathfrak{h}_\mathfrak{k}$. It follows in particular that Γ permutes the $\mathfrak{h}_\mathfrak{k}$-root spaces of $\mathfrak{k}$.

As $\mathfrak{h}^\Gamma := \mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}^\Gamma = \mathfrak{h}_\mathfrak{k}^\Gamma$ contains a regular element of $\mathfrak{k}$, it also is a splitting Cartan subalgebra of $\mathfrak{k}^\Gamma$. If $\Delta_\mathfrak{k}$ is the root system of $\mathfrak{k}$ and Δ_0 the root system of $\mathfrak{k}^\Gamma$, then clearly $\Delta_0 \subseteq \Delta_\mathfrak{k}|_{\mathfrak{h}^\Gamma}$, but it may happen that the latter set still is a root system.

Example I.10. Let Γ be a finite group of automorphisms of $\mathfrak{k}$ preserving the Cartan subalgebra $\mathfrak{h}_\mathfrak{k}$ and such that the action on the dual space preserves a positive system $\Delta_\mathfrak{k}^+$ of roots. By averaging over the orbit of an element $x \in \mathfrak{h}_\mathfrak{k}$ on which all positive roots are positive, we then obtain an element fixed by Γ on which all positive roots are positive, so that this element is regular in $\mathfrak{k}$.

Typical examples for this situation come from cyclic groups of diagram automorphisms which are discussed below. A *diagram automorphism* is an automorphism φ of $\mathfrak{g}_\Delta$ for which there exists a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, elements $x_{\pm\alpha_i} \in \mathfrak{g}_{\Delta,\pm\alpha_i}$ with $[x_{\alpha_i}, x_{-\alpha_i}] = \check{\alpha}_i$, and a map $\underline{\varphi}\colon \Pi \to \Pi$ such that

$$\varphi(x_{\pm\alpha_i}) = x_{\pm\underline{\varphi}(\alpha_i)}.$$

(a) For type A_{2r-1} we have

$$\Delta_\mathfrak{k} = \{\pm(\varepsilon_i - \varepsilon_j)\colon i > j \in \{1, \ldots, 2r\}\}$$

on $\mathfrak{h}_\mathfrak{k} \cong \mathbb{K}^{2r}$. The non-trivial diagram automorphism σ is an involution satisfying

$$\sigma(x_1, \ldots, x_{2r}) = (-x_{2r}, \ldots, -x_1) \quad \text{and} \quad \sigma(\varepsilon_i) = -\varepsilon_{2r+1-i}.$$

We identify

$$\mathfrak{h}^\Gamma = \{(x_1, \dots, x_r, -x_r, \dots, -x_1) \colon x_i \in \mathbb{K}\}$$

with $\mathbb{K}^r$ by forgetting the last r entries. If $R \colon \mathfrak{h}_\mathfrak{k}^* \to (\mathfrak{h}^\Gamma)^*$ is the restriction map, then

$$\alpha_j := R(\varepsilon_j - \varepsilon_{j+1}), \quad j = 1, \dots, r,$$

is a basis for the root system

$$R(\Delta_\mathfrak{k}) = \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_j \colon 1 \leq j < i \leq r, 1 \leq j \leq r\}$$

of type C_r.
(b) For type D_{r+1}, $r \geq 4$, we have

$$\Delta_\mathfrak{k} = \{\pm(\varepsilon_i \pm \varepsilon_j) \colon i \neq j \in \{1, \dots, r+1\}\}$$

on $\mathfrak{h}_\mathfrak{k} \cong \mathbb{K}^{r+1}$. A non-trivial diagram automorphism σ is the involution

$$\sigma(x_1, \dots, x_{r+1}) = (x_1, \dots, x_r, -x_{r+1}).$$

We identify $\mathfrak{h}^\Gamma = \{(x_1, \dots, x_r, 0\}$ with $\mathbb{K}^r$ by forgetting the last entry. Then

$$R(\Delta_\mathfrak{k}) = \{\pm(\varepsilon_i \pm \varepsilon_j) \colon i \neq j \in \{1, \dots, r\}\} \cup \{\varepsilon_j \colon j = 1, \dots, r\}$$

is a root system of type B_r.
(c) For the triality automorphism of D_4 of order 3, we obtain a root system Δ_0 of type G_2.
(d) For the diagram involution of E_6 we obtain a root system Δ_0 of type F_4.

It is not hard to verify that for all cases (a)–(d) above $R(\Delta_\mathfrak{k})$ is the root system of $\mathfrak{k}^\Gamma$. ∎

Now let $\mathfrak{k}$ and Γ be such that $R(\Delta_\mathfrak{k})$ is the root system of $\mathfrak{k}^\Gamma$ and assume, in addition, that $\mathfrak{k}^\Gamma$ is simple with root system Δ. We write $\mathfrak{g}_\Delta := \mathfrak{k}^\Gamma$, $\mathfrak{h} := \mathfrak{h}^\Gamma$ and assume that Δ coincides with $R(\Delta_\mathfrak{k})$, which is the case for all cyclic groups of diagram automorphisms of type (a)–(d) above. Note that this excludes in particular the diagram automorphism of A_{2r} for which $R(\Delta_\mathfrak{k})$ is not reduced.

Further let A be a locally convex commutative unital associative algebra on which Γ acts by continuous automorphisms. Then Γ also acts on the Lie algebra $A \otimes \mathfrak{k}$ via $\gamma.(a \otimes x) := \gamma.a \otimes \gamma.x$. We consider the Lie subalgebra

$$\mathfrak{g} := (A \otimes \mathfrak{k})^\Gamma$$

of Γ-fixed points in $A \otimes \mathfrak{k}$. We clearly have $\mathfrak{g} \supseteq A^\Gamma \otimes \mathfrak{g}_\Delta \supseteq \mathbf{1} \otimes \mathfrak{g}_\Delta$. Moreover, the action of $\mathfrak{h} = \mathfrak{h}_\mathfrak{k}^\Gamma$ on $A \otimes \mathfrak{k}$ commutes with the action of Γ, and our assumption implies that the $\mathfrak{h}$-weights of $\mathfrak{h}$ on $A \otimes \mathfrak{k}$ coincide with the root system Δ. This implies that $\mathfrak{g}$ satisfies (R1)–(R3) with respect to the subalgebra $\mathfrak{g}_\Delta$, and therefore that the closure of the subalgebra generated by the root spaces is Δ-graded.

Example I.11. This construction covers in particular all twisted loop algebras. In this case $A = C^\infty(\mathbb{T}, \mathbb{C})$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and if $\Gamma = \langle\sigma\rangle$ is generated by a diagram automorphism σ of order m, then we define the action of Γ on A by $\sigma(f)(z) = f(z\zeta)$, where ζ is a primitive m-th root of unity.

For Δ_k of type A_{2r-1}, D_{r+1}, E_6 and D_4, we thus obtain the twisted loop algebras of type $A_{2r-1}^{(2)}, D_{r+1}^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$, and the corresponding root systems Δ are of type B_r, C_r, F_4 and G_2 ([Ka90]). ■

I.4 (Δ, Δ_0)-graded Lie algebras

Let Δ be a reduced irreducible root system and $\Delta_l \subseteq \Delta$ be the subset of long roots. Suppose that $\alpha, \beta \in \Delta_l$ with $\gamma := \alpha + \beta \in \Delta$. Then $\gamma \in \Delta_l$. Since α and β generate a subsystem of Δ whose rank is at most two, this can be verified by direct inspection of the cases A_2, $B_2 \cong C_2$ and G_2. Alternatively, we can observe that if $(\cdot, \cdot)$ denote the euclidean scalar product on $\mathrm{span}_{\mathbb{R}} \Delta \subseteq \mathfrak{h}^*$, then

$$\beta(\check{\alpha}) = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = 2\frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)}\sqrt{(\beta, \beta)}}$$

equals $2 \cdot \cos\delta$, where δ is the angle between α and β. On the other hand $\beta(\check{\alpha}) \in \mathbb{Z}$, so that the only possible values are $\{0, \pm 1, \pm 2\}$, where ± 2 only arises for $\beta = \pm\alpha$ which is excluded if $\alpha + \beta \in \Delta$. Therefore

$$(\alpha, \alpha) \geq (\gamma, \gamma) = (\alpha, \alpha) + (\beta, \beta) + 2(\alpha, \beta) = 2(\alpha, \alpha) + 2(\alpha, \beta) = 2(\alpha, \alpha) \pm (\alpha, \alpha)$$

implies $(\alpha, \alpha) = (\gamma, \gamma)$, hence that γ is long.

We conclude that Δ_l satisfies

$$(\Delta_l + \Delta_l) \cap \Delta \subseteq \Delta_l,$$

and hence that we have an inclusion

$$\mathfrak{g}_{\Delta_l} \hookrightarrow \mathfrak{g}_\Delta.$$

It follows in particular that each Δ-graded Lie algebra $\mathfrak{g}$ can also be viewed as a (Δ, Δ_l)-graded Lie algebra and that each Δ-graded Lie algebra contains the Δ_l-graded Lie algebra

$$\mathfrak{g}_0 + \sum_{\alpha \in \Delta_l} \mathfrak{g}_\alpha.$$

The following table describes the systems Δ_l for the non-simply laced root systems.

Δ	B_r	C_r	F_4	G_2
Δ_l	D_r	$(A_1)^r$	D_4	A_2

In many cases the subalgebra $\mathfrak{g}_{\Delta_l}$ of $\mathfrak{g}_\Delta$ also has a description as the fixed point algebra of an automorphism γ fixing $\mathfrak{h}$ pointwise. Such an automorphism is given by a morphism

$$\chi\colon \mathbb{Z}[\Delta] \to \mathbb{K}^\times$$

of abelian groups via

$$\gamma.x_\alpha = \chi(\alpha)x_\alpha, \quad x_\alpha \in (\mathfrak{g}_\Delta)_\alpha.$$

For

$$\Delta = B_r = \{\pm(\varepsilon_i \pm \varepsilon_j)\colon i \neq j \in \{1,\ldots,r\}\} \cup \{\varepsilon_j\colon j = 1,\ldots,r\}$$

we define

$$\widetilde{\chi}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}, \quad \sum_i n_i\varepsilon_i \mapsto \sum_i n_i.$$

Then

$$\widetilde{\chi}^{-1}(0) \cong A_{r-1}, \quad \Delta_s = \widetilde{\chi}^{-1}(2\mathbb{Z}+1) \quad \text{and} \quad \Delta_l = \widetilde{\chi}^{-1}(2\mathbb{Z}).$$

Therefore $\chi := (-1)^{\widetilde{\chi}}$ yields an involution γ_χ of $\mathfrak{g}_\Delta$ whose fixed point set is the subalgebra $\mathfrak{g}_{\Delta_l}$.

We likewise obtain for $\Delta = G_2$ a homomorphism $\widetilde{\chi}\colon \mathbb{Z}[\Delta] \to \mathbb{Z}$ with

$$\Delta_l = \widetilde{\chi}^{-1}(3\mathbb{Z}).$$

If $1 \neq \zeta \in \mathbb{K}^\times$ satisfies $\zeta^3 = 1$, we then obtain via $\chi := \zeta^{\widetilde{\chi}}$ an automorphism γ_χ of order 3 whose fixed point set is $\mathfrak{g}_{\Delta_l} \cong \mathfrak{sl}_3(\mathbb{K})$.

Problem I. Determine a systematic theory of (Δ,Δ_0)-graded Lie algebras for suitable classes of pairs (Δ,Δ_0). ■

II The coordinate algebra of a root graded Lie algebra

After having seen various examples of root graded locally convex Lie algebras in Section I, we now take a more systematic look at the structure of root graded Lie algebras. The main point of the present section is to associate to a Δ-graded Lie algebra $\mathfrak{g}$ a locally convex algebra $\mathcal{A}$, its coordinate algebra, together with a locally convex Lie algebra D (the centralizer of $\mathfrak{g}_\Delta$ in $\mathfrak{g}$), acting continuously by derivations on $\mathcal{A}$, and a continuous bilinear map $\delta^D\colon \mathcal{A}\times\mathcal{A} \to D$. The triple $(\mathcal{A}, D, \delta^D)$ is called the coordinate structure of $\mathfrak{g}$. The bracket of $\mathfrak{g}$ is completely determined by the coordinate structure and the root system Δ. The type of the coordinate algebra $\mathcal{A}$ (associative, alternative, Jordan etc.) and the map $\delta_{\mathcal{A}}\colon \mathcal{A}\times\mathcal{A} \to \mathrm{der}(\mathcal{A})$ determined by δ^D, is determined by the type of the root system Δ. These results will be refined in Section IV, where we discuss isogeny classes of locally convex root graded Lie algebras and describe the universal covering Lie algebra of $\mathfrak{g}$ in terms of the coordinate structure $(\mathcal{A}, D, \delta^D)$.

The algebraic results of this section are known; new is only that they still remain true in the context of locally convex Lie algebras, which requires additional arguments in several places and a more coordinate free approach, because in the topological context we can never argue with bases of vector spaces. We also tried to put an emphasis on those arguments which can be given for general root graded Lie algebras without any case by case analysis, as f.i. in Theorem II.13. We do not go into the details of the exceptional and the low-dimensional cases. For the arguments leading to the coordinate algebra, we essentially follow the expositions in [ABG00], [BZ96] (see also [Se76] which already contains many of the key ideas and arguments).

Let $\mathfrak{g}$ be a locally convex root graded Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathfrak{g}_\Delta$ a grading subalgebra. We consider the adjoint representation of $\mathfrak{g}_\Delta$ on $\mathfrak{g}$. From (R3) we immediately derive that $\mathfrak{g}$ is a $\mathfrak{g}_\Delta$-weight module in the sense that the action of $\mathfrak{h}$ is diagonalized by the Δ-grading. Moreover, the set of weights is $\Delta \cup \{0\}$ and therefore finite, so that Proposition A.2 leads to:

Theorem II.1. *The Lie algebra $\mathfrak{g}$ is a semisimple $\mathfrak{g}_\Delta$-weight module with respect to $\mathfrak{h}$. All simple submodules are finite-dimensional highest weight modules. There are only finitely many isotypic components $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$, and for each isotypic component the projection $p_i \colon \mathfrak{g} \to \mathfrak{g}_i$ can be realized by an element of the center of $U(\mathfrak{g}_\Delta)$. In particular, each p_i is continuous.* ■

Now we take a closer look at the isotypic components of the Lie algebra $\mathfrak{g}$. Let $\Delta_l \subseteq \Delta$ denote the subset of long roots and $\Delta_s \subseteq \Delta$ the subset of short roots, where we put $\Delta_l := \Delta$ if all roots have the same length. Then the Weyl group $\mathcal{W}$ of Δ acts transitively on the sets of short and long roots, so that it has at most three orbits in $\Delta \cup \{0\}$. Hence only three types of simple $\mathfrak{g}_\Delta$-modules may contribute to $\mathfrak{g}$. First we have the adjoint module $\mathfrak{g}_\Delta$, and each root vector in $\mathfrak{g}_\alpha$ for a long root α generates a highest weight module isomorphic to $\mathfrak{g}_\Delta$. Therefore the weight set of each other type of non-trivial simple $\mathfrak{g}_\Delta$-module occurring in $\mathfrak{g}$ must be smaller than $\Delta \cup \{0\}$, which already implies that it coincides with $\Delta_s \cup \{0\}$. The corresponding simple $\mathfrak{g}_\Delta$-module is the *small adjoint module* $V_s \cong L(\lambda_s, \mathfrak{g}_\Delta)$, i.e., the simple module whose highest weight is the highest short root λ_s with respect to a positive system Δ^+. In view of Theorem II.1, we therefore have a $\mathfrak{g}_\Delta$-module decomposition

$$\mathfrak{g} \cong (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D, \tag{2.1}$$

where

$$A := \operatorname{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g}), \quad B := \operatorname{Hom}_{\mathfrak{g}_\Delta}(V_s, \mathfrak{g}), \quad \text{and} \quad D := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_\Delta) \cong \operatorname{Hom}_{\mathfrak{g}_\Delta}(\mathbb{K}, \mathfrak{g})$$

are multiplicity spaces. We have

$$\mathfrak{g}_\alpha \cong \begin{cases} A & \text{for } \alpha \in \Delta_l \\ A \oplus B & \text{for } \alpha \in \Delta_s. \end{cases}$$

Our next goal is to construct an algebra structure on the topological direct sum $\mathcal{A} := A \oplus B$. This *coordinate algebra* will turn out to be an important structural feature of $\mathfrak{g}$.

For each finite-dimensional simple $\mathfrak{g}_\Delta$-module M the space $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M, \mathfrak{g})$ is a closed subspace of $\mathrm{Hom}(M, \mathfrak{g}) \cong M^* \otimes \mathfrak{g} \cong \mathfrak{g}^{\dim M}$, hence inherits a natural locally convex topology from the one on $\mathfrak{g}$, and the evaluation map

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(M, \mathfrak{g}) \otimes M \to \mathfrak{g}, \quad \varphi \otimes m \mapsto \varphi(m)$$

is an embedding of locally convex spaces onto the M-isotypic component of $\mathfrak{g}$. In this sense we think of $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$ as topological subspaces of $\mathfrak{g}$. We conclude that the addition map

$$(A \otimes \mathfrak{g}_\Delta) \times (B \otimes V_s) \times D \to \mathfrak{g}, \quad (a \otimes x, b \otimes y, d) \mapsto a \otimes x + b \otimes y + d$$

is a continuous bijection of locally convex spaces. That its inverse is also continuous follows from Theorem II.1 which ensures that the isotypic projections of $\mathfrak{g}$ are continuous linear maps. Therefore the decomposition (2.1) is a direct sum decomposition of locally convex spaces. If $\mathfrak{g}$ is a Fréchet space, we do not have to use Theorem II.1 because we can argue with the Open Mapping Theorem.

It is clear that the subspace $D = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_\Delta)$ is a closed Lie subalgebra. To obtain an algebra structure on $A \oplus B$. The following lemma is crucial for our analysis.

Lemma II.2. *Let M_j, $j = 1, 2, 3$, be finite-dimensional simple $\mathfrak{g}_\Delta$-modules and V_j, $j = 1, 2, 3$, locally convex spaces considered as trivial $\mathfrak{g}_\Delta$-modules. We consider the locally convex spaces $V_j \otimes M_j$ as $\mathfrak{g}_\Delta$-modules. Let $\beta_1, \ldots, \beta_k$ be a basis of* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_1 \otimes M_2, M_3)$ *and*

$$\alpha \colon (V_1 \otimes M_1) \times (V_2 \otimes M_2) \to V_3 \otimes M_3$$

a continuous invariant bilinear map. Then there exist continuous bilinear maps

$$\gamma_1, \ldots, \gamma_k \colon V_1 \times V_2 \to V_3$$

with

$$\alpha(v_1 \otimes m_1, v_2 \otimes m_2) = \sum_{i=1}^k \gamma_i(v_1, v_2) \otimes \beta_i(m_1, m_2).$$

Proof. Fix $v_1 \in V_1$ and $v_2 \in V_2$. Then the map

$$\alpha_{v_1,v_2} \colon (m_1, m_2) \mapsto \alpha(v_1 \otimes m_1, v_2 \otimes m_2)$$

is an invariant bilinear map $M_1 \times M_2 \to V_3 \otimes M_3$. As the image of α_{v_1,v_2} is finite-dimensional, there exist $w_1, \ldots, w_m \in V_3$ such that

$$\alpha_{v_1,v_2} = \sum_{j=1}^{m} \sum_{i=1}^{k} w_j \otimes \beta_i = \sum_{i=1}^{k} \sum_{j=1}^{m} w_j \otimes \beta_i.$$

This shows that there are bilinear maps $\gamma_1, \ldots, \gamma_k \colon V_1 \times V_2 \to V_3$ with $\alpha = \sum_{i=1}^k \gamma_i \otimes \beta_i$. For each i there exists an element $a_i := \sum_\ell m_1^\ell \otimes m_2^\ell \in M_1 \otimes M_2$ with $\beta_i(a_i) \neq 0$ and $\beta_j(a_i) = 0$ for $i \neq j$. Then

$$\sum_\ell \alpha(v_1 \otimes m_1^\ell, v_2 \otimes m_2^\ell) = \gamma_i(v_1, v_2) \otimes \beta_i(a_i)$$

shows that each map γ_i is continuous. ■

Remark II.3. If $M_1 := \mathfrak{g}_\Delta$, $M_2 := V_s$, $M_3 = \mathbb{K}$ and $V_i := \mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i, \mathfrak{g})$, then the Lie bracket on $\mathfrak{g}$ induces a family of $\mathfrak{g}_\Delta$-equivariant continuous bilinear maps

$$V_i \otimes M_i \times V_j \otimes M_j \to M_k \otimes V_k.$$

To apply Lemma II.2, we therefore have to analyze the spaces $\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i \otimes M_j, M_k)$.

The case $3 \in \{i, j\}$ is trivial because $D = \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_\Delta)$ commutes with the action of $\mathfrak{g}_\Delta$, so that the bracket map induces continuous bilinear maps

$$D \times A \to A, \quad (d, a) \mapsto d.a \quad \text{and} \quad D \times B \to B, \quad (d, b) \mapsto d.b$$

with

$$[d, a \otimes x] = d.a \otimes x \quad \text{and} \quad [d, b \otimes y] = d.b \otimes y.$$

Interpreting A as the space $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g})$, the action of D on this space corresponds to

$$d.\varphi := (\mathrm{ad}\, d) \circ \varphi,$$

and likewise for $B = \mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s, \mathfrak{g})$.

We may therefore assume that $i, j \in \{1, 2\}$. For $k = 3$, i.e., $M_k = \mathbb{K}$, the space

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i \otimes M_j, \mathbb{K}) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(M_i, M_j^*)$$

is trivial for $i \neq j$ because M_1 and M_2 have different dimensions. For $M_1 = \mathfrak{g}_\Delta$ we have

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, \mathbb{K}) = \mathbb{K}\kappa,$$

where κ is the Cartan-Killing form. As V_s and V_s^* have the same weight set $\Delta_s = -\Delta_s$, they are isomorphic, and [Bou90, Ch. VIII, §7, no. 5, Prop. 12] implies that, for $i = j = 2$,

$$\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, \mathbb{K}) = \mathbb{K}\kappa_{V_s}$$

for a non-zero invariant symmetric bilinear form κ_{V_s} on V_s. The symmetry of the form follows from the fact that the highest weight λ_s of V_s is an integral linear combination of the base roots of Δ. ■

The complete information on the relevant Hom-spaces is given in Theorem II.6 below. We have to prepare the statement of this theorem with the discussion of some special cases.

Definition II.4. (a) On the space $M_n(\mathbb{K})$ of $n \times n$-matrices the matrix product is equivariant with respect to the adjoint action of the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$. Hence the product $(x, y) \mapsto xy + yx$ does also have this property, and therefore the map

$$\mathfrak{sl}_n(\mathbb{K}) \times \mathfrak{sl}_n(\mathbb{K}) \to \mathfrak{sl}_n(\mathbb{K}), \quad (x, y) \mapsto x * y := xy + yx - \frac{2\,\mathrm{tr}(xy)}{n}\mathbf{1}$$

is equivariant with respect to the adjoint action of $\mathfrak{sl}_n(\mathbb{K})$. In the following $x * y$ will always denote this product.
(b) Let Ω be the non-degenerate alternating form on $\mathbb{K}^{2r}$ given by $\Omega(x, y) = (x, y)J(x, y)^\top$, where $J = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ (cf. Example I.7). For $X^\sharp := JX^\top J^{-1}$ we then have

$$\mathfrak{sp}_{2r}(\mathbb{K}) \cong \{X \in \mathfrak{gl}_{2r}(\mathbb{K}) \colon X^\sharp = -X\} \quad \text{and} \quad V_s \cong \{X \in \mathfrak{gl}_{2r}(\mathbb{K}) \colon X^\sharp = X, \mathrm{tr}\, X = 0\}.$$

This follows easily by decomposing $\mathfrak{gl}_{2r}(\mathbb{K})$ into weight spaces with respect to a Cartan subalgebra of $\mathfrak{sp}_{2r}(\mathbb{K})$. Here we use $(XY)^\sharp = Y^\sharp X^\sharp$ to see that V_s is invariant under brackets with $\mathfrak{sp}_{2r}(\mathbb{K})$ and satisfies $[V_s, V_s] \subseteq \mathfrak{sp}_{2r}(\mathbb{K})$. Moreover, the $*$-product restricts to $\mathfrak{sp}_{2r}(\mathbb{K})$-equivariant symmetric bilinear maps

$$\beta^V_{\mathfrak{g}} \colon \mathfrak{sp}_{2r}(\mathbb{K}) \times \mathfrak{sp}_{2r}(\mathbb{K}) \to V_s \quad \text{and} \quad \beta^V_V \colon V_s \times V_s \to V_s. \qquad \blacksquare$$

Remark II.5. For $\Delta = A_r$, $r \geq 2$, the product $*$ is an equivariant symmetric product on $\mathfrak{g}_\Delta = \mathfrak{sl}_{r+1}(\mathbb{K})$. Of course, the same formula also yields for $r = 1$ a symmetric product, but in this case we have $x * y = 0$ (Lemma I.8). ■

Theorem II.6. *For the* Hom-*spaces of the different kinds of Lie algebras we have:*

(1) *For Δ not of type A_r, $r \geq 2$, the space* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, \mathfrak{g}_\Delta)$ *is one-dimensional and generated by the Lie bracket. For Δ of type A_r, $r \geq 2$, this space is two-dimensional and a second generator is the symmetric product $*$ on $\mathfrak{g}_\Delta \cong \mathfrak{sl}_{r+1}(\mathbb{K})$.*

(2) *If Δ is not of type C_r, $r \geq 2$, then* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, \mathfrak{g}_\Delta) = \{0\}$. *For Δ of type C_r, $r \geq 2$, and $\mathfrak{g}_\Delta \cong \mathfrak{sp}_{2r}(\mathbb{K})$ the space* $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s)$ *is generated by the $*$-product.*

(3) $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, \mathfrak{g}_\Delta) \cong \mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, V_s)$ *is one-dimensional and generated by the module structure on V_s. For Δ of type C_r, a basis of the first space is given by the bracket map on $\mathfrak{gl}_{2r}(\mathbb{K})$, restricted to V_s.*

(4) $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, V_s)$ *is one-dimensional for* C_n, $n \geq 3$, F_4 *and* G_2, *and vanishes for* B_n, $n \geq 2$. *For* Δ *of type* C_n, *a basis of this space is given by the* $*$*-product.*

Proof. All these statements follow from Definition II.4 and the explicit decomposition of the tensor products, which are worked out in detail in [Se76, §A.2] (see also the Appendix of [BZ96] for a list of the decompositions). ■

Before we turn to a more explicit description of the Lie bracket on $\mathfrak{g}$, we have to fix a notation for the basis elements of the Hom-spaces mentioned above.

Definition II.7. First we recall the symmetric invariant bilinear form κ_{V_s} on V_s from Remark II.3. Let $\beta_{\mathfrak{g}}^V$ be a basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta, V_s)$ if this space is non-zero, and $\beta_{\mathfrak{g},V}^{\mathfrak{g}}$ the corresponding basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta \otimes V_s, \mathfrak{g}_\Delta)$ which is related to $\beta_{\mathfrak{g}}^V$ by the relation

$$\kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,y), v) = \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v), y), \quad x, y \in \mathfrak{g}_\Delta, v \in V_s.$$

Let $\beta_V^{\mathfrak{g}} \colon V_s \otimes V_s \to \mathfrak{g}_\Delta$ be the equivariant map defined by

$$\kappa_{V_s}(x.v, v') = \kappa(\beta_V^{\mathfrak{g}}(v, v'), x), \quad v, v' \in V_s, x \in \mathfrak{g}_\Delta.$$

Then

$$\kappa_{V_s}(x.v, v') = -\kappa_{V_s}(v, x.v') = -\kappa_{V_s}(x.v', v)$$

(cf. Remark II.3 for the symmetry of κ_{V_s}) implies that $\beta_V^{\mathfrak{g}}$ is skew-symmetric . We further write β_V^V for a basis element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(V_s \otimes V_s, V_s)$.

For Δ of type C_r, $r \geq 2$, we take

$$\kappa_{V_s}(v, w) = \theta \,\mathrm{tr}(vw),$$

where the factor $\theta = 2(r+1)$ is determined by $\kappa(x, y) = \theta \,\mathrm{tr}(xy)$ ([Bou90, Ch. VIII]). We further put

$$\beta_{\mathfrak{g}}^V(x,y) := x * y, \quad \beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v) = x * v, \quad \beta_V^{\mathfrak{g}}(v,w) = [v,w], \quad \beta_V^V(v,w) = v * w,$$

and observe that from the embedding $\mathfrak{sp}_{2r}(\mathbb{K}) \hookrightarrow \mathfrak{sl}_{2r}(\mathbb{K})$ we get for $v \in V_s$:

$$\begin{aligned} \kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,y), v) &= \theta \,\mathrm{tr}((x*y)\cdot v) = \theta \,\mathrm{tr}((xy+yx)\cdot v) \\ &= \theta \,\mathrm{tr}((vx+xv)\cdot y) = \theta \,\mathrm{tr}((x*v)\cdot y) = \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v), y). \end{aligned}$$

This calculation implies that our special definitions for type C_r are compatible with the general requirements on the relation between $\beta_{\mathfrak{g}}^V$ and $\beta_{\mathfrak{g},V}^{\mathfrak{g}}$. ■

In view of Lemma II.2 and Theorem II.6, there exist continuous bilinear maps

$\gamma^A_\pm : A \times A \to A$, $\gamma^B_A \colon A \times A \to B$, $\gamma^A_{A,B} \colon A \times B \to A$, $\gamma^B_{A,B} \colon A \times B \to B$,
$\gamma^A_B : B \times B \to A$, $\gamma^B_B \colon B \times B \to B$, $\delta^D_A \colon A \times A \to D$, $\delta^D_B \colon B \times B \to D$,

such that the Lie bracket on

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D$$

satisfies

(B1) $[a \otimes x, a' \otimes x'] = \gamma^A_+(a,a') \otimes [x,x'] + \gamma^A_-(a,a') \otimes x * x' + \gamma^B_A(a,a') \otimes \beta^V_{\mathfrak{g}}(x,x')$
$+\kappa(x,x')\delta^D_A(a,a')$, for $a, a' \in A, x, x' \in \mathfrak{g}_\Delta$,

(B2) $[a \otimes x, b \otimes v] = \gamma^A_{A,B}(a,b) \otimes \beta^{\mathfrak{g}}_{\mathfrak{g},V}(x,v) + \gamma^B_{A,B}(a,b) \otimes x.v$,

for $a \in A, b \in B, x \in \mathfrak{g}_\Delta, v \in V_s$, and for $b, b' \in B$ and $v, v' \in V_s$:

(B3) $[b \otimes v, b' \otimes v'] = \gamma^A_B(b,b') \otimes \beta^{\mathfrak{g}}_V(v,v') + \gamma^B_B(b,b') \otimes \beta^V_V(v,v') + \kappa_{V_s}(v,v')\delta^D_B(b,b')$.

From the skew-symmetry of the Lie bracket and the symmetry of $*$, it follows that γ^A_+ is symmetric and γ^A_- is alternating. Further the symmetry of κ and κ_{V_s} implies that δ^D_A and δ^D_B are alternating. The skew-symmetry of $\beta^{\mathfrak{g}}_V$ implies that γ^A_B is symmetric and likewise the symmetry of $\beta^V_{\mathfrak{g}}$ entails that γ^B_A is skew-symmetric.

If Δ is not of type $A_r, r \geq 2$, then we put $\gamma^A_- = 0$. In all cases where the β-map vanishes, we define the corresponding γ-map to be zero.

Definition II.8. (The coordinate algebra $\mathcal{A}$ of $\mathfrak{g}$) (a) On A we define an algebra structure by

$$ab := \gamma^A_+(a,b) + \gamma^A_-(a,b),$$

and observe that

$$\gamma^A_+(a,b) = \frac{ab + ba}{2} \quad \text{and} \quad \gamma^A_-(a,b) = \frac{ab - ba}{2}.$$

We define a (not necessarily associative) algebra structure on $\mathcal{A} := A \oplus B$ by defining the product on $A \times A$ by $\gamma^A_+ + \gamma^A_- + \gamma^B_A$, on $A \times B$ by $\gamma^A_{A,B} + \gamma^B_{A,B}$, on $B \times B$ by $\gamma^A_B + \gamma^B_B$, and on $B \times A$ by

$$ba := \gamma^B_{A,B}(a,b) - \gamma^A_{A,B}(a,b) = ab - 2\gamma^A_{A,B}(a,b).$$

Then

$$\gamma^A_{A,B}(a,b) = \frac{1}{2}[a,b] = \frac{1}{2}(ab - ba) \quad \text{and} \quad \gamma^B_{A,B}(a,b) = \frac{1}{2}(ab + ba).$$

(b) The space $D = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_\Delta)$ is a Lie subalgebra of $\mathfrak{g}$ which acts by derivations on $\mathcal{A}$ preserving both subspaces A and B. This easily follows from the fact that the actions of D and $\mathfrak{g}_\Delta$ on $\mathfrak{g}$ commute.

We combine the two maps δ^D_A and δ^D_B to an alternating bilinear map

$$\delta^D \colon \mathcal{A} \times \mathcal{A} \to D, \quad (a + b, a' + b') \mapsto \delta^D_A(a,a') + \delta^D_B(b,b')$$

vanishing on $A \times B$. ■

Example II.9. Below we briefly explain how the relations (B1)–(B3) simplify for the two classes of Lie algebras that we obtain if we distinguish Lie algebras of type A_r or C_r and all others. In some sense the information is more explicit for A_r and C_r. We first discuss the other cases.
(a) For Δ not of type A_r, $r \geq 2$, we have $\gamma^A_- = 0$, and for Δ not of type C_r, $r \geq 2$, we have $\gamma^B_A = \gamma^A_{A,B} = 0$ (Theorem II.6.(2)). If these two conditions are satisfied, then the product on $\mathcal{A}$ is given by

$$\begin{aligned}(a,b)\cdot(a',b') &= (\gamma^A_+(a,a') + \gamma^A_B(b,b'), \gamma^B_{A,B}(a,b') + \gamma^B_{A,B}(a',b) + \gamma^B_B(b,b')) \\ &= (aa' + \gamma^A_B(b,b'), ab' + ba' + \gamma^B_B(b,b')).\end{aligned}$$

In this case the Lie bracket in $\mathfrak{g}$ can be written as

$$[a\otimes x, a'\otimes x'] = aa'\otimes[x,x'] + \kappa(x,x')\delta^D_A(a,a'), \quad a,a'\in A, x,x'\in\mathfrak{g}_\Delta,$$

$$[a\otimes x, b\otimes v] = ab\otimes x.v, \quad a\in A, b\in B, x\in\mathfrak{g}_\Delta, v\in V_s,$$

and

$$[b\otimes v, b'\otimes v'] = \gamma^A_B(b,b')\otimes\beta^{\mathfrak{g}}_V(v,v') + \gamma^B_B(b,b')\otimes\beta^V_V(v,v') + \kappa_{V_s}(v,v')\delta^D_B(b,b').$$

(b) If Δ is of type A_r, $r \geq 1$, then $B = \{0\}$ and $\mathcal{A} = A$.

For Δ of type C_r, $r \geq 2$, we have $\beta^V_V(v,v') = v * v'$, which is symmetric. Therefore γ^B_B is skew-symmetric. In view of

$$bb' = \gamma^A_B(b,b') + \gamma^B_B(b,b'),$$

this implies

$$\gamma^A_B(b,b') = \frac{bb' + b'b}{2} \quad\text{and}\quad \gamma^B_B(b,b') = \frac{1}{2}[b,b'] := \frac{bb' - b'b}{2}.$$

For $r = 2$ we have $\beta^V_V = 0$ and therefore $\gamma^B_B = 0$ (Theorem II.6(4)). In this case $C_2 \cong B_2$ implies that V_s can be viewed as the representation of $\mathfrak{so}_{3,2}(\mathbb{K})$ on $\mathbb{K}^5$.

In contrast to the formulas under (a), we have for Δ of type A_r and C_r the unifying formulas

$$\begin{aligned}[a\otimes x, a'\otimes x'] &= \frac{aa'+a'a}{2}\otimes[x,x'] + \underbrace{\gamma^A_-(a,a')}_{=0 \text{ for } C_r}\otimes x*x' \\ &\quad + \underbrace{\gamma^B_A(a,a')}_{=0 \text{ for } A_r}\otimes x*x' + \kappa(x,x')\delta^D_A(a,a'), \\ &= \frac{aa'+a'a}{2}\otimes[x,x'] + \frac{1}{2}[a,a']\otimes x*x' + \kappa(x,x')\delta^D_A(a,a')\end{aligned}$$

for $a, a' \in A, x, x' \in \mathfrak{g}_\Delta$, where we use that

$$[a,a'] = aa' - a'a = 2(\gamma^A_- + \gamma^B_A)(a,a'), \quad a,a'\in A.$$

We further have for C_r:

$$[a\otimes x, b\otimes v] = \frac{1}{2}[a,b]\otimes x * v + \frac{1}{2}(ab+ba)\otimes[x,v], \quad a\in A, b\in B, x\in\mathfrak{g}_\Delta, v\in V_s,$$

and

$$[b\otimes v, b'\otimes v'] = \frac{1}{2}(bb'+b'b)\otimes[v,v'] + \frac{1}{2}[b,b']\otimes v * v' + \kappa_{V_s}(v,v')\delta_B^D(b,b'). \quad \blacksquare$$

Remark II.10. (Involution on $\mathcal{A}$) On the space $\mathcal{A} = A\oplus B$ we have a natural continuous involution $\sigma(a,b) := (a,-b)$ with

$$A = \mathcal{A}^\sigma := \{a\in A\colon a^\sigma = a\} \quad \text{and} \quad B = \mathcal{A}^{-\sigma} := \{a\in A\colon a^\sigma = -a\}.$$

The map σ is an algebra involution, i.e., $\sigma(xx') = \sigma(x')\sigma(x)$ for $x,x'\in\mathcal{A}$, if and only if

(I1) $\sigma(aa') = a'a$ for $a,a'\in A$, i.e., $\gamma_-^A = 0$,

(I2) $\sigma(ab) = -ba$ for $a\in A$, $b\in B$, which is always the case because $[a,b]\in A$, and

(I3) $\sigma(bb') = b'b$ for $b,b'\in B$, which means that γ_B^A is symmetric and γ_B^B is skew-symmetric.

Condition (I1) is satisfied for any Δ not of type A_r, $r\geq 2$. For condition (I3), we recall that γ_B^A is symmetric because $\beta_V^{\mathfrak{g}}$ is skew-symmetric (Definition II.7). That γ_B^B is skew-symmetric means that β_V^V is symmetric, which is the case for Δ of type C_n, where $\beta_V^V(v,v') = v * v'$. It is also the case for Δ of type F_4, but not for type G_2, where it is the Malcev product on the pure octonions (cf. [ABG00, p.521]). $\blacksquare$

Remark II.11. (a) (The identity in $\mathcal{A}$) The inclusion $\mathfrak{g}_\Delta \hookrightarrow \mathfrak{g}$ is an element of $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta,\mathfrak{g}) = A\subseteq\mathcal{A}$ which we call $\mathbf{1}$. It satisfies

$$[\mathbf{1}\otimes x, a\otimes y] = x.(a\otimes y) = a\otimes[x,y], \quad \text{and} \quad [\mathbf{1}\otimes x, b\otimes v] = b\otimes x.v.$$

This means that

$$\mathbf{1}a = a\mathbf{1} = a \quad \text{and} \quad \delta^D(\mathbf{1},a) = 0 \quad \text{for all} \quad a\in\mathcal{A}.$$

In particular, $\mathbf{1}$ is an identity element in $\mathcal{A}$.
(b) The subspace A is a subalgebra of $\mathcal{A}$ if and only if $\gamma_A^B = 0$. If this map is non-zero, then $\beta_{\mathfrak{g}}^V \neq 0$ and Δ is of type C_r, $r\geq 2$ (Theorem II.6(2)). In all other cases A is a subalgebra of $\mathcal{A}$, and this subalgebra is commutative if and only if γ_-^A vanishes, which in turn is the case if Δ is not of type A_r or $C_r, r\geq 2$. $\blacksquare$

Remark II.12. (a) Axiom (R4) for a locally convex root graded Lie algebra is equivalent to the condition that the D-parts of the brackets $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ span a dense subspace of D. First we observe that only brackets of the type (B1) and (B3) have a non-zero D-part. Using the coordinate structure (B1)–(B3) of $\mathfrak{g}$, we can therefore translate (R4) into the fact that $\mathrm{im}(\delta^D_A) + \mathrm{im}(\delta^D_B) = \mathrm{im}(\delta^D)$ spans a dense subspace of D.
(b) Recall from Remark II.5 that for each root α we have $x_\alpha * x_{-\alpha} = 0$, and therefore, for all $a, a' \in A$, the simplification

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] = \gamma^A_+(a, a') \otimes [x_\alpha, x_{-\alpha}] + \kappa(x_\alpha, x_{-\alpha})\delta^D_A(a, a').$$

Hence

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] - [a' \otimes x_\alpha, a \otimes x_{-\alpha}] = 2\kappa(x_\alpha, x_{-\alpha})\delta^D_A(a, a'). \qquad \blacksquare$$

Theorem II.13. *The alternating map $\delta^D \colon \mathcal{A} \times \mathcal{A} \to D$ satisfies the cocycle condition*

$$\delta^D(aa', a'') + \delta^D(a'a'', a) + \delta^D(a''a, a') = 0, \quad a, a', a'' \in \mathcal{A}, \tag{2.2}$$

and

$$\delta^D(d.a, a') + \delta^D(a, d.a') = [d, \delta^D(a, a')] \quad d \in D, a, a' \in \mathcal{A}. \tag{2.3}$$

Proof. The plan of the proof is as follows. We will use the fact that (B1)–(B3) satisfy the Jacobi identity to obtain four relations for δ^D, which then will lead to the required cocycle condition for δ^D, where $0, 1, 2, 3$ elements among a, a', a'' are contained in A, and the others in B.

Step 1: For $a, a', a'' \in A$ and $x, x', x'' \in \mathfrak{g}_\Delta$, we use (B1) to see that the D-component of

$$[[a \otimes x, a' \otimes x'], a'' \otimes x'']$$

is

$$\kappa([x, x'], x'')\delta^D_A(\gamma^A_+(a, a'), a'') + \kappa(x * x', x'')\delta^D_A(\gamma^A_-(a, a'), a''). \tag{2.4}$$

From the invariance and the symmetry of κ, we derive

$$\kappa([x, x'], x'') = \kappa(x, [x', x'']) = \kappa([x', x''], x),$$

i.e., the cyclic invariance of $\kappa([x, x'], x'')$. If Δ is not of type A_r, $r \geq 2$, then $x * x' = 0$, and the second summand in (2.4) vanishes. But for Δ of type A_r we have $\kappa(x, x') = 2(r+1)\,\mathrm{tr}(xx')$ and therefore

$$\begin{aligned}&\kappa(x * x', x'')\\ &= 2(r+1)\,\mathrm{tr}\Big(\Big(xx' + x'x - \frac{2\,\mathrm{tr}(xx')}{r+1}\mathbf{1}\Big) \cdot x''\Big) = 2(r+1)\Big(\mathrm{tr}(xx'x'') + \mathrm{tr}(x'xx'')\Big).\end{aligned}$$

Hence we get in all cases the cyclic invariance of $\kappa(x*x', x'')$. Therefore the Jacobi identity in $\mathfrak{g}$, applied to the D-components of the form (2.4), leads to

$$\begin{aligned} 0 &= \sum_{\text{cycl.}} \Big(\kappa([x,x'],x'')\delta_A^D(\gamma_+^A(a,a'),a'') + \kappa(x*x',x'')\delta_A^D(\gamma_-^A(a,a'),a'')\Big) \\ &= \kappa([x,x'],x'')\sum_{\text{cycl.}} \delta_A^D(\gamma_+^A(a,a'),a'') + \kappa(x*x',x'')\sum_{\text{cycl.}} \delta_A^D(\gamma_-^A(a,a'),a''). \end{aligned}$$

For $x \in \mathfrak{g}_\alpha$ and $x' \in \mathfrak{g}_{-\alpha}$ with $[x,x'] = \check{\alpha}$ we have $x * x' = 0$ (Remark II.5), and we thus obtain

$$\sum_{\text{cycl.}} \delta_A^D(\gamma_+^A(a,a'),a'') = 0.$$

Choosing x, x', x'' such that $\kappa(x*x', x'') \neq 0$, we also obtain $\sum_{\text{cycl.}} \delta_A^D(\gamma_-^A(a,a'),a'') = 0$. Adding these two identities leads to

$$\sum_{\text{cycl.}} \delta_A^D(aa', a'') = 0.$$

Step 2: For $a, a' \in A$, $b \in B$, and $x, x' \in \mathfrak{g}_\Delta$, $v \in V_s$, we get for the D-component of

$$0 = [[a \otimes x, a' \otimes x'], b \otimes v] + [[a' \otimes x', b \otimes v], a \otimes x] + [[b \otimes v, a \otimes x], a' \otimes x']$$

the relation

$$\begin{aligned} 0 &= \kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,x'),v)\delta_B^D(\gamma_A^B(a,a'),b) + \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x',v),x)\delta_A^D(\gamma_{A,B}^A(a',b),a) \\ &\quad -\kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v),x')\delta_A^D(\gamma_{A,B}^A(a,b),a') \\ &= \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v),x')\Big(\delta_B^D(\gamma_A^B(a,a'),b) + \delta_A^D(\gamma_{A,B}^A(a',b),a) - \delta_A^D(\gamma_{A,B}^A(a,b),a')\Big) \\ &= \kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v),x')\Big(\delta(aa',b) + \delta(a'b,a) + \delta(ba,a')\Big) \end{aligned}$$

because δ^D vanishes on $A \times B$, the A-component $\gamma_{A,B}^A(a,b)$ of ab is skew-symmetric in a and b, and

$$\kappa(\beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v),x') = \kappa_{V_s}(\beta_{\mathfrak{g}}^V(x,x'),v)$$

is symmetric in x and x' (Definition II.7). We conclude that

$$\delta^D(aa',b) + \delta^D(a'b,a) + \delta^D(ba,a') = 0.$$

Step 3: For $a \in A$, $b, b' \in B$, and $x \in \mathfrak{g}_\Delta$, $v, v' \in V_s$, we get from the D-components of

$$0 = [[b \otimes v, b' \otimes v'], a \otimes x] + [[b' \otimes v', a \otimes x], b \otimes v] + [[a \otimes x, b \otimes v], b' \otimes v']$$

the relation

$$\begin{aligned} 0 &= \kappa(\beta_V^{\mathfrak{g}}(v,v'),x)\delta_A^D(\gamma_B^A(b,b'),a) - \kappa_{V_s}(x.v',v)\delta_B^D(\gamma_{A,B}^B(a,b'),b) \\ &\quad +\kappa_{V_s}(x.v,v')\delta_B^D(\gamma_{A,B}^B(a,b),b') \\ &= \kappa_{V_s}(x.v,v')\Big(\delta_A^D(\gamma_B^A(b,b'),a) + \delta_B^D(\gamma_{A,B}^B(a,b'),b) + \delta_B^D(\gamma_{A,B}^B(a,b),b')\Big) \\ &= \kappa_{V_s}(x.v,v')\Big(\delta^D(bb',a) + \delta^D(b'a,b) + \delta^D(ab,b')\Big) \end{aligned}$$

because δ^D vanishes on $A \times B$ and the B-component $\gamma_{A,B}^B(a,b)$ of ab is symmetric in a and b. We conclude that

$$0 = \delta^D(bb',a) + \delta^D(b'a,b) + \delta^D(ab,b').$$

Step 4**:** For $b, b', b'' \in A$ and $v, v', v'' \in V_s$, the D-component of $[[b \otimes v, b' \otimes v'], b'' \otimes v'']$ is

$$\kappa_{V_s}(\beta_V^V(v,v'),v'')\delta_B^D(\gamma_B^B(b,b'),b'').$$

We claim that $F(v,v',v'') := \kappa_{V_s}(\beta_V^V(v,v'),v'')$ satisfies

$$F(v,v',v'') = F(v',v'',v) \quad \text{for} \quad v,v',v'' \in V_s.$$

Fix $v', v'' \in V_s$. Then the map

$$V_s \to \mathbb{K}, \quad v \mapsto \kappa_{V_s}(\beta_V^V(v,v'),v'') = F(v,v',v'')$$

can be written as

$$V_s \to \mathbb{K}, \quad v \mapsto \kappa_{V_s}(T(v',v''),v)$$

for a unique element $T(v',v'') \in V_s$. From the $\mathfrak{g}_\Delta$-equivariance properties and the uniqueness, we derive that $T\colon V_s \times V_s \to V_s$ is $\mathfrak{g}_\Delta$-equivariant, hence of the form $\lambda\beta_V^V$ for some $\lambda \in \mathbb{K}$ (Theorem II.6). As F is symmetric or skew-symmetric in the first two arguments, F is an eigenvector for the action of S_3 on $\mathrm{Lin}(V \otimes V \otimes V, \mathbb{K})$. Then F is fixed by the commutator subgroup of S_3, hence fixed under cyclic rotations, and this implies $\lambda = 1$.

Therefore the Jacobi identity in $\mathfrak{g}$, applied to the D-components above, leads to

$$0 = \sum_{\text{cycl.}} \delta_B^D(\gamma_B^B(b,b'),b'') = \sum_{\text{cycl.}} \delta^D(bb',b'').$$

Combining all four cases, we see that δ^D satisfies the cocycle identity (2.2) because the function

$$G\colon \mathcal{A}^3 \to D, \quad (a,b,c) \mapsto \delta^D(ab,c) + \delta^D(bc,a) + \delta^D(ca,b)$$

is cyclically invariant and trilinear, so that it suffices to verify it in the four cases we dealt with above.

To verify the relation (2.3), we first use (B1) and (B3) to see that a comparison of the D-components of the brackets

$$[d,[a\otimes x,a'\otimes x']] = [d.a\otimes x,a'\otimes x'] + [a\otimes x,d.a'\otimes x'],\quad a,a'\in A, x,x'\in\mathfrak{g}_\Delta$$

and

$$[d,[b\otimes v,b'\otimes v']] = [d.b\otimes v,b'\otimes v'] + [b\otimes v,d.b'\otimes v'],\quad b,b'\in B, v,v'\in V_s$$

leads to (2.3). ■

Definition II.14. Let $\mathfrak{g}$ be a Δ-graded Lie algebra. From the isotypic decomposition of $\mathfrak{g}$ with respect to $\mathfrak{g}_\Delta$, we then obtain three items which, in view of (B1)–(B3), completely encode the structure of $\mathfrak{g}$:

(1) the coordinate algebra $\mathcal{A} = A\oplus B$,

(2) the Lie algebra D and its representation by derivations on $\mathcal{A}$ preserving the subspaces A and B, and

(3) the cocycle $\delta^D\colon \mathcal{A}\times\mathcal{A}\to D$ (Theorem II.13).

All other data that enters the description of the bracket in $\mathfrak{g}$ only depends on the Lie algebra $\mathfrak{g}_\Delta$ and the module V_s (Theorem II.6). We therefore call the triple $(\mathcal{A}, D, \delta^D)$ the *coordinate structure of the Δ-graded Lie algebra $\mathfrak{g}$.* ■

Theorem II.15. *Let $\mathfrak{g}$ be a root graded Lie algebra with coordinate structure $(\mathcal{A}, D, \delta^D)$. Further let $\widehat{D}$ be a locally convex Lie algebra acting by derivations preserving A and B on $\mathcal{A}$, and*

$$\delta^{\widehat{D}}\colon \mathcal{A}\times\mathcal{A}\to\widehat{D}$$

a continuous alternating bilinear map such that

(1) $\delta^{\widehat{D}}(aa',a'') + \delta^{\widehat{D}}(a'a'',a) + \delta^{\widehat{D}}(a''a,a') = 0$ *for* $a,a',a''\in\mathcal{A}$,

(2) *the map* $\widehat{D}\times\mathcal{A}\to\mathcal{A}, (d,a)\mapsto d.a$ *is continuous,*

(3) $[d,\delta^{\widehat{D}}(a,a')] = \delta^{\widehat{D}}(d.a,a') + \delta^{\widehat{D}}(a,d.a')$ *for* $a,a'\in\mathcal{A}$, $d\in\widehat{D}$, *and*

(4) $\delta^{\widehat{D}}(a,a').a'' = \delta^D(a,a').a''$ *for* $a,a',a''\in\mathcal{A}$, *and*

(5) $\delta^{\widehat{D}}(A\times B) = \{0\}$.

Then we obtain on

$$\widehat{\mathfrak{g}} := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widehat{D}$$

a Lie bracket by

$$[d, a \otimes x + b \otimes v + d'] = d.a \otimes x + d.b \otimes v + [d, d'],$$

and

$$\begin{aligned}
[a \otimes x, a' \otimes x'] &= \gamma_+^A(a, a') \otimes [x, x'] + \gamma_-^A(a, a') \otimes x * x' + \gamma_A^B(a, a') \otimes \beta_{\mathfrak{g}}^V(x, x') \\
&\quad + \kappa(x, x')\delta^{\widehat{D}}(a, a'), \\
[a \otimes x, b \otimes v] &= \frac{ab - ba}{2} \otimes \beta_{\mathfrak{g},V}^{\mathfrak{g}}(x, v) + \frac{ab + ba}{2} \otimes x.v, \\
[b \otimes v, b' \otimes v'] &= \gamma_B^A(b, b') \otimes \beta_V^{\mathfrak{g}}(v, v') + \gamma_B^B(b, b') \otimes \beta_V^V(v, v') + \kappa_{V_s}(v, v')\delta^{\widehat{D}}(b, b').
\end{aligned}$$

If $\mathrm{im}(\delta^{\widehat{D}})$ *is dense in* $\widehat{D}$*, then* $\widehat{\mathfrak{g}}$ *is a* Δ*-graded Lie algebra with coordinate structure* $(\mathcal{A}, \widehat{D}, \delta^{\widehat{D}})$.

Proof. From the definition and condition (3) it directly follows that the operators $\operatorname{ad} d$, $d \in \widehat{D}$, are derivations for the bracket. Therefore it remains to verify the Jacobi identity for triples of elements in $A \otimes \mathfrak{g}_\Delta$ or $B \otimes V_s$. In view of (4) and the fact that the Jacobi identity is satisfied in $\mathfrak{g}$, it suffices to consider the $\widehat{D}$-components of triple brackets. Reading the proof of Theorem II.13 backwards, it is easy to see that (1) and (4), applied to the four cases corresponding to how many among the a, a', a'' are contained in A, resp., B, lead to the Jacobi identity for triple brackets of elements in $A \otimes \mathfrak{g}_\Delta$, resp., $B \otimes V_s$.

For this argument one has to observe that in the case $a, a', a'' \in A$ the relation (1) for all a, a', a'' also implies

$$\begin{aligned}
&\delta^{\widehat{D}}(\gamma_+^A(a, a'), a'') + \delta^{\widehat{D}}(\gamma_+^A(a', a''), a) + \delta^{\widehat{D}}(\gamma_+^A(a'', a), a') \\
&\quad = \frac{1}{2}\Big(\delta^{\widehat{D}}(aa', a'') + \delta^{\widehat{D}}(a'a'', a) + \delta^{\widehat{D}}(a''a, a') + \delta^{\widehat{D}}(a'a, a'') \\
&\qquad + \delta^{\widehat{D}}(aa'', a') + \delta^{\widehat{D}}(a''a', a)\Big) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\delta^{\widehat{D}}(\gamma_-^A(a, a'), a'') + \delta^{\widehat{D}}(\gamma_-^A(a', a''), a) + \delta^{\widehat{D}}(\gamma_-^A(a'', a), a') \\
&\quad = \frac{1}{2}\Big(\delta^{\widehat{D}}(aa', a'') + \delta^{\widehat{D}}(a'a'', a) + \delta^{\widehat{D}}(a''a, a') - \delta^{\widehat{D}}(a'a, a'') \\
&\qquad - \delta^{\widehat{D}}(aa'', a') - \delta^{\widehat{D}}(a''a', a)\Big) = 0.
\end{aligned}$$

■

Examples II.16. We now take a second look at the examples in Section I.
(a) For the algebras of the type $\mathfrak{g} = A \otimes \mathfrak{g}_\Delta$ (Example I.4), it is clear that $\mathcal{A} = A$ is the corresponding coordinate algebra, and $B = D = \{0\}$.
(b) For $\mathfrak{g} = \mathfrak{sl}_n(A)$ (Example I.5), formula (1.1) for the bracket shows that $\mathcal{A} = A$ is the coordinate algebra of $\mathfrak{g}$, $D = \overline{[A, A]} \otimes \mathbf{1} \cong \overline{[A, A]}$, and

$$\delta^D(a, b) = \frac{1}{2n^2}[a, b]$$

because $\kappa(x, y) = 2n \operatorname{tr}(xy)$ for $x, y \in \mathfrak{sl}_n(\mathbb{K})$.
(c) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)'$ (Example I.7), which is of type C_n, we see with the formula in Example II.9(b) that $A = \mathcal{A}^\sigma$, $B = \mathcal{A}^{-\sigma}$, $D = \overline{[\mathcal{A}, \mathcal{A}]}^{-\sigma} \otimes \mathbf{1} \cong \overline{[\mathcal{A}, \mathcal{A}]}^{-\sigma}$, and that $\mathcal{A}$ is the coordinate algebra.

From $\kappa(x, y) = \theta \operatorname{tr}(xy)$, $\kappa_{V_s}(x, y) = \theta \operatorname{tr}(xy)$ $(\theta = 2(n + 1))$,

$$\kappa(x, x')\delta_A^D(a, a') = [a, a'] \otimes \frac{\operatorname{tr}(xx')}{2n}\mathbf{1}, \quad \text{and} \quad \kappa_{V_s}(v, v')\delta_B^D(b, b') = [b, b'] \otimes \frac{\operatorname{tr}(vv')}{2n}\mathbf{1},$$

we get

$$\delta^D(\alpha, \beta) = \frac{1}{2\theta n}\frac{1}{2}([\alpha, \beta] - [\alpha, \beta]^\sigma) \otimes \mathbf{1} = \frac{1}{4\theta n}([\alpha, \beta] + [\alpha^\sigma, \beta^\sigma]) \otimes \mathbf{1},$$

because

$$[a + b, a' + b'] = \underbrace{[a, a'] + [b, b']}_{\in \mathcal{A}^{-\sigma}} + \underbrace{[a, b'] + [b, a']}_{\in \mathcal{A}^{\sigma}}, \quad a \in \mathcal{A}^\sigma, b \in \mathcal{A}^{-\sigma}.$$

(d) For $\mathfrak{g} = \operatorname{TKK}(J)$ for a Jordan algebra J (Example I.9), we also see directly from the definition that J is the coordinate algebra of $\mathfrak{g}$ and $D = \langle J, J \rangle$. We have $\kappa(x, y) = 4 \operatorname{tr}(xy)$ for $x, y \in \mathfrak{sl}_2(\mathbb{K})$, and therefore

$$\delta^D(a, b) = \delta_J(a, b) = \frac{1}{4}\langle a, b \rangle. \qquad \blacksquare$$

The following proposition deals with the special case where B is trivial and the root system is not of type A_r. In this case it contains complete information on the possibilities of the coordinate algebra. For the root systems Δ of type D_r, $r \geq 4$, and E_r, it provides a full description of all Δ-graded Lie algebras (cf. [BM92] for the algebraic version of this result).

Proposition II.17. (a) *If $B = \{0\}$ and Δ is not of type A_r, $r \geq 1$, then the bracket of $\mathfrak{g}$ is of the form*

$$[a \otimes x, a' \otimes x'] = ab \otimes [x, x'] + \kappa(x, x')\delta^D(a, a'),$$

where A is a commutative associative unital algebra and D is central in $\mathfrak{g}$, i.e., D acts trivially on A.

(b) *If, conversely, $\widehat{D}$ is a locally convex space, A a locally convex unital commutative associative algebra and the continuous alternating bilinear map $\delta^{\widehat{D}}\colon A \times A \to \widehat{D}$ satisfies*

$$\delta^{\widehat{D}}(aa', a'') + \delta^{\widehat{D}}(a'a'', a) + \delta^{\widehat{D}}(a''a, a') = 0, \quad a, a', a'' \in A,$$

then

$$\widehat{\mathfrak{g}} := (A \otimes \mathfrak{g}_\Delta) \oplus \widehat{D}$$

is a Lie algebra with respect to the bracket

$$[a \otimes x + d, a' \otimes x' + d'] = aa' \otimes [x, x'] + \kappa(x, x')\delta^{\widehat{D}}(a, a').$$

Proof. (a) Our assumption that Δ is not of type A_1 means that $\dim \mathfrak{h} \geq 2$, so that there exist roots α and β with $\beta \neq \pm\alpha$. Moreover, the exclusion of A_r, $r \geq 2$, implies $\gamma^A_- = 0$, so that by consideration of the $A \otimes \mathfrak{g}_\Delta$-component of the cyclic sum $\sum_{\text{cycl.}}[[a \otimes x, a' \otimes x'], a'' \otimes x'']$, the Jacobi identity in $\mathfrak{g}$ implies

$$\sum_{\text{cycl.}} (aa')a'' \otimes [[x, x'], x''] + \delta^D(a, a').a'' \otimes \kappa(x, x')x'' = 0 \tag{2.6}$$

for $a, a', a'' \in A$ and $x, x', x'' \in \mathfrak{g}_\Delta$.

Let $x \in \mathfrak{g}_\alpha$, $x' \in \mathfrak{g}_\beta$, and $x'' \in \mathfrak{h}$. Then $\kappa(x, x') = \kappa(x', x'') = \kappa(x'', x) = 0$, and therefore

$$\begin{aligned}
&(aa')a'' \otimes [[x, x'], x''] + (a'a'')a \otimes [[x', x''], x] + (a''a)a' \otimes [[x'', x], x'] \\
&\quad = \; -(\alpha + \beta)(x'')(aa')a'' \otimes [x, x'] - \beta(x'')(a'a'')a \otimes [x', x] + \alpha(x'')(a''a)a' \otimes [x, x'] \\
&\quad = \; \Big(-(\alpha + \beta)(x'')(aa')a'' + \beta(x'')(a'a'')a + \alpha(x'')(a''a)a' \Big) \otimes [x, x'].
\end{aligned}$$

For $\beta(x'') = 0$ and $\alpha(x'') = 1$, we now get

$$(aa')a'' = (a'a'')a = a(a'a'').$$

Therefore the commutative algebra A is associative.

It remains to see that D is central. We consider the identity (2.6) with $x \in \mathfrak{g}_\alpha$, $x' \in \mathfrak{g}_{-\alpha}$ and $x'' = \check{\alpha}$. Then $\kappa(x, x') \neq 0 = \kappa(x, x'') = \kappa(x', x'')$. Further

$$\sum_{\text{cycl.}} (aa')a'' \otimes [[x, x'], x''] = (aa')a'' \otimes \sum_{\text{cycl.}} [[x, x'], x''] = 0$$

follows from the fact that A is commutative and associative, and the Jacobi identity in $\mathfrak{g}_\Delta$. Hence (2.6) leads to $\delta^D(a, a').a'' = 0$. This means that $\delta^D(A, A)$ is central in $\mathfrak{g}$, and since this set spans a dense subspace of D (Remark II.12(a)), the subalgebra D of $\mathfrak{g}$ is central.

(b) For the converse, we first observe that the map

$$\omega\colon (A\otimes\mathfrak{g}_\Delta)\times(A\otimes\mathfrak{g}_\Delta)\to\widehat{D},\quad \omega(a\otimes x,a'\otimes x')\to\kappa(x,x')\delta^{\widehat{D}}(a,a')$$

is a Lie algebra cocycle because

$$\begin{aligned}&\sum_{\text{cycl.}}\omega([a\otimes x,a'\otimes x'],a''\otimes x'')\\ &=\ \sum_{\text{cycl.}}\kappa([x,x'],x'')\delta^{\widehat{D}}(aa',a'')=\kappa([x,x'],x'')\sum_{\text{cycl.}}\delta^{\widehat{D}}(aa',a'')=0.\end{aligned}$$

From this the Jacobi identity of $\widehat{\mathfrak{g}}$ follows easily, and the map $\widehat{\mathfrak{g}}\to A\otimes\mathfrak{g}_\Delta$ with kernel $\widehat{D}$ defines a central extension of the Lie algebra $A\otimes\mathfrak{g}_\Delta$ by $\widehat{D}$ (cf. Example I.4). ∎

Definition II.18. (The Weyl group of $\mathfrak{g}$) To the simple split Lie algebra $\mathfrak{g}_\Delta$ we associate the subgroup $G_\Delta\subseteq\mathrm{Aut}(\mathfrak{g}_\Delta)$ generated by the automorphisms $e^{\mathrm{ad}\,x}$, $x\in\mathfrak{g}_{\Delta,\alpha}$, $\alpha\in\Delta$, which are defined because the operators $\mathrm{ad}\,x$ are nilpotent and the characteristic of $\mathbb{K}$ is zero. Since the set of $\mathfrak{h}$-weights of V_S is contained in the set of roots of $\mathfrak{g}_\Delta$, it follows from the theory of reductive algebraic groups that G_Δ also has a representation on V_S, compatible with the representation ρ_{V_S} of the Lie algebra $\mathfrak{g}_\Delta$ in the sense that $e^{\mathrm{ad}\,x}$, $x\in\mathfrak{g}_{\Delta,\alpha}$, acts by $e^{\rho_V(x)}$. This implies that G_Δ also acts in a natural way on the root graded Lie algebra $\mathfrak{g}$, and that it is isomorphic to the subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the automorphisms $e^{\mathrm{ad}\,x}$ of $\mathfrak{g}$. From now on we identify G_Δ with the corresponding subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Let $\alpha\in\Delta$ and fix $x_{\pm\alpha}\in\mathfrak{g}_{\pm\alpha}$ such that $[x_\alpha,x_{-\alpha}]=\check{\alpha}$. We consider the automorphism

$$\sigma_\alpha:=e^{\mathrm{ad}\,x_\alpha}e^{-\mathrm{ad}\,x_{-\alpha}}e^{\mathrm{ad}\,x_\alpha}\in G_\Delta\subseteq\mathrm{Aut}(\mathfrak{g}).$$

If $h\in\ker\alpha\subseteq\mathfrak{h}$, then h commutes with $x_{\pm\alpha}$, so that $\sigma_\alpha.h=h$. We claim that $\sigma_\alpha.\check{\alpha}=-\check{\alpha}$.

In $\mathrm{SL}_2(\mathbb{K})$ we have

$$S:=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

As $\sigma_\alpha|_{\mathfrak{g}_\Delta}$ corresponds to conjugation with S in $\mathfrak{sl}_2(\mathbb{K})$, we obtain

$$\sigma_\alpha.\check{\alpha}=-\check{\alpha},\quad \sigma_\alpha.x_\alpha=-x_{-\alpha}\quad\text{and}\quad \sigma_\alpha.x_{-\alpha}=-x_\alpha.$$

We conclude that $\sigma_\alpha|_{\mathfrak{h}}$ coincides with the reflection in the hyperplane $\check{\alpha}^\perp$:

$$\sigma_\alpha(h)=h-\alpha(h)\check{\alpha}\quad\text{for } h\in\mathfrak{h}$$

(cf. [MP95, Props. 4.1.3, 6.1.8]). The corresponding reflection on $\mathfrak{h}^*$ is given by

$$r_\alpha\colon\mathfrak{h}^*\to\mathfrak{h}^*,\quad \beta\mapsto\beta-\beta(\check{\alpha})\alpha.$$

This leads to

$$\sigma_\alpha(\mathfrak{g}_\beta) = \mathfrak{g}_{r_\alpha.\beta}, \quad \beta \in \Delta \cup \{0\}.$$

We call

$$\mathcal{W} := \langle r_\alpha \colon \alpha \in \Delta \rangle \subseteq \mathrm{GL}(\mathfrak{h})$$

the *Weyl group of* $\mathfrak{g}$.

From the preceding calculation we obtain in particular that

$$\sigma_\alpha \in N_{G_\Delta}(\mathfrak{h}) := \{\varphi \in G_\Delta \colon \varphi(\mathfrak{h}) = \mathfrak{h}\}.$$

This group contains the subgroup

$$Z_{G_\Delta}(\mathfrak{h}) = \{\varphi \in G_\Delta \colon \varphi|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\},$$

and each automorphism φ in this group is given by a group homomorphism

$$\chi \in \mathrm{Hom}(\mathbb{Z}[\Delta], \mathbb{K}^\times) \cong (\mathbb{K}^\times)^r$$

in the sense that $\varphi(x) = \chi(\alpha)x$ for all $\alpha \in \Delta$ and $x \in \mathfrak{g}_\alpha$. We therefore have a group extension

$$\Gamma \hookrightarrow \widehat{\mathcal{W}} \twoheadrightarrow \mathcal{W},$$

where $\widehat{\mathcal{W}} \subseteq N_{G_\Delta}(\mathfrak{h})$ is the inverse image of $\mathcal{W}$ under the restriction homomorphism to $\mathfrak{h}$ and $\Gamma \subseteq (\mathbb{K}^\times)^r$ is a subgroup. This extension does not split for $\Delta(\check{\Delta}) \not\subseteq 2\mathbb{Z}$ because in this case there exists a root α with $1 \in \Delta(\check{\alpha})$, which implies that σ_α is of order 4, as we see from the even-dimensional simple modules of $\mathrm{SL}_2(\mathbb{K})$. ■

Example II.19. (cf. [Ti62]) We take a closer look at the case $\Delta = A_1 = \{\pm\alpha\}$. We write

$$\mathfrak{g}_\Delta = \mathrm{span}\{\check{\alpha}, x_\alpha, x_{-\alpha}\}$$

with

$$x_\alpha \in \mathfrak{g}_\alpha, \quad x_{-\alpha} \in \mathfrak{g}_{-\alpha}, \quad \check{\alpha} = [x_\alpha, x_{-\alpha}].$$

Then formula (B1) for the product on A leads to

$$[a \otimes x_\alpha, [\mathbf{1} \otimes x_{-\alpha}, b \otimes x_\alpha]] = [a \otimes x_\alpha, -b \otimes \check{\alpha}] = ab \otimes [\check{\alpha}, x_\alpha] = 2ab \otimes x_\alpha,$$

and hence to

$$ab \otimes x_\alpha = \frac{1}{2}[a \otimes x_\alpha, [\mathbf{1} \otimes x_{-\alpha}, b \otimes x_\alpha]] = \frac{1}{2}[a \otimes x_\alpha, [x_{-\alpha}, b \otimes x_\alpha]].$$

Identifying A via the map $a \mapsto a \otimes x_\alpha$ with $\mathfrak{g}_\alpha$, the product on A is given by

$$ab := \frac{1}{2}[a, [x_{-\alpha}, b]].$$

We recall from Definition II.18 the automorphism σ_α of $\mathfrak{g}$. From the $\mathfrak{g}_\Delta$-module decomposition of $\mathfrak{g}$ it follows directly that $\sigma_\alpha^2 = \mathrm{id}_\mathfrak{g}$ because the restriction of σ_α to $\mathfrak{g}_\Delta$ is an involution. Moreover, $\sigma_\alpha(x_\alpha) = -x_{-\alpha}$. To see that the product on $\mathfrak{g}_\alpha$ defines a Jordan algebra structure on A, we first observe that Theorem C.3 (a) implies that

$$\{x, y, z\} := \frac{1}{2}[[x, \sigma_\alpha.y], z]$$

defines a Jordan triple structure on $\mathfrak{g}_\alpha$, and hence that $ab = \{a, -x_\alpha, b\}$ defines a Jordan algebra structure by Theorem C.4(b).

The quadratic operators of the Jordan triple structure are given by

$$P(x).y = \{x, y, x\} = -\frac{1}{2}(\mathrm{ad}\, x)^2 \circ \sigma_\alpha.y.$$

We claim that

$$P(-x_\alpha) = -\frac{1}{2}(\mathrm{ad}\, x_\alpha)^2 \circ \sigma_\alpha = -\,\mathrm{id}_{\mathfrak{g}_\alpha}.$$

Since the action of $\mathrm{ad}\, x_\alpha$ and σ_α is given by the $\mathfrak{g}_\Delta$-module structure of $\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus D$, the claim follows from

$$-\frac{1}{2}(\mathrm{ad}\, x_\alpha)^2 \circ \sigma_\alpha.x_\alpha = \frac{1}{2}(\mathrm{ad}\, x_\alpha)^2.x_{-\alpha} = \frac{1}{2}[x_\alpha, \check{\alpha}] = -x_\alpha.$$

We now conclude from Theorem C.4(b) that the Jordan triple structure associated to the Jordan algebra structure is given by $-\{\cdot,\cdot,\cdot\}$.

This permits us to determine δ_A. First we recall that

$$[a \otimes x_\alpha, a' \otimes x_{-\alpha}] = aa' \otimes \check{\alpha} + \delta^D(a, a')\kappa(x_\alpha, x_{-\alpha}) = aa' \otimes \check{\alpha} + 4\delta^D(a, a'),$$

which leads to

$$\begin{aligned} &2(aa')a'' \otimes x_\alpha + 4\delta_A(a, a').a'' \otimes x_\alpha \\ &= \Big[[a \otimes x_\alpha, a' \otimes x_{-\alpha}], a'' \otimes x_\alpha\Big] \\ &= -\Big[[a \otimes x_\alpha, \sigma_\alpha(a' \otimes x_\alpha)], a'' \otimes x_\alpha\Big] = -2\{a, a', a''\} \otimes x_\alpha \\ &= 2((aa')a'' + a(a'a'') - a'(aa'')) \otimes x_\alpha. \end{aligned}$$

From that we immediately get

$$\delta_A(a, a') = \frac{1}{2}[L_a, L_{a'}]. \qquad \blacksquare$$

The following theorem contains some refined information on the type of the coordinate algebras. We define

$$\delta_\mathcal{A}(\alpha, \beta).\gamma := \delta^D(\alpha, \beta).\gamma, \quad \alpha, \beta, \gamma \in \mathcal{A}.$$

Theorem II.20. (Coordinatization Theorem) *The coordinate algebra $\mathcal{A}$ of a Δ-graded Lie algebra $\mathfrak{g}$ is:*

(1) *a Jordan algebra for* Δ *of type* A_1*, and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{2}[L_\alpha, L_\beta].$$

(2) *an alternative algebra for* Δ *of type* A_2*, and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{3}(L_{[\alpha,\beta]} - R_{[\alpha,\beta]} - 3[L_\alpha, R_\beta]).$$

(3) *an associative algebra for* Δ *of type* A_r*,* $r \geq 3$*, and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{r+1}\,\mathrm{ad}[\alpha, \beta].$$

(4) *an associative commutative algebra for* Δ *of type* D_r*,* $r \geq 4$*, and* E_6, E_7 *and* E_8*, and* $\delta_{\mathcal{A}}(\alpha, \beta) = 0$*.*

(5) *an associative algebra* $(\mathcal{A}, \sigma)$ *with involution for* Δ *of type* C_r*,* $r \geq 4$*, and*

$$\delta_{\mathcal{A}}(\alpha, \beta) = \frac{1}{4r}(\mathrm{ad}[\alpha, \beta] + \mathrm{ad}[\alpha^\sigma, \beta^\sigma]).$$

(6) *a Jordan algebra associated to a symmetric bilinear form* $\beta\colon B \times B \to A$ *for* Δ *of type* B_r*,* $r \geq 3$*, and* $\delta_{\mathcal{A}}(\alpha, \beta) = -[L_\alpha, L_\beta]$*.*

Proof. (1) follows from the discussion in Example II.19 (see also [Ti62] and [BZ96]).
(2)–(4) [BM92]; see also Appendix B for some information on alternative algebras and Proposition II.17 for a proof of (4).
(5), (6) [BZ96] (cf. Lemma B.7 for Jordan algebras associated to symmetric bilinear forms and the discussion in Example I.9(d)). ■

The scalar factors in the formulas for $\delta_{\mathcal{A}}$ are due to the normalization of the invariant bilinear forms κ and κ_{V_s}.

For the details on the coordinate algebras for Δ of type C_3 (an alternative algebra with involution containing A in the associative center (the nucleus), i.e., left, resp., right multiplications with elements of A commute with all other right, resp., left multiplications), C_2 (a Peirce half space of a unital Jordan algebra containing a triangle), F_4 (an alternative algebra over A with normalized trace mapping satisfying the Cayley–Hamilton identity ch_2) and G_2 (a Jordan algebra over A with a normalized trace mapping satisfying the Cayley-Hamilton identity ch_3), we refer to [ABG00], [BZ96] and [Neh96]. For all these types of coordinate algebras one has natural derivations $\delta_{\mathcal{A}}(\alpha, \beta)$ given by explicit formulas.

III Universal covering Lie algebras and isogeny classes

In this section we discuss the concept of a generalized central extension of a locally convex Lie algebra. It generalizes central extensions $\widehat{\mathfrak{g}} \to \mathfrak{g}$, i.e., quotient maps with central kernel. Its main advantage is that it permits us to construct for a topologically perfect locally convex Lie algebra $\mathfrak{g}$ a universal generalized central extension $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$. This is remarkable because universal central extensions do not always exist, not even for topologically perfect Banach–Lie algebras.

III.1 Generalized central extensions

Definition III.1. Let $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ be locally convex Lie algebras. A continuous Lie algebra homomorphism $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with dense range is called a *generalized central extension* if there exists a continuous bilinear map $b \colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ with

(3.1) $$b(q(x), q(y)) = [x, y] \quad \text{for} \quad x, y \in \widehat{\mathfrak{g}}.$$

We observe that, since q has dense range, the map b is uniquely determined by (3.1) and that (3.1) implies that $\ker q$ is central in $\widehat{\mathfrak{g}}$. ■

Remark III.2. If $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a quotient homomorphism of locally convex Lie algebras with central kernel, i.e., a *central extension*, then $q \times q \colon \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{g}$ also is a quotient map. Therefore the Lie bracket of $\widehat{\mathfrak{g}}$ factors through a continuous bilinear map $b \colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $b(q(x), q(y)) = [x, y]$ for $x, y \in \widehat{\mathfrak{g}}$, showing that q is a generalized central extension of $\mathfrak{g}$. ■

Definition III.3. (a) Let $\mathfrak{z}$ be a locally convex space and $\mathfrak{g}$ a locally convex Lie algebra. A *continuous $\mathfrak{z}$-valued Lie algebra 2-cocycle* is a continuous skew-symmetric bilinear function $\omega \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ satisfying

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \quad x, y, z \in \mathfrak{g}.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$ with $\omega(x, y) = \alpha([x, y])$ for all $x, y \in \mathfrak{g}$. We write $Z^2(\mathfrak{g}, \mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2(\mathfrak{g}, \mathfrak{z})$ for the subspace of coboundaries. We define the *second continuous Lie algebra cohomology space* as

$$H^2(\mathfrak{g}, \mathfrak{z}) := Z^2(\mathfrak{g}, \mathfrak{z}) / B^2(\mathfrak{g}, \mathfrak{z}).$$

(b) If ω is a continuous $\mathfrak{z}$-valued 2-cocycle on $\mathfrak{g}$, then we write $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for the locally convex Lie algebra whose underlying locally convex space is the topological product $\mathfrak{g} \times \mathfrak{z}$, and whose bracket is defined by

$$[(x, z), (x', z')] = \Big([x, x'], \omega(x, x')\Big).$$

Then $q \colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}, (x, z) \mapsto x$ is a central extension and $\sigma \colon \mathfrak{g} \to \mathfrak{g} \oplus_\omega \mathfrak{z}, x \mapsto (x, 0)$ is a continuous linear section of q. ■

Lemma III.4. *For a generalized central extension $q\colon \widehat{\mathfrak g} \to \mathfrak g$ with the corresponding map b the following assertions hold:*

(1) $[x,y] = q(b(x,y))$ *for all* $x, y \in \mathfrak g$.

(2) $[\mathfrak g, \mathfrak g] \subseteq \mathrm{im}(q)$ *and* $\ker q \subseteq \mathfrak z(\widehat{\mathfrak g})$.

(3) $b \in Z^2(\mathfrak g, \widehat{\mathfrak g})$, *i.e.,* $b([x,y],z) + b([y,z],x) + b([z,x],y) = 0$ *for* $x, y, z \in \mathfrak g$.

(4) *For* $x \in \mathfrak g$ *we define*

$$\widehat{\mathrm{ad}}(x)\colon \widehat{\mathfrak g} \to \widehat{\mathfrak g}, \quad y \mapsto b(x, q(y)).$$

Then $\widehat{\mathrm{ad}}$ *defines a continuous representation of* $\mathfrak g$ *on* $\widehat{\mathfrak g}$ *by derivations for which* q *is equivariant with respect to the adjoint representation of* $\mathfrak g$ *on* $\mathfrak g$.

(5) *If* $\widehat{\mathfrak g}$ *is topologically perfect, then* $q^{-1}(\mathfrak z(\mathfrak g)) = \mathfrak z(\widehat{\mathfrak g})$.

Proof. (1) If $x = q(a)$ and $y = q(b)$ holds for $a, b \in \widehat{\mathfrak g}$, then

$$[x,y] = [q(a), q(b)] = q([a,b]) = q(b(x,y)).$$

Therefore the Lie bracket on $\mathfrak g$ coincides on the dense subset $\mathrm{im}(q) \times \mathrm{im}(q)$ of $\mathfrak g \times \mathfrak g$ with the continuous map $q \circ b$, so that (1) follows from the continuity of both maps.
(2) follows from (1).
(3) In view of (3.1), the Jacobi identity in $\widehat{\mathfrak g}$ leads to

$$\begin{aligned} 0 &= [[x,y],z] + [[y,z],x] + [[z,x],y] \\ &= b(q([x,y]), q(z)) + b(q([y,z]), q(x)) + b(q([z,x]), q(y)) \\ &= b([q(x),q(y)], q(z)) + b([q(y),q(z)], q(x)) + b([q(z),q(x)], q(y)). \end{aligned}$$

Therefore the restriction of b to $\mathrm{im}(q)$ is a Lie algebra cocycle, and since $\mathrm{im}(q)$ is dense and b is continuous, it is a Lie algebra cocycle on $\mathfrak g$.
(4) First we observe that the bilinear map $\mathfrak g \times \widehat{\mathfrak g} \to \widehat{\mathfrak g}, (x,y) \mapsto b(x, q(y))$ is continuous. Moreover, (1) implies

$$q(\widehat{\mathrm{ad}}(x).y) = q(b(x, q(y))) = [x, q(y)],$$

i.e., $q \circ \widehat{\mathrm{ad}}(x) = \mathrm{ad}\, x \circ q$.

From the cocycle identity

$$b([x,y],z) + b([y,z],x) + b([z,x],y) = 0, \quad x, y, z \in \mathfrak g,$$

we derive in particular for $x \in \mathfrak{g}$ and $y, z \in \widehat{\mathfrak{g}}$:

$$\begin{aligned} 0 &= b([x, q(y)], q(z)) + b([q(y), q(z)], x) + b([q(z), x], q(y)) \\ &= b(q(\widehat{\mathrm{ad}}(x)y), q(z)) + b(q([y, z]), x) - b(q(\widehat{\mathrm{ad}}(x).z), q(y)) \\ &= [\widehat{\mathrm{ad}}(x)y, z] - \widehat{\mathrm{ad}}(x)[y, z] - [\widehat{\mathrm{ad}}(x)z, y]. \end{aligned}$$

Therefore each $\widehat{\mathrm{ad}}(x)$ is a derivation of $\widehat{\mathfrak{g}}$. On the other hand, the cocycle identity for b leads for $x, y \in \mathfrak{g}$ and $z \in \widehat{\mathfrak{g}}$ to

$$\begin{aligned} 0 &= b([x, y], q(z)) + b([y, q(z)], x) + b([q(z), x], y) \\ &= \widehat{\mathrm{ad}}([x, y])z + b(q(\widehat{\mathrm{ad}}(y)z), x) - b(q(\widehat{\mathrm{ad}}(x)z), y) \\ &= \widehat{\mathrm{ad}}([x, y])z - \widehat{\mathrm{ad}}(x)\widehat{\mathrm{ad}}(y)z + \widehat{\mathrm{ad}}(y)\widehat{\mathrm{ad}}(x)z, \end{aligned}$$

so that $\widehat{\mathrm{ad}}\colon \mathfrak{g} \to \mathrm{der}(\widehat{\mathfrak{g}})$ is a representation of $\mathfrak{g}$ by derivations of $\widehat{\mathfrak{g}}$, and the map q is equivariant with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.
(5) Let $\widehat{\mathfrak{z}}(\mathfrak{g}) := q^{-1}(\mathfrak{z}(\mathfrak{g}))$. We first observe that $[\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]$ is contained in $\ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$ because

$$q([\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]) \subseteq [\mathfrak{z}(\mathfrak{g}), \mathfrak{g}] = \{0\}.$$

This leads to

$$[\widehat{\mathfrak{z}}(\mathfrak{g}), [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]] \subseteq [\widehat{\mathfrak{g}}, [\widehat{\mathfrak{z}}(\mathfrak{g}), \widehat{\mathfrak{g}}]] \subseteq [\widehat{\mathfrak{g}}, \ker q] = \{0\}.$$

If $\widehat{\mathfrak{g}}$ is topologically perfect, we obtain $\widehat{\mathfrak{z}}(\mathfrak{g}) \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$. The other inclusion follows from the density of the image of q. ■

The following proposition shows that generalized central extensions can be characterized as certain closed subalgebras of central extensions defined by cocycles.

Proposition III.5. (a) *If $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a generalized central extension and $b\colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ the corresponding cocycle, then the map*

$$\psi\colon \widehat{\mathfrak{g}} \to \mathfrak{g} \oplus_b \widehat{\mathfrak{g}}, \quad x \mapsto (q(x), x)$$

is a topological embedding of $\widehat{\mathfrak{g}}$ onto a closed ideal of $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$ containing the commutator algebra.

If $|\mathfrak{g}|$ denotes the space $\mathfrak{g}$ considered as an abelian Lie algebra, then the map

$$\eta\colon \mathfrak{g} \oplus_b \widehat{\mathfrak{g}} \to |\mathfrak{g}|, \quad (x, y) \mapsto x - q(y)$$

is a quotient morphism of Lie algebras whose kernel is $\mathrm{im}(\psi) \cong \widehat{\mathfrak{g}}$.
(b) *If $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$ is a continuous 2-cocycle, $p\colon \mathfrak{g} \oplus_\omega \mathfrak{z} \to \mathfrak{g}$ the projection onto $\mathfrak{g}$ of the corresponding central extension, and $\widehat{\mathfrak{g}} \subseteq \mathfrak{g} \oplus_\omega \mathfrak{z}$ is a closed subalgebra for which $p(\widehat{\mathfrak{g}})$ is dense in $\mathfrak{g}$, then $q := p|_{\widehat{\mathfrak{g}}} : \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a generalized central extension with $b(x, y) = ([x, y], \omega(x, y))$ for $x, y \in \mathfrak{g}$.*

Proof. (a) We recall from Definition III.3 that the bracket in $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$ is given by

$$[(x,y),(x',y')] = ([x,x'], b(x,x')).$$

Now

$$\begin{aligned}[\psi(x),\psi(x')] &= [(q(x),x),(q(x'),x')] = ([q(x),q(x')], b(q(x),q(x'))) \\ &= (q([x,x']),[x,x']) = \psi([x,x'])\end{aligned}$$

implies that the continuous linear map ψ is a morphism of Lie algebras. As the graph of the continuous linear map q, the image of ψ is a closed subspace of $\mathfrak{g} \oplus_b \widehat{\mathfrak{g}}$, and the projection onto the second factor is a continuous linear map. Therefore ψ is a topological embedding onto a closed subalgebra.

Moreover, the formula for the bracket, together with $q(b(x,x')) = [x,x']$ shows that $\mathrm{im}(\psi)$ contains all brackets, hence is an ideal. Therefore the map $\eta\colon \mathfrak{g} \oplus_b \widehat{\mathfrak{g}} \to |\mathfrak{g}|$ whose kernel is $\mathrm{im}(\psi)$ is a morphism of Lie algebras. That it is a quotient map follows from the fact that its restriction to the subspace $\mathfrak{g}$ is a topological isomorphism.

(b) The range of q is dense by the assumption that $p(\widehat{\mathfrak{g}})$ is dense in $\mathfrak{g}$. It is also clear that $b \circ (p \times p)$ is the bracket on $\mathfrak{g} \oplus_\omega \mathfrak{z}$, but it remains to show that $\mathrm{im}(b) \subseteq \widehat{\mathfrak{g}}$.

For $x = q(x'), y = q(y')$ in $\mathrm{im}(q) = p(\widehat{\mathfrak{g}})$ we have

$$b(x,y) = b(q(x'),q(y')) = [x',y'] = ([x,y],\omega(x,y)) \in \widehat{\mathfrak{g}}.$$

Now the continuity of b, the density of $\mathrm{im}(q)$ in $\mathfrak{g}$, and the closedness of $\widehat{\mathfrak{g}}$ imply that $\mathrm{im}(b) \subseteq \widehat{\mathfrak{g}}$. ■

III.2 Full cyclic homology of locally convex algebras

In this subsection we define cyclic 1-cocycles for locally convex algebras $\mathcal{A}$ which are not necessarily associative. This includes in particular Lie algebras, where cyclic 1-cocycles are Lie algebra 2-cocycles. It also covers the more general coordinate algebras of root graded locally convex Lie algebras (see Section IV). In particular, we associate to $\mathcal{A}$ a locally convex space $\langle \mathcal{A}, \mathcal{A} \rangle$ in such a way that continuous cyclic 1-cocycles are in one-to-one correspondence to linear maps on $\langle \mathcal{A}, \mathcal{A} \rangle$. Moreover, we will discuss a method to obtain Lie algebra structures on $\langle \mathcal{A}, \mathcal{A} \rangle$, which will be crucial in Section IV for the construction of the universal covering algebra of a root graded Lie algebra.

Definition III.6. (a) Let $\mathcal{A}$ be a locally convex algebra (not necessarily associative or with unit). We endow the tensor product $\mathcal{A} \otimes \mathcal{A}$ with the projective tensor product topology and denote this space by $\mathcal{A} \otimes_\pi \mathcal{A}$. Let

$$I := \overline{\mathrm{span}\{a \otimes a, ab \otimes c + bc \otimes a + ca \otimes b : a,b,c \in \mathcal{A}\}} \subseteq \mathcal{A} \otimes_\pi \mathcal{A}.$$

We define

$$\langle \mathcal{A}, \mathcal{A} \rangle := (\mathcal{A} \otimes_\pi \mathcal{A})/I,$$

endowed with the quotient topology, which turns it into a locally convex space. We write $\langle a, b\rangle$ for the image of $a \otimes b$ in the quotient space $\langle \mathcal{A}, \mathcal{A}\rangle$.
(b) Our definition of $\langle \mathcal{A}, \mathcal{A}\rangle$ in (a) is the one corresponding to the category of locally convex spaces, resp., algebras. In the category of complete locally convex spaces we write $\langle \mathcal{A}, \mathcal{A}\rangle$ for the completion of the quotient space $(\mathcal{A} \otimes_\pi \mathcal{A})/I$, and in the category of sequentially complete spaces for the smallest sequentially closed subspace of the completion, i.e., its sequential completion.

In the category of Fréchet spaces, the completed version of $\langle \mathcal{A}, \mathcal{A}\rangle$ can be obtained more directly by first replacing $\mathcal{A} \otimes_\pi \mathcal{A}$ by its completion $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}$. If $\overline{I}$ denotes the closure of I in the completion $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}$, then the quotient space $\mathcal{A}\widehat{\otimes}_\pi\mathcal{A}/\overline{I}$ is automatically complete, hence a Fréchet space.
(c) For a locally convex space $\mathfrak{z}$ the continuous linear maps $\langle \mathcal{A}, \mathcal{A}\rangle \to \mathfrak{z}$ correspond to those alternating continuous bilinear maps $\omega\colon \mathcal{A} \times \mathcal{A} \to \mathfrak{z}$ satisfying

$$\omega(ab, c) + \omega(bc, a) + \omega(bc, a) = 0, \quad a, b, c \in \mathcal{A}.$$

These maps are called *cyclic* 1*-cocycles*. We write $Z^1(\mathcal{A}, \mathfrak{z})$ for the space of continuous cyclic 1-cocycles $\mathcal{A} \times \mathcal{A} \to \mathfrak{z}$ and note that

$$Z^1(\mathcal{A}, \mathfrak{z}) \cong \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A}\rangle, \mathfrak{z}).$$

The identity $\mathrm{id}_{\langle \mathcal{A}, \mathcal{A}\rangle}$ corresponds to the *universal cocycle*

$$\omega_u\colon \mathcal{A} \times \mathcal{A} \to \langle \mathcal{A}, \mathcal{A}\rangle, \quad (a, b) \mapsto \langle a, b\rangle. \qquad \blacksquare$$

Remark III.7. Lie algebra 2-cocycles $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ (Definition III.3) are the same as cyclic 1-cocycles of the algebra $\mathfrak{g}$.

In particular we have

$$Z^2(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\langle \mathfrak{g}, \mathfrak{g}\rangle, \mathfrak{z})$$

for any locally convex space $\mathfrak{z}$. ■

Remark III.8. Let $\mathcal{A}$ be a locally convex associative algebra, $\mathcal{A}_L$ the corresponding Lie algebra with the commutator bracket $[a, b] = ab - ba$, and $\mathcal{A}_J$ the corresponding Jordan algebra with the product $a \circ b := \frac{1}{2}(ab + ba)$. In $\mathcal{A} \otimes \mathcal{A}$ we have the relations

$$[a, b] \otimes c + [b, c] \otimes a + [c, a] \otimes b = ab \otimes c + bc \otimes a + ca \otimes b - (ba \otimes c + cb \otimes a + ac \otimes b)$$

and

$$2(a \circ b \otimes c + b \circ c \otimes a + c \circ a \otimes b) = ab \otimes c + bc \otimes a + ca \otimes b + ba \otimes c + cb \otimes a + ac \otimes b.$$

Therefore we have natural continuous linear maps

$$\langle \mathcal{A}_L, \mathcal{A}_L\rangle \to \langle \mathcal{A}, \mathcal{A}\rangle, \ \langle a, b\rangle \mapsto \langle a, b\rangle \quad \text{and} \quad \langle \mathcal{A}_J, \mathcal{A}_J\rangle \to \langle \mathcal{A}, \mathcal{A}\rangle, \ \langle a, b\rangle \mapsto \langle a, b\rangle.$$

■

A remarkable point of the following proposition is that it applies without any assumption on the algebra $\mathcal{A}$, such as associativity etc.

Proposition III.9. *Let $\mathcal{A}$ be a locally convex algebra and*

$$\delta\colon \langle \mathcal{A}, \mathcal{A}\rangle \to \mathrm{der}(\mathcal{A}), \quad \langle a, b\rangle \mapsto \delta(a, b)$$

be a cyclic 1*-cocycle for which the map $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}, (a, b, c) \mapsto \delta(a, b).c$ is continuous. As* der($\mathcal{A}$) *acts naturally on $\langle \mathcal{A}, \mathcal{A}\rangle$ by*

$$d.\langle a, b\rangle = \langle d.a, b\rangle + \langle a, d.b\rangle, \quad d \in \mathrm{der}(\mathcal{A}), a, b \in \mathcal{A},$$

we obtain a well-defined continuous bilinear map

$$\begin{aligned}[\cdot,\cdot]\colon \langle \mathcal{A}, \mathcal{A}\rangle \times \langle \mathcal{A}, \mathcal{A}\rangle \to \langle \mathcal{A}, \mathcal{A}\rangle, \quad [\langle a, b\rangle, \langle c, d\rangle] &= \delta(a, b).\langle c, d\rangle \\ &= \langle \delta(a, b).c, d\rangle + \langle c, \delta(a, b).d\rangle.\end{aligned}$$

Suppose that

(1) $\delta(\delta(a, b).\langle c, d\rangle) = [\delta(a, b), \delta(c, d)]$*, and*

(2) $\delta(a, b).\langle c, d\rangle = -\delta(c, d).\langle a, b\rangle$ *for* $a, b, c, d \in \mathcal{A}$.

Then $[\cdot,\cdot]$ defines on $\langle \mathcal{A}, \mathcal{A}\rangle$ the structure of a locally convex Lie algebra and δ is a homomorphism of Lie algebras.

Proof. According to our continuity assumption on δ, the quadrilinear map

$$\mathcal{A}\times\mathcal{A}\times\mathcal{A}\times\mathcal{A} \to \langle \mathcal{A}, \mathcal{A}\rangle, \quad (a, b, c, d) \mapsto \delta(a, b).\langle c, d\rangle = \langle \delta(a, b).c, d\rangle + \langle c, \delta(a, b).d\rangle$$

is continuous. That δ is a cyclic cocycle implies that it factors through a continuous bilinear map

$$[\cdot,\cdot]\colon \langle \mathcal{A}, \mathcal{A}\rangle \times \langle \mathcal{A}, \mathcal{A}\rangle \to \langle \mathcal{A}, \mathcal{A}\rangle, \quad (\langle a, b\rangle, \langle c, d\rangle) \mapsto \delta(a, b).\langle c, d\rangle.$$

Condition (2) means that the bracket on $\langle \mathcal{A}, \mathcal{A}\rangle$ is alternating. In view of (1), the Jacobi identity follows from

$$\begin{aligned}[[\langle a, b\rangle, \langle c, d\rangle], \langle u, v\rangle] &= \delta(\delta(a, b).\langle c, d\rangle).\langle u, v\rangle = [\delta(a, b), \delta(c, d)].\langle u, v\rangle \\ &= [\langle a, b\rangle, [\langle c, d\rangle, \langle u, v\rangle]] - [\langle c, d\rangle, [\langle a, b\rangle, \langle u, v\rangle]].\end{aligned}$$

Finally, we observe that (1) means that δ is a homomorphism of Lie algebras. ■

Example III.10. Typical examples where Proposition III.9 applies are
(1) Lie algebras: If $\mathfrak{g}$ is a locally convex Lie algebra and $\delta(x,y) = \mathrm{ad}[x,y]$, then the Jacobi identity implies that δ is a cocycle. That δ is equivariant with respect to the action of $\mathrm{der}(\mathfrak{g})$ follows for $d \in \mathrm{der}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$ from

$$\delta(d.x,y)+\delta(x,d.y) = \mathrm{ad}([d.x,y]+[x,d.y]) = \mathrm{ad}(d.[x,y]) = [d, \mathrm{ad}[x,y]] = [d, \delta(x,y)].$$

We also have in $\langle \mathfrak{g}, \mathfrak{g} \rangle$ the relation:

$$\begin{aligned} \delta(x,y).\langle x', y' \rangle &= \langle [[x,y],x'], y' \rangle + \langle x', [[x,y],y'] \rangle \\ &= -\langle [x',y'], [x,y] \rangle - \langle [y',[x,y]], x' \rangle + \langle x', [[x,y],y'] \rangle \\ &= \langle [x,y], [x',y'] \rangle, \end{aligned}$$

which implies $\delta(x,y).\langle x', y' \rangle = -\delta(x',y').\langle x, y \rangle$. Moreover, the bracket map

$$b_{\mathfrak{g}} \colon \langle \mathfrak{g}, \mathfrak{g} \rangle \to \mathfrak{g}, \quad \langle x, y \rangle \mapsto [x,y]$$

is a homomorphism of Lie algebras because

$$b_{\mathfrak{g}}([\langle x,y \rangle, \langle x', y' \rangle]) = [[x,y],[x',y']] = [b_{\mathfrak{g}}(\langle x,y \rangle), b_{\mathfrak{g}}(\langle x', y' \rangle)].$$

(2) Associative algebras: If $\mathcal{A}$ is an associative algebra, then the commutator bracket

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (a,b) \mapsto [a,b] = ab - ba$$

is a cyclic cocycle because

$$[ab,c] + [bc,a] + [ca,b] = abc - cab + bca - abc + cab - bca = 0.$$

Therefore $\delta(x,y) = \mathrm{ad}[x,y]$ defines a cocycle $\mathcal{A} \times \mathcal{A} \to \mathrm{der}(\mathcal{A})$. That δ is equivariant with respect to the action of $\mathrm{der}(\mathcal{A})$ follows with the same calculations as in (1) above. Alternatively, we can observe that if $\mathcal{A}_L$ denotes the Lie algebra $\mathcal{A}$ with the commutator bracket, then $\langle \mathcal{A}, \mathcal{A} \rangle$ is a quotient of $\langle \mathcal{A}_L, \mathcal{A}_L \rangle$ (Remark III.8).
(3) If $\mathcal{A}$ is a Jordan algebra and $\delta_{\mathcal{A}}(a,b) = [L(a), L(b)]$, then we have

$$\delta_{\mathcal{A}}(d.\langle a, b \rangle) = [d, \delta_{\mathcal{A}}(a,b)]$$

for all derivations $d \in \mathrm{der}(\mathcal{A})$, hence (1) in Proposition III.9. To verify (2), we calculate

$$\begin{aligned}
\delta_{\mathcal{A}}(a,a').\langle b,b'\rangle &= \langle \delta_{\mathcal{A}}(a,a').b,b'\rangle + \langle b,\delta_{\mathcal{A}}(a,a').b'\rangle \\
&= \langle a(a'b)-a'(ab),b'\rangle + \langle b,a(a'b')-a'(ab')\rangle \\
&= \langle a(a'b),b'\rangle - \langle a'(ab),b'\rangle + \langle b,a(a'b')\rangle - \langle b,a'(ab')\rangle \\
&= -\langle (a'b)b',a\rangle - \langle b'a,a'b\rangle + \langle (ab)b',a'\rangle + \langle b'a',ab\rangle \\
&\quad -\langle a,(a'b')b\rangle - \langle a'b',ba\rangle + \langle a',(ab')b\rangle + \langle ab',ba'\rangle \\
&= -\langle b'(ba'),a\rangle - \langle b'a,a'b\rangle + \langle b'(ba),a'\rangle + \langle b'a',ab\rangle \\
&\quad -\langle a,b(b'a')\rangle - \langle b'a',ab\rangle + \langle a',b(b'a)\rangle + \langle b'a,a'b\rangle \\
&= -\langle b'(ba'),a\rangle + \langle b'(ba),a'\rangle - \langle a,b(b'a')\rangle + \langle a',b(b'a)\rangle \\
&= \langle \delta_{\mathcal{A}}(b',b).a,a'\rangle + \langle a,\delta_{\mathcal{A}}(b',b).a'\rangle \\
&= -\delta_{\mathcal{A}}(b,b').\langle a,a'\rangle.
\end{aligned}$$

■

III.3 The universal covering of a locally convex Lie algebra

We call a generalized central extension $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ of a locally convex Lie algebra $\mathfrak{g}$ *universal* if for any generalized central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ there exists a unique morphism of locally convex Lie algebras $\alpha\colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ with $q \circ \alpha = q_{\mathfrak{g}}$.

Theorem III.11. *A locally convex Lie algebra $\mathfrak{g}$ has a universal generalized central extension if and only if it is topologically perfect. If this is the case, then the universal generalized central extension is given by the natural Lie algebra structure on $\widetilde{\mathfrak{g}} := \langle \mathfrak{g}, \mathfrak{g}\rangle$ satisfying*

(3.2) $$[\langle x,x'\rangle, \langle y,y'\rangle] = \langle [x,x'],[y,y']\rangle \qquad \textit{for} \quad x,x',y,y' \in \mathfrak{g},$$

and the natural homomorphism

$$q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}, \quad \langle x,y\rangle \mapsto [x,y]$$

is given by the Lie bracket on $\mathfrak{g}$.

Proof. Suppose first that $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a universal generalized central extension. We consider the trivial central extension $\widehat{\mathfrak{g}} := \mathfrak{g} \times \mathbb{K}$ with $q(x,t) = x$. According to the universal property, there exists a unique morphism of locally convex Lie algebras $\alpha\colon \widetilde{\mathfrak{g}} \to \mathfrak{g} \times \mathbb{K}$ with $q \circ \alpha = q_{\mathfrak{g}}$. For each Lie algebra homomorphism $\beta\colon \widetilde{\mathfrak{g}} \to \mathbb{K}$ the sum $\alpha + \beta\colon \widetilde{\mathfrak{g}} \to \mathfrak{g} \times \mathbb{K}$ also is a homomorphism of Lie algebras with $q \circ (\alpha + \beta) = q_{\mathfrak{g}}$. Hence the uniqueness implies that $\beta = 0$. That all morphisms $\widetilde{\mathfrak{g}} \to \mathbb{K}$ are trivial means that $\widetilde{\mathfrak{g}}$ is topologically perfect, and therefore $\mathfrak{g}$ is topologically perfect.

Conversely, we assume that $\mathfrak{g}$ is topologically perfect and construct a universal generalized central extension. Using Proposition III.9 and Example III.10(1), we see that $\langle \mathfrak{g}, \mathfrak{g} \rangle$ carries a locally convex Lie algebra structure with

$$[\langle x, y \rangle, \langle z, u \rangle] = \langle [x, y], [z, u] \rangle, \quad x, y, z, u \in \mathfrak{g}.$$

Next we observe that $\mathrm{im}(q_{\mathfrak{g}})$ is dense because $[\mathfrak{g}, \mathfrak{g}]$ is dense in $\mathfrak{g}$. The corresponding bracket map on $\widetilde{\mathfrak{g}}$ is given by the universal cocycle

$$\omega_u \colon \mathfrak{g} \times \mathfrak{g} \to \widetilde{\mathfrak{g}}, \quad (x, y) \mapsto \langle x, y \rangle.$$

In fact, for $x, x', y, y' \in \mathfrak{g}$ we have

$$\omega_u(q_{\mathfrak{g}}(\langle x, x' \rangle), q_{\mathfrak{g}}(\langle y, y' \rangle)) = \omega_u([x, x'], [y, y']) = \langle [x, x'], [y, y'] \rangle = [\langle x, x' \rangle, \langle y, y' \rangle].$$

Since the elements of the form $\langle x, x' \rangle$ span a dense subspace of $\widetilde{\mathfrak{g}}$, equation (3.1) holds for $q = q_{\mathfrak{g}}$.

Now let $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ be another generalized central extension with the corresponding map $b \colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$. Then Lemma III.4(3) and Remark III.7 imply the existence of a unique continuous linear map $\alpha \colon \widetilde{\mathfrak{g}} = \langle \mathfrak{g}, \mathfrak{g} \rangle \to \widehat{\mathfrak{g}}$ with

$$b(x, y) = \alpha(\langle x, y \rangle), \quad x, y \in \mathfrak{g}.$$

For $x = q(a)$, $x' = q(a')$, $y = q(b)$ and $y' = q(b')$ we then have

$$\begin{aligned} \alpha([\langle x, x' \rangle, \langle y, y' \rangle]) &= \alpha(\langle [x, x'], [y, y'] \rangle) = b([x, x'], [y, y']) = b(q([a, a']), q([b, b'])) \\ &= [[a, a'], [b, b']] = [b(x, x'), b(y, y')] = [\alpha(\langle x, x' \rangle), \alpha(\langle y, y' \rangle)]. \end{aligned}$$

Now the fact that $\mathrm{im}(q)$ is dense in $\mathfrak{g}$ implies that α is a homomorphism of Lie algebras. Further,

$$q(\alpha(\langle x, y \rangle)) = q(b(x, y)) = [x, y] = q_{\mathfrak{g}}(\langle x, y \rangle),$$

again with the density of $\mathrm{im}(q)$ in $\mathfrak{g}$, leads to $q \circ \alpha = q_{\mathfrak{g}}$.

To see that α is unique, we first observe that $\widetilde{\mathfrak{g}}$ is topologically perfect because $\mathfrak{g}$ is topologically perfect. If $\beta \colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ is another homomorphism with $q \circ \beta = q_{\mathfrak{g}}$, then $\gamma := \beta - \alpha$ is a continuous linear map $\widetilde{\mathfrak{g}} \to \ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$. Moreover, for $x, y \in \langle \mathfrak{g}, \mathfrak{g} \rangle = \widetilde{\mathfrak{g}}$,

$$\begin{aligned} \gamma([x, y]) &= \beta([x, y]) - \alpha([x, y]) = [\beta(x), \beta(y)] - [\alpha(x), \alpha(y)] \\ &= [\beta(x) - \alpha(x), \beta(y)] + [\alpha(x), \beta(y)] - [\alpha(x), \alpha(y)] \\ &= [\gamma(x), \beta(y)] + [\alpha(x), \gamma(y)] = 0 \end{aligned}$$

because the values of γ are central. Now $\gamma = 0$ follows from the topological perfectness of $\widetilde{\mathfrak{g}}$. ■

Definition III.12. For a topologically perfect locally convex Lie algebra $\mathfrak{g}$ the Lie algebra $\widetilde{\mathfrak{g}} = \langle \mathfrak{g}, \mathfrak{g} \rangle$ is called the *universal generalized central extension of* $\mathfrak{g}$ or the *(topological) universal covering Lie algebra of* $\mathfrak{g}$.

We call two topologically perfect Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ *centrally isogenous* if $\widetilde{\mathfrak{g}}_1 \cong \widetilde{\mathfrak{g}}_2$.

In the category of sequentially complete, resp., complete locally convex Lie algebras we define $\widetilde{\mathfrak{g}}$ as $\langle \mathfrak{g}, \mathfrak{g} \rangle$ in the sense of Definition III.6(b). Then the same arguments as in the proof of Theorem III.11 show that $\widetilde{\mathfrak{g}}$ is a universal generalized central extension in the corresponding category. ■

We call a central extension $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ of a locally convex Lie algebra $\mathfrak{g}$ *universal* if for any central extension $q' \colon \widehat{\mathfrak{g}}' \to \mathfrak{g}$ there exists a unique morphism of locally convex Lie algebras $\alpha \colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}'$ with $q' \circ \alpha = q$. The following corollary clarifies the relation between universal central extensions and universal generalized central extensions. In particular it implies that the existence of a universal central extension is a quite rare phenomenon.

Corollary III.13. *A locally convex Lie algebra $\mathfrak{g}$ has a universal central extension if and only if it is topologically perfect and the universal covering map $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a quotient map. Then $q_{\mathfrak{g}}$ is a universal central extension.*

Proof. Suppose first that $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a universal central extension. Then the same argument as in the proof of Theorem III.11 implies that $\widehat{\mathfrak{g}}$ is topologically perfect, which implies that $\mathfrak{g}$ is topologically perfect. Therefore the universal generalized central extension $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ exists by Theorem III.11. Its universal property implies the existence of a unique morphism $\widetilde{q} \colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ with $q \circ \widetilde{q} = q_{\mathfrak{g}}$. If $\widehat{b} \colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ is the unique continuous bilinear map for which $\widehat{b} \circ (q \times q)$ is the bracket on $\widehat{\mathfrak{g}}$, the construction in the proof of Theorem III.11 implies that

$$\widetilde{q} \circ \omega_u = \widehat{b}$$

for the universal cocycle $\omega_u(x, y) = \langle x, y \rangle$.

Now let $q_u \colon \mathfrak{g} \oplus_{\omega_u} \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the central extension of $\mathfrak{g}$ by $\widetilde{\mathfrak{g}}$, considered as an abelian Lie algebra, defined by the universal cocycle. Then the universal property of $\widehat{\mathfrak{g}}$ implies the existence of a unique morphism

$$\psi \colon \widehat{\mathfrak{g}} \to \mathfrak{g} \oplus_{\omega_u} \widetilde{\mathfrak{g}}$$

with $q_u \circ \psi = q$. This means that $\psi(x) = (q(x), \alpha(x))$, where $\alpha \colon \widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ is a continuous linear map. That ψ is a Lie algebra homomorphism means that

$$(q([x, y]), \alpha([x, y])) = \psi([x, y]) = [\psi(x), \psi(y)] = ([q(x), q(y)], \langle q(x), q(y) \rangle),$$

which implies that

$$\alpha(\widehat{b}(q(x), q(y))) = \alpha([x, y]) = \langle q(x), q(y) \rangle, \quad x, y \in \widehat{\mathfrak{g}},$$

and hence

$$\alpha \circ \widehat{b} = \omega_u.$$

For the continuous linear maps $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ corresponding to these cocycles, we obtain

$$\alpha \circ \widetilde{q} = \mathrm{id}_{\widetilde{\mathfrak{g}}}.$$

We also have

$$\widetilde{q} \circ \alpha \circ \widehat{b} = \widetilde{q} \circ \omega_u = \widehat{b},$$

and since $\mathrm{im}(\widehat{b})$ spans a dense subspace of the topologically perfect Lie algebra $\widehat{\mathfrak{g}}$, it follows that

$$\widetilde{q} \circ \alpha = \mathrm{id}_{\widehat{\mathfrak{g}}}.$$

Therefore $\widetilde{q}$ is an isomorphism of locally convex spaces, hence an isomorphism of locally convex Lie algebras, and this implies that $q_{\mathfrak{g}}$ is a central extension.

If, conversely, $\mathfrak{g}$ is topologically perfect and $q_{\mathfrak{g}}$ is a central extension, its universal property as a generalized central extension implies that it is a universal central extension. ∎

Comparing the construction above with the universal central extensions investigated in [Ne02c], it appears that generalized central extensions are more natural in the topological context because one does not have to struggle with the problem that closed subspaces of locally convex spaces do not always have closed complements, which causes many problems if one works only with central extensions defined by cocycles (cf. Definition III.3). Moreover, universal generalized central extensions do always exist for topologically perfect locally convex algebras, whereas there are Banach–Lie algebras which do not admit a universal central extension ([Ne01, Ex. II.18, III.9] and Proposition III.19 below, combined with Corollary III.13). The typical example is the Lie algebra of Hilbert–Schmidt operators on an infinite-dimensional Hilbert space discussed in some detail below.

We now address the question for which Lie algebra the universal covering morphism $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism. At the end of this section we will in particular describe examples, where $q_{\widetilde{\mathfrak{g}}} \colon \widetilde{\widetilde{\mathfrak{g}}} \to \widetilde{\mathfrak{g}}$ is not an isomorphism.

Proposition III.14. *For a topologically perfect locally convex Lie algebra $\mathfrak{g}$ the following are equivalent:*

(1) $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ *is an isomorphism of Lie algebras.*

(2) $H^2(\mathfrak{g}, \mathfrak{z}) = \{0\}$ *for each locally convex space* $\mathfrak{z}$.

If, in addition, $\mathfrak{g}$ is a topologically perfect Banach–Lie algebra, then (1) *and* (2) *are equivalent to*

(3) $H^2(\mathfrak{g}, \mathbb{K}) = \{0\}$.

Proof. (1) ⇒ (2): Let $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$ be a continuous Lie algebra cocycle $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$. According to Remark III.7, there exists a continuous linear map $\alpha\colon \widetilde{\mathfrak{g}} \to \mathfrak{z}$ with

$$\omega(x, y) = \alpha(\langle x, y\rangle) = \alpha \circ q_{\mathfrak{g}}^{-1}([x, y])$$

for $x, y \in \mathfrak{g}$, and this means that ω is a coboundary.
(2) ⇒ (1): The triviality of $H^2(\mathfrak{g}, \widetilde{\mathfrak{g}})$ implies that there exists a continuous linear map $\alpha\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ with

$$\langle x, y\rangle = \alpha([x, y]), \quad x, y \in \mathfrak{g}. \tag{3.3}$$

Then

$$(q_{\mathfrak{g}} \circ \alpha)([x, y]) = q_{\mathfrak{g}}(\langle x, y\rangle) = [x, y],$$

so that the density of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ leads to $q_{\mathfrak{g}} \circ \alpha = \mathrm{id}_{\mathfrak{g}}$. On the other hand, (3.3) can also be read as $\alpha \circ q_{\mathfrak{g}} = \mathrm{id}_{\widetilde{\mathfrak{g}}}$. Therefore $q_{\mathfrak{g}}$ is an isomorphism of locally convex spaces, hence of locally convex Lie algebras.

Now we assume that $\mathfrak{g}$ is a topologically perfect Banach–Lie algebra. It is clear that (2) implies (3).
(3) ⇒ (1): (cf. [Ne02c, Prop. 3.5]) Let $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the universal covering map. The condition $H^2(\mathfrak{g}, \mathbb{K}) = \{0\}$ means that each 2-cocycle is a coboundary, i.e., that the adjoint map

$$q_{\mathfrak{g}}^*\colon \operatorname{Lin}(\mathfrak{g}, \mathbb{K}) \to \operatorname{Lin}(\widetilde{\mathfrak{g}}, \mathbb{K}) \cong Z^2(\mathfrak{g}, \mathbb{K})$$

is surjective. Since $\mathfrak{g}$ is topologically perfect, it is also injective, hence bijective. The surjectivity of $q_{\mathfrak{g}}^*$ implies in particular that $q_{\mathfrak{g}}$ is injective. Further the Closed Range Theorem ([Ru73, Th. 4.14]) implies that the image of $q_{\mathfrak{g}}$ is closed, and hence that $q_{\mathfrak{g}}$ is bijective. Finally the Open Mapping Theorem implies that $q_{\mathfrak{g}}$ is an isomorphism. ■

A topologically perfect locally convex Lie algebra satisfying the two equivalent conditions of Proposition III.14 is called *centrally closed*. This means that $\mathfrak{g}$ is its own universal covering algebra, or, equivalently, that the Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a universal Lie algebra cocycle.

Remark III.15. (a) Let $\mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_3$ be topologically perfect locally convex Lie algebras and $q_1\colon \mathfrak{g}_1 \to \mathfrak{g}_2$, $q_2\colon \mathfrak{g}_2 \to \mathfrak{g}_3$ generalized central extensions. Then $q := q_2 \circ q_1\colon \mathfrak{g}_1 \to \mathfrak{g}_3$ is a morphism of locally convex Lie algebras with dense range. Moreover, Lemma III.4(5) implies that

$$\ker q = q_1^{-1}(\ker q_2) \subseteq q_1^{-1}(\mathfrak{z}(\mathfrak{g}_2)) = \mathfrak{z}(\mathfrak{g}_1).$$

Unfortunately, we cannot conclude in general that q is a generalized central extension. The bilinear map $b_1\colon \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathfrak{g}_1$ for which $b_1 \circ (q_1 \times q_1)$ is the Lie bracket of $\mathfrak{g}_1$ is a Lie algebra cocycle, which implies that

$$b_1(\ker q_2, \mathfrak{g}_2) \subseteq b_1(\mathfrak{z}(\mathfrak{g}_2), \overline{[\mathfrak{g}_2, \mathfrak{g}_2]}) = \{0\}.$$

Therefore b_1 factors through a bilinear map

$$b\colon \operatorname{im}(q_2) \times \operatorname{im}(q_2) \to \mathfrak{g}_1, \quad (q_2(x), q_2(y)) \mapsto b_1(x,y)$$

with

$$b(q(x), q(y)) = b_1(q_1(x), q_1(y)) = [x,y], \quad x, y \in \mathfrak{g}_1.$$

If b is continuous, it extends to a continuous bilinear map $\mathfrak{g}_3 \times \mathfrak{g}_3 \to \mathfrak{g}_1$ with the required properties, and q is a generalized central extension, but unfortunately, there is no reason for this to be the case.

(b) If q_2 is a quotient map, i.e., a central extension, then b is continuous. This shows that in the context of topologically perfect locally convex Lie algebras a generalized central extension of a central extension is a generalized central extension. This means in particular that if the universal covering map $q_{\mathfrak{g}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a quotient map, then $\widetilde{\mathfrak{g}}$ is centrally closed. ∎

Proposition III.16. *Let $\widetilde{q}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be a generalized central extension, $\mathfrak{z} \subseteq \mathfrak{z}(\mathfrak{g})$ a closed subspace and $p_{\mathfrak{z}}\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$ the quotient map. Then the composition map $q_{\mathfrak{z}} := p_{\mathfrak{z}} \circ \widetilde{q}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}/\mathfrak{z}$ is a generalized central extension. If q is universal, then $q_{\mathfrak{z}}$ is universal, too.*

Proof. From Remark III.15(b) we derive in particular that $q_{\mathfrak{z}}$ is a generalized central extension.

Now we assume that $\widetilde{q}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is universal. So let $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}/\mathfrak{z}$ be a generalized central extension and consider the pullback Lie algebra

$$\mathfrak{h} := \{(x,y) \in \widehat{\mathfrak{g}} \oplus \mathfrak{g}\colon q(x) = p_{\mathfrak{z}}(y)\},$$

on which we have two coordinate projections $p_{\mathfrak{g}}\colon \mathfrak{h} \to \mathfrak{g}$ and $p_{\widehat{\mathfrak{g}}}\colon \mathfrak{h} \to \widehat{\mathfrak{g}}$. We claim that $p_{\mathfrak{g}}$ is a generalized central extension. Its range is the inverse image $p_{\mathfrak{z}}^{-1}(\operatorname{im} q) \subseteq \mathfrak{g}$. If $U \subseteq \mathfrak{g}$ is an open subset intersecting $p_{\mathfrak{z}}^{-1}(\operatorname{im} q)$ trivially, then the open subset $p_{\mathfrak{z}}(U) \subseteq \mathfrak{g}/\mathfrak{z}$ intersects $\operatorname{im}(q)$ trivially, and therefore $U = \emptyset$. Hence $\operatorname{im}(p_{\mathfrak{g}})$ is dense in $\mathfrak{g}$. Let $b_{\mathfrak{z}}\colon \mathfrak{g}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{z} \to \widehat{\mathfrak{g}}$ denote a continuous bilinear map for which $b_{\mathfrak{z}} \circ (q \times q)$ is the Lie bracket on $\widehat{\mathfrak{g}}$. Then the map

$$b\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}, \quad (y, y') \mapsto (b_{\mathfrak{z}}(p_{\mathfrak{z}}(y), p_{\mathfrak{z}}(y')), [y, y'])$$

satisfies

$$\begin{aligned} b(p_{\mathfrak{g}}(x,y), p_{\mathfrak{g}}(x',y')) &= b(y,y') = (b_{\mathfrak{z}}(p_{\mathfrak{z}}(y), p_{\mathfrak{z}}(y')), [y,y']) \\ &= (b_{\mathfrak{z}}(q(x), q(x')), [y,y']) = ([x,x'], [y,y']) = [(x,y),(x',y')]. \end{aligned}$$

Hence $p_{\mathfrak{g}}$ is a generalized central extension, and the universal property of $\widetilde{q}$ implies the existence of a unique Lie algebra morphism $\alpha\colon \widetilde{\mathfrak{g}} \to \mathfrak{h}$ with $p_{\mathfrak{g}} \circ \alpha = \widetilde{q}$. This means that

$$\alpha(x) = (\beta(x), \widetilde{q}(x))$$

for some Lie algebra morphism $\beta\colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ satisfying $q_{\mathfrak{z}} = p_{\mathfrak{z}} \circ \widetilde{q} = q \circ \beta$. This argument shows that $q_{\mathfrak{z}}\colon \widetilde{\mathfrak{g}} \to \mathfrak{g}/\mathfrak{z}$ is a universal generalized central extension of $\mathfrak{g}/\mathfrak{z}$. ∎

III.4 Schatten classes as interesting examples

Lemma III.17. *Let H be a Hilbert space and $\mathfrak{sl}_0(H)$ the Lie algebra of all continuous finite rank operators of zero trace on H. For each derivation*

$$\Delta\colon \mathfrak{sl}_0(H) \to \mathfrak{sl}_0(H)$$

there exists a continous operator $D \in B(H)$ with $\Delta(x) = [D, x]$ for each $x \in \mathfrak{sl}_0(H)$. The operator D is unique up to an element in $\mathbb{K}\mathbf{1}$.

Proof. ([dlH72]) **Step 1:** For each finite subset F of $\mathfrak{sl}_0(H)$ there exists a finite-dimensional subspace $E \subseteq H$ such that

$$F \subseteq \mathfrak{sl}(E) := \{\varphi \in \mathfrak{sl}_0(H)\colon \varphi(E) \subseteq E, \varphi(E^\perp) = \{0\}\}.$$

The Lie algebra $\mathfrak{sl}(E) \cong \mathfrak{sl}_{|E|}(\mathbb{K})$ is simple and the restriction Δ_E of Δ to $\mathfrak{sl}(E)$ is a linear map $\mathfrak{sl}(E) \to \mathfrak{sl}_0(H)$ satisfying

$$\Delta_E([x, y]) = [\Delta_E(x), y] + [x, \Delta_E(y)].$$

This means that $\Delta_E \in Z^1(\mathfrak{sl}(E), \mathfrak{sl}_0(H))$, where $\mathfrak{sl}(E)$ acts on $\mathfrak{sl}_0(H)$ by the adjoint action. Since this action turns $\mathfrak{sl}_0(H)$ into a locally finite module, Lemma A.3 implies that the cocycle Δ_E is trivial, i.e., there exists an element $D_E \in \mathfrak{sl}_0(H)$ with $\Delta_E(x) = [D_E, x]$ for all $x \in \mathfrak{sl}(E)$. Suppose that D'_E is another element in $\mathfrak{sl}_0(H)$ with this property. Then we write

$$D_E - D'_E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as a block matrix according to the decomposition $H = E \oplus E^\perp$. As $D_E - D'_E$ commutes with $\mathfrak{sl}(E)$, it preserves the subspaces $\mathfrak{sl}(E).H = E$ and $E^\perp = \{x \in H\colon \mathfrak{sl}(E).x = \{0\}\}$. Therefore $b = c = 0$, and $a \in \mathbb{K}\,\mathrm{id}_E$. This proves that $D_E\,|_E - D'_E\,|_E \in \mathbb{K}\,\mathrm{id}_E$. If we require, in addition, $D_E.v \perp v$ for some non-zero vector $v \in E$, then the restriction of D_E to E is uniquely determined.

Step 2: We may assume that $\dim H \geq 2$, otherwise the assertion is trivial. Fix $0 \neq v \in H$. As in Step 1, we find for each finite-dimensional subspace $E \subseteq H$ an operator D_E as above with $D_E.v \perp v$. For $E \subseteq E'$ the operator $D_{E'}$ also satisfies $D_{E'}.v \perp v$ and $\Delta_E(x) = [D_{E'}, x]$ for $x \in \mathfrak{sl}(E) \subseteq \mathfrak{sl}(E')$. Therefore $D_{E'}\,|_E = D_E$, so that we obtain a well-defined operator

$$D\colon H \to H, \quad D.w := D_E.w \quad \text{for} \quad w \in E.$$

This operator satisfies

$$\Delta(x) = [D, x] \quad \text{for all} \quad x \in \mathfrak{sl}_0(H).$$

Step 3: D is continuous: For $x, y \in H$ we consider the rank-one-operator $P_{x,y}.v = \langle v, y\rangle x$. Then $\operatorname{tr} P_{x,y} = \langle x, y\rangle$ vanishes if $x \bot y$. Then $P_{x,y} \in \mathfrak{sl}_0(H)$, and

$$[D, P_{x,y}](v) = P_{D.x,y}.v - \langle D.v, y\rangle x.$$

As for each $y \in H$ there exists an element x orthogonal to y, it follows that all functionals

$$v \mapsto \langle D.v, y\rangle$$

are continuous, i.e., that the adjoint operator D^* of the unbounded operator D is everywhere defined, and therefore that D has a closed graph ([Ne99, Th. A.II.8]). Now the Closed Graph Theorem implies that D is continuous.

Step 4: Uniqueness: We have to show that if an operator D on H commutes with $\mathfrak{sl}_0(H)$, then it is a multiple of the identity. The condition $[D, P_{x,y}] = 0$ for $x \bot y$ implies that

$$\langle v, y\rangle D.x = \langle D.v, y\rangle x, \quad v \in H.$$

It follows in particular that each $x \in H$ is an eigenvector, and hence that $D \in \mathbb{K}\mathbf{1}$.

■

Definition III.18. Let H be an infinite-dimensional Hilbert space. For each $p \in [1, \infty]$ we write $B_p(H)$ for the corresponding Schatten ideal in $B(H)$, where $B_\infty(H)$ denotes the space of compact operators (cf. [dlH72], [GGK00]). Each operator $A \in B_p(H)$ is compact, and if we write the non-zero eigenvalues of the positive operator $\sqrt{A^*A}$ (counted with multiplicity) in a sequence $(\lambda_n)_{n\in\mathbb{N}}$ (which might also contain zeros), the norm on $B_p(H)$ is given by

$$\|A\|_p = \Big(\sum_{n\in\mathbb{N}} \lambda_n^p\Big)^{\frac{1}{p}}.$$

According to [GGK00, Th. IV.11.2], we then have the estimate

$$\|AB\|_p \leq \|A\|_{p_1}\|B\|_{p_2} \quad \text{for} \quad \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}.$$

It follows in particular that each $B_p(H)$ is a Banach algebra. We also have

$$\|ABC\| \leq \|A\|\|B\|_p\|C\|, \quad B \in B_p(H), A, C \in B(H).$$

For $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$B_p(H)' \cong B_q(H),$$

where the pairing is induced by the trace $\langle x, y\rangle = \operatorname{tr}(xy)$. Here we use that $B_p(H)B_q(H) \subseteq B_1(H)$, and that the trace extends to a continuous linear functional $\operatorname{tr}\colon B_1(H) \to \mathbb{K}$ (cf. [dlH72, p.113]). We have

$$B_1(H) \subseteq B_p(H) \subseteq B_{p'}(H) \subseteq B_\infty(H)$$

for $p \leq p'$.

For $p = 1$ the elements of $B_1(H)$ are the *trace class operators* and for $p = 2$ the elements of $B_2(H)$ are the *Hilbert-Schmidt operators.* As the trace is a continuous linear functional on $B_1(H)$ vanishing on all commutators, the subspace

$$\mathfrak{sl}(H) := \{x \in B_1(H) \colon \operatorname{tr} x = 0\}$$

is a Lie algebra hyperplane ideal. ■

Proposition III.19. *For $1 \leq p \leq \infty$ let $\mathfrak{gl}_p(H)$ be the Banach-Lie algebra obtained from $B_p(H)$ with the commutator bracket. Then $\mathfrak{gl}_p(H)$ is topologically perfect if and only if $p > 1$. The universal covering map is given by the inclusion maps*

$$\mathfrak{sl}(H) \hookrightarrow \mathfrak{gl}_p(H) \quad \text{for } 1 < p \leq 2, \qquad \text{and} \qquad \mathfrak{gl}_{\frac{p}{2}}(H) \hookrightarrow \mathfrak{gl}_p(H) \quad \text{for } 2 < p = \infty.$$

The Lie algebra $\mathfrak{sl}(H)$ is topologically perfect and centrally closed.

Proof. That $\mathfrak{gl}_1(H)$ is not topologically perfect follows from the fact that the trace vanishes on all brackets. Assume that $p > 1$. Then an elementary argument with diagonal matrices implies that $\mathfrak{sl}_0(H)$ is dense in $B_p(H)$ with respect to $\|\cdot\|_p$. Since $\mathfrak{sl}_0(H)$ is a perfect Lie algebra, $\mathfrak{gl}_p(H)$ is topologically perfect.

Let $\omega \colon \mathfrak{gl}_p(H) \times \mathfrak{gl}_p(H) \to \mathbb{K}$ be a continuous Lie algebra cocycle. Then there exists a unique continuous linear map

$$\Delta \colon \mathfrak{gl}_p(H) \to \mathfrak{gl}_q(H) \cong \mathfrak{gl}_p(H)', \quad \frac{1}{p} + \frac{1}{q} = 1,$$

with $\operatorname{tr}(\Delta(x)y) = \omega(x,y)$ for all $x, y \in \mathfrak{gl}_p(H)$, and the cocycle identity for ω implies that Δ is a derivation, i.e.,

$$\Delta([x,y]) = [\Delta(x), y] + [x, \Delta(y)], \quad x, y \in \mathfrak{gl}_p(H).$$

The Lie algebra $\mathfrak{sl}_0(H)$ is a perfect ideal in $\mathfrak{gl}(H)$ and hence in each $\mathfrak{gl}_p(H)$. Therefore it is invariant under Δ, and Lemma III.17 implies the existence of a continuous operator $D \in B(H)$ with $\Delta(x) = [D, x]$ for all $x \in \mathfrak{sl}_0(H)$. As both sides describe continuous linear maps $\mathfrak{gl}_p(H) \to \mathfrak{gl}(H)$ which coincide on the dense subspace $\mathfrak{sl}_0(H)$, we have $\Delta = \operatorname{ad} D$ on $\mathfrak{gl}_p(H)$.

For $1 \leq p \leq 2$ we have $q \geq 2 \geq p$, so that each bounded operator $D \in B(H)$ satisfies $\operatorname{ad} D(\mathfrak{gl}_p(H)) \subseteq \mathfrak{gl}_p(H) \subseteq \mathfrak{gl}_q(H)$. For $p > 2$ the dual space $\mathfrak{gl}_q(H)$ is a proper subspace of $\mathfrak{gl}_p(H)$, and it is shown in [dlH72, p.141] that

$$\{D \in \mathfrak{gl}(H) \colon [D, \mathfrak{gl}_p(H)] \subseteq \mathfrak{gl}_q(H)\} = \mathfrak{gl}_r(H) \quad \text{for} \quad \frac{1}{r} = \frac{1}{q} - \frac{1}{p} = 1 - \frac{2}{p} = \frac{p-2}{p}.$$

The cocycle associated to an operator D is given by

$$\omega(x,y) = \operatorname{tr}([D,x]y) = \operatorname{tr}(D[x,y]), \quad x,y \in \mathfrak{gl}_p(H).$$

That the trace on the right hand side makes sense follows from $B_p(H)B_p(H) \subseteq B_1(H)$ for $p \leq 2$ and $B_p(H)B_p(H) \subseteq B_{\frac{p}{2}}(H)$ and $D \in B_{\frac{p}{2}}(H)'$ for $p > 2$.

For $p \leq 2$ we have

$$[\mathfrak{gl}_p(H), \mathfrak{gl}_p(H)] \subseteq [\mathfrak{gl}_2(H), \mathfrak{gl}_2(H)] \subseteq \overline{[\mathfrak{sl}_0(H), \mathfrak{sl}_0(H)]} = \overline{\mathfrak{sl}_0(H)} = \mathfrak{sl}(H),$$

where the closure refers to the trace norm $\|\cdot\|_1$. An operator $D \in \mathfrak{gl}(H) \cong \mathfrak{gl}_1(H)'$ represents the cocycle 0 if and only if it is orthogonal to the hyperplane $\mathfrak{sl}(H)$, which means that $D \in \mathbb{K}\mathbf{1}$. For $p > 2$ an operator $D \in \mathfrak{gl}_r(H)$ is never a multiple of $\mathbf{1}$, so that we obtain

$$Z^2(\mathfrak{gl}_p(H), \mathbb{K}) \cong \begin{cases} \mathfrak{pgl}(H) := \mathfrak{gl}(H)/\mathbb{K}\mathbf{1} & \text{for } 1 \leq p \leq 2 \\ \mathfrak{gl}_{\frac{p}{2}}(H)' \cong \mathfrak{gl}_r(H) & \text{for } 2 < p. \end{cases} \tag{3.4}$$

Now let $q(\langle x,y\rangle) = [x,y]$ denote the bracket map

$$q \colon \widetilde{\mathfrak{gl}}_p(H) \cong \langle \mathfrak{gl}_p(H), \mathfrak{gl}_p(H)\rangle \to \begin{cases} \mathfrak{sl}(H) & \text{for } 1 \leq p \leq 2 \\ \mathfrak{gl}_{\frac{p}{2}}(H) & \text{for } 2 < p. \end{cases}$$

Then q is a continuous morphism of Banach–Lie algebras. Further

$$Z^2(\mathfrak{gl}_p(H), \mathbb{K}) \cong \operatorname{Lin}(\widetilde{\mathfrak{gl}}_p(H), \mathbb{K}),$$

and (3.4) imply that the adjoint map q^* is bijective. That q^* is injective implies that q has dense range and the surjectivity of q^* implies in particular that q is injective. Further the Closed Range Theorem ([Ru73, Th. 4.14]) implies that the image of q is closed, and hence that q is bijective. Finally the Open Mapping Theorem implies that q is an isomorphism.

It remains to show that $\mathfrak{sl}(H)$ is centrally closed. That it is topologically perfect follows immediately from the density of the perfect ideal $\mathfrak{sl}_0(H)$. Since the dual space of $\mathfrak{gl}_1(H)$ can be identified with the full operator algebra $\mathfrak{gl}(H)$ via the trace pairing, and the annihilator of the closed hyperplane $\mathfrak{sl}(H)$ is the center $\mathbb{K}\mathbf{1} \subseteq \mathfrak{gl}(H)$, the dual space $\mathfrak{sl}(H)'$ can be identified in a natural way with the quotient $\mathfrak{pgl}(H) = \mathfrak{gl}(H)/\mathbb{K}\mathbf{1}$. Let $\omega \in Z^2(\mathfrak{sl}(H), \mathbb{K})$ be a continuous cocycle. As above, there exists a continuous derivation $\Delta \colon \mathfrak{sl}(H) \to \mathfrak{sl}(H)'$ with $\operatorname{tr}(\Delta(x)y) = \omega(x,y)$ for $x,y \in \mathfrak{sl}(H)$, where we use that $\operatorname{tr}(ab) := \operatorname{tr}(a'b)$ is well defined for $a = a' + \mathbb{K}\mathbf{1} \in \mathfrak{pgl}(H)$ and $b \in \mathfrak{sl}(H)$. From the invariance of the perfect ideal $\mathfrak{sl}_0(H)$ under Δ, we obtain with Lemma III.17 the existence of $D \in \mathfrak{gl}(H)$ with $\Delta(x) = [D,x]$ for all $x \in \mathfrak{sl}_0(H)$, and the density of $\mathfrak{sl}_0(H)$ implies that $\Delta = \operatorname{ad} D$. Therefore

$$\omega(x,y) = \operatorname{tr}([D,x]y) = \operatorname{tr}(D[x,y])$$

is a coboundary, which leads to $H^2(\mathfrak{sl}(H), \mathbb{K}) = \{0\}$, and thus $\mathfrak{sl}(H)$ is centrally closed by Proposition III.14. ∎

Remark III.20. From the preceding proposition, we obtain in particular examples of Lie algebras where the universal covering algebra is not centrally closed. For example each $\mathfrak{gl}_p(H)$ with $p > 2$ has this property. For $p < 2 \leq 4$ we have

$$\widetilde{\mathfrak{gl}}_p(H) \cong \mathfrak{gl}_{\frac{p}{2}}(H) \quad \text{and} \quad \widetilde{\widetilde{\mathfrak{gl}}}_p(H) \cong \mathfrak{sl}(H),$$

but for $2^k < p \leq 2^{k+1}$ we need to pass $k+1$-times to the universal covering Lie algebra until we reach $\mathfrak{sl}(H)$ which is centrally closed. ■

In Section IV below we shall see many other concrete examples of universal central extensions, when we discuss root graded locally convex Lie algebras.

IV Universal coverings of locally convex root graded Lie algebras

In this section we describe the universal covering Lie algebra $\widetilde{\mathfrak{g}}$ of a locally convex root graded Lie algebra $\mathfrak{g}$. In particular, we shall see that it can be constructed directly from its coordinate structure $(\mathcal{A}, D, \delta^D)$. For the class of the so called regular root graded Lie algebras, the universal covering algebra does not depend on D, hence has a particularly nice structure. Since not every root graded Lie algebra $\mathfrak{g}$ is regular, the description of $\widetilde{\mathfrak{g}}$ is more involved than in the algebraic context ([ABG00]). A key point is that the concept of a generalized central extension provides the natural framework to translate the algebraic structure of the universal covering algebra into the locally convex context.

IV.1 Generalized central extensions of root graded Lie algebras

Proposition IV.1. *Let $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ be a generalized central extension for which $\widehat{\mathfrak{g}}$ is topologically perfect. If $\mathfrak{g}$ is Δ-graded, then $\widehat{\mathfrak{g}}$ is Δ-graded and vice versa.*

Proof. (a) First we assume that $\mathfrak{g}$ is Δ-graded. On $\widehat{\mathfrak{g}}$ we consider the $\mathfrak{g}_\Delta$-module structure given by $\widehat{\mathrm{ad}}$ (Lemma III.4). Then the corestriction $\widehat{\mathfrak{g}} \to \mathrm{im}(q)$ is an extension of the locally finite $\mathfrak{g}_\Delta$-module $\mathrm{im}(q)$ by the trivial module $\ker q$, hence a trivial extension (Proposition A.4). It follows in particular that $\widehat{\mathfrak{g}}$ is an $\mathfrak{h}$-weight module. The weights occurring in this module are identical with those occurring in $\mathrm{im}(q) \supseteq [\mathfrak{g}, \mathfrak{g}]$ (Lemma III.4(1)). This implies that we have an $\mathfrak{h}$-weight decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in \Delta} \widehat{\mathfrak{g}}_\alpha$$

with $q(\widehat{\mathfrak{g}}_\alpha) = \mathfrak{g}_\alpha$ for $\alpha \neq 0$. As the central Lie algebra extension $q^{-1}(\mathfrak{g}_\Delta) \to \to \mathfrak{g}_\Delta$ is trivial, its commutator algebra $\widehat{\mathfrak{g}}_\Delta$ is a subalgebra which is mapped by

q isomorphically onto $\mathfrak{g}_\Delta$. Therefore (R1)–(R3) are satisfied for $\widehat{\mathfrak{g}}_\Delta$ as a grading subalgebra in $\widehat{\mathfrak{g}}$.

As the bracket in $\widehat{\mathfrak{g}}$ is given by $[x,y] = b(q(x),q(y))$, the topological perfectness of $\widehat{\mathfrak{g}}$ implies that the image of b spans a dense subspace of $\widehat{\mathfrak{g}}$. Therefore

$$b(\mathfrak{g}_0,\mathfrak{g}_0) + \sum_{0\neq\alpha} b(\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}) = b(\mathfrak{g}_0,\mathfrak{g}_0) + \sum_{0\neq\alpha} [\widehat{\mathfrak{g}}_\alpha,\widehat{\mathfrak{g}}_{-\alpha}]$$

is dense in $\widehat{\mathfrak{g}}_0$. For $x_{\pm\alpha} \in \widehat{\mathfrak{g}}_{\pm\alpha}$ and $x_{\pm\beta} \in \widehat{\mathfrak{g}}_{\pm\beta}$ we further have

$$b([q(x_\alpha),q(x_{-\alpha})],[q(x_\beta),q(x_{-\beta})]) = [[x_\alpha,x_{-\alpha}],[x_\beta,x_{-\beta}]] \in [\widehat{\mathfrak{g}}_0,[\widehat{\mathfrak{g}}_\beta,\widehat{\mathfrak{g}}_{-\beta}]] \subseteq [\widehat{\mathfrak{g}}_\beta,\widehat{\mathfrak{g}}_{-\beta}].$$

Hence

$$b([\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}],[\mathfrak{g}_\beta,\mathfrak{g}_{-\beta}]) \subseteq [\widehat{\mathfrak{g}}_\beta,\widehat{\mathfrak{g}}_{-\beta}],$$

so that (R4) holds for $\mathfrak{g}$, and the relation $q(\widehat{\mathfrak{g}}_\alpha) = \mathfrak{g}_\alpha$ for $\alpha \neq 0$ imply that $b(\mathfrak{g}_0,\mathfrak{g}_0)$ is contained in the closure of the sum of the spaces $[\widehat{\mathfrak{g}}_\alpha,\widehat{\mathfrak{g}}_{-\alpha}]$, $\alpha \neq 0$. This implies (R4) for $\widehat{\mathfrak{g}}$.
(b) Now we assume that $\widehat{\mathfrak{g}}$ is Δ-graded with grading subalgebra $\widehat{\mathfrak{g}}_\Delta$. Then $\ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$, so that $\mathfrak{g}_\Delta := q(\widehat{\mathfrak{g}}_\Delta) \cong \widehat{\mathfrak{g}}_\Delta$. Clearly $\mathfrak{g}$ carries a natural $\mathfrak{g}_\Delta$-module structure.

From $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{im}(q)$ (Lemma III.4(2)) we derive that $\mathfrak{g}/\mathrm{im}(q)$ is a trivial $\mathfrak{g}_\Delta$-module. Moreover, $\mathrm{im}(q) \cong \widehat{\mathfrak{g}}/\ker(q)$ is a locally finite $\mathfrak{g}_\Delta$-module. Therefore Proposition A.4 implies that $\mathfrak{g}$ is a locally finite $\mathfrak{g}_\Delta$-module which is a direct sum of $q(\widehat{\mathfrak{g}})$ and a trivial module Z. This immediately leads to a weight decomposition of $\mathfrak{g}$ with weight system Δ, and it is obvious that (R1)–(R3) are satisfied.

As $\mathfrak{h}$ acts on $\mathfrak{g}$ by continuous operators, the projection $\mathfrak{g} \to \mathfrak{g}_0$ along the sum of the other root spaces is continuous, so that the density of the image of q in $\mathfrak{g}$ implies that $q(\widehat{\mathfrak{g}}_0)$ is dense in $\mathfrak{g}_0$. We further have

$$[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] = q(b(\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha})) = q(b(q(\widehat{\mathfrak{g}}_\alpha),q(\widehat{\mathfrak{g}}_{-\alpha}))) = q([\widehat{\mathfrak{g}}_\alpha,\widehat{\mathfrak{g}}_{-\alpha}]),$$

so that (R4) for $\widehat{\mathfrak{g}}$ implies (R4) for $\mathfrak{g}$. ■

Corollary IV.2. *If $\mathfrak{g}$ is Δ-graded with grading subalgebra $\mathfrak{g}_\Delta$, then $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}_\mathfrak{g}(\mathfrak{g}_\Delta) \subseteq \mathfrak{z}_\mathfrak{g}(\mathfrak{h}) = \mathfrak{g}_0$, and $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathrm{ad}\,\mathfrak{g}$ is a Δ-graded Lie algebra. The quotient map* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ *is a morphism of Δ-graded Lie algebras.* ■

Lemma IV.3. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be locally convex Δ-graded Lie algebras with coordinate structures $(\mathcal{A}_i = A_i \oplus B_i, D_i, \delta^{D_i})$, and $\eta_i\colon \mathfrak{g}_\Delta \to \mathfrak{g}$ the corresponding embeddings that we use to identify $\mathfrak{g}_\Delta$ with a subalgebra of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. If $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a morphism of locally convex Lie algebra with $\varphi \circ \eta_1 = \eta_2$, then there exist continuous linear maps*

$$\varphi_A\colon A_1 \to A_2, \quad \varphi_B\colon B_1 \to B_2 \quad \text{and} \quad \varphi_D\colon D_1 \to D_2$$

such that

$$\varphi(a \otimes x + b \otimes v + d) = \varphi_A(a) \otimes x + \varphi_B(b) \otimes v + \varphi_D(d) \tag{4.1}$$

for $a \in A_1, b \in B_1, d \in D_1, x \in \mathfrak{g}_\Delta$ *and* $v \in V_s$*, and*

$$\varphi_{\mathcal{A}} := \varphi_A \oplus \varphi_B \colon \mathcal{A}_1 \to \mathcal{A}_2$$

is a continuous algebra homomorphism with

$$\delta^{D_2} \circ (\varphi_{\mathcal{A}} \times \varphi_{\mathcal{A}}) = \varphi_D \circ \delta^{D_1}. \tag{4.2}$$

Proof. The condition $\varphi \circ \eta_1 = \eta_2$ means that φ is equivariant with respect to the representations of $\mathfrak{g}_\Delta$ on $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Identifying A_1 with $\mathrm{Hom}_{\mathfrak{g}_\Delta}(\mathfrak{g}_\Delta, \mathfrak{g}_1)$, the equivariance of φ with respect to $\mathfrak{g}_\Delta$ permits us to define $\varphi_A(a) := \varphi \circ a$. We likewise define φ_B and φ_D. Then (4.1) is satisfied. Now (4.2) and that $\varphi_{\mathcal{A}}$ defines an algebra homomorphism follow directly from (B1)–(B3), because the algebra structure on $\mathcal{A}_1$, resp., $\mathcal{A}_2$ is completely determined by the Lie bracket. ■

Remark IV.4. The preceding lemma applies in particular to generalized central extensions $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$. In this case the proof of Proposition IV.1 implies that $q_{\mathcal{A}}$ is a topological isomorphism, hence an isomorphism of locally convex algebras. We therefore may assume that $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ have the same coordinate algebra $\mathcal{A}$. In this sense we write

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D \quad \text{and} \quad \widehat{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widehat{D},$$

and $q_D \colon \widehat{D} \to D$ is a map with dense range, $q_D \circ \delta^{\widehat{D}} = \delta^D$, and since q is a generalized central extension, restricting the $\mathfrak{g}_\Delta$-equivariant corresponding map $b \colon \mathfrak{g} \times \mathfrak{g} \to \widehat{\mathfrak{g}}$ to $D \times D$ leads to a continuous bilinear map $b_D \colon D \times D \to \widehat{D}$ with $b_D(q_D(d), q_D(d')) = [d, d']$ for $d, d' \in \widehat{D}$. We conclude that $q_D \colon \widehat{D} \to D$ also is a generalized central extension.

This applies in particular to the universal covering algebra, which we write as

$$\widetilde{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widetilde{D}.$$

In the following subsection we will see how $\widetilde{D}$ can be described directly in terms of the coordinate algebra $\mathcal{A}$ and $\delta_{\mathcal{A}}$. ■

IV.2 The universal covering of a Δ-graded locally convex Lie algebra

To describe the universal covering Lie algebra $\widetilde{\mathfrak{g}}$ of a locally convex root graded Lie algebra $\mathfrak{g}$, we first consider its coordinate structure $(\mathcal{A} = A \oplus B, D, \delta^D)$ (Definition II.14). We consider the locally convex space

$$\langle \mathcal{A}, \mathcal{A} \rangle^\sigma := \langle \mathcal{A}, \mathcal{A} \rangle / \overline{\langle A, B \rangle}$$

and write the image of $\langle a, a' \rangle \in \langle \mathcal{A}, \mathcal{A} \rangle$ in $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ also as $\langle a, a' \rangle$.

Theorem IV.5. *For each root system Δ, a corresponding coordinate algebra $\mathcal{A}$, and the natural map $\delta_{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \to \mathrm{der}(\mathcal{A})$, the derivations $\delta_{\mathcal{A}}(a,b)$ preserve the subspace $\overline{\langle A, B\rangle}$ of $\langle \mathcal{A}, \mathcal{A}\rangle$, and we obtain on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ the structure of a locally convex Lie algebra by*

$$[\langle a, a'\rangle, \langle b, b'\rangle] := \delta_{\mathcal{A}}(a, a').\langle b, b'\rangle.$$

The map $\delta_{\mathcal{A}}$ factors through a Lie algebra homomorphism $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \mathrm{der}(\mathcal{A})$.

Proof. Since the map $\mathcal{A}^3 \to \mathcal{A}, (a,b,c) \mapsto \delta^D(a,b).c$ is continuous, and δ^D is a cyclic 1-cocycle vanishing on $A \times B$ (Theorem II.13), it defines a continuous linear map

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D, \quad \langle a, b\rangle \mapsto \delta^D(a,b).$$

Now define

$$\delta_{\mathcal{A}} \colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \mathrm{der}(\mathcal{A}), \quad \delta_{\mathcal{A}}(a,b).c := \delta^D(a,b).c,$$

and observe that the bilinear map

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \times \mathcal{A} \to \mathcal{A}, \quad (\langle a, b\rangle, c) \mapsto \delta_{\mathcal{A}}(a,b).c$$

is continuous.

From (2.3) in Theorem II.13 we further derive that

$$\delta_{\mathcal{A}}(\delta_{\mathcal{A}}(a,b).\langle c,d\rangle) = \delta_{\mathcal{A}}(\delta_{\mathcal{A}}(a,b).c, d) + \delta_{\mathcal{A}}(c, \delta_{\mathcal{A}}(a,b).d) = [\delta_{\mathcal{A}}(a,b), \delta_{\mathcal{A}}(c,d)] \tag{4.3}$$

for $a, b, c, d \in \mathcal{A}$.

As the operators $\delta(a,b) \in \mathrm{der}(\mathcal{A})$ all preserve the subspaces A and B of $\mathcal{A}$, the subspace $\langle A, B\rangle \subseteq \langle \mathcal{A}, \mathcal{A}\rangle$ is invariant under all these operators with respect to the natural action of $\mathrm{der}(\mathcal{A})$ on $\langle \mathcal{A}, \mathcal{A}\rangle$, and we therefore obtain a well-defined bracket on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ with

$$[\langle a, a'\rangle, \langle b, b'\rangle] := \delta_{\mathcal{A}}(a, a').\langle b, b'\rangle.$$

As in Proposition III.9, the Jacobi identity for this bracket is a direct consequence of (4.3). That the bracket is alternating is equivalent to the relation

$$\delta_{\mathcal{A}}(a, a').\langle b, b'\rangle = -\delta_{\mathcal{A}}(b, b').\langle a, a'\rangle \tag{4.4}$$

for $a, a', b, b' \in \mathcal{A}$. This relation can be verified case by case for the coordinate algebras associated to the different types of root systems (see [ABG00, p.521]; cf. also Theorem II.20 and the subsequent comments). Formula (4.3) immediately shows that $\delta_{\mathcal{A}}$ is a morphism of Lie algebras.

For the case where $\mathcal{A}$ is an associative or a Jordan algebra, (4.4) can be obtained as in Example III.10(2), (3). In this case we already have on $\langle \mathcal{A}, \mathcal{A}\rangle$ a natural Lie algebra structure, and since $\langle A, B\rangle$ is invariant under the operators $\delta_{\mathcal{A}}(a,b)$, it is a Lie algebra ideal, so that $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ simply is the quotient Lie algebra. ■

Definition IV.6. Let D be a locally convex Lie algebra and $D \times \mathcal{A} \to \mathcal{A}$ a continuous action by derivations on $\mathcal{A}$ which preserves the subspaces A and B of $\mathcal{A}$. Since the map

$$D \times \mathcal{A} \times \mathcal{A} \to \langle \mathcal{A}, \mathcal{A} \rangle, \quad (d, a, b) \mapsto d.\langle a, b \rangle = \langle d.a, b \rangle + \langle a, d.b \rangle$$

is trilinear and continuous, it induces a continuous bilinear map

$$D \times \langle \mathcal{A}, \mathcal{A} \rangle \to \langle \mathcal{A}, \mathcal{A} \rangle.$$

Therefore the semidirect product $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma \rtimes D$ carries a natural structure of a locally convex Lie algebra.

Using Example III.10(1) and Proposition III.9, we obtain a locally convex Lie algebra structure on $\langle D, D \rangle$ with

$$[\langle d, d' \rangle, \langle e, e' \rangle] = \langle [d, d'], [e, e'] \rangle$$

such that the bracket map $b_D \colon \langle D, D \rangle \to D, \langle d, d' \rangle \mapsto [d, d']$ is a morphism of locally convex Lie algebras.

Combining this with the semidirect product construction from above, we obtain a semidirect product $D_1 := \langle \mathcal{A}, \mathcal{A} \rangle^\sigma \rtimes \langle D, D \rangle$. In this Lie algebra the closed subspace I generated by the elements of the form

$$(d.\langle a, a' \rangle, -\langle d, \delta^D(a, a') \rangle), \quad a, a' \in \mathcal{A}, d \in D$$

is an ideal because I commutes with the ideal $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$, and D acts in a natural way by derivations on D_1 preserving I. Since $\mathrm{im}(\delta^D)$ spans a dense subspace of D, the ideal I is also generated by the elements of the form

$$([\langle a, a' \rangle, \langle b, b' \rangle], -\langle \delta^D(a, a'), \delta^D(b, b') \rangle), \quad a, a', b, b' \in \mathcal{A}.$$

We define

$$\widetilde{D} := (\langle \mathcal{A}, \mathcal{A} \rangle^\sigma \rtimes \langle D, D \rangle)/I.$$

This is a locally convex Lie algebra that will be needed in the description of the universal covering algebra $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ of a root graded Lie algebra $\mathfrak{g}$ with coordinate structure $(\mathcal{A}, D, \delta^D)$. We write $[(x, y)]$ for the image of the pair $(x, y) \in D_1$ in the quotient Lie algebra $\widetilde{D}$. ∎

Lemma IV.7. *The map*

$$q_{\mathfrak{g},D} \colon \widetilde{D} \to D, \quad [(\langle a, a' \rangle, \langle d, d' \rangle)] \mapsto \delta^D(a, a') + [d, d']$$

is a well-defined generalized central extension.

Proof. First we observe that $\delta^D_{\mathcal{A}} \colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D$ is a morphism of Lie algebras because

$$\delta^D_{\mathcal{A}}([\langle a,b\rangle, \langle c,d\rangle]) = \delta^D_{\mathcal{A}}(\delta_{\mathcal{A}}(a,b).\langle c,d\rangle) = [\delta^D_{\mathcal{A}}(\langle a,b\rangle), \delta^D_{\mathcal{A}}(\langle c,d\rangle)]$$

(Theorem II.13). Therefore

$$\delta^D_{\mathcal{A}}([\langle a,a'\rangle, \langle b,b'\rangle]) = [\delta^D(a,a'), \delta^D(b,b')],$$

which implies that $q_{\mathfrak{g},D}$ is well-defined. The equivariance of $q_{\mathfrak{g},D}$ with respect to the action of D by derivations on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ and $\langle D, D\rangle$ implies that $q_{\mathfrak{g},D}$ is a morphism of Lie algebras.

Its range contains the image of δ^D, hence is dense in D. Moreover, the continuous bilinear map

$$b \colon D \times D \to \widetilde{D}, \quad (d,d') \mapsto [(0, \langle d,d'\rangle)]$$

satisfies

$$b([d,d'],[e,e']) = [(0, \langle [d,d'],[e,e']\rangle)] = \Big[[(0,\langle d,d'\rangle)], [(0,\langle e,e'\rangle)]\Big],$$

$$b(\delta^D(a,a'), \delta^D(b,b')) = [(0, \langle \delta^D(a,a'), \delta^D(b,b')\rangle)] = [([\langle a,a'\rangle, \langle b,b'\rangle],0)],$$

and

$$\begin{aligned} b([d,d'], \delta^D(a,a')) &= [(0, -\langle \delta^D(a,a'), [d,d']\rangle)] = [([d,d'].\langle a,a'\rangle, 0)] \\ &= \Big[[(0,\langle d,d'\rangle)], [(\langle a,a'\rangle, 0)]\Big]. \end{aligned}$$

This implies that $b \circ (q_{\mathfrak{g},D} \times q_{\mathfrak{g},D})$ is the Lie bracket on $\widetilde{D}$, and hence that $q_{\mathfrak{g},D}$ is a generalized central extension. ∎

Note that, in general, $\widetilde{D}$ is not the universal covering Lie algebra because D might be abelian, so that it has no universal covering algebra.

The following theorem is the locally convex version of the description of the universal covering Lie algebra (cf. [ABG00] for the algebraic case).

Theorem IV.8. *Let $\mathfrak{g}$ be a Δ-graded locally convex Lie algebra with coordinate structure $(\mathcal{A}, D, \delta^D)$. Then the Lie algebra $\widetilde{D}$ acts continuously by derivations on $\mathcal{A}$ via*

$$(\langle a,b\rangle, \langle d,d'\rangle).c := \delta^D(a,b).c + [d,d'].c,$$

and we have a continuous bilinear map

$$\delta^{\widetilde{D}} \colon \mathcal{A} \times \mathcal{A} \to \widetilde{D}, \quad (a,b) \mapsto [(\langle a,b\rangle, 0)].$$

The Lie algebra

$$\widetilde{\mathfrak{g}} := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widetilde{D}$$

with the Lie bracket given by

$$[d, a \otimes x + b \otimes v + d'] = d.a \otimes x + d.b \otimes v + [d, d'],$$

and

$$\begin{aligned}
[a \otimes x, a' \otimes x'] &= \gamma^A_+(a, a') \otimes [x, x'] + \gamma^A_-(a, a') \otimes x * x' + \gamma^B_A(a, a') \otimes \beta^V_{\mathfrak g}(x, x') \\
&\quad + \kappa(x, x')\delta^{\widetilde D}(a, a'), \\
[a \otimes x, b \otimes v] &= \frac{ab + ba}{2} \otimes \beta^{\mathfrak g}_{\mathfrak g, V}(x, v) + \frac{ab - ba}{2} \otimes x.v, , \\
[b \otimes v, b' \otimes v'] &= \gamma^A_B(b, b') \otimes \beta^{\mathfrak g}_V(v, v') + \gamma^B_B(b, b') \otimes \beta^V_V(v, v') + \kappa_{V_s}(v, v')\delta^{\widetilde D}(b, b')
\end{aligned}$$

is the universal covering Lie algebra of $\mathfrak g$ *with respect to the map*

$$q_{\mathfrak g}(a \otimes x + b \otimes v + d) = a \otimes x + b \otimes v + q_{\mathfrak g, D}(d),$$

where

$$q_{\mathfrak g, D} \colon \widetilde D \to D, \quad [(\langle a, a' \rangle, \langle d, d' \rangle)] \mapsto \delta^D(a, a') + [d, d'].$$

Proof. In view of the comments in Definition IV.6, the Lie algebra $\widetilde D$ together with the map $\delta^{\widetilde D} \colon \widetilde D \to \mathrm{der}(\mathcal A)$ satisfy all assumptions of Theorem II.15, and we obtain on

$$\widetilde{\mathfrak g} := (A \otimes \mathfrak g_\Delta) \oplus (B \otimes V_s) \oplus \widetilde D$$

a Lie bracket as described above for which $\widetilde{\mathfrak g}$ is a Δ-graded Lie algebra with coordinate structure $(\mathcal A, \widetilde D, \delta_{\widetilde D})$, and $q_{\mathfrak g} \colon \widetilde{\mathfrak g} \to \mathfrak g$ is a morphism of Lie algebras. Since the range of $q_{\mathfrak g, D}$ contains the image of δ^D, the range of $q_{\mathfrak g}$ is dense.

To see that $q_{\mathfrak g}$ is a generalized central extension, we observe that the formulas for the bracket in Theorem II.15 show how to define a continuous bilinear map $b_{\mathfrak g} \colon \mathfrak g \times \mathfrak g \to \widetilde{\mathfrak g}$ for which $b \circ (q_{\mathfrak g} \times q_{\mathfrak g})$ is the bracket of $\widetilde{\mathfrak g}$ (cf. Lemma IV.7). We only have to replace $\delta^{\widehat D}$ by $\delta^{\widetilde D}$ and define $b_{\mathfrak g}$ on $D \times \mathfrak g$ by

$$b_{\mathfrak g}(d, a \otimes x + b \otimes v + d') := d.a \otimes x + d.b \otimes v + [(0, \langle d, d' \rangle)].$$

The main point in the complicated construction of the Lie algebra $\widetilde D$ was the need for the bilinear map $b_{\mathfrak g}$ on $D \times D$. This proves that $q_{\mathfrak g}$ is a generalized central extension.

To prove the universality of $q_{\mathfrak g}$, let $q \colon \widehat{\mathfrak g} \to \mathfrak g$ be a generalized central extension, where we write $\widehat{\mathfrak g}$ as

$$\widehat{\mathfrak g} = (A \otimes \mathfrak g_\Delta) \oplus (B \otimes V_s) \oplus \widehat D$$

and recall that $q_D \colon \widehat D \to D$ also is a generalized central extension, so that there exists a continuous bilinear map $b_D \colon D \times D \to \widehat D$ such that $b_D \circ (q_D \times q_D)$ is the Lie bracket on $\widehat D$ (Remark IV.4). Then the corresponding map $\delta^{\widehat D}_{\mathcal A} \colon \langle \mathcal A, \mathcal A \rangle^\sigma \to \widehat D$ is a continuous homomorphism of Lie algebras because

$$\delta^{\widehat D}_{\mathcal A}([\langle a, b \rangle, \langle c, d \rangle]) = \delta^{\widehat D}_{\mathcal A}(\delta_{\mathcal A}(a, b).\langle c, d \rangle) = [\delta^{\widehat D}_{\mathcal A}(a, b), \delta^{\widehat D}_{\mathcal A}(c, d)]$$

(Theorem II.13). This homomorphism is equivariant with respect to the action of D on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ and $\widehat{D}$, where the action of D on $\widehat{D}$ is given by factorization of the adjoint representation of $\widehat{D}$ to an action of D on $\widehat{D}$ (Lemma III.4). Further b_D induces a continuous Lie algebra homomorphism

$$b_D \colon \langle D, D\rangle \to \widehat{D}$$

(Lemma III.4.3) because for $d, d', e, e' \in D$ we have

$$\begin{aligned}[b_D(\langle d, d'\rangle), b_D(\langle e, e'\rangle)] &= b_D(q_D(b_D(\langle d, d'\rangle)), q_D(b_D(\langle e, e'\rangle))) \\ &= b_D(\langle [d, d'], [e, e']\rangle) = b_D([\langle d, d'\rangle, \langle e, e'\rangle]).\end{aligned}$$

Combining b_D with $\delta^{\widehat{D}}_{\mathcal{A}}$, we get a continuous Lie algebra morphism

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \rtimes \langle D, D\rangle \to \widehat{D}, \quad (\langle a, a'\rangle, \langle d, d'\rangle) \mapsto \delta^{\widehat{D}}(a, a') + b_D(d, d'),$$

and this morphism maps

$$[\langle a, a'\rangle, \langle b, b'\rangle] - \langle \delta^D(a, a'), \delta^D(b, b')\rangle$$

to

$$[\delta^{\widehat{D}}(a, a'), \delta^{\widehat{D}}(b, b')] - b_D(\delta^D(a, a'), \delta^D(b, b')) = 0$$

because $q_D \circ \delta^{\widehat{D}} = \delta^D$. Hence it factors through a morphism

$$q_{\mathfrak{g},D} \colon \widetilde{D} \to \widehat{D}, \quad [(\langle a, a'\rangle, \langle d, d'\rangle)] \mapsto \delta^{\widehat{D}}(a, a') + b_D(d, d').$$

We now obtain a continuous linear map

$$q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \widehat{\mathfrak{g}}, \quad a \otimes x + b \otimes v + d \mapsto a \otimes x + b \otimes v + q_{\mathfrak{g},D}(d),$$

and (B1)–(B3) together with the relation $q_D \circ \delta^{\widehat{D}} = \delta^D$ ((4.2) in Lemma IV.3) imply that this map is a continuous morphism of Lie algebras satisfying $q \circ q_{\mathfrak{g}} = \widetilde{q}$. This proves that $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a universal covering Lie algebra of $\mathfrak{g}$. ■

Definition IV.9. We call a Δ-graded Lie algebra $\mathfrak{g}$ with coordinate structure $(\mathcal{A}, D, \delta^D)$ *regular* if the natural map

$$\delta^D_{\mathcal{A}} \colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D, \quad \langle a, b\rangle \mapsto \delta^D(a, b)$$

is a generalized central extension, i.e., there exists a continuous bilinear map $b_D \colon D \times D \to \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ such that $b_D \circ (\delta^D_{\mathcal{A}} \times \delta^D_{\mathcal{A}})$ is the bracket on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$. ■

Examples IV.10. We continue the discussion from Examples II.16 by showing that all Lie algebras discussed there are regular.
(a) For the algebras of the type $\mathfrak{g} = A \otimes \mathfrak{g}_\Delta$ we have $D = \{0\}$, so that they are regular.
(b) For $\mathfrak{g} = \mathfrak{sl}_n(A)$ we have $\mathcal{A} = A$, $D \cong \overline{[A, A]}$, and

$$\delta^D(a, b) = \frac{1}{2n^2}[a, b].$$

The corresponding Lie bracket on $\langle A, A\rangle$ is given by

$$\begin{aligned}[\langle a, b\rangle, \langle a', b'\rangle] &= \delta^D(a, b).\langle a', b'\rangle = \frac{1}{2n^2}(\langle[[a, b], a'], b'\rangle + \langle a, [[a, b], b']\rangle) \\ &= \frac{1}{2n^2}\langle[a, b], [a', b']\rangle.\end{aligned}$$

Therefore the bilinear map

$$b\colon D \times D \to \langle A, A\rangle, \quad (a, b) \mapsto 2n^2\langle a, b\rangle$$

satisfies

$$b(\delta^D(a, b), \delta^D(a', b')) = \frac{1}{4n^4}b([a, b], [a', b']) = \frac{1}{2n^2}\langle[a, b], [a', b']\rangle = [\langle a, b\rangle, \langle a', b'\rangle],$$

which implies that $\delta^D_A\colon \langle A, A\rangle \to D$ is a generalized central extension, and therefore $\mathfrak{sl}_n(A)$ is regular.
(c) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)'$ we have with the notation from Example II.16

$$D \cong \overline{[\mathcal{A}, \mathcal{A}]}^{-\sigma} \quad \text{and} \quad \delta^D(a, b) = \mu_n([a, b] + [a^\sigma, b^\sigma])$$

for some $\mu_n \in \mathbb{K}$. The corresponding Lie bracket on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ is given by

$$\begin{aligned}[\langle a, b\rangle, \langle a', b'\rangle] &= \delta^D(a, b).\langle a', b'\rangle = \mu_n\langle[a, b] - [a, b]^\sigma, [a', b']\rangle \\ &= \frac{\mu_n}{2}\langle[a, b] - [a, b]^\sigma, [a', b'] - [a', b']^\sigma\rangle.\end{aligned}$$

Therefore the bilinear map

$$b\colon D \times D \to \langle A, A\rangle, \quad (a, b) \mapsto \frac{1}{2\mu_n}\langle a, b\rangle$$

satisfies

$$\begin{aligned}b(\delta^D(a, b), \delta^D(a', b')) &= \mu_n^2 b([a, b] - [a, b]^\sigma, [a', b'] - [a', b']^\sigma) \\ &= \frac{\mu_n}{2}\langle[a, b] - [a, b]^\sigma, [a', b'] - [a', b']^\sigma\rangle = [\langle a, b\rangle, \langle a', b'\rangle],\end{aligned}$$

which implies that $\delta^{\widetilde{D}}\colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D$ is a generalized central extension, so that $\mathfrak{sp}_{2n}(A)'$ is regular.
(d) For $\mathfrak{g} = \mathrm{TKK}(J)$ for a Jordan algebra J we have $D = \langle J, J\rangle \cong \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$, so that $\delta^{\widetilde{D}} = \mathrm{id}$ implies that $\mathfrak{g}$ is regular. ■

The following lemma provides a handy criterion for regularity.

Lemma IV.11. *The Δ-graded Lie algebra $\mathfrak{g}$ with coordinate structure $(\mathcal{A}, D, \delta^D)$ is regular if and only if the natural map*

$$\delta_{\mathcal{A}}^{\widetilde{D}}\colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \widetilde{D}, \quad \langle a, b\rangle \mapsto [(\langle a, b\rangle, 0)]$$

is an isomorphism.

Proof. According to Lemma IV.7, the map $q_{\mathfrak{g},D}\colon \widetilde{D} \to D$ is a generalized central extension. If $\delta_{\mathcal{A}}^{\widetilde{D}}$ is an isomorphism, the composed map $\delta_{\mathcal{A}}^{D}\colon \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to D$ also is a generalized central extension.

If, conversely, $\mathfrak{g}$ is regular, i.e., $\delta_{\mathcal{A}}^{D}$ is a generalized central extension, and $b_D\colon D \times D \to \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$ a continuous bilinear map for which $b_D \circ (\delta_{\mathcal{A}}^{D} \times \delta_{\mathcal{A}}^{D})$ is the bracket on $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma$, then we define

$$\varphi\colon \widetilde{D} \to \langle \mathcal{A}, \mathcal{A}\rangle^\sigma, \quad [(\langle a, a'\rangle, \langle d, d'\rangle)] \mapsto \langle a, a'\rangle + b_D(d, d').$$

That this map is well-defined follows from

$$\begin{aligned} &[\langle a, a'\rangle, \langle b, b'\rangle] - b_D(\delta^D(a, a'), \delta^D(b, b')) \\ &= \ [\langle a, a'\rangle, \langle b, b'\rangle] - b_D(\delta^{\widetilde{D}}(\langle a, a'\rangle), \delta^{\widetilde{D}}(\langle b, b'\rangle)) = 0 \end{aligned}$$

for $a, a', b, b' \in \mathcal{A}$. Moreover, φ is a morphism of Lie algebras:

$$\begin{aligned} &[\langle a, a'\rangle + b_D(d, d'), \langle b, b'\rangle + b_D(e, e')] \\ &= \ [\langle a, a'\rangle, \langle b, b'\rangle] + [d, d'] .\langle b, b'\rangle - [e, e'] .\langle a, a'\rangle + [b_D(d, d'), b_D(e, e')] \\ &= \ [\langle a, a'\rangle, \langle b, b'\rangle] + [d, d'] .\langle b, b'\rangle - [e, e'] .\langle a, a'\rangle + b_D([\langle d, d'\rangle, \langle e, e'\rangle]) \\ &= \ \varphi\Big(\big[[(\langle a, a'\rangle, \langle d, d'\rangle)] , [(\langle b, b'\rangle, \langle e, e'\rangle)] \big]\Big). \end{aligned}$$

We have $\varphi \circ \delta_{\mathcal{A}}^{\widetilde{D}} = \mathrm{id}_{\langle \mathcal{A}, \mathcal{A}\rangle^\sigma}$ and

$$(\delta_{\mathcal{A}}^{\widetilde{D}} \circ \varphi)\big([(\langle a, a'\rangle, \langle d, d'\rangle)]\big) = [(\langle a, a'\rangle + b_D(d, d'), 0)].$$

For $d = \delta^D(a, b)$ and $d' = \delta^D(a', b')$ we have

$$b_D(d, d') = b_D(\delta^D(a, b), \delta^D(a', b')) = b_D(\delta^{\widetilde{D}}(\langle a, b\rangle), \delta^{\widetilde{D}}(\langle a', b'\rangle)) = [\langle a, b\rangle, \langle a', b'\rangle],$$

which, as an element of $\widetilde{D}$, equals $\langle \delta^D(a, b), \delta^D(a', b')\rangle = \langle d, d'\rangle$. Since the image of δ^D spans a dense subspace of D, it follows that

$$[(\langle d, d'\rangle, 0)] = [(0, b_D(d, d'))]$$

for all $d, d' \in D$, and hence that $\delta_{\mathcal{A}}^{\widetilde{D}} \circ \varphi = \mathrm{id}_{\widetilde{D}}$. Therefore $\delta_{\mathcal{A}}^{\widetilde{D}}$ is an isomorphism of locally convex Lie algebras whose inverse is φ. ■

Remark IV.12. (a) The preceding lemma shows that if $\mathfrak{g}$ is regular, then its universal covering Lie algebra is given by

$$\widetilde{\mathfrak{g}} \cong \widetilde{\mathfrak{g}}(\Delta, \mathcal{A}) := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \langle \mathcal{A}, \mathcal{A} \rangle^\sigma$$

with the Lie bracket given by

$$[d, a \otimes x + b \otimes v + d'] = d.a \otimes x + d.b \otimes v + [d, d'],$$

and

$$\begin{aligned}
[a \otimes x, a' \otimes x'] &= \gamma_+^A(a,a') \otimes [x,x'] + \gamma_-^A(a,a') \otimes x * x' + \gamma_A^B(a,a') \otimes \beta_{\mathfrak{g}}^V(x,x') \\
&\quad + \kappa(x,x')\delta_{\mathcal{A}}(a,a'), \\
[a \otimes x, b \otimes v] &= \frac{ab+ba}{2} \otimes \beta_{\mathfrak{g},V}^{\mathfrak{g}}(x,v) + \frac{ab-ba}{2} \otimes x.v, \\
[b \otimes v, b' \otimes v'] &= \gamma_B^A(b,b') \otimes \beta_V^{\mathfrak{g}}(v,v') + \gamma_B^B(b,b') \otimes \beta_V^V(v,v') + \kappa_{V_s}(v,v')\delta_{\mathcal{A}}(b,b')
\end{aligned}$$

with

$$q_{\mathfrak{g}}(a \otimes x + b \otimes v + d) = a \otimes x + b \otimes v + \delta_{\mathcal{A}}^D(d),$$

where $\delta_{\mathcal{A}}^D(\langle a, b \rangle) = \delta^D(a,b)$ for $a, b \in \mathcal{A}$.

(b) If $\mathfrak{g}$ is not regular, then we can still consider the Lie algebra

$$\mathfrak{g}^\sharp := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \langle \mathcal{A}, \mathcal{A} \rangle^\sigma$$

with the coordinate structure $(\mathcal{A}, \langle \mathcal{A}, \mathcal{A} \rangle^\sigma, \delta^{\langle \mathcal{A}, \mathcal{A} \rangle^\sigma})$, where $\delta_{\langle \mathcal{A}, \mathcal{A} \rangle^\sigma}(a,b) = \langle a, b \rangle$, and $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ acts on $\mathcal{A}$ via $\delta_{\mathcal{A}}$ (Theorem II.15). Then the map

$$q^\sharp \colon \mathfrak{g}^\sharp \to \mathfrak{g}, \quad a \otimes x + b \otimes v + d \mapsto a \otimes x + b \otimes v + \delta_{\mathcal{A}}^D(d)$$

is a morphism of locally convex Lie algebras with dense range. The subspace $\ker q^\sharp = \ker \delta_{\mathcal{A}}^D \subseteq \langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ acts trivially on $\mathcal{A}$, hence on $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$, and therefore on $\mathfrak{g}^\sharp$. This means that $\ker q^\sharp$ is central. If $\mathfrak{g}$ is not regular, then $q^\sharp$ is not a generalized central extension.

Nevertheless, $q^\sharp$ has the following universal property: If $q \colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ is a generalized central extension with

$$\widehat{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \widehat{D}$$

(cf. Remark IV.4), then $q_D \colon \widehat{D} \to D$ also is a generalized central extension. As in the proof of Theorem IV.8, we see that the corresponding map $\delta_{\mathcal{A}}^{\widehat{D}} \colon \langle \mathcal{A}, \mathcal{A} \rangle^\sigma \to \widehat{D}$ is a continuous homomorphism of Lie algebras and that we obtain a continuous morphism of Lie algebras

$$\varphi \colon \mathfrak{g}^\sharp \to \widehat{\mathfrak{g}}, \quad a \otimes x + b \otimes v + d \mapsto a \otimes x + b \otimes v + \delta_{\mathcal{A}}^{\widehat{D}}(d)$$

with $q \circ \varphi = q^\sharp$. As $\ker q$ is central, the uniqueness of φ follows from the fact that all Lie algebra homomorphisms $\mathfrak{g}^\sharp \to \ker q \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$ are trivial because $\mathfrak{g}^\sharp$ is topologically perfect. ■

Corollary IV.13. *If $\mathfrak{g}$ is a regular Δ-graded locally convex Lie algebra, then its universal covering Lie algebra $\widetilde{\mathfrak{g}}$ only depends on the pair $(\mathcal{A}, \delta_{\mathcal{A}})$, which in turn is completely determined by the coordinate algebra $\mathcal{A}$ and the type of Δ. If we write $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ for $\widetilde{\mathfrak{g}}$, then the assignment*

$$\mathcal{A} \mapsto \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$$

defines a functor from the category of locally convex algebras determined by the root system Δ to the category of locally convex Lie algebras. ■

Corollary IV.14. *Each Lie algebra $\widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ is centrally closed and in particular regular.*

Proof. For the Lie algebra $\mathfrak{g} := \widetilde{\mathfrak{g}}(\Delta, \mathcal{A})$ we have $D = \langle \mathcal{A}, \mathcal{A} \rangle^\sigma$, so that $\delta_{\mathcal{A}}^D = \mathrm{id}_D$, which trivially is a generalized central extension. Therefore the explicit description of $\widetilde{\mathfrak{g}}$ in Theorem IV.8 implies that $\mathfrak{g}$ is its own universal covering Lie algebra because the universal covering Lie algebra has the same coordinate algebra $\mathcal{A}$. ■

We shall see in Example IV.24 below that there are examples of root graded Lie algebras for which $\widetilde{\mathfrak{g}}$ is not centrally closed.

Remark IV.15. Let $\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D$ be a root graded locally convex Banach–Lie algebra. Let $D_p := \overline{\mathrm{im}(\delta^D)} \subseteq \mathrm{der}(\mathcal{A})$ (the p stands for "projective"), where the closure is to be taken with respect to the norm topology on $\mathrm{der}(\mathcal{A}) \subseteq B(\mathcal{A})$. Then Theorem II.15 applies to the natural corestriction $\delta^{D_p} \colon \mathcal{A} \times \mathcal{A} \to \widehat{D}$, and we obtain a root graded Lie algebra

$$\mathfrak{pg}(\Delta, \mathcal{A}) := (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D_p$$

with the coordinate structure $(\mathcal{A}, D_p, \delta^{D_p})$. It is clear from the construction that the center of the Lie algebra $\mathfrak{pg}(\Delta, \mathcal{A})$ is trivial because D_p acts faithfully on $\mathcal{A}$. Moreover, the adjoint representation $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{der}(\mathfrak{g})$ factors through a continuous linear map

$$\mathfrak{g} \to \mathfrak{pg}(\Delta, \mathcal{A}) \to \mathrm{der}(\mathfrak{g}),$$

and it is easy to see that

$$\mathfrak{pg}(\Delta, \mathcal{A}) \cong \overline{\mathrm{ad}(\mathfrak{g})}$$

because the natural action of $\mathfrak{g}_\Delta$ on $\overline{\mathrm{ad}(\mathfrak{g})}$ directly leads to the structure of a Δ-graded Lie algebra on $\overline{\mathrm{ad}(\mathfrak{g})}$ with coordinate structure $(\mathcal{A}, D_p, \delta^{D_p})$.

This implies that for a Banach–Lie algebra $\mathfrak{g}$, the Lie algebra $\overline{\mathrm{ad}(\mathfrak{g})}$ only depends on $\mathcal{A}$ and Δ, which justifies the notation $\mathfrak{pg}(\Delta, \mathcal{A})$, the *projective Lie algebra associated to Δ and $\mathcal{A}$.*

The Lie algebra $\mathfrak{g}$ is now caught in a diagram of the form

$$\widetilde{\mathfrak{g}}(\Delta, \mathcal{A}) \xrightarrow{p_{\mathfrak{g}}} \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{pg}(\Delta, \mathcal{A})$$

with morphisms with dense range and central kernel which need not be generalized central extensions. ■

IV.3 Lie algebra cocycles on root graded Lie algebras

Proposition IV.16. *Every continuous Lie algebra cocycle on a root graded Lie algebra $\mathfrak{g}$ is equivalent to a $\mathfrak{g}_\Delta$-invariant one.*

Proof. As a module of $\mathfrak{g}_\Delta$, the Lie algebra $\mathfrak{g}$ decomposes topologically as

$$\mathfrak{g} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus D,$$

and therefore

$$\mathfrak{g} \otimes \mathfrak{g} \cong (\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta) \otimes (A \otimes A) \oplus (\mathfrak{g}_\Delta \otimes V_s) \otimes (A \otimes B) + \cdots$$

is the decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}_\Delta$-module, where A, B and D are considered as trivial modules. We conclude that for each trivial locally convex $\mathfrak{g}_\Delta$-module $\mathfrak{z}$ we have

$$\operatorname{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z}) \cong (\mathfrak{g}_\Delta \otimes \mathfrak{g}_\Delta)^* \otimes \operatorname{Lin}(A \otimes A, \mathfrak{z}) \oplus (\mathfrak{g}_\Delta \otimes V_s)^* \otimes \operatorname{Lin}(A \otimes B, \mathfrak{z}) + \cdots$$

Since $\mathfrak{g}_\Delta$ and V_s are finite-dimensional, $\operatorname{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z})$ is a locally finite $\mathfrak{g}_\Delta$-module, hence semisimple. This property is in particular inherited by the submodule $Z^2(\mathfrak{g}, \mathfrak{z}) \subseteq \operatorname{Lin}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{z})$ of continuous Lie algebra cocycles. Hence the decomposition into trivial and effective part yields

$$Z^2(\mathfrak{g}, \mathfrak{z}) = Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \oplus \mathfrak{g}_\Delta.Z^2(\mathfrak{g}, \mathfrak{z}).$$

For the representation ρ of $\mathfrak{g}$ on the space $C^2(\mathfrak{g}, \mathfrak{z})$ of continuous Lie algebra 2-cochains we have the Cartan formula

$$\rho(x) = i_x \circ d + d \circ i_x, \quad x \in \mathfrak{g},$$

which implies that on 2-cocycles we have $\rho(x).\omega = d(i_x.\omega)$ and hence $\mathfrak{g}.Z^2(\mathfrak{g}, \mathfrak{z}) \subseteq B^2(\mathfrak{g}, \mathfrak{z})$. We conclude that each element of $H^2(\mathfrak{g}, \mathfrak{z})$ has a $\mathfrak{g}_\Delta$-invariant representative. ■

Proposition IV.17. *If $\mathfrak{g}$ is a regular Δ-graded Lie algebra, then the $\mathfrak{g}_\Delta$-invariant Lie algebra cocycles $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta}$ are in one-to-one correspondence with the elements of the space $\operatorname{Lin}(\langle \mathcal{A}, \mathcal{A} \rangle^\sigma, \mathfrak{z})$, where we obtain from $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \operatorname{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta}$ a function $\omega_\mathcal{A}$ on $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ by restricting to the subspace $\langle \mathcal{A}, \mathcal{A} \rangle^\sigma$ of $\widetilde{\mathfrak{g}}$.*

The cocycle ω is a coboundary if and only if $\omega_\mathcal{A}$ can be written as $\alpha \circ \delta_\mathcal{A}^D$ for an $\alpha \in \operatorname{Lin}(D, \mathfrak{z})$, so that

$$H^2(\mathfrak{g}, \mathfrak{z}) \cong \operatorname{Lin}(\langle \mathcal{A}, \mathcal{A} \rangle^\sigma, \mathfrak{z}) / \operatorname{Lin}(D, \mathfrak{z}) \circ \delta_\mathcal{A}^D.$$

Proof. If $q_{\mathfrak{g}} \colon \widetilde{\mathfrak{g}} \cong \langle \mathfrak{g}, \mathfrak{g} \rangle \to \mathfrak{g}$ is the universal covering Lie algebra, then we have for each locally convex space $\mathfrak{z}$ a natural isomorphism $Z^2(\mathfrak{g}, \mathfrak{z}) \cong \mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})$ (Remark III.7). As $q_{\mathfrak{g}}$ is equivariant with respect to the action of $\mathfrak{g}_\Delta$, this leads to

$$Z^2(\mathfrak{g}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta}$$

for the invariant Lie algebra cocycles. On the other hand

$$\widetilde{\mathfrak{g}} = (A \otimes \mathfrak{g}_\Delta) \oplus (B \otimes V_s) \oplus \langle \mathcal{A}, \mathcal{A} \rangle^\sigma$$

implies that $\mathrm{Lin}(\widetilde{\mathfrak{g}}, \mathfrak{z})^{\mathfrak{g}_\Delta} \cong \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A} \rangle^\sigma, \mathfrak{z})$.

If $\alpha \in \mathrm{Lin}(D, \mathfrak{z})$, then we extend α to a continuous linear map $\alpha_{\mathfrak{g}} \colon \mathfrak{g} \to \mathfrak{z}$ by zero on the subspaces $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$. Then $d\alpha(x, y) = \alpha([y, x])$ is a $\mathfrak{g}_\Delta$-invariant cocycle on $\mathfrak{g}$, and the corresponding function $(d\alpha)_{\widetilde{\mathfrak{g}}}$ on $\widetilde{\mathfrak{g}} \cong \langle \mathfrak{g}, \mathfrak{g} \rangle$ satisfies $(d\alpha)_{\widetilde{\mathfrak{g}}} = -\alpha \circ b_{\mathfrak{g}}$ which implies that

$$(d\alpha)_{\mathcal{A}} = -\alpha \circ b_{\mathfrak{g}}|_{\langle \mathcal{A}, \mathcal{A} \rangle^\sigma} = -\alpha \circ \delta^D_{\mathcal{A}}.$$

If, conversely, $\omega = d\alpha$ is a $\mathfrak{g}_\Delta$-invariant coboundary, then the same argument as in the proof of Proposition IV.16 implies that we may choose α as a $\mathfrak{g}_\Delta$-invariant function on $\mathfrak{g}$, which means that α vanishes on $A \otimes \mathfrak{g}_\Delta$ and $B \otimes V_s$, hence is of the form discussed above. We conclude that

$$\mathrm{Lin}(D, \mathfrak{z}) \circ \delta^D_{\mathcal{A}} \subseteq \mathrm{Lin}(\langle \mathcal{A}, \mathcal{A} \rangle^\sigma, \mathfrak{z})$$

corresponds to the $\mathfrak{g}_\Delta$-invariant coboundaries. This completes the proof. ∎

The preceding proposition describes the cohomology of $\mathfrak{g}$ with values in a trivial module $\mathfrak{z}$ in terms of the coordinate algebra. For the topological homology space we get

$$H_2(\mathfrak{g}) := \ker q_{\mathfrak{g}} \cong \ker \delta^D_{\mathcal{A}} \subseteq \langle \mathcal{A}, \mathcal{A} \rangle^\sigma,$$

which describes $H_2(\mathfrak{g})$ completely in terms of the coordinate algebra and D.

Definition IV.18. Motivated by the corresponding concept for associative algebras with involution (Appendix D), we define the *full skew dihedral homology of* $\mathcal{A}$, resp., the pair $(\mathcal{A}, \delta_{\mathcal{A}})$ as

$$HF(\mathcal{A}) := \ker \delta_{\mathcal{A}} \subseteq \langle \mathcal{A}, \mathcal{A} \rangle^\sigma.$$ ∎

Proposition IV.19. *If $\mathfrak{g}$ is a regular Δ-graded locally convex Lie algebra, then the centerfree Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is also Δ-graded with the same coordinate algebra and the same universal covering algebra, and*

$$H_2(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) \cong HF(\mathcal{A}).$$

Proof. The first two assertions follow from Corollary IV.2, Proposition IV.17 and Proposition III.16.

With respect to the $\mathfrak{g}_\Delta$-isotypical decomposition of $\mathfrak{g}$, we have

$$\mathfrak{z}(\mathfrak{g}) = \{d \in D \colon (\forall a \in \mathcal{A})\ d.a = 0\},$$

which implies that

$$H_2(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) = \ker q_{\mathfrak{g}/\mathfrak{z}(\mathfrak{g})} = q_{\mathfrak{g}}^{-1}(\mathfrak{z}(\mathfrak{g})) = \mathfrak{z}(\widetilde{\mathfrak{g}}) = \ker \delta_{\mathcal{A}} = HF(\mathcal{A}). \qquad \blacksquare$$

Example IV.20. (a) Let $\mathcal{A}$ be an associative algebra with involution σ, $A := \mathcal{A}^\sigma$, $B := \mathcal{A}^{-\sigma}$, and consider the modified bracket map defined by

$$b_\sigma(x,y) := [x,y] - [x,y]^\sigma = [x,y] - [y^\sigma, x^\sigma] = [x,y] + [x^\sigma, y^\sigma].$$

Then b_σ defines a continuous linear map $\langle \mathcal{A}, \mathcal{A}\rangle^\sigma \to \mathcal{A}^{-\sigma}$, and

$$HD_1'(\mathcal{A},\sigma) := \ker b_\sigma \subseteq \langle \mathcal{A}, \mathcal{A}\rangle^\sigma$$

is called the *first skew-dihedral homology space* of $(\mathcal{A},\sigma)$ (see Appendix D for more information on skew-dihedral homology). The corresponding full dihedral homology space is

$$HF(\mathcal{A}) = b_\sigma^{-1}(Z(\mathcal{A})) = \{x \in \langle \mathcal{A}, \mathcal{A}\rangle^\sigma \colon \operatorname{ad}(b_\sigma(x)) = 0\}.$$

(b) If $\mathcal{A} = A$ is an associative algebra, $B = \{0\}$, and $\delta_A(a,b) = \operatorname{ad}([a,b])$, then

$$\langle \mathcal{A}, \mathcal{A}\rangle^\sigma = \langle A, A\rangle$$

with the Lie algebra structure

$$[\langle a,b\rangle, \langle c,d\rangle] = \langle [a,b],[c,d]\rangle$$

defined in Example III.10(2). If $b_A \colon \langle A, A\rangle \to A, \langle a,b\rangle \mapsto [a,b]$ is the commutator bracket, then

$$HC_1(A) := \ker b_A$$

is the *first cyclic homology of* A, and in this case the full skew-dihedral homology space is the *full cyclic homology space:*

$$HF(A) = b_A^{-1}(Z(A)) \supseteq HC_1(A),$$

where $Z(A)$ is the center of A.

By corestriction of the bracket map b_A, we obtain a generalized central extension of locally convex Lie algebras

$$HC_1(A) \hookrightarrow \langle A, A\rangle \to \overline{[A,A]}.$$

We also have a generalized central extension of locally convex Lie algebras

$$HF(A) \hookrightarrow \langle A, A\rangle \to \overline{[A,A]}/\big(Z(A) \cap \overline{[A,A]}\big).$$

(c) If A is commutative and associative, then $b_A = 0$, so that

$$HF(A) = HC_1(A) = \langle A, A\rangle.$$

A more direct description of this space can be given as follows. Let M be a locally convex A-module in the sense that the module structure $A \times M \to M$ is continuous. A *derivation* $D\colon A \to M$ is a continuous linear map with $D(ab) = a.D(b) + b.D(a)$ for $a, b \in A$. One can show that for each locally convex commutative associative algebra there exists a *universal differential module* $\Omega^1(A)$, which is endowed with a derivation $d\colon A \to \Omega^1(A)$ which has the universal property that for each derivation $D\colon A \to M$ there exists a continuous linear module homomorphism $\varphi\colon \Omega^1(A) \to M$ with $\varphi \circ d = D$ (cf. [Ma02]). We consider the quotient space $\Omega^1(A)/\overline{dA}$ endowed with the locally convex quotient topology. Then we have a natural isomorphism

$$\langle A, A\rangle \to \Omega^1(A)/\overline{dA}, \quad \langle a, b\rangle \mapsto [a \cdot db]. \qquad \blacksquare$$

Example IV.21. (a) Let $n \geq 4$. If $\mathfrak{g} = \mathfrak{sl}_n(A)$ for a locally convex unital associative algebra, then Examples IV.10 and the preceding considerations imply that

$$H_2(\mathfrak{sl}_n(A)) \cong HC_1(A) \quad \text{and} \quad H_2(\mathfrak{psl}_n(A)) \cong HF(A), \tag{4.5}$$

where

$$\mathfrak{psl}_n(A) := \mathfrak{sl}_n(A)/\mathfrak{z}(\mathfrak{sl}_n(A)) \cong \mathfrak{sl}_n(A)/(Z(A) \cap \overline{[A, A]}).$$

If $n = 3$, then $\mathfrak{g}$ is A_2-graded, and we have to consider A as an alternative algebra. Since A is associative, the left and right multiplications L_a and R_b on A commute, so that

$$L_{[a,b]} - R_{[a,b]} - 3[L_a, R_b] = \mathrm{ad}[a, b].$$

This implies that $\langle A, A\rangle$ carries the same Lie algebra structure, regardless of whether we consider it as an associative or an alternative algebra. We conclude that (4.5) remains true for $n = 3$.

For $n = 2$ the coordinate algebra of $\mathfrak{sl}_2(A)$ is the Jordan algebra $\mathcal{A} = A_J$ with the product $a \circ b = \frac{ab+ba}{2}$. Let $L_a(x) = ax$ and $R_a(x) = xa$ denote the left and right multiplications in the associative algebra A, and $L_a^J(x) = \frac{1}{2}(L_a + R_a)$ the left multiplication in the corresponding Jordan algebra. Then

$$\begin{aligned} 8\delta_{A_J}(a, b) &= 4[L_a^J, L_b^J] = [L_a + R_a, L_b + R_b] = [L_a, L_b] + [R_a, R_b] \\ &= L_{[a,b]} - R_{[a,b]} = \mathrm{ad}[a, b]. \end{aligned}$$

For $\mathfrak{g} = \mathfrak{sl}_2(A)$ we also have $D = \overline{[A, A]}$ and

$$\delta_{A_J}^D(a, b) = \frac{1}{2}[a, b]$$

(Example II.16(b)). We therefore obtain

$$H_2(\mathfrak{sl}_2(A)) \cong \ker \delta^D_{A_J} \quad \text{and} \quad H_2(\mathfrak{psl}_2(A)) \cong HF(A_J).$$

In the algebraic context, the preceding results have been obtained for $n = 2$ by Gao ([Gao93]), and for $n \geq 3$ by Kassel and Loday ([KL82]).
(b) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)$ (Example I.7, Example II.16(c)) the coordinate algebra is an associative algebra $\mathcal{A}$ with involution. For

$$\mathfrak{psp}_{2n}(\mathcal{A}, \sigma) := \mathfrak{sp}_{2n}(\mathcal{A}, \sigma)/\mathfrak{z}(\mathfrak{sp}_{2n}(\mathcal{A}, \sigma)),$$

we therefore obtain

$$H_2(\mathfrak{psp}_{2n}(\mathcal{A}, \sigma)) \cong HF(\mathcal{A})$$

and $H_2(\mathfrak{sp}_{2n}(\mathcal{A}, \sigma))$ is isomorphic to the kernel of the map

$$\langle \mathcal{A}, \mathcal{A} \rangle^\sigma \to \overline{[\mathcal{A}, \mathcal{A}]}^{-\sigma}, \quad \langle a, b \rangle \mapsto [a, b] + [a^\sigma, b^\sigma].$$

(c) If J is a Jordan algebra, then it follows from the construction in Example I.9 and our explicit description of the centrally closed Δ-graded Lie algebras in this section that $\widetilde{\mathrm{TKK}}(J)$ is centrally closed, hence the notation. In the sense of Corollary IV.13, we could also write $\widetilde{\mathrm{TKK}}(J) = \widetilde{\mathfrak{g}}(A_2, J)$. ■

Example IV.22. In general it is not always easy to determine the space $HC_1(A)$ for a concrete commutative locally convex algebra. The following cases are of particular interest for applications:
(1) $\Omega^1(A) = \{0\}$ for any commutative C^*-algebra A (Johnson, 1972; see [BD73, Prop. VI.14]).
(2) If M is a connected finite-dimensional smooth manifold and $A = C^\infty(M, \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then A is a Fréchet algebra (a Fréchet space with continuous algebra multiplication). If $\Omega^1(M, \mathbb{K})$ is the space of smooth $\mathbb{K}$-valued 1-forms on M, then the differential

$$d\colon C^\infty(M, \mathbb{K}) \to \Omega^1(M, \mathbb{K}), \quad f \mapsto df$$

has the universal property, and therefore

$$\Omega^1(A) \cong \Omega^1(M, \mathbb{K}) \quad \text{and} \quad HC_1(A) \cong \Omega^1(M, \mathbb{K})/dC^\infty(M, \mathbb{K})$$

([Ma02]).

A similar result holds for the locally convex algebra $A = C^\infty_c(M, \mathbb{K})$ of smooth functions with compact support, endowed with the locally convex direct limit topology with respect to the Fréchet spaces $C^\infty_K(M, \mathbb{K})$ of all those functions whose support is contained in a fixed compact subset $K \subseteq M$. In this case we have

$$\Omega^1(A) \cong \Omega^1_c(M, \mathbb{K}) \quad \text{and} \quad HC_1(A) \cong \Omega^1_c(M, \mathbb{K})/dC^\infty_c(M, \mathbb{K})$$

([Ma02], [Ne02d]).

(3) If M is a complex manifold, then the algebra $A := \mathcal{O}(M)$ of $\mathbb{C}$-valued holomorphic functions is a Fréchet algebra with respect to the topology of uniform convergence on compact subsets of M. Assume that M can be realized as an open submanifold of a closed submanifold of some $\mathbb{C}^n$, i.e., as an open subset of a Stein manifold. Let $\Omega^1_{\mathcal{O}}(M)$ be the space of holomorphic 1-forms on M. Then it is shown in [NW03] that the differential

$$d\colon \mathcal{O}(M) \to \Omega^1_{\mathcal{O}}(M), \quad f \mapsto df$$

has the universal property, and therefore

$$\Omega^1(A) \cong \Omega^1_{\mathcal{O}}(M) \quad \text{and} \quad HC_1(A) \cong \Omega^1_{\mathcal{O}}(M)/d\mathcal{O}(M). \qquad \blacksquare$$

Example IV.23. We construct two root graded Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which are isogenous, non-isomorphic, but have trivial center.

Let A be a locally convex associative unital algebra with $A = \overline{[A, A]} \oplus \mathbb{K}\mathbf{1}$ and $Z(A) = \mathbb{K}\mathbf{1}$. Then the center of

$$\mathfrak{sl}_n(A) \cong A \otimes \mathfrak{sl}_n(\mathbb{K}) \oplus \overline{[A, A]} \otimes \mathbf{1}$$

is trivial.

For the associative Banach algebra $B_2(H)$ of Hilbert-Schmidt operators on an infinite-dimensional Hilbert space H we consider the associated unital Banach algebra $A := B_2(H) + \mathbb{K}\mathbf{1}$. Then

$$\langle A, A\rangle = \langle B_2(H), B_2(H)\rangle$$

follows from $\langle A, \mathbf{1}\rangle = \{0\}$. If $\mathfrak{gl}_2(H) := B_2(H)_L$ is the Lie algebra obtained from $B_2(H)$ via the commutator bracket, then we have seen in Proposition III.19 that $\widetilde{\mathfrak{gl}}_2(H) = \langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle \cong \mathfrak{sl}(H)$, and the universal Lie algebra cocycle is the commutator bracket

$$\omega_u\colon \mathfrak{gl}_2(H) \times \mathfrak{gl}_2(H) \to \mathfrak{sl}(H).$$

On the other hand the discussion in Example III.10(2) shows that the space $\langle B_2(H), B_2(H)\rangle$ obtained from the associative algebra structure is a quotient of $\langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle$. As the bracket map $q_{\mathfrak{gl}_2(H)}\colon \langle \mathfrak{gl}_2(H), \mathfrak{gl}_2(H)\rangle \to \mathfrak{gl}_2(H)$ is injective, $\langle B_2(H), B_2(H)\rangle$ must be the quotient by the trivial subspace, and therefore the bracket map

$$\langle B_2(H), B_2(H)\rangle \to \mathfrak{sl}(H), \quad \langle a, b\rangle \mapsto [a, b]$$

is an isomorphism of Banach spaces.

Let $n \geq 3$. Then the natural morphism

$$\widetilde{\mathfrak{sl}}_n(A) \cong (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus \langle A, A \rangle \to \mathfrak{sl}_n(A)$$

is injective, and hence $\widetilde{\mathfrak{sl}}_n(A)$ has trivial center. As the map $\mathfrak{sl}(H) \to B_2(H)$ is not surjective, the two A_{n-1}-graded Lie algebras $\widetilde{\mathfrak{sl}}_n(A)$ and $\mathfrak{sl}_n(A)$ both have trivial center but are not isomorphic. ∎

Example IV.24. We describe examples of non-regular locally convex root graded Lie algebras. As in the preceding example, we consider the associative algebra $\mathcal{A} := A := \mathbb{K}\mathbf{1} + B_2(H)$, where H is a $\mathbb{K}$-Hilbert space. Then for each $p > 1$ the Lie algebra $D := \mathfrak{gl}_p(H)$ of operators of Schatten class p acts continuously by derivations on A via $d.a := [d, a]$ (Definition III.18). Moreover, the bracket defines a continuous bilinear map

$$\delta^D \colon A \times A \to D, \quad (a, b) \mapsto [a, b].$$

Applying Theorem II.15 to the A_{n-1}-graded Lie algebra $\mathfrak{sl}_n(A)$ for $n \geq 3$, we obtain an A_{n-1}-graded Lie algebra

$$\mathfrak{g} = (A \otimes \mathfrak{sl}_n(\mathbb{K})) \oplus D$$

with the coordinate structure (A, D, δ^D).

We have seen in Example IV.22 that $\langle A, A \rangle \cong \mathfrak{sl}(H)$, where the bracket map corresponds to the natural inclusion $\mathfrak{sl}(H) \hookrightarrow B_2(H) \hookrightarrow A$. Further Proposition III.19 shows that the universal covering Lie algebra $\langle D, D \rangle$ of D is $\mathfrak{sl}(H)$ for $1 < p \leq 2$ and $\mathfrak{gl}_{\frac{p}{2}}(H)$ for $p > 2$. This determines the Lie algebra $\langle A, A \rangle \rtimes \langle D, D \rangle$. The ideal I is generated by the elements of the form

$$(d.\langle a, a' \rangle, -\langle d, \delta^D(a, a') \rangle) = ([d, [a, a']], -[d, [a, a']]), \quad a, a' \in A, d \in D.$$

As the subset $[D, \mathfrak{sl}(H)]$ is dense in $\mathfrak{sl}(H)$, it follows that

$$I = \{(x, -x) \colon x \in \mathfrak{sl}(H)\},$$

which implies that

$$\widetilde{D} \cong \langle D, D \rangle$$

is the universal covering algebra of D.

Now Theorem IV.8 implies that the universal covering algebra $\widetilde{\mathfrak{g}}$ has the coordinate structure $(A, \widetilde{D}, \delta^{\widetilde{D}})$. For $p > 2$ the map

$$\delta^D_{\mathcal{A}} \colon \langle A, A \rangle \cong \mathfrak{sl}(H) \to D$$

is an inclusion with dense range, but not a generalized central extension, because there exists no continuous projection $\mathfrak{gl}_{\frac{p}{2}}(H) \to \mathfrak{sl}(H)$. Hence $\mathfrak{g}$ is not regular for $p > 2$. Furthermore, $\widetilde{\mathfrak{g}}$ is a Lie algebra of the same type as $\mathfrak{g}$, so that we can iterate the preceding arguments to determine $\widetilde{\widetilde{\mathfrak{g}}}$. Now Proposition III.19 shows that $\widetilde{\mathfrak{g}}$ is not centrally closed for $2 < p < \infty$. For $p = \infty$ we have $\widetilde{D} \cong \langle D, D \rangle \cong D$, so that $\widetilde{\mathfrak{g}}$ is centrally closed. For $p = 1$ we obtain the Lie algebra $\widetilde{\mathfrak{g}}(A_{n-1}, A)$ which is centrally closed by Corollary IV.14. ∎

V Perspectives: Root graded Lie groups

In this section we briefly discuss some aspects of the global Lie theory of root graded Lie algebras, namely root graded Lie groups.

An infinite-dimensional Lie group G is a manifold modeled on a locally convex space $\mathfrak{g}$ which carries a group structure for which the multiplication and the inversion map are smooth ([Mi83], [Gl01a], [Ne02b]). The space of left invariant vector fields on G is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. Identifying elements of the tangent space $\mathfrak{g} := T_{\mathbf{1}}(G)$ of G in the identity $\mathbf{1}$ with left invariant vector fields, we obtain on $\mathfrak{g}$ the structure of a *locally convex Lie algebra* $\mathbf{L}(G)$. That the so obtained Lie bracket on $\mathfrak{g}$ is continuous follows most easily from the observation that if we consider the group multiplication in local coordinates, where the identity element $\mathbf{1} \in G$ corresponds to $0 \in \mathfrak{g}$, then the first two terms of its Taylor expansion are given by

$$x * y = x + y + b(x, y) + \cdots ,$$

where the quadratic term $b\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear with

$$[x, y] = b(x, y) - b(y, x).$$

We call a locally convex Lie algebra $\mathfrak{g}$ *integrable* if there exists a Lie group G with $\mathbf{L}(G) = \mathfrak{g}$. A Lie group G is said to be Δ-*graded* if its Lie algebra $\mathbf{L}(G)$ is Δ-graded. The question when a root graded Lie algebra $\mathfrak{g}$ is integrable can be quite difficult.

According to Lie's Third Theorem, every finite-dimensional Lie algebra is integrable, but this is no longer true for infinite-dimensional locally convex Lie algebras. If $\mathfrak{g}$ is a Banach–Lie algebra, then the Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ always is integrable. Let $\mathrm{PG}(\mathfrak{g})$ denote a corresponding connected Lie group. Then there is a natural homomorphism of abelian groups, called the *period homomorphism*

$$\mathrm{per}_{\mathfrak{g}}\colon \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\mathfrak{g}),$$

and $\mathfrak{g}$ is integrable if and only if the image of $\mathrm{per}_{\mathfrak{g}}$ is discrete. For general locally convex Lie algebras the situation is more complicated, but if $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g} = \mathbf{L}(G)$ is a central extension with a sequentially complete locally convex space $\mathfrak{z}$ as kernel and a continuous linear section, then there is a period homomorphism

$$\mathrm{per}\colon \pi_2(G) \to \mathfrak{z},$$

and the existence of a Lie group $\widehat{G}$ with $\mathbf{L}(\widehat{G}) = \widehat{\mathfrak{g}}$ depends on the discreteness of the image of per ([Ne02a], [Ne03a]). For finite-dimensional groups these obstructions are vacuous because $\pi_2(G)$ always vanishes by a theorem of É. Cartan ([Mim95, Th. 3.7]).

For the class of root graded Banach–Lie algebras the situation can be described very well by period maps. In this case the Lie algebra $\mathfrak{g}$ is integrable if and only if the image of $\mathrm{per}_{\mathfrak{g}}$ is discrete. As the universal covering $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ also is a universal covering of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \widetilde{\mathfrak{g}}/\mathfrak{z}(\widetilde{\mathfrak{g}})$ (Remark III.15), we obtain a similar criterion for the integrability of $\widetilde{\mathfrak{g}}$ via a period map

$$\mathrm{per}_{\widetilde{\mathfrak{g}}} \colon \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\widetilde{\mathfrak{g}}) = HF(\mathcal{A}),$$

where $\mathcal{A}$ is the coordinate algebra of $\mathfrak{g}$ and $HF(\mathcal{A})$ is its full skew-dihedral homology. If $\mathfrak{g}_1$ is a quotient of $\widetilde{\mathfrak{g}}$ by a central subspace and $\widetilde{\mathfrak{g}}$ is integrable, then $\mathfrak{g}_1$ is integrable if and only if the period map

$$\mathrm{per}_{\mathfrak{g}_1} \colon \pi_2(\mathrm{PG}(\mathfrak{g})) \to \mathfrak{z}(\mathfrak{g}_1)$$

obtained by composing $\mathrm{per}_{\widetilde{\mathfrak{g}}}$ with the natural map $\mathfrak{z}(\widetilde{\mathfrak{g}}) \to \mathfrak{z}(\mathfrak{g}_1)$ has discrete image.

For general locally convex root graded Lie algebras which are not Banach–Lie algebras the situation is less clear, but there are many important classes of locally convex root graded Lie algebras, to which many results from the Banach context can be extended, namely the Lie algebras related to matrix algebras over continuous inverse algebras. A *unital continuous inverse algebra* (CIA) is a unital locally convex algebra A for which the unit group $A^\times$ is open and the inversion is a continuous map $A^\times \to A, a \mapsto a^{-1}$. Typical associated root graded Lie algebras are the A_{n-1}-graded Lie algebra $\mathfrak{sl}_n(A)$, and for a commutative CIA the Lie algebras of the type $\mathfrak{g} = A \otimes \mathfrak{g}_\Delta$ (cf. [Gl01b]). Further examples are the Lie algebras $\mathfrak{sp}_{2n}(A,\sigma)$ and $\mathfrak{o}_{n,n}(A,\sigma)$ discussed in Section I. For Jordan algebras the situation is more complicated, but in this context there also is a natural concept of a *continuous inverse Jordan algebra*, which is studied in [BN03], and can be applied to show that certain related Lie algebras are integrable.

Both classes lead to interesting questions in non-commutative geometry because for a sequentially complete CIA the discreteness of the image of the period map for $\widetilde{\mathfrak{sl}}_n(A)$ follows from the discreteness of the image of a natural homomorphism

$$P_A^3 \colon K_3(A) \to HC_1(A) \cong H_2(\mathfrak{sl}_n(A)),$$

where $K_3(A) := \varinjlim \pi_2(\mathrm{GL}_n(A))$ is the third topological K-group of the algebra A. If, in addition, A is complex, Bott periodicity implies that

$$K_3(A) \cong K_1(A) := \varinjlim \pi_0(\mathrm{GL}_n(A)),$$

and the latter group is much better accessible. In particular, we get a period map

$$P_A^1 \colon K_1(A) \to HC_1(A).$$

One can show that this homomorphism is uniquely determined as a natural transformation between the functors K_1 and HC_1, which permits us to evaluate it for many concrete CIAs ([Ne03a]). If P_A has discrete image, then $\widetilde{\mathfrak{sl}}_n(A)$ is integrable, but the converse is not clear and might even be false. Nevertheless, one can construct certain Fréchet CIAs which are quantum tori of dimension three, for which the Lie algebra $\widetilde{\mathfrak{sl}}_n(A)$ is not integrable. For the details of these constructions we refer to [Ne03a].

There is also a purely algebraic approach to groups corresponding to root graded Lie algebras. Here we associate to a root graded Lie algebra $\mathfrak{g}$ the corresponding *projective group*

$$\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g}) := \langle e^{\operatorname{ad}\mathfrak{g}_\alpha} \colon \alpha \in \Delta\rangle \subseteq \operatorname{Aut}(\mathfrak{g}).$$

As each derivation $\operatorname{ad} x$, $x \in \mathfrak{g}_\alpha$, of $\mathfrak{g}$ is nilpotent, the operator $e^{\operatorname{ad} x}$ is a well-defined automorphism of $\mathfrak{g}$ (cf. [Ti66], [Ze94]). The group $\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g})$ can easily seen to be perfect, so that it has a universal covering group (a universal central extension) $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g})$. Let $\mathrm{PG}(\mathfrak{g})$ be a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. There are many interesting problems associated with these groups:

(1) Describe $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g})$ by generators and relations.

(2) Show that $\mathrm{PG}(\mathfrak{g})$ is a topologically perfect group. When is it perfect?

(3) Suppose that $\widetilde{\mathrm{G}}(\mathfrak{g})$ is a Lie group with Lie algebra $\widetilde{\mathfrak{g}}$. Describe the kernel of the universal covering $\widetilde{\mathrm{G}}(\mathfrak{g}) \to \mathrm{PG}(\mathfrak{g})$ in terms of the coordinate algebra.

(4) Is there a homomorphism $\mathrm{PG}^{\mathrm{alg}}(\mathfrak{g}) \to \mathrm{PG}(\mathfrak{g})$?

(5) Is there a homomorphism $\widetilde{\mathrm{G}}^{\mathrm{alg}}(\mathfrak{g}) \to \widetilde{\mathrm{G}}(\mathfrak{g})$?

It is an interesting project to clarify the precise relation between the Lie theoretic (analytic) approach to root graded groups and the algebraic one.

Appendix A. Some generalities on representations

In this section we collect some material on finite-dimensional representations of reductive Lie algebras, which is used in Sections II and III of this paper. All results in this appendix are valid over any field $\mathbb{K}$ of characteristic zero.

Let $\mathfrak{r}$ be a finite-dimensional split reductive Lie algebra over the field $\mathbb{K}$ of characteristic zero and $\mathfrak{h} \subseteq \mathfrak{r}$ a splitting Cartan subalgebra. We fix a positive system Δ^+ of roots of $\mathfrak{r}$ with respect to $\mathfrak{h}$ and write $L(\lambda)$ for the simple $\mathfrak{r}$-module of highest weight $\lambda \in \mathfrak{h}^*$ with respect to Δ^+. We write $Z := Z(U(\mathfrak{r}))$ for the center of the enveloping algebra $U(\mathfrak{r})$ of $\mathfrak{r}$. Recall that for each highest weight module V we have $\operatorname{End}_{\mathfrak{r}}(V) = \mathbb{K}\mathbf{1}$ because the highest weight space is one-dimensional and cyclic. Therefore Z acts by scalar multiples of the identity on $L(\lambda)$, and we obtain for each λ an algebra homomorphism $\chi_\lambda\colon Z \to \mathbb{K}$, the corresponding central character.

The following theorem permits us to see immediately that certain modules are locally finite. We call an $\mathfrak{r}$-module an $\mathfrak{h}$-*weight module* if it is the direct sum of the common $\mathfrak{h}$-eigenspaces. An $\mathfrak{h}$-weight module V of a split reductive Lie algebra $\mathfrak{r}$ is called *integrable* if for each $x_\alpha \in \mathfrak{r}_\alpha$ the operator $\operatorname{ad} x_\alpha$ is locally nilpotent.

Theorem A.1. *For an $\mathfrak{h}$-weight module V of the finite-dimensional split reductive Lie algebra $\mathfrak{r}$ with splitting Cartan subalgebra $\mathfrak{h}$ the following assertions hold:*

(1) *If V is integrable, then V is locally finite and semisimple.*

(2) *If $\operatorname{supp}(V) := \{\alpha \in \mathfrak{h}^*\colon V_\alpha \neq \{0\}\}$ is finite, then V is integrable.*

Proof. (1) Let V be an integrable $\mathfrak{r}$-module and $\Delta := \{\alpha_1, \ldots, \alpha_m\}$. Then

$$\mathfrak{r} = \mathfrak{h} \oplus \mathfrak{r}_{\alpha_1} \oplus \ldots \oplus \mathfrak{r}_{\alpha_m},$$

so that the Poincaré–Birkhoff–Witt Theorem implies

$$U(\mathfrak{r}) = U(\mathfrak{h})U(\mathfrak{r}_{\alpha_1}) \cdots U(\mathfrak{r}_{\alpha_m}).$$

Since V is integrable, it is by definition a locally finite module for each of the one-dimensional Lie algebras $\mathfrak{r}_\alpha$, $\alpha \in \Delta$. Hence for each vector $v \in V$ we see inductively that the space

$$U(\mathfrak{r}_{\alpha_j}) \cdots U(\mathfrak{r}_{\alpha_m}).v$$

is finite-dimensional for $j = m, m-1, \ldots, 1$, and finally that $U(\mathfrak{r}).v$ is finite-dimensional. Therefore V is a locally finite $\mathfrak{r}$-module.

Let $F \subseteq V$ be a finite-dimensional submodule. Since F is a weight module, it is a direct sum of the common eigenspaces for $\mathfrak{z}(\mathfrak{r}) \subseteq \mathfrak{h}$, which are $\mathfrak{r}$-submodules. According to Weyl's Theorem, these common eigenspaces are semisimple modules of the semisimple Lie algebra $\mathfrak{r}' := [\mathfrak{r}, \mathfrak{r}]$, hence also of $\mathfrak{r} = \mathfrak{r}' + \mathfrak{z}(\mathfrak{r})$. Therefore F is a sum of simple submodules, and the same conclusion holds for the locally finite module V. As a sum of simple submodules, the module V is semisimple ([La93, XVII, §2]).

(2) If $\operatorname{supp}(V)$ is finite, then $x_\alpha.V_\beta \subseteq V_{\beta+\alpha}$ for $\beta \in \operatorname{supp}(V)$ and $\alpha \in \Delta$ imply that the root vectors x_α act as locally nilpotent operators on V. ■

The preceding theorem is a special case of a much deeper theorem on Kac–Moody algebras. According to the Kac–Peterson Theorem, each integrable module in category $\mathcal{O}$ is semisimple ([MP95, Th. 6.5.1]). This implies in particular that integrable modules of finite-dimensional split reductive Lie algebras are semisimple.

Proposition A.2. *Let V be an $\mathfrak{h}$-weight module of $\mathfrak{r}$ for which* $\mathrm{supp}(V)$ *is finite. Then the following assertions hold:*

(1) *V is a semisimple $\mathfrak{r}$-module with finitely many isotypic components $V_1, \ldots, V_n$.*

(2) *The simple submodules of V are finite-dimensional highest weight modules $L(\lambda_1), \ldots, L(\lambda_n)$.*

(3) *For each $j \in \{1, \ldots, n\}$ there exists a central element z_j in $U(\mathfrak{g}_\Delta)$ with $\chi_{\lambda_k}(z_j) = \delta_{jk}$. In particular, z_j acts on V as the projection onto the isotypic component V_j.*

Proof. (1), (2) First Theorem A.1 implies that V is semisimple. Moreover, each simple submodule is a finite-dimensional weight module, hence isomorphic to some $L(\lambda)$. As $\mathrm{supp}(V)$ is finite, there are only finitely many possibilities for the highest weights λ.
(3) According to Harish-Chandra's Theorem ([Dix74, Prop. 7.4.7]), for $\lambda, \mu \in \mathfrak{h}^*$ we have

$$\chi_\lambda = \chi_\mu \quad \Leftrightarrow \quad \mu + \rho \in \mathcal{W}.(\lambda + \rho),$$

where $\mathcal{W}$ is the Weyl group of $(\mathfrak{r}, \mathfrak{h})$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. If $L(\lambda)$ and $L(\mu)$ are finite-dimensional, then λ and μ are dominant integral. Therefore $\lambda+\rho$ and $\mu+\rho$ are dominant, so that $\mu+\rho \in \mathcal{W}.(\lambda+\rho)$ implies $\lambda = \mu$. Hence two non-isomorphic finite-dimensional highest weight modules $L(\lambda)$ and $L(\mu)$ have different central characters.

This proves that the central characters $\chi_{\lambda_1}, \ldots, \chi_{\lambda_n}$ corresponding to the isotypic components of V are pairwise different. As the kernel of a character is a hyperplane ideal, this means that for $i \neq j$ we have

$$\ker \chi_{\lambda_i} + \ker \chi_{\lambda_j} = Z.$$

Now the Chinese Remainder Theorem ([La93, Th. II.2.1]) implies that the map

$$\chi\colon Z \to \mathbb{K}^n, \quad z \mapsto (\chi_{\lambda_1}(z), \ldots, \chi_{\lambda_n}(z))$$

is surjective. Finally (3) follows with $z_i := \chi^{-1}(e_i)$, where $e_1, \ldots, e_n \in \mathbb{K}^n$ are the standard basis vectors. ∎

For the following lemma, we recall the definition of Lie algebra cohomology from [We95].

Lemma A.3. *If $\mathfrak{s}$ is a finite-dimensional semisimple Lie algebra and V a locally finite $\mathfrak{s}$-module, then*

$$H^p(\mathfrak{s}, V) = \{0\} \quad \textit{for} \quad p = 1, 2.$$

Proof. As V is a direct sum of finite-dimensional modules V_j, $j \in J$, the relations

$$C^p(\mathfrak{s}, V) \cong \bigoplus_{j \in J} C^p(\mathfrak{s}, V_j) \quad \text{easily lead to} \quad H^p(\mathfrak{s}, V) \cong \bigoplus_{j \in J} H^p(\mathfrak{s}, V_j),$$

so that the assertion follows from the Whitehead Lemmas ([We95, Cor. 7.8.10/12]), saying that $H^p(\mathfrak{s}, V_j)$ vanishes for each j and $p = 1, 2$. ■

Proposition A.4. *Let $\mathfrak{s}$ be a semisimple finite-dimensional Lie algebra $\mathfrak{s}$.*

(1) *Each extension $Z \hookrightarrow \widehat{M} \xrightarrow{q} M$ of a locally finite $\mathfrak{s}$-module M by a trivial module Z is trivial.*

(2) *Each extension $M \hookrightarrow \widehat{M} \xrightarrow{q} Z$ of a trivial $\mathfrak{s}$-module Z by a locally finite $\mathfrak{s}$-module M is trivial.*

Proof. (1) If $\widehat{M}$ is locally finite, then Weyl's Theorem implies that it is semisimple, and therefore that the extension of M by Z splits. Hence it suffices to show that $\widehat{M}$ is locally finite. Let $v \in \widehat{M}$. We have to show that v generates a finite-dimensional submodule. Since the $\mathfrak{s}$-submodule of M generated by $q(v)$ is finite-dimensional, we may replace M by this module and hence assume that M is finite-dimensional. Now

$$\operatorname{Ext}(M, Z) \cong H^1(\mathfrak{s}, \operatorname{Hom}(M, Z))$$

([We95, Ex. 7.4.5]), and $\operatorname{Hom}(M, Z) \cong M^* \otimes Z$ is a locally finite module, so that

$$H^1(\mathfrak{s}, \operatorname{Hom}(M, Z)) = \{0\}$$

(Lemma A.3). Therefore the module extension splits, and in particular $\widehat{M}$ is locally finite.

(2) First we show that $\widehat{M}$ is locally finite. Let $v \in \widehat{M}$. To see that v generates a finite-dimensional submodule, we may assume that Z is one-dimensional. Then $\operatorname{Hom}(Z, M) \cong M$ is a locally finite $\mathfrak{s}$-module, and the same argument as in (1) above implies that the extension $\widehat{M} \to Z$ is trivial. In particular, we conclude that $\widehat{M}$ is locally finite.

Returning to the general situation, we obtain from Weyl's Theorem that the locally finite module $\widehat{M}$ is semisimple, hence in particular that $\widehat{M} = \mathfrak{g}.\widehat{M} \oplus \widehat{M}^{\mathfrak{g}}$. As Z is trivial, we have $\mathfrak{g}.\widehat{M} \subseteq M$, so that each subspace of $\widehat{M}^{\mathfrak{g}}$ complementing $M \cap \widehat{M}^{\mathfrak{g}}$ yields a module complement to M. ■

Appendix B. Jordan algebras and alternative algebras

In this appendix we collect some elementary results on Jordan algebras.

Jordan algebras

Definition B.1. A finite dimensional vector space J over a field $\mathbb{K}$ is said to be a *Jordan algebra* if it is endowed with a bilinear map $J \times J \to J, (x,y) \mapsto xy$ satisfying:
(JA1) $xy = yx$.
(JA2) $x(x^2y) = x^2(xy)$, where $x^2 := xx$. ■

In this section J denotes a Jordan algebra and $(a,b) \mapsto L(a)b := ab = ba$ the multiplication of J. Then (JA2) means that

$$[L(a), L(a^2)] = 0 \quad \text{for all} \quad a \in J.$$

Proposition B.2. *For a Jordan algebra J over a field $\mathbb{K}$ with $\{2,3\} \subseteq \mathbb{K}^\times$ the following assertions hold for $x,y,z \in J$.*
(1) $[L(x), L(yz)] + [L(y), L(zx)] + [L(z), L(xy)] = 0$.
(2) $L(x(yz) - y(xz)) = [[L(x), L(y)], L(z)]$.

Proof. Passing to the first derivative of (JA2) with respect to x in the direction of z leads to

$$z(x^2y) + 2x((xz)y) = 2(xz)(xy) + x^2(zy)$$

for $x,y,z \in J$. Passing again to the derivative with respect to x in the direction of u leads to

$$z((xu)y) + u((xz)y) + x((uz)y) = (uz)(xy) + (xz)(uy) + (xu)(zy)$$

for $u,x,y,z \in J$. This means that

$$[L(z), L(xu)] + [L(u), L(xz)] + [L(x), L(uz)] = 0,$$

or, by interpreting each term as a function of u,

$$L(xy)L(z) + L(zx)L(y) + L(yz)L(x) = L(z)L(y)L(x) + L\Big((zx)y\Big) + L(x)L(y)L(z).$$

Note that the expression

$$L(xy)L(z) + L(zx)L(y) + L(yz)L(x)$$

is invariant under any permutation of x,y,z. By exchanging x and y and subtracting, we therefore obtain

$$[[L(x), L(y)], L(z)] = L\Big((zy)x\Big) - L\Big((zx)y\Big) = L\Big(x(yz) - y(xz)\Big).$$

■

Corollary B.3.

$$[L(J), L(J)] \subseteq \operatorname{der}(J) := \{D \in \operatorname{End}(J) \colon (\forall x, y \in J) D.(xy) = (D.x)y + x(D.y)\}.$$

Proof. This means that for $x, y \in J$ the operator $D := [L(x), L(y)]$ is a derivation of J, which in turn means that

$$[D, L(z)] = L(D.z), \quad z \in J.$$

This is a reformulation of Proposition B.2(2). ■

Jordan algebras associated to bilinear forms

Lemma B.4. *Let A be a commutative associative algebra, B an A-module and $\beta \colon B \times B \to A$ a symmetric bilinear form which is invariant in the sense that*

$$a\beta(b, b') = \beta(ab, b') = \beta(b, ab'), \quad a \in A, b, b' \in B.$$

Then $\mathcal{A} := A \oplus B$ is a Jordan algebra with respect to

$$(a, b)(a', b') := (aa' + \beta(b, b'), ab' + a'b).$$

Proof. First we note that

$$L(a, 0)(a', b') = (aa', ab') \quad \text{and} \quad L(0, b)(a', b') = (\beta(b, b'), a'b).$$

The set $L(A, 0) \subseteq \operatorname{End}(\mathcal{A})$ is commutative because A is a commutative algebra. Further

$$L(0, b)L(a, 0)(a', b') = (\beta(b, ab'), aa'b) = L(a, 0)L(0, b)(a', b')$$

implies that $L(A, 0)$ commutes with $L(0, B)$, so that $L(A, 0)$ is central in the subspace $L(\mathcal{A})$ of $\operatorname{End}(\mathcal{A})$.

It is clear that $\mathcal{A}$ is commutative. To see that it is a Jordan algebra, we have to verify that each $L(a, b)$ commutes with

$$L((a, b)^2) = L(a^2 + \beta(b, b), 2ab).$$

As $L(A, 0)$ is central in $L(\mathcal{A})$, it suffices to show that $L(0, b)$ commutes with $L(0, ab)$, which follows from

$$\begin{aligned} L(0, b)L(0, ab)(x, y) &= L(0, b)(\beta(ab, y), xab) = (\beta(b, xab), \beta(ab, y)b) \\ &= (\beta(xb, ab), \beta(b, y)ab) = L(0, ab)(\beta(b, y), xb) \\ &= L(0, ab)L(0, b)(x, y). \end{aligned}$$

■

Alternative algebras

Lemma B.5. *Let A be a (non-associative) algebra. For $a, b, c \in A$ we define the associator*

$$(a, b, c) := (ab)c - a(bc).$$

Then the associator is an alternating function if and only if for $a, b \in A$ we have

$$a^2 b = a(ab) \quad \text{and} \quad ab^2 = (ab)b. \tag{B.1}$$

Proof. First we assume that the associator is alternating. Then

$$a^2 b - a(ab) = (a, a, b) = 0 \quad \text{and} \quad ab^2 - (ab)b = -(a, b, b) = 0.$$

Suppose, conversely, that (B.1) is satisfied. The derivative of the function

$$f_c(a) := a^2 c - a(ac)$$

in the direction of b is given by

$$df_c(a)(b) = (ab + ba)c - b(ac) - a(bc),$$

which leads to the identity

$$(a, b, c) = (ab)c - a(bc) = b(ac) - (ba)c = -(b, a, c).$$

We likewise obtain from $a(c^2) = (ac)c$ the identity

$$(a, b, c) = (ab)c - a(bc) = a(cb) - (ac)b = -(a, c, b).$$

As the group S_3 is generated by the transpositions (12) and (23), the associator is an alternating function. ■

We call an algebra A *alternative* if the conditions from Lemma B.5 are satisfied. For $L_a(b) := ab =: R_b(a)$ this means that

$$L_a^2 = L_{a^2} \quad \text{and} \quad R_{b^2} = R_b^2.$$

Theorem B.6. (Artin) *An algebra is alternative if every subalgebra generated by two elements is associative.*

Proof. In view of (B.1), the algebra A is alternative if any pair (a, b) of elements generates an associative subalgebra. For the converse we refer to [Sch66, Th. 3.1]. ■

Lemma B.7. *Each alternative algebra is a Jordan algebra with respect to $a \circ b := \frac{1}{2}(ab + ba)$.*

Proof. Let $L_a^J(b) := a \circ b$, $L_a(b) = ab$ and $R_a(b) := ba$. Since A is alternative, we have

$$0 = (a, b, a) = (ab)a - a(ba)$$

which means that $[L_a, R_a] = 0$. Therefore the associative subalgebra of $\mathrm{End}(A)$ generated by L_a and R_a is commutative. Since $L_a^J = \frac{1}{2}(L_a + R_a)$ commutes with

$$L_{a^2}^J = \frac{1}{2}\Big(L_{a^2} + R_{a^2}\Big) = \frac{1}{2}\Big(L_a^2 + R_a^2\Big),$$

$(A, \circ)$ is a Jordan algebra. ■

Appendix C. Jordan triple systems

The natural bridge between Lie algebras and Jordan algebras is formed by Jordan triple systems. In this appendix we briefly recall how this bridge works. We are using this correspondence in particular in Section III to see that for each A_1-graded Lie algebra the coordinate algebra is a Jordan algebra.

Definition C.1. (a) A finite dimensional vector space V over a field $\mathbb{K}$ is said to be a *Jordan triple system* (JTS) if it is endowed with a trilinear map $\{\cdot\}\colon V \times V \times V \to V$ satisfying:

(JT1) $\{x, y, z\} = \{z, y, x\}$.

(JT2) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$ for all $a, b, x, y, z \in V$.

For $x, y \in V$ we define the operator $x\Box y$ by $(x\Box y).z := \{x, y, z\}$ and put $P(x)(y) := \{x, y, x\}$. Then (JT2) is equivalent to

$$(\mathrm{JT2}') \qquad [a\Box b, x\Box y] = \Big((a\Box b).x\Big)\Box y - x\Box\Big((b\Box a).y\Big).$$

It follows in particular that the subspace $V\Box V \subseteq \mathrm{End}_{\mathbb{K}}(V)$ spanned by the elements $x\Box y$ is a Lie algebra. This Lie algebra is denoted $\mathfrak{istr}(V)$ and called the *inner structure algebra* of V.

If $2 \in \mathbb{K}^\times$, then (JT1) implies that the trilinear map $\{\cdot, \cdot, \cdot\}$ can be reconstructed from the quadratic maps $P(x)$ via polarization of $P(x).y = \{x, y, x\}$, i.e., by taking derivatives w.r.t. x in the direction of z. Therefore the Jordan triple structure is completely determined by the maps $P(x)$, $x \in V$. ■

Lemma C.2. *If $3 \in \mathbb{K}^\times$ and $(V, \{\cdot, \cdot, \cdot\})$ is a Jordan triple system, then the following formulas hold for $x, y, z \in V$:*

(1) $P(x).\{y, x, z\} = \{P(x).y, z, x\} = \{x, y, P(x).z\}$.

(2) $P(x)(y\Box x) = (x\Box y)P(x)$.

(3) $[P(x)P(y), x\Box y] = 0$.

Proof. (1) From the Jordan triple identity

$$x\Box y.\{a, b, c\} = \{x\Box y.a, b, c\} - \{a, y\Box x.b, c\} + \{a, b, x\Box y.c\}$$

we derive

$$\begin{aligned}
\{x, y, \{x, z, x\}\} &= \{\{x, y, x\}, z, x\} - \{x, \{y, x, z\}, x\} + \{x, z, \{x, y, x\}\} \\
&= 2\{\{x, y, x\}, z, x\} - \{x, \{y, x, z\}, x\} \\
&= 2\{x, y, \{x, z, x\}\} - 2\{x, \{y, x, z\}, x\} + 2\{\{x, z, x\}, y, x\} \\
&\quad -\{x, \{y, x, z\}, x\} \\
&= 4\{x, y, \{x, z, x\}\} - 3\{x, \{y, x, z\}, x\}.
\end{aligned}$$

This implies

$$3\{x, y, \{x, z, x\}\} = 3\{x, \{y, x, z\}, x\},$$

so that $3 \in \mathbb{K}^\times$ leads to

$$\{x, y, \{x, z, x\}\} = \{x, \{y, x, z\}, x\}.$$

This proves that the first and third term are equal. The equality of the first and the second term now follows from (JT1).
(2) follows directly from (1).
(3) is an immediate consequence of (2). ∎

Theorem C.3. (a) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ is a 3-graded Lie algebra with an involutive automorphism τ satisfying $\tau(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j = 0, \pm 1$, then $V := \mathfrak{g}_1$ is a Jordan triple system with respect to $\{x, y, z\} := \Big[[x, \tau.y], z\Big]$.*
(b) *If, conversely, V is a Jordan triple system for which there exists an involution σ on $\mathfrak{istr}(V)$ with $\sigma(a\Box b) = -b\Box a$ for $a, b \in V$, then $\mathfrak{g} := V \times \mathfrak{istr}(V) \times V$ is a Lie algebra with respect to the bracket*

$$[(a, x, d), (a', x', d')] = (x.a' - x'.a, a\Box d' - a'\Box d + [x, x'], \sigma(x).d' - \sigma(x').d)$$

and $\tau(a, b, c) := (c, \sigma(b), a)$ is an involutive automorphism of $\mathfrak{g}$.

Proof. (a) Since $\mathfrak{g}$ is graded, we have $[\mathfrak{g}_1, \mathfrak{g}_1] = \{0\}$, and this implies that $[\operatorname{ad} x, \operatorname{ad} y] = 0$ for $x, y \in \mathfrak{g}_1$, hence (JT1). To verify (JT2), we first observe that $a\Box b = \operatorname{ad}[a, \tau.b]$. We have

$$\begin{aligned} \Big[[a, \tau.b], [c, \tau.d]\Big] &= \Big[[[a, \tau.b], c], \tau.d\Big] + \Big[c, [[a, \tau.b], \tau.d]\Big] \\ &= \Big[[[a, \tau.b], c], \tau.d\Big] + \Big[c, \tau.[[\tau.a, b], d]\Big] \\ &= \Big[[[a, \tau.b], c], \tau.d\Big] - \Big[c, \tau.[[b, \tau.a], d]\Big]. \end{aligned}$$

Therefore (JT2) follows from

$$\begin{aligned} [a\Box b, c\Box d] &= \operatorname{ad}\Big[[a, \tau.b], [c, \tau.d]\Big] = \operatorname{ad}\Big[[[a, \tau.b], c], \tau.d\Big] - \operatorname{ad}\Big[c, \tau.[[b, \tau.a], d]\Big] \\ &= (a\Box b).c\Box d - c\Box(b\Box a).d. \end{aligned}$$

(b) One observes directly that τ is an involution preserving the bracket. It is clear that the bracket is skew symmetric, so that

$$J(x, y, z) := \Big[[x, y], z\Big] + \Big[[y, z], x\Big] + \Big[[z, x], y\Big]$$

is an alternating trilinear function on $\mathfrak{g}$. We have to show that J vanishes.

Let $\mathfrak{g}_1 := V \times \{(0, 0)\}$, $\mathfrak{g}_0 = \{0\} \times \mathfrak{istr}(V) \times \{0\}$, and $\mathfrak{g}_{-1} := \{(0, 0)\} \times V$. It is easy to check that $J(x, y, z) = 0$ if all entries are contained either in $\mathfrak{g}_0 + \mathfrak{g}_1$

or in $\mathfrak{g}_0 + \mathfrak{g}_{-1}$. We identify $x \in V$ with $(x, 0, 0)$ and write $\widetilde{x} = (0, 0, x)$ for the corresponding element of $\mathfrak{g}_{-1}$. Then we may assume that the first entry is $x \in \mathfrak{g}_1$ and the second one is $\widetilde{y} \in \mathfrak{g}_{-1}$. For $z \in V \cong \mathfrak{g}_1$ we then obtain

$$J(x, \widetilde{y}, z) = \Big[[\widetilde{y}, z], x\Big] + \Big[[x, \widetilde{y}], z\Big] = (x\Box y).z - (z\Box y).x = \{x, y, z\} - \{z, y, x\} = 0.$$

If $z \in \mathfrak{g}_{-1}$, the assertion follows from $\tau.J(x, \widetilde{y}, z) = J(\tau.x, \tau.\widetilde{y}, \tau.z) = 0$. Finally, let $z \in \mathfrak{g}_0$. We may assume that $z = a\Box b$. Then (JT2) implies that $[z, x\Box y] = [z, x]\Box y + x\Box\sigma(z).y$. This leads to

$$\begin{aligned} J(x, \widetilde{y}, z) &= \Big[[\widetilde{y}, z], x\Big] + \Big[[z, x], \widetilde{y}\Big] + \Big[[x, \widetilde{y}], z\Big] \\ &= -[(\sigma(z).y)\widetilde{\ }, x] + [z.x, \widetilde{y}] + [x\Box y, z] \\ &= x\Box(\sigma(z).y) + (z.x)\Box y - [z, x\Box y] = 0. \end{aligned}$$

■

We conclude this section with the connection between Jordan algebras and Jordan triple systems.

Theorem C.4. *Suppose that* $2, 3 \in \mathbb{K}^\times$.
(a) *If* J *is a Jordan algebra, then* J *is a Jordan triple system with respect to*

$$(C.1) \quad \{x, y, z\} = (xy)z + x(yz) - y(xz), \qquad i.e., \quad x\Box y = L(xy) + [L(x), L(y)],$$

where we write $L(x)y := xy$ *for the left multiplications in* J.
(b) *If* V *is a JTS and* $a \in V$, *then*

$$x \cdot_a y := \{x, a, y\}$$

defines on V *the structure of a Jordan algebra. The Jordan triple structure determined by the Jordan product* $\cdot_a$ *is given by*

$$\{x, y, z\}_a = \{x, \{a, y, a\}, z\} = \{x, P(a).y, z\}.$$

It coincides with the original one if $P(a) = \mathbf{1}$.
(c) *Let* J *be a Jordan algebra which we endow with the Jordan triple structure from* (a). *If* $e \in J$ *is an identity element, then* $x \cdot_e y = xy$ *reconstructs the Jordan algebra structure from the Jordan triple structure.*

Proof. (a) From (JA1) it immediately follows that (C.1) satisfies (JT1). The proof of (JT2) requires Lemma B.2.

In view of Corollary B.3, $D := [L(x), L(y)]$ is a derivation of J, so that

$$D.\{a, b, c\} = \{D.a, b, c\} + \{a, D.b, c\} + \{a, b, D.c\}.$$

Therefore (C.1) shows that to prove (JT2), it suffices to show that for each $x \in J$ we have

$$L(x).\{a, b, c\} = \{L(x).a, b, c\} - \{a, L(x).b, c\} + \{a, b, L(x).c\},$$

i.e.,

$$L(x).(a\Box b) = (xa)\Box b - a\Box(xb) + (a\Box b)L(x),$$

which in turn means that

$$\begin{aligned} L(x)L(ab) + L(x)[L(a), L(b)] &= L((xa)b) + [L(xa), L(b)] - L(a(bx)) \\ &\quad -[L(a), L(xb)] + L(ab)L(x) + [L(a), L(b)]L(x), \end{aligned}$$

i.e.,

$$[L(x), L(ab)]+[L(a), L(xb)]+[L(b), L(ax)] = [[L(a), L(b)], L(x)]+L((xa)b)-L(a(bx)).$$

This identity follows from Lemma B.2, because both sides of this equation vanish separately.
(b) Put $xy := x \cdot_a y$, so that $L(x) = x\Box a$. The identity (JA1) follows directly from (JT1). To verify (JA2), we observe that

$$\begin{aligned} L(x^2).y &= \{\{x, a, x\}, a, y\} = \{y, a, \{x, a, x\}\} \\ &= \{\{y, a, x\}, a, x\} - \{x, \{a, y, a\}, x\} + \{x, a, \{y, a, x\}\} \\ &= 2(x\Box a)^2.y - P(x)P(a).y. \end{aligned}$$

Therefore Lemma C.2(3) implies

$$[L(x^2), L(x)] = [2(x\Box a)^2 - P(x)P(a), x\Box a] = [x\Box a, P(x)P(a)] = 0.$$

The quadratic operator $P^a(x)$ associated to the Jordan triple structure defined by $\cdot_a$ in the sense of (a) is given by

$$P^a(x) = 2L(x)^2 - L(x^2) = 2(x\Box a)^2 - \Big(2(x\Box a)^2 - P(x)P(a)\Big) = P(x)P(a).$$

Therefore the Jordan triple structure associated to $\cdot_a$ is given by $\{x, y, z\}_a = \{x, P(a).y, z\}$.
(c) is trivial. ■

Example C.5. (Jordan triple systems associated to a quadratic form) Let A be a commutative algebra with $2 \in A^\times$ and M an A-module. A *quadratic form* $q\colon M \to A$ is a map for which the map

$$M \times M \to A, \quad (x, y) \mapsto q(x, y) := \frac{1}{2}(q(x + y) - q(x) - q(y))$$

is A-bilinear. Note that $q(x, x) = q(x)$.

In the following we assume that $2 \in A^\times$. We claim that

$$\{x, y, z\} := -q(x, y)z - q(z, y)x + q(x, z)y$$

defines on M the structure of an A-Jordan triple system. In fact, in Lemma B.4 we have seen that $J(M) := A \oplus M$ is a Jordan algebra with respect to the multiplication

$$(a,m)(a',m') = (aa' - q(m,m'), am' + a'm).$$

For the corresponding Jordan triple structure we have

$$\begin{aligned}\{m,m',m''\} &= (m\Box m').m'' = (mm')m'' + m(m'm'') - m'(mm'') \\ &= -q(m,m')m'' - q(m',m'')m + q(m,m'')m',\end{aligned}$$

so that, with respect to the Jordan triple structure defined above, M is a sub-Jordan triple system of the Jordan algebra $J(M)$.

Note that the operators $x\Box y$ satisfy

$$\begin{aligned}q((x\Box y).m,m') &= q(-q(x,y)m - q(m,y)x + q(x,m)y, m') \\ &= -q(x,y)q(m,m') - q(m,y)q(x,m') + q(x,m)q(y,m') \\ &= q(m, -q(x,y)m' - q(x,m')y + q(y,m')x) = q(m,(y\Box x).m').\end{aligned}$$

This implies that the operators $x\Box y$ belong to the conformal linear Lie algebra of the quadratic module (M,q):

$$\{X \in \mathrm{End}_A(M) \colon (\exists \lambda \in A) q(X.m,m') + q(m,X.m') = \lambda q(m,m')\}.$$

We can also view $J(M)$ as an A-module, and consider the quadratic form defined by the bilinear form

$$\widetilde{q}((a,m),(a',m')) = aa' - q(m,m') = p_A((a,m)(a',m')),$$

where $p_A \colon J(M) \to A$ is the projection onto the A-component. Then $\widetilde{q}$ is an A-invariant symmetric bilinear form because the Jordan multiplication on $J(M)$ is A-bilinear and commutative. This process can be continued inductively and leads to a sequence of quadratic modules

$$(M,q),\ \ (A\oplus M, q_A \oplus -q),\ \ (A^2 \oplus M, q_A \oplus -q_A \oplus q),$$
$$(A^2, q_A \oplus -q_A) \oplus (A \oplus M, q_A \oplus -q) \ldots,$$

where we write $q_A(a) = a^2$ for $a \in A$. This means that two steps of this process produce a direct factor which is a hyperbolic A-plane $(A^2, q_A \oplus -q_A)$.

For $m \in M$, considered as a Jordan triple system, the operator $P(m)$ is given by

$$P(m).x = \{m,x,m\} := -q(m,x)m - q(m,x)m + q(m,m)x = q(m,m)x - 2q(m,x)m.$$

If $q(m,m) \in A^\times$, then

$$q(m,m)^{-1}P(m).x = x - 2\frac{q(m,x)}{q(m,m)}m$$

is the orthogonal reflection in the A-submodule $m^\perp$ of M, which implies that $P(m)$ is invertible. ■

Appendix D. Skew dihedral homology

In this section we briefly recall the definition of skew dihedral homology of associative algebras, which is the background for the definition of the full skew-dihedral homology spaces defined in Section IV.

Definition D.1. Let $\mathcal{A}$ be a unital associative algebra and $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$ the $(n+1)$-fold tensor product of $\mathcal{A}$ with itself. We define a *boundary operator*

$$b_n \colon C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A}) \quad \text{for} \quad n \in \mathbb{N}$$

and $b_0 \colon C_0(\mathcal{A}) \to \{0\}$ by

$$\begin{aligned} & b_n(a_0 \otimes \ldots \otimes a_n) \\ := \ & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Then $b_n b_{n+1} = 0$ for each $n \in \mathbb{N}_0$, and the corresponding homology spaces $HH_*(\mathcal{A})$ are called the *Hochschild homology of* $\mathcal{A}$. ■

Of particular interest for Lie algebras is the first Hochschild homology group $HH_1(\mathcal{A})$. The map $b_1 \colon C_1(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} \to C_0(\mathcal{A}) \cong \mathcal{A}$ is given by

$$b_1(x \otimes y) = xy - yx = [x, y],$$

so that $Z_1(\mathcal{A}) = \ker b \subseteq C_1(\mathcal{A})$ is the kernel of the bracket map. The space $B_1(\mathcal{A})$ of boundaries is spanned by elements of the type

$$b_2(x \otimes y \otimes z) = xy \otimes z - x \otimes yz + zx \otimes y.$$

Note in particular that $b_2(x \otimes \mathbf{1} \otimes \mathbf{1}) = x \otimes \mathbf{1}$, so that $\mathcal{A} \otimes \mathbf{1} \subseteq B_1(\mathcal{A})$.

Definition D.2. Let $(\mathcal{A}, \sigma)$ be an associative algebra with involution $\sigma \colon \mathcal{A} \to \mathcal{A}, a \mapsto a^\sigma$. Then we obtain a natural action of the dihedral group D_{n+1} on the space $C_n(\mathcal{A})$ as follows. We present D_{n+1} as the group generated by x_n and y_n subject to the relations

$$x_n^{n+1} = y_n^2 = \mathbf{1} \quad \text{and} \quad y_n x_n y_n^{-1} = x_n^{-1},$$

and define the action of x_n and y_n on $C_n(\mathcal{A})$ by

$$x_n(a_0 \otimes \ldots \otimes a_n) := (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

and

$$y_n(a_0 \otimes \ldots \otimes a_n) := -(-1)^{\frac{n(n+1)}{2}} a_0^\sigma \otimes a_n^\sigma \otimes a_{n-1}^\sigma \ldots \otimes a_2^\sigma \otimes a_1^\sigma.$$

These operators are compatible with the boundary operators in the sense that the operators b_n induce on the spaces $C'_n(\mathcal{A})$ of coinvariants for the D_{n+1}-action boundary operators

$$b'_n \colon C'_n(\mathcal{A}) \to C'_{n-1}(\mathcal{A}).$$

The corresponding homology is called the *skew-dihedral homology* $HD'_n(\mathcal{A}, \sigma)$ *of the algebra with involution* $(\mathcal{A}, \sigma)$ (cf. [Lo98, 10.5.4; Th. 5.2.8]). ■

In the present paper we only need the space $HD_1'(\mathcal{A},\sigma)$. We observe that

$$x_1.(a_0 \otimes a_1) = -a_1 \otimes a_0 \quad \text{and} \quad y_1.(a_0 \otimes a_1) = a_0^\sigma \otimes a_1^\sigma.$$

Writing the image of $a_0 \otimes a_1$ in $C_1'(\mathcal{A})$ as $\langle a, b\rangle$, this means that

$$\langle a_0, a_1\rangle = -\langle a_1, a_0\rangle = \langle a_0^\sigma, a_1^\sigma\rangle, \quad a_0, a_1 \in \mathcal{A}.$$

It follows in particular that $\langle \mathcal{A}^\sigma, \mathcal{A}^{-\sigma}\rangle = \{0\}$, and further that

$$C_1'(\mathcal{A}) \cong \Lambda^2(\mathcal{A}^\sigma) \oplus \Lambda^2(\mathcal{A}^{-\sigma}).$$

Moreover,

$$\begin{aligned} &b_2'(\langle a_0, a_1, a_2\rangle) \\ &\quad = \langle a_0a_1, a_2\rangle - \langle a_0, a_1a_2\rangle + \langle a_2a_0, a_1\rangle = \langle a_0a_1, a_2\rangle + \langle a_1a_2, a_0\rangle + \langle a_2a_0, a_1\rangle, \end{aligned}$$

and these elements span the space $B_1'(\mathcal{A}) \subseteq C_1'(\mathcal{A})$ of skew-dihedral 1-boundaries.

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Spectral synthesis for orbits of compact groups in the dual of certain generalized $\mathcal{L}^1$–algebras

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Abstract

Our objects of study are generalized $\mathcal{L}^1$–algebras $\mathcal{L}^1(K, Q)$, where K is a closed normal subgroup of the compact group L, and Q is a commutative Banach algebra whose Gelfand space is a transitive L–space. The main result tells that L–orbits in the dual of $\mathcal{L}^1(K, Q)$ are sets of synthesis, i.e., there is a unique closed two–sided ideal in $\mathcal{L}^1(K, Q)$ whose hull coincides with a given L–orbit. Also the empty set is a set of synthesis, which means that each proper closed two–sided ideal is contained in the kernel of an irreducible involutive representation. To this end, L–fixed projections in $\mathcal{L}^1(K, Q)$ are constructed. Such projections are also useful in other circumstances.

Introduction

The notion of sets of synthesis (or Wiener sets) is best known in the case of $\mathcal{L}^1(G)$, G a locally compact abelian group. In this case the (Gelfand) structure space $\mathcal{L}^1(G)^\wedge$ of the commutative Banach algebra $\mathcal{L}^1(G)$ can be identified with the Pontryagin dual $G^\wedge$. With each closed ideal $\mathcal{I}$ in $\mathcal{L}^1(G)$, one can associate a closed subset of $G^\wedge$, namely the hull $h(\mathcal{I}) := \{\chi \in G^\wedge \mid \ker_{\mathcal{L}^1(G)} \chi \supset \mathcal{I}\}$. A closed subset A of $G^\wedge$ is called a *set of synthesis* if there is only one closed ideal $\mathcal{I}$ in $\mathcal{L}^1(G)$ with $h(\mathcal{I}) = A$. In this case $\mathcal{I}$ is necessarily equal to the kernel $k(A) := \bigcap_{\chi \in A} \ker_{\mathcal{L}^1(G)} \chi$ of A. For some results on sets of synthesis in the case of abelian groups compare [13, 14].

Usually, the kernel is defined in the equivalent way: $k(A) = \{f \in \mathcal{L}^1(G) \mid \widehat{f} = 0 \text{ on } A\}$. We have chosen the above formulation, because then all the introduced notions generalize immediately to arbitrary Banach algebras, as soon as one agrees on the structure space to be considered. In the present article we study algebras of the following type:

Let K be as closed normal subgroup of a compact group L, and let Q be a symmetric semi–simple involutive commutative Banach algebra. Symmetry means in the commutative case that $\omega(q^*) = \overline{\omega(q)}$ for all $q \in Q$ and all ω in the Gelfand space $Q^\wedge$, i.e., for all multiplicative linear functionals $\omega : Q \to \mathbb{C}$. Suppose that L acts strongly continuously (from the right) on Q with the usual properties:

$$(\lambda q + r)^\ell = \lambda q^\ell + r^\ell, (qr)^\ell = q^\ell r^\ell, (q^*)^\ell = (q^\ell)^*, q^{\ell m} = (q^\ell)^m,$$

$\|q^\ell\| = \|q\|$, and $L \ni \ell \longmapsto q^\ell \in Q$ is continuous. Then L acts on $Q^\wedge$, $(\ell\omega)(q) = \omega(q^\ell)$, and we suppose that this action is *transitive. These assumptions are retained throughout the article.*

Now one can form our object of study, the generalized $\mathcal{L}^1$–algebra $\mathcal{L}^1(K, Q)$, compare [7], multiplication and involution being given by

$$\begin{aligned} (f \star g)(a) &= \int_K f(a\,b)^{b^{-1}} g(b^{-1}) d\,b \\ f^*(a) &= f(a^{-1})^{*a} \end{aligned}$$

for $a \in K$, $f, g \in \mathcal{L}^1(K, Q)$. $\mathcal{L}^1(K, Q)$ carries a natural L–action,

$$f^\ell(a) = f(\ell\, a\, \ell^{-1})^\ell$$

satisfying the usual properties (as written above for the pair (L, Q)).

As structure space of $\mathcal{L}^1(K, Q)$ we take the collection $\mathrm{Priv}_* \mathcal{L}^1(K, Q)$ of kernels of all irreducible involutive representations of $\mathcal{L}^1(K, Q)$ in Hilbert spaces equipped with the Jacobson topology. This space carries a natural L–action. Our main goal is to show that L–orbits are sets of synthesis. En passant, we also prove that the empty set is a set of synthesis (sometimes called Wiener property, see [8, 10]), i.e., each proper closed ideal in $\mathcal{L}^1(K, Q)$ is contained in the kernel of an involutive irreducible representation, that there exist operators of finite rank in the image of irreducible representations, and that $\mathrm{Priv}_*\mathcal{L}^1(K, Q)$ coincides with $\mathrm{Priv}\, \mathcal{L}^1(K, Q)$, the collection of primitive ideals in $\mathcal{L}^1(K, Q)$.

The proofs are more or less exercises in representation theory of compact groups, based on the existence of the Haar measure, particularly on the following easy, but useful lemma, whose proof is omitted.

Lemma 0.1. *Let $\iota : E \longrightarrow F$ be a bounded linear dense injection of Banach spaces. Suppose that a compact group G acts continuously on E and F by linear isometries, and that ι intertwines the action. If either all G–isotypical components in E or in F are finite–dimensional then ι induces an isomorphism of each of the components and, as a consequence, an isomorphism from the collection $E^{(G)}$ of G–finite vectors onto $F^{(G)}$. Moreover, $E^{(G)}$ resp. $F^{(G)}$ is dense in E resp. F.*

1 The C^*–hull and the irreducible involutive representations of $\mathcal{L}^1(K, Q)$

Let us fix a base point $\omega \in Q^\wedge$. Then $Q^\wedge$ can be identified with the space L/L_ω of cosets, where, of course, L_ω denotes the stabilizer of ω; for $x \in L$ we denote by $[x] = x\,L_\omega$ the corresponding coset. The Gelfand transform can be identified with an injective map $\mathcal{G} : Q \longrightarrow \mathcal{C}(L/L_\omega)$; it is L–equivariant if $\ell \in L$ acts on $\varphi \in \mathcal{C}(L/L_\omega)$ via $\varphi^\ell([x]) = \varphi([\ell x])$. The map $\mathcal{G}$ induces an injective morphism of involutive Banach algebras.

$$\mathcal{L}^1(K,Q) \to \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)). \tag{1.1}$$

Each involutive representation π of $\mathcal{L}^1(K,Q)$ is given by a covariance pair (π', π''), π' being a continuous unitary representation of K, π'' an involutive representation of Q:

$$\pi(f)\xi = \int_K \pi'(a)\pi''(f(a))\xi da \tag{1.2}$$

for $f \in \mathcal{L}^1(K,Q)$.

As π'' extends to a representation of $\mathcal{C}(L/L_\omega)$, so does π, i.e., the C^*–hull of $\mathcal{L}^1(K,Q)$ is the C^*–transformation algebra $C^*(K, \mathcal{C}(L/L_\omega)$. We obtain three dense continuous inclusions

$$\begin{array}{ccccc} \mathcal{L}^1(K,Q) & \longrightarrow & \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)) & \longrightarrow & C^*(K, \mathcal{C}(L/L_\omega)) \\ & & \uparrow & & \\ & & \mathcal{C}(K \times L/L_\omega) & & \end{array} \tag{1.3}$$

where $\mathcal{C}(K \times L/L_\omega)$ is equipped with the uniform norm. The compact group $G := L \ltimes (K \times K)$ with multiplication law

$$(\ell, k_1, k_2)(t, a_1, a_2) = (\ell\, t, t^{-1}k_1 t\, a_1, f^{-1}k_2 t\, a_2) \tag{1.4}$$

acts on all these four spaces, on $\mathcal{L}^1(K,Q)$ via

$$\sigma(\ell, k_1, k_2)f = \left(\varepsilon_{k_1} \star f \star \varepsilon_{k_2^{-1}}\right)^{\ell^{-1}}, \tag{1.5}$$

where

$$(\varepsilon_k \star f)(a) = f(k^{-1}a), (f \star \varepsilon_k)(a) = f(a\, k^{-1})^k \text{ for } a, k \in K, f \in \mathcal{L}^1(K,Q), \tag{1.6}$$

on $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$ via

$$(\sigma(\ell, k_1, k_2)\varphi)(a, [x]) = \varphi\left(k_1^{-1}\ell^{-1}a\, \ell\, k_2, \left[k_2^{-1}\ell^{-1}x\right]\right) \tag{1.7}$$

for $\varphi \in \mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$, $a \in K$, $[x] \in L/L_\omega$.

The action on $\mathcal{C}(K \times L/L_\omega)$ is obtained by restriction, the action on $C^*(K, \mathcal{C}(L/L_\omega))$ by functoriality, but it is not explicitly needed. By its very construction all three inclusions are G–invariant.

Sometimes the isomorphic copy $G' = (K \times K) \rtimes L$ of G is also useful, where the multiplication is given by $(k_1, k_2, \ell)(a_1, a_2, t) = (k_1\ell\, a_1\ell^{-1}, k_2\ell\, a_2\ell^{-1}, \ell\, t)$. Via the canonical isomorphism

$$\delta : G' \longrightarrow G, \delta(k_1, k_2, \ell) = (\ell, \ell^{-1}k_1\ell, \ell^{-1}k_2\ell) \tag{1.8}$$

one obtains representations σ' of G' in the above four spaces, for instance

$$\sigma'(k_1, k_2, \ell)\varphi(a, [x]) = \varphi\left(\ell^{-1}k_1^{-1}a\, k_2\ell, \left[\ell^{-1}k_2^{-1}a\right]\right) \tag{1.9}$$

for $\varphi \in \mathcal{C}(K \times L/L_\omega)$.

If we restrict σ' to the subgroup $H' := K \rtimes L = (K \times \{e\}) \rtimes L \leq G'$ we just get the left regular representation of H' in $\mathcal{C}(H'/L_\omega = K \times L/L_\omega)$, whose H'–isotypical components are finite–dimensional. We conclude that the isotypical components of σ in $\mathcal{C}(K \times L/L_\omega)$ are finite–dimensional as well. The Lemma in the introduction tells us:

Proposition 1.10. *All the G–isotypical components in the four spaces $\mathcal{L}^1(K, Q)$, $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega)$, $\mathcal{C}(K \times L/L_\omega)$, $C^*(K, \mathcal{C}(L/L_\omega)$ are finite–dimensional. In fact, they coincide as well as the collections of G–finite vectors, which are dense in the respective spaces.*

For later use we define here the group $H := L \ltimes K = L \ltimes (K \times \{e\}) \leq G$ which is isomorphic to H' via

$$\gamma : H' \longrightarrow H \ , \ \gamma(k, \ell) = (\ell, \ell^{-1}k\, \ell). \tag{1.11}$$

The group L acts on involutive representations π of $\mathcal{L}^1(K, Q)$ (or of $C^*(K, \mathcal{C}(L/L_\omega))$ via

$$(\ell\, \pi)(f) = \pi(f^\ell) = \pi(\sigma(\ell)^{-1}(f)). \tag{1.12}$$

For a continuous irreducible unitary representation α of $K_\omega = L_\omega \cap K$ in V_α we define an irreducible involutive representation π_α of $C^*(K, \mathcal{C}(L/L_\omega)$ in the Hilbert space

$$\begin{aligned} \mathfrak{H}_\alpha = \mathcal{L}^2_{K_\omega}(K, V_\alpha) := \{\xi : K \longrightarrow V_\alpha \mid \xi \text{ is measurable}, \\ \xi(k\, a) = \alpha(a)^{-1}\xi(k) \text{ for } a \in K_\omega, k \in K, \text{ and } \int_{K/K_\omega} \|\xi(k)\|^2 dk < \infty\} \end{aligned} \tag{1.13}$$

by the covariance pair $(\pi'_\alpha, \pi''_\alpha)$

$$\begin{aligned} (\pi'_\alpha(k)\xi)(a) &= \xi(k^{-1}a) \\ (\pi''_\alpha(\psi)\xi)(a) &= \psi([a])\xi(a). \end{aligned}$$

In particular, for $\varphi \in \mathcal{C}(K \times L/L_\omega)$ and $\xi \in \mathfrak{H}_\alpha$ the vector $\pi_a(\varphi)\xi$ is given by

$$(\pi_\alpha(\varphi)\xi)(b) = \int_K \varphi(y, [y^{-1}b])\xi(y^{-1}b)dy. \tag{1.14}$$

If π is any irreducible representation of $C^*(K, \mathcal{C}(L/L_\omega)$ given by the covariance pair (π', π'') then π'' is supported by a K–orbit in L/L_ω as $K\backslash L/L_\omega$ is Hausdorff.

For a suitable $\ell \in L$, $(\ell\pi)''$ is supported by the K–orbit through the origin, i.e., $\ell\pi$ may be considered as a representation of $C^*(K,\mathcal{C}(K/K_\omega))$. Such algebras, actually in much higher generality, were studied in [5]. In particular, we know that $C^*(K,\mathcal{C}(K/K_\omega))$ is liminal, and that $\ell\pi$ is equivalent to one of the above π_α. Therefore, we have

1.15. *Each irreducible involutive representation of $C^*(K,\mathcal{C}(L/L_\omega))$ is equivalent to one of the collection $\ell\pi_\alpha, \ell \in L, \alpha \in K_\omega^\wedge$. Moreover, $(\ell\pi_\alpha)(C^*(K,\mathcal{C}(L/L_\omega))$ is equal to algebra $\mathcal{K}(\mathfrak{H}_\alpha)$ of compact operators on $\mathfrak{H}_\alpha$.*

Next, we investigate, which of those representations are equivalent. If $\ell_1\pi_\alpha \sim \ell_2\pi_\beta$ then $\pi_\beta \sim \ell_2^{-1}\ell_1\pi_\alpha$. If follows that their restrictions to $\mathcal{C}(L/L_\omega)$ (i.e., their second components considered as covariance pairs) must be carried by the same K–orbit which means $\ell_2^{-1}\ell_1 \in L_\omega K$. Write $\ell_2^{-1}\ell_1 = \ell_0 k$ with $k \in K$ and $\ell_0 \in L_\omega$. As $k\pi_\alpha \sim \pi_\alpha$ because $k\pi_\alpha(f) = \pi_\alpha(f^k) = \pi_\alpha(\varepsilon_{k^{-1}} \star f \star \varepsilon_k) = \pi'_\alpha(k^{-1})\pi_\alpha(f)\pi'_\alpha(k)$, compare (1.6), we find that $\ell_0\pi_\alpha \sim \pi_\beta$. Further, it is easy so see:

1.16. *For $\ell_0 \in L_\omega$ one has $\ell_0\pi_\alpha \sim \pi_\beta$ if and only if $\ell_0\alpha \sim \beta$ (as representations of K_ω, $(\ell_0\alpha)(b) = \alpha(\ell_0^{-1} b\, \ell_0)$ for $b \in K_\omega$).*

Also for later use we write down an intertwining operator explicitly. If $\mathcal{U} : V_\alpha \to V_\beta$ is a unitary operator with $\mathcal{U}\alpha(\ell_0^{-1} b\, \ell_0) = \beta(b)\mathcal{U}$ for all $b \in K_\omega$ then define $\mathcal{U}' : \mathfrak{H}_\alpha \to \mathfrak{H}_\beta$ by

$$(\mathcal{U}'\xi)(k) = \mathcal{U}(\xi(\ell_0^{-1} k\, \ell_0)), k \in K. \tag{1.17}$$

These arguments work also the other way around, and we conclude:

1.18. *$\ell_1\pi_\alpha \sim \ell_2\pi_\beta$ means that $\ell_2^{-1}\ell_1$ can be written in the form $\ell_2^{-1}\ell_1 = \ell_0 k$ with $k \in K$ and $\ell_0 \in L_\omega$ satisfying $\ell_0\alpha \sim \beta$.*

In view of this observation we choose an indexed set of representatives of the L_ω–orbits in $K_\omega^\wedge$, i.e., we take a collection α_j, $j \in J$, of concrete continuous irreducible unitary representations of K_ω in V_j with the following properties.

1.19. *Each continuous irreducible unitary representation of K_ω is equivalent to $\ell\alpha_j$ for some $j \in J$, $\ell \in L_\omega$. If $\ell\alpha_j \sim \ell'\alpha_{j'}$, for $\ell, \ell' \in L_\omega$ and $j, j' \in J$ then $j = j'$.*

With those representations α_j we construct as above the representations $\pi_j := \pi_{\alpha_j}$ of $C^*(K,\mathcal{C}(L/L_\omega))$ or of $\mathcal{L}^1(K,Q)$ in $\mathfrak{H}_j := \mathfrak{H}_{\alpha_j}$. Our discussion shows:

Proposition 1.20. *Each of the irreducible involutive representations of $\mathcal{L}^1(K,Q)$ is equivalent to one of the form $\ell\pi_j$, $\ell \in L$, $j \in J$. For $\ell_1, \ell_2 \in L$ and $i, j \in J$ the condition $\ell_1\pi_i \sim \ell_2\pi_j$ is equivalent to $i = j$ and $\ell_2^{-1}\ell_1 \in L^j K$, where L^j denotes the stabilizer of α_j in L_ω. (L^j is of finite index in L_ω.) In other words, the set $\mathcal{L}^1(K,Q)^\wedge$ of equivalence classes is a disjoint union of the L–orbits $L\pi_j$, $j \in J$, and the L–stabilizer of π_j is $L^j K$.*

Remark 1.21. *This description can be used to write down all the members of* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$*, however, it is not clear at present that inequivalent representations yield different kernels. But they do as we shall see later.*

Remark 1.22. *The description of* $\mathcal{L}^1(K,Q)^\wedge$ *given in (1.20) is the one we are going to use in the sequel. A little more canonical is the following one, also suggested by the above discussion. The group* $\mathfrak{U}_\omega := L_\omega K$ *acts from the left on* $K_\omega^\wedge$*: For* $\ell_0 k$, $\ell_0 \in L_\omega$, $k \in K$ *and* $\alpha \in K_\omega^\wedge$ *the element* $\alpha' = \ell_0 k \cdot \alpha \in K_\omega^\wedge$ *is given by* $\alpha'(v) = \alpha(\ell_0^{-1} v \ell_0)$*. And* $\mathfrak{U}_\omega$ *acts also from the left on* L *by right translations:* $u \cdot x = xu^{-1}$ *for* $x \in L$, $u \in \mathfrak{U}_\omega$*. Thus,* $\mathfrak{U}_\omega$ *acts on* $L \times K_\omega^\wedge$*. By (1.15), there is a surjection* $L \times K_\omega^\wedge \longrightarrow \mathcal{L}^1(K,Q)^\wedge$*, and by (1.16) the fibers of this map are exactly the* $\mathfrak{U}_\omega$*–orbits. Moreover, the* L*–action on* $\mathcal{L}^1(K,Q)^\wedge$ *corresponds to translation on* $L \times K_\omega^\wedge$ *in the first variable. Clearly, the space of* $\mathfrak{U}_\omega$*–orbits in* $L \times K_\omega^\wedge$ *can be identified with the disjoint union* $\bigcup_{j \in J} L/L^j K$ *in an obvious manner, respecting the* L*–action.*

2 The kernel operators for the representations $\ell\pi_j$, a surjectivity theorem

Many questions in harmonic analysis depend on an appropriate description of the image of the Fourier transform; this principle applies also to non–commutative situations. We shall write down the kernel functions which give the operators $(\ell\pi_j)(\varphi)$ of the previous section, and shall prove a surjectivity theorem describing the image $(\ell\pi_j)(\varphi)$, $\ell \in L$, $\varphi \in \mathcal{C}(K \times L/L_\omega)$. Using G–equivariance we shall obtain a result for $\mathcal{C}(K \times L/L_\omega)^{(G)} = \mathcal{L}^1(K,Q)^{(G)}$.

Given j, ℓ, φ as above, we recall, (1.7), (1.14), that $(\ell\pi_j)(\varphi)$ in $\mathcal{B}(\mathfrak{H}_j)$ is given by

$$\begin{aligned}
[(\ell\pi_j)(\varphi)\xi](b) &= [\pi_j(\varphi^\ell = \sigma(\ell^{-1})\varphi)\xi](b) = \int_K \varphi(\ell\, y\, \ell^{-1}, [\ell y^{-1} b])\xi(y^{-1}b)dy \\
&= \int_K \varphi(\ell\, bc^{-1}\ell^{-1}, [\ell\, c])\xi(c)dc = \int_{K/K_\omega} (R_j\varphi)(\ell,b,c)\xi(c)dc,
\end{aligned}$$

where $R_j\varphi : L \times K \times K \to \mathcal{B}(V_j)$ is defined by

(2.1) $$(R_j\varphi)(\ell,b,c) = \int_{K_\omega} \varphi(\ell\, b\, s^{-1} c^{-1} \ell^{-1}, [\ell\, c])\alpha_j(s)^{-1} ds.$$

Clearly, the functions $R_j\varphi$ are continuous, but they share also three covariance properties, which are most easily expressed by viewing $R_j\varphi$ as a function on $G = L \ltimes (K \times K)$. To this end, we choose intertwining operators between $\ell\alpha_j$ and α_j, $\ell \in L^j$:

2.2. *Let* $X_j(\ell)$, $\ell \in L^j$, *be a unitary operator on* V_j *satisfying* $\alpha_j(\ell^{-1} r\, \ell) = X_j(\ell)^* \alpha_j(r) X_j(\ell)$ *for all* $r \in K_\omega$*.*

For later use we remark:

2.3. *For each $\ell_0 \in L^j$ there exists a function c defined in a neighborhood of ℓ_0 in L^j with values in $\mathbb{T}$ such that $c(\ell_0) = 1$ and $\ell \longmapsto c(\ell)X_j(\ell)$ is continuous on this neighborhood.*

To see (2.3) define $F(\ell)$, $\ell \in L^j$, by $F(\ell) = \int_{K_\omega} \alpha_j(x)X_j(\ell_0)(\ell\,\alpha_j)(x)^{-1}dx \in \mathcal{B}(V_j)$. Clearly, F is continuous, and $F(\ell_0) = X_j(\ell_0)$. For ℓ sufficiently close to ℓ_0 the operator $F(\ell)$ is invertible (or, equivalently, different from 0 as it is an intertwining operator). Then $F'(\ell) = \|F(\ell)\|^{-1}F(\ell)$ is unitary, F' is still continuous, and $F'(\ell) = c(\ell)X_j(\ell)$ for suitable numbers $c(\ell) \in \mathbb{T}$.

□

2.4. *The $\mathcal{B}(V_j)$–valued function $R_j\varphi$ on G satisfies: $(R_j\varphi)(t\,k, b, c) = (R_j\varphi)(t, k\,b, k\,c)$ for all $(t, b, c) \in G$, all $k \in K$ or, equivalently,*

(i) $$(R_j\varphi)(g\,x) = R_j\varphi(g)$$

for all $g \in G$, all $x \in \Delta := \{(k^{-1}, k, k) \mid k \in K\}$. Observe that Δ is a subgroup of G.

(ii) $$(R_j\varphi)(g(\ell, 1, 1)) = X_j(\ell)^* R_j\varphi(g)X_j(\ell)$$

for all $g \in G$, $\ell \in L^j$.

(iii) $$(R_j\varphi)(g(1, u, v)) = \alpha_j(u)^{-1}(R_j\varphi)(g)\alpha_j(v)$$

for all $g \in G$ and $u, v \in K_\omega$.

Denote the space of all continuous $\mathcal{B}(V_j)$–valued functions on G satisfying (i), (ii), (iii) by $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$.

All these properties are easy to check as well as

2.5. *For all $x, g \in G$ one has $R_j(\sigma(g)\varphi)(x) = (\tau_j(g)R_j\varphi)(x)$, where $\tau_j(g)$ on $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$ is just left translation, i.e., $(\tau_j(g)\Phi)(x) = \Phi(g^{-1}x)$.*

Property (i) of (2.4) implies that the members Φ of $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$ are completely determined by their restrictions $\rho(\Phi)$ to $H = L \ltimes K = L \ltimes (K \times \{1\}) \leq G$, in other words,

2.6. *ρ is an isomorphism from $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$ onto the space $\mathcal{C}_{L^j,K_\omega}(H, \mathcal{B}(V_j))$ of all continuous functions $\Lambda : H \to \mathcal{B}(V_j)$ satisfying*

(a) $\Lambda(h(\ell, 1)) = X_j(\ell)^{-1}\Lambda(h)X_j(\ell)$ *for all $h \in H$, $\ell \in L^j$.*

(b) $\Lambda(h(1, u)) = \alpha_j(u)^{-1}\Lambda(h)$ *for all $h \in H$, $u \in K_\omega$.*

To transform this space into a space of functions on $H' = K \rtimes L$ along the canonical isomorphism $\gamma : H' \to H$, see (1.11), we first note:

2.7. *The subgroup $M^j := K_\omega \rtimes L^j$ of H' has a canonical continuous (irreducible) representation β_j in the space $\mathcal{B}(V_j)$ given by*

$$\beta_j(x, \ell_0)^{-1}(A) = X_j(\ell_0)^* \alpha_j(x)^{-1} A X_j(\ell_0)$$

for $A \in \mathcal{B}(V_j)$.

2.8. *γ induces an isomorphism $\widetilde{\gamma}, \widetilde{\gamma}(\Lambda)(k, \ell) = \Lambda(\ell, \ell^{-1} k\, \ell)$, from $\mathcal{C}_{L^j, K_\omega}(H, \mathcal{B}(V_j))$ onto the space $\mathcal{C}_{M^j}(H', \mathcal{B}(V_j))$ of all continuous functions $\Psi : H' \to \mathcal{B}(V_j)$ satisfying*

$$\Psi(h(x, \ell_0)) = \beta_j(x, \ell_0)^{-1} \Psi(h)$$

for all $h \in H'$, $(x, \ell_0) \in M^j$.

These easy, but a little confusing changes of viewpoints simplify the proof of the following surjectivity theorem. In fact, they are not absolutely necessary, but they shed some light on the structure of the elements in $\mathcal{C}_{\Delta, L^j, K_\omega, K_\omega}(G, \mathcal{B}(V_j))$.

Theorem 2.9. *Given $j \in J$ and $\Phi \in \mathcal{C}_{\Delta, L^j, K_\omega, K_\omega}(G, \mathcal{B}(V_j))$ there exists $\varphi \in \mathcal{C}(K \times L/L_\omega)$ satisfying $R_j\varphi = \Phi$ and $R_i\varphi = 0$ for all $i \in J$, $i \neq j$.*

Proof. By what we have seen above, it is enough to find φ with $(\widetilde{\gamma} \circ \rho)(\Phi) = (\widetilde{\gamma} \circ \rho)(R_j\varphi)$ and $(\widetilde{\gamma} \circ \rho)(R_i\varphi) = 0$ for $i \in J$, $i \neq j$. For short we put $\Psi = (\widetilde{\gamma} \circ \rho)(\Phi)$. We choose an orthonormal basis $v_1, \ldots, v_n$ of V_j, and identify, for fixed $h = (k, \ell) \in H'$, the operator $\Psi(h)$ with an $n \times n$–matrix,

$$\Psi(h) v_t = \sum_{s=1}^{n} \Psi(h)_{st} v_s$$

for $1 \leq t \leq n$.

In terms of this basis we construct explicitly a function $\widetilde{\psi}$ on K_ω such that $\int_{K_\omega} \widetilde{\psi}(u) \alpha_j(u) du = \Psi(h)$, namely

$$\widetilde{\psi}(h; u) := n \sum_{s,r=1}^{n} \langle \alpha_j(u)^{-1} v_s, v_r \rangle \Psi(h)_{sr} \quad \text{for} \quad u \in K_\omega. \tag{2.10}$$

The orthogonality relations, see e.g. [4, p. 278], readily imply

$$\Psi(h) = \int_{K_\omega} \widetilde{\psi}(h; u) \alpha_j(u) du. \tag{2.11}$$

Indeed, as $\langle \alpha_j(u)^{-1} v_s, v_r \rangle$ is the entry in the matrix corresponding to $\alpha_j(u)^{-1}$ at position r, s, the sum $\sum_{s=1}^{n} \langle \alpha_j(u)^{-1} v_s, v_r \rangle \Psi(h)_{sr}$ is nothing but the entry in the matrix corresponding to $\alpha_j(u)^{-1} \Psi(h)$ at position r, r.

But $\alpha_j(u)^{-1}\Psi(h) = \Psi(h(u,1))$, hence

$$\widetilde{\psi}(h;u) = n\,\mathrm{Tr}\,\Psi(h(u,1)). \tag{2.12}$$

In particular, the function $K_\omega \ni u \longmapsto \widetilde{\psi}(h(u^{-1},1);u)$ is constant for all $h \in H'$. From (2.8) and (2.12) it follows that $\widetilde{\psi}(h(1,\ell);1) = \widetilde{\psi}(h;1)$ for all $h \in H'$, all $\ell \in L^j$. Therefore,

$$\varphi(k,[t]) := \sum_{\ell \in L_\omega/L^j} \widetilde{\psi}(k, t\,\ell; 1) \tag{2.13}$$

is a function on $K \times L/L_\omega$, and we claim that this φ has the desired properties.

At a point $(b,t) \in H' = K \rtimes L$ one finds for an $i \in J$:

$$\begin{aligned} [(\widetilde{\gamma} \circ \rho)(R_i\varphi)]\,(b,t) &= \int_{K_\omega} \varphi(btst^{-1},[t])\alpha_i(s)ds \\ &= \sum_{\ell \in L_\omega/L^j} \int_{K_\omega} ds\,\widetilde{\psi}((b,t)(s,\ell);1)\alpha_i(s)ds. \end{aligned}$$

We introduce artificially another integration

$$\widetilde{\psi}((b,t)(s,\ell);1) = \int_{K_\omega} du\,\widetilde{\psi}((b,t)(s,\ell)(u^{-1},1);u).$$

As $(s,\ell)(u^{-1},1) = (s\,\ell\,u^{-1}\ell^{-1},\ell)$, with the new integration variable $s' = s\,\ell u^{-1}\ell^{-1} \in K_\omega$ one gets

$$[(\widetilde{\gamma} \circ \rho)(R_i\varphi)](b,t) = \sum_{\ell \in L_\omega/L^j} \int_{K_\omega} ds \int_{K_\omega} du\,\widetilde{\psi}((b,t)(s,\ell);u)\alpha_i(s)\alpha_i(\ell\,u\,\ell^{-1}).$$

If $i \neq j$, then the representation $\ell^{-1}a_i$, $\ell \in L_\omega$, is not equivalent to α_j, hence by the construction of $\widetilde{\psi}$, (2.10), and the orthogonality relations the integral $\int_{K_\omega} du\,\widetilde{\psi}(h;u)\,(\ell^{-1}\alpha_i)\,(u)$ vanishes for all $h \in H'$. Therefore, $(\widetilde{\gamma} \circ \rho)(R_i\varphi)$ is equal to zero.

If $i = j$, for the same reason the above integral over u vanishes if ℓ is outside L^j. Thus, we obtain

$$[(\widetilde{\gamma} \circ \rho)(R_j\varphi)](b,t) = \int_{K_\omega} ds \int_{K_\omega} du\ \widetilde{\psi}((b,t)(s,1);u)\alpha_j(s)\alpha_j(u).$$

The integration over u can be carried out using (2.11),

$$[(\widetilde{\gamma} \circ \rho)(R_j\varphi)](b,t) = \int_{K_\omega} ds\,\alpha_j(s)\Psi((b,t)(s,1)).$$

But the integrand is constant, hence $[(\widetilde{\gamma}\circ\rho)(R_j\varphi)](b,t) = \Psi(b,t) = [(\widetilde{\gamma}\circ\rho)(\Phi)](b,t)$ as desired. □

Corollary 2.14. *Given $j \in J$, for each G–finite kernel function $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ there exists a G–finite function $f \in \mathcal{L}^1(K,Q)$ such that the operator $(\ell\pi_j)(f) = \pi_j(f^\ell)$ is given by the kernel $\Phi(\ell,-,-)$, and that $(\ell\pi_i)(f) = 0$ for all $\ell \in L$, $i \in J$, $i \neq j$.*

Proof. First of all, by (2.9) there exists an $\varphi \in \mathcal{C}(K \times L/L_\omega)$, considered as a subspace of $\mathcal{L}^1(K,\mathcal{C}(L/L_\omega))$, with the corresponding properties as stated in the Corollary. As the map $\Pi_{i\in J}R_i : \mathcal{C}(K \times L/L_\omega) \to \Pi_{i\in J}\mathcal{C}_{\Delta,L^i,K_\omega,K_\omega}(G,\mathcal{B}(V_i))$ is injective ("uniqueness of the Fourier transform") and G–equivariant, the function φ has necessarily to be G–finite, whence it is "contained" in $\mathcal{L}^1(K,Q)^{(G)}$, see (1.10). □

Corollary 2.15. *If $i,j \in J$ and $\ell_1,\ell_2 \in L$ have the property that $\ell_1\pi_i$ is not equivalent to $\ell_2\pi_j$ then there exists a (G–finite) $f \in \mathcal{L}^1(K,Q)$ such that $(\ell_1\pi_i)(f) = 0$, but $(\ell_2\pi_j)(f) \neq 0$. In view of (1.20), see also (1.21), this means that the canonical map from the set of equivalence classes of irreducible involutive representations of $\mathcal{L}^1(K,Q)$ onto* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is a bijection. Thus, a parametrization of* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is obtained.*

Proof. If $i \neq j$, then, in view of Corollary 0.1, one only has to note the evident fact that there is a G–finite vector $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ with $\Phi(\ell_2,-,-) \not\equiv 0$.

If $i = j$, then $\ell_2^{-1}\ell_1 \notin L^jK$ by (1.20). Choose $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))^{(G)}$ such that $\Phi(\ell_2,-,-) \not\equiv 0$. Next choose a representative function μ in $\mathcal{R}(G/L^jK \ltimes (K \times K))$, i.e., a G–finite function which is constant on $L^jK \ltimes (K\times K)$–cosets, such that $\mu(\ell_2,1,,1) = 1$, but $\mu(\ell_1,1,1) = 0$. This can be done because $(\ell_1,1,1)(L^jK \ltimes (K \times K)) \neq (\ell_2,1,1)(L^jK \ltimes (K \times K))$. Then $\mu\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))^{(G)}$ satisfies $(\mu\Phi)(\ell_2,-,-) \neq 0$, but $(\mu\Phi)(\ell_1,-,-) = 0$. By (2.14) there exists $f \in \mathcal{L}^1(K,Q)$ such that the operator $(\ell\pi_j)(f)$ is represented by the kernel function $(\mu\Phi)(\ell,-,-)$. □

Corollary 2.16. *The L–orbits in* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *are open and closed with respect to the Jacobson topology.*

Proof. Any L–orbit is of the form $\Omega_j := \{\ker \ell\pi_j \mid \ell \in L\}$ for a certain $j \in J$. To show that Ω_j is open, take the kernel of $\Omega'_j := \{\ker \ell\pi_i \mid i \in J,\, i \neq j',\, \ell \in L\}$, i.e., $\mathcal{I}'_j := \{f \in \mathcal{L}^1(K,Q) \mid (\ell\pi_i)(f) = 0 \text{ for all } \ell \in L,\, i \in J,\, i \neq j\}$.

Then, by definition, the hull of $\mathcal{I}'_j$ is closed, and $h(\mathcal{I}'_j)$ contains Ω'_j. On the other hand, for any $\ell \in L$, by (2.14) there exists $f \in \mathcal{I}'_j$ such that $(\ell\pi_j)(f) \neq 0$, which implies that $h(\mathcal{I}'_j) = \Omega'_j$, whence Ω_j is open.

To show that Ω_j is closed, take the kernel of Ω_j, i.e., $\mathcal{I}_j := \{f \in \mathcal{L}^1(K,Q) \mid (\ell\pi_j)(f) = 0 \text{ for all } \ell \in L\}$. Clearly, the closed set $h(\mathcal{I}_j)$ contains Ω_j. On the other hand, if $\ell \in L$, $i \in J$, $i \neq j$, then by (2.14) there exists $f \in \mathcal{L}^1(K,Q)$ such that $(\ell\pi_i)(f) \neq 0$, but $(\ell'\pi_j)(f) = 0$ for all $\ell' \in L$. This means that $f \in \mathcal{I}_j$, and $\ker \ell\pi_i \notin h(\mathcal{I}_j)$, whence $\Omega_j = h(\mathcal{I}_j)$ is closed. □

Remark 2.17. *We did not claim anything on the internal (Jacobson) topology of the various L–orbits in* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$. *Presumably, if one assumes that Q is regular, i.e., the Gelfand topology coincides with the Jacobson topology on the structure space $Q^\wedge$, then those orbits carry their natural topology, which would imply that* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is homeomorphic to* $\mathrm{Priv}_*C^*(K,\mathcal{C}(L/L_\omega))$, *i.e.,* $\mathcal{L}^1(K,Q)$ *is $*$–regular in the sense of [1], where originally this class of groups/algebras was denoted by* $[\Psi]$. *But I must admit that I did not study this circle of questions seriously. Certainly G–finite functions are too algebraic in nature in order to separate arbitrary closed sets in* $\mathrm{Priv}_*C^*(K,\mathcal{C}(L/L_\omega))$ *from points.*

The consideration in this section can also be used to "compute" the C^*–hull of $\mathcal{L}^1(K,Q)$, which is the same as the C^*–hull of $\mathcal{L}^1(K,\mathcal{C}(L/L_\omega))$. Given $j \in J$ and chosen intertwining operators $X_j(\ell)$, $\ell \in L^j$, as in (2.2), we define unitary operators $Y_j(\ell)$ on $\mathfrak{H}_j$, compare (1.17), by

$$(Y_j(\ell)\xi)(k) = X_j(\ell)\xi(\ell^{-1}k\ell) \tag{2.18}$$

for $\ell \in L^j$, $k \in K$, $\xi \in \mathfrak{H}_j$ satisfying

$$(\ell\pi_j)(f) = \pi_j(f^\ell) = Y_j(\ell)^*\pi_j(f)Y_j(\ell)$$

for $f \in \mathcal{L}^1(K,Q)$ or in $C^*(K,\mathcal{C}(L/L_\omega))$.

Using these operators we define a space of continuous functions from L into the algebra $\mathcal{K}(\mathfrak{H}_j)$ of compact operators on $\mathfrak{H}_j$ as follows.

2.19. $\mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ *consists of all continuous functions $T : L \to \mathcal{K}(\mathfrak{H}_j)$ satisfying*

$$\begin{aligned} T(\ell\,\ell') &= Y_j(\ell')^*T(\ell)Y_j(\ell') \text{ for } \ell' \in L^j, \text{ and} \\ T(\ell\,k) &= \pi_j'(k)^*T(\ell)\pi_j'(k) \text{ for } k \in K. \end{aligned}$$

Observe that each $Y_j(\ell')$ normalizes $\pi_j'(K)$. Each $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ yields an element $T_\Phi \in \mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ via

$$\left(T_\Phi(\ell)\xi\right)(a) = \int_{K/K_\omega} \Phi(\ell,a,b)\xi(b)db \tag{2.20}$$

for $a \in K$, $\xi \in \mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K,V_j)$.

As a matter of fact, if $\Phi = R_j\varphi$, $\varphi \in \mathcal{C}(K \times L/L_\omega)$, one has

$$T_{R_j\varphi}(\ell) = (\ell\pi_j)(\varphi). \tag{2.21}$$

There is an action ν_j of the group $G = L \ltimes (K \times K)$ on $\mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ given by

$$(\nu_j(\ell,k_1,k_2)T)(\ell') = \pi_j'(\ell'^{-1}\ell\,k_1\ell^{-1}\ell')T(\ell^{-1}\ell')\pi_j'(\ell'^{-1}\ell\,k_2^{-1}\ell^{-1}\ell') \tag{2.22}$$

for $(\ell, k_1, k_2) \in G$, $\ell' \in L$.

The map $\Phi \longmapsto T_\Phi$ of (2.20) is G-equivariant, i.e.,

$$T_{\tau_j(g)\Phi} = \nu_j(g) T_\Phi. \tag{2.23}$$

Furthermore, it is a matter of routine to check that this map is in fact dense (and injective) if both spaces are equipped with the uniform norm.

By means of (2.9) we conclude that the map

$$\mathcal{C}(K \times L/L_\omega) \ni \varphi \longmapsto T_{R_j\varphi} \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j)) \tag{2.24}$$

has a dense image. Moreover, this map is multiplicative, if the multiplication in $\mathcal{C}_{L^jK}\ (L, \mathcal{K}(\mathfrak{H}_j))$ is defined pointwise, and if $\mathcal{C}(K \times L/L_\omega)$ is considered as a subalgebra of the crossed product $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$.

Finally, we define the space (C^*–algebra) $\mathcal{D}$ consisting of all functions ψ on $J \times L$ such that the value $\psi(j, \ell)$ is contained in $\mathcal{K}(\mathfrak{H}_j)$, in fact $\psi(j, -) \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$, and that ψ vanishes at infinity, i.e., for all $\varepsilon > 0$ there exists a finite subset $J_\varepsilon \subset J$ such that

$$\|\psi(j, \ell)\|_{\mathcal{K}(\mathfrak{H}_j)} < \varepsilon \text{ for all } \ell \in L, j \in J \setminus J_\varepsilon. \tag{2.25}$$

On $\mathcal{D}$, a norm is defined by

$$\|\psi\| = \sup_{(j,\ell) \in J \times L} \|\psi(j, \ell)\|_{\mathcal{K}(\mathfrak{H}_j)}.$$

The operations on $\mathcal{D}$ are defined pointwise, in particular the multiplication.

The previous discussion and (2.9) imply:

Proposition 2.26. *The map $\mathcal{C}(K \times L/L_\omega) \ni \varphi \longmapsto \psi \in \mathcal{D}, \psi(j, \ell) = T_{R_j\varphi}(\ell)$, extends to an isomorphism from $C^*(K, \mathcal{C}(L/L_\omega))$ onto $\mathcal{D}$.* □

Using once more the Lemma of the introduction, resp. its consequence (1.10) we obtain:

Corollary 2.27. *The map $\mathcal{L}^1(K, Q) \ni f \longmapsto \psi \in \mathcal{D}$, $\psi(j, \ell) = (\ell\pi_j)(f)$, yields an isomorphism from $\mathcal{L}^1(K, Q)^{(G)}$ onto $\mathcal{D}^{(G)}$, where the action of G on the various pieces $\mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$ of $\mathcal{D}$ is given in (2.22).* □

3 Invariant projectors of finite rank, applications to the ideal theory in $\mathcal{L}^1(K, Q)$

The first purpose of this section is to show that each $(\ell\pi_j)(\mathcal{L}^1(K, Q))$ contains projectors of finite rank, in fact realized by L–invariant functions. Further, we show that $\mathrm{Priv}_*\mathcal{L}^1(K,Q) = \mathrm{Priv}\mathcal{L}^1(K,Q)$, that $\mathcal{L}^1(K,Q)$ has the Wiener property, and that L–orbits are sets of synthesis.

A basic ingredient of the proofs is the fact that a certain subgroup of the unitary group $\mathfrak{U}(\mathfrak{H}_j)$, $j \in J$, is actually compact.

It is easy to check that for any $j \in J$ the subset

$$\mathfrak{U}_j := \{t\, Y_j(\ell)\pi'_j(k) \mid t \in \mathbb{T}, \ell \in L^j, k \in K\} \tag{3.1}$$

of the unitary group $\mathfrak{U}(\mathfrak{H}_j)$ is a subgroup, in fact, the $Y_j(\ell)$ normalize $\pi'_j(K)$, as we observed earlier.

Lemma 3.2. *$\mathfrak{U}_j$ is compact w.r.t. the strong operator topology.*

Proof. Let a sequence (or net) $t_nY_j(\ell_n)\pi_j(k_n)$ be given. W.l.o.g. we may assume that (ℓ_n) converges to ℓ_0, and that (k_n) converges to k_0. Choosing, as in (2.3), a function c on a neighborhood of ℓ_0, with values in $\mathbb{T}$, one can arrange that $\ell \longmapsto c(\ell)Y_j(\ell)$ is (locally) continuous w.r.t. the strong operator topology.

As $t_nY_j(\ell_n)\pi_j(k_n) = (c(\ell_n)^{-1}t_n)c(\ell_n)Y_j(\ell_n)\pi_j(k_n)$, passing once more to a subnet we can get that $(c(\ell_n)^{-1}t_n)$ converges in $\mathbb{T}$ to t_0. But then the subnet converges to $t_0Y_j(\ell_0)\pi_j(k_0)$. □

As a further application of (2.14) (and of the previous lemma which shows that $\mathfrak{H}_j$ decomposes into an orthogonal sum of finite–dimensional $\mathfrak{U}_j$–invariant subspaces) we obtain the following theorem.

Theorem 3.3. *Given $j \in J$ for each $\mathfrak{U}_j$–invariant finite–dimensional subspace $\mathfrak{F}$ of $\mathfrak{H}_j$ there exists an L–fixed G–finite vector $\mathfrak{p} = \mathfrak{p}_{j,\mathfrak{F}}$ in $\mathcal{L}^1(K,Q)$ such that $(\ell\pi_j)(\mathfrak{p}) = \pi_j(\mathfrak{p})$ is the orthogonal projection on $\mathfrak{F}$, and $(\ell\pi_i)(\mathfrak{p}) = 0$ for all $\ell \in L$, $i \in J$, $i \neq j$. Moreover, $\mathfrak{p}^* = \mathfrak{p}$ and $\mathfrak{p} * \mathfrak{p} = \mathfrak{p}$.*

Proof. Denote by $p : \mathfrak{H}_j \to \mathfrak{H}_j$ the orthogonal projection onto $\mathfrak{F}$. From the $\mathfrak{U}_j$–invariance of $\mathfrak{F}$ it follows that the constant map $L \ni \ell \longmapsto p \in \mathcal{K}(\mathfrak{H}_j)$ is contained in $\mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$. Hence the function ψ_p on $J \times L$ given by

$$\psi_p(i,\ell) = \delta_{ij}\, p \tag{3.4}$$

is contained in the C^*–algebra $\mathcal{D}$. Actually, ψ_p is a G–finite vector. The details of the proof of this statement are left to the reader. We just remark the following crucial fact. Define a representation of K in $\mathfrak{F}$ by restricting π'_j, i.e.,

$$\widetilde{\pi}_j(k) = \pi'_j(k)\Big|_{\mathfrak{F}}, k \in K. \tag{3.5}$$

Then there is a subgroup $L_{\mathfrak{F}}$ of L of finite index such that $\ell\,\widetilde{\pi}_j$ is unitarily equivalent to $\widetilde{\pi}_j$ for all $\ell \in L_{\mathfrak{F}}$.

Corollary (2.27) delivers a (unique) element $\mathfrak{p}$ in $\mathcal{L}^1(K,Q)^{(G)}$ with $(\ell\pi_i)(\mathfrak{p}) = \psi_p(i,\ell)$ for all $i \in J$, $\ell \in L$. □

Retaining the previous notations, we consider the corner $\psi_p \mathcal{D} \psi_p$ in $\mathcal{D}$, which clearly can be identified with

$$\mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j)) \cap \mathcal{C}(L, \mathcal{B}(\mathfrak{F})) =: \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F})), \tag{3.6}$$

i.e., with the space of all continuous functions $L \to \mathcal{B}(\mathfrak{F})$ satisfying the transformation properties w.r.t. L^jK analogous to (2.19); of course, π_j' has to be replaced by $\widetilde{\pi}_j$.

All the irreducible representations of $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ are given by evaluation at points $\ell \in L$, in particular, they live in $\mathfrak{F}$, and $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))^\wedge$ can be identified with L/L^jK.

The map

$$\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p} \ni f \longmapsto (\ell \longmapsto (\ell\pi_j)(f)) \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F})) = \psi_p \mathcal{D} \psi_p \tag{3.7}$$

yields a dense embedding of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ into the C^*–algebra $\psi_P \mathcal{D} \psi_p$. Further, this map is L–equivariant if we let L act on $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ by left translations. In particular, the L–finite vectors in the two spaces coincide. We are going to show that the map of (3.7) is actually the C^*–completion of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$. The proof of this rests on the fact that

$$\mathcal{L}^1(K, Q) \text{ is symmetric} \tag{3.8}$$

as a "compact extension" of the symmetric algebra Q by [9, Theorem 1]. Here a few words on symmetric Banach algebras are in order, for more information see [2, 11, 14]. (There are also more recent contributions, for instance by Pták, but we do not need these results.) In [11] such algebras are called completely symmetric, in [2] hermitean. An involutive Banach algebra $\mathcal{A}$ is called symmetric if for all $a \in \mathcal{A}$ the spectrum of a^*a is contained in $[0, \infty)$. By the theorem of Shirali–Ford, see [2, p. 226], this is equivalent to the fact that all hermitean elements, i.e., $a^* = a$, have a real spectrum. The latter property is evidently conserved by adding a unit to the algebra. This observation is here important because we wish to use some results of [11], which were there only formulated for algebras with unit. Also we note that closed involutive subalgebras of symmetric algebras are symmetric (the spectra of hermitean elements in the subalgebra can be "computed" by means of [2, Prop. 14, p. 25]. In particular, the subalgebra $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ of $\mathcal{L}^1(K, Q)$ is symmetric.

Proposition 3.9. *The C^*–hull of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ is $\psi_p \mathcal{D} \psi_p \cong \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$.*

Proof. We have to show that each irreducible involutive representation θ of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ in some Hilbert space $\mathfrak{K}$ can be lifted along the map of (3.7). By the symmetry of $\mathcal{L}^1(K, Q)$ the representation θ can be extended to an irreducible involutive representation μ of $\mathcal{L}^1(K, Q)$ in some Hilbert space $\mathfrak{H} \supset \mathfrak{K}$, compare [11, Thm. 1, p. 311, Ch. V, § 23], i.e., restricting μ to $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ and $\mathfrak{K}$ yields the original θ. By (1.20) there exist $i \in J$ and $\ell \in L$ such that $\mathfrak{H} = \mathfrak{H}_i$ and $\mu = \ell\pi_i$. Since $(\ell\pi_i)(\mathfrak{p} * \mathcal{L}^1(K,Q) * \mathfrak{p}) = 0$ for $i \neq j$, we have to have $i = j$. Moreover, $\mathfrak{K}$ is contained in $\mathfrak{F} = (\ell\pi_j)(\mathfrak{p})$. As $(\ell\pi_j)(\mathcal{L}^1(K,Q))$ is dense in $\mathcal{K}(\mathfrak{H}_j)$, we conclude that $(\ell\pi_j)(\mathfrak{p} * \mathcal{L}^1(K,Q) * \mathfrak{p}) = p(\ell\pi_j)(\mathcal{L}^1(K,Q))p$, which may be considered as a subspace of $\mathcal{B}(\mathfrak{F})$, has in fact to coincide with $\mathcal{B}(\mathfrak{F})$. It follows that $\mathfrak{K} = \mathfrak{F}$, and $\theta = \ell\pi_j\big|_{\mathfrak{p}*\mathcal{L}^1(K,Q)*\mathfrak{p},\mathfrak{F}}$ which clearly implies what we had to show. □

The construction of the element $\mathfrak{p}$ can be made more explicit in terms of kernel functions. Actually, the following considerations will be a little more general giving the kernel functions for all elements in $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ (still j and $\mathfrak{F}$ are fixed).

For $k \in K$ define $B(k) : \mathfrak{F} \to V_j$ by $B(k)\xi = \xi(k)$. This map is well–defined as $\mathfrak{F}$ consists of continuous functions, because $\mathfrak{F}$ is finite–dimensional and invariant under $\pi'_j(K)$. The latter invariance and the transformation property of the members of $\mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K, V_j)$ yield

$$B(ku) = \alpha_j(u)^{-1}B(k), B(ak) = B(k)\widetilde{\pi}_j(a)^{-1} \tag{3.10}$$

for all $a, k \in K$, $u \in K_\omega$.

In particular, the operator $B(k)$ depends continuously on $k \in K$.

Moreover, the invariance of $\mathfrak{F}$ under $Y_j(\ell), \ell \in L^j$, leads to

$$B(\ell^{-1}k\,\ell) = X_j(\ell)^* B(k) Y_j(\ell), \ell \in L^j. \tag{3.11}$$

For the definition of $Y_j(\ell)$ see (2.18).

With the family $B(k)$ of operators we also have their adjoints $B(k)^* : V_j \to \mathfrak{F}$, and it is easy to check that the projection $p : \mathfrak{H}_j \to \mathfrak{F}$ may be written as

$$p\,\xi = \int_{K/K_\omega} B(k)^* \xi(k) dk. \tag{3.12}$$

This means that p is given by the kernel function

$$F(a,b) = B(a)B(b)^*, a, b \in K. \tag{3.13}$$

The above transformation laws for B imply

$$F(ka, kb) = F(a,b), F(\ell^{-1}a\ell, \ell^{-1}b\ell) = X_j(\ell)^* F(a,b) X_j(\ell) \tag{3.14}$$

for $a, b, k \in K$, $\ell \in L^j$.

From (3.14) it follows that the function $\Phi : G \to \mathcal{B}(V_j)$ defined by $\Phi(\ell, a, b) = F(a,b)$ is contained in $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$, see (2.4). Thus, the function Φ yields $\psi_p \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$, which is nothing but the constant function with value $\mathrm{Id}_{\mathfrak{F}}$, and we are looking for $\varphi \in \mathcal{C}(K \times L/L_\omega)$ with $R_j\varphi = \Phi$. More generally, we take any $A \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$, and form $\Phi_A : G \to \mathcal{B}(V_j)$,

$$\Phi_A(\ell, a, b) = B(a)A(\ell)B(b)^*. \tag{3.15}$$

The operator given by this kernel, i.e., $\xi \longmapsto (a \longmapsto \int_{K/K_\omega} \Phi(\ell, a, b)\xi(b)db)$ is just $A(\ell)$. By a straightforward computation it follows from (3.10), (3.11) and the definition of $\mathcal{C}_{L^jK}(L, \mathcal{B}(V_j))$, see (3.6), that $\Phi_A \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$. Of course, in general Φ_A is not a G–finite vector; it is if the left L–translates of A are sitting in a finite–dimensional subspace, as it happens for instance in the case $A \equiv \mathrm{Id}_{\mathfrak{F}}$, which corresponds to the projector ψ_p. To find a $\varphi_A \in \mathcal{C}(K \times L/L_\omega)$ with $R_j\varphi_A = \Phi_A$ we use the recipe of (2.10) – (2.13). Again, for short we define $\Psi : H' \to \mathcal{B}(V_j)$ by

$$\Psi = (\tilde{\gamma} \circ \rho)(\Phi), \Psi(k, \ell) = \Phi_A(\ell, \ell^{-1}k\ell, 1) = B(\ell^{-1}k\ell)A(\ell)B(1)^*,$$

and get

$$\begin{aligned} \varphi_A(k, [t]) &= \sum_{\ell \in L_\omega/L^j} n\,\mathrm{Tr}\,\Psi(k, t\,\ell) \\ &= \sum_{\ell \in L_\omega/L^j} n\,\mathrm{Tr}\,(B(\ell^{-1}t^{-1}k\,t\,\ell)A(t\,\ell)B(1)^*), \end{aligned} \tag{3.16}$$

where $n = \dim V_j$.

In particular, if $\mathfrak{p}' \in \mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$ denotes the element corresponding to $\mathfrak{p} \in \mathcal{L}^1(K, Q)$, then $\mathfrak{p}'$ is a continuous function on $K \times L/L_\omega$ given by $\mathfrak{p}'(k, [t]) = \sum_{\ell \in L_\omega/L^j} n\,\mathrm{Tr}(B(\ell^{-1}t^{-1}k\,t\,\ell)B(1)^*)$. Our previous discussion yields the following result.

Proposition 3.17. *The subalgebra* $\mathfrak{p}' * \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)) * \mathfrak{p}'$ *is equal to* $\mathfrak{p}' * \mathcal{C}(K \times L/L_\omega) * \mathfrak{p}'$ *and isomorphic to the* C^**–algebra* $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$. □

Next, we are going to show that $\mathrm{Priv}_* \mathcal{L}^1(K, Q) = \mathrm{Priv}\,\mathcal{L}^1(K, Q)$, where $\mathrm{Priv}\,\mathcal{L}^1(K, Q)$ denotes the collection of primitive ideals in $\mathcal{L}^1(K, Q)$. Recall that for any (complex) algebra $\mathcal{A}$ a primitive ideal is, by definition, the annihilator of a simple left $\mathcal{A}$–module E. Simple means that $\mathcal{A}\,\xi = E$ for all non–zero $\xi \in E$; for a little more information on this notion see for instance the first pages of [12].

It is a general fact for symmetric Banach algebras $\mathcal{A}$ that $\mathrm{Priv}\,\mathcal{A}$ is contained in $\mathrm{Priv}_* \mathcal{A}$, because each simple module E can be "unitarized", see [11, Cor. 1, p. 307], i.e., there exists an irreducible involutive representation π in some Hilbert space and a non–zero $\mathcal{A}$–intertwining operator $E \to \mathfrak{H}$, necessarily with dense image, which implies that the annihilator of E is equal to $\ker \pi$.

On the other hand, if π is an irreducible involutive representation of an involutive Banach algebra $\mathcal{A}$ in $\mathfrak{H}$ such that $\pi(\mathcal{A})$ contains at least one non–zero operator of finite rank, then one may form the two–sided ideal $\mathcal{I}_\pi := \{a \in \mathcal{A} \mid \pi(a)$ is of finite rank $\}$; and, according to the arguments of [3, Théorème 2], the (dense) subspace $\mathfrak{H}' := \mathcal{I}_\pi\mathfrak{H}$ of $\mathfrak{H}$ is a simple $\mathcal{A}$–module, whose annihilator equals $\ker \pi$.

Since the above assumptions are met by $\mathcal{L}^1(K, Q)$ we obtain the following proposition. For this, it is not necessary to use the above constructed L–invariant projections, because in fact **all** G–finite vectors in $\mathcal{L}^1(K,Q)$ yield finite rank operators when represented irreducibly.

Proposition 3.18. $\operatorname{Priv}\mathcal{L}^1(K,Q) = \operatorname{Priv}_*\mathcal{L}^1(K,Q)$. □

3.19. *For each $i \in J$ we choose and fix a $\mathfrak{U}_i$–invariant, for the definition of $\mathfrak{U}_i$ see (3.2), finite–dimensional subspace $\mathfrak{F}_i$ of $\mathfrak{H}_i$, to which we find $\mathfrak{p}_i \in \mathcal{L}^1(K,Q)$ according to (3.3).*

Next, we prove the Wiener property of $\mathcal{L}^1(K,Q)$; for more information on this notion see [8, 10].

Theorem 3.20. *If $\mathcal{I}$ is a proper closed two–sided ideal in $\mathcal{L}^1(K,Q)$ then there exist $\ell \in L$ and $j \in J$ such that $\mathcal{I}$ is contained in $\ker_{\mathcal{L}^1(K,Q)} \ell\pi_j$.*

Proof. For short, put $\mathcal{A} := \mathcal{L}^1(K,Q)$. Suppose to the contrary that for any ℓ, j the image $(\ell\pi_j)(\mathcal{I})$ is non–zero. Then the closure of $(\ell\pi_j)(\mathcal{I})$ in $\mathcal{K}(\mathfrak{H}_j)$ is a non–trivial ideal, hence equal to $\mathcal{K}(\mathfrak{H}_j)$.

For any $i \in J$, fixed for the moment, consider the (closed) ideal $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i = \mathcal{I} \cap \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$ in $\mathcal{A}_i := \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$. For any $\ell \in L$ the image $(\ell\pi_i)(\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i)$ is dense in $\mathcal{B}(\mathfrak{F}_i)$, hence equal to $\mathcal{B}(\mathfrak{F}_i)$. In particular, there exists $g_\ell \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $(\ell\pi_i)(g_\ell) = \mathrm{Id}_{\mathfrak{F}_i}; f_\ell := g_\ell^* * g_\ell$ has the same properties.

By continuity there exists a neighborhood W_ℓ of ℓ such that $\langle(\ell'\pi_i)(f_\ell)\xi,\xi\rangle \geq \frac{1}{2} < \xi,\xi >$ for all $\xi \in \mathfrak{F}_i$, $\ell' \in W_\ell$. Using the compactness of L we find a finite cover $(W_\mu)_{1\leq\mu\leq m}$ of L and positive elements f_μ in $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $\langle(\ell\pi_i)(f_\mu)\xi,\xi\rangle \geq \frac{1}{2} < \xi,\xi >$ for all $\xi \in \mathfrak{F}_i$, $\ell \in W_\mu$. The element $f := \sum_{\mu=1}^m f_\mu \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ has the property that

$$\langle(\ell\pi_i)(f)\xi,\xi\rangle \geq \frac{1}{2} < \xi,\xi > \text{ for all } \ell \in L, \xi \in \mathfrak{F}_i. \tag{3.21}$$

We claim that f is invertible in the algebra $\mathcal{A}_i$ with unit $\mathfrak{p}_i$. Suppose to the contrary that 0 is in the spectrum of f. Then 0 is in the left and in the right spectrum of f as $(\mathcal{A}_i f)^* = f^*\mathcal{A}_i = f\mathcal{A}_i$, f being hermitean. By [11, Ch. V, § 23, p. 311, V.] there exist an irreducible involutive representation θ of $\mathcal{A}_i$ in some Hilbert space $\mathfrak{K}$, and a non–zero $\eta \in \mathfrak{K}$ with $\theta(f)\eta = 0$. Observe that $\mathcal{A}_i$ is symmetric as a closed involutive subalgebra of $\mathcal{A}$. Above, (3.7) and (3.9), we computed the irreducible representations of $\mathcal{A}_i$. Hence, we may suppose that $\mathfrak{K} = \mathfrak{F}_i$ and $\theta = \ell\pi_i|_{\mathcal{A}_i,\mathfrak{F}_i}$ for some $\ell \in L$. The equation $\theta(f)\eta = 0$ contradicts (3.21).

Since $f \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ is invertible in $\mathcal{A}_i$ it follows that $\mathfrak{p}_i \in \mathcal{I}$ (for all $i \in J$), hence $\mathcal{P} := \{\sum_{i\in J} \mathcal{A} * \mathfrak{p}_i * \mathcal{A}\}^- \subseteq \mathcal{I}$. The image of the two–sided ideal $\mathcal{P}$ in the C^*–hull $C^*(\mathcal{A}) = C^*(K, \mathcal{C}(L/L_\omega))$ is dense, because there are no (irreducible) involutive representations annihilating $\mathcal{P}$ (all C^*–algebras have the Wiener property). Again, by (1.10) and the Lemma in the introduction, observe that $\mathcal{P}$ is G–invariant, $\mathcal{P}^{(G)} = C^*(K, \mathcal{C}(L, L_\omega))^{(G)} = \mathcal{A}^{(G)}$. But $\mathcal{A}^{(G)}$ is dense in $\mathcal{A}$, whence $\mathcal{A} = \mathcal{P} = \mathcal{I}$. □

The proof that L–orbits are sets of synthesis follows the same trail.

Theorem 3.22. *Let $\mathcal{I}$ be a closed two–sided ideal in $\mathcal{A} = \mathcal{L}^1(K,Q)$. Suppose that the hull $h(\mathcal{I})$, i.e., $h(\mathcal{I}) := \{\mathcal{P} \in \mathrm{Priv}_*\mathcal{A} \mid \mathcal{P} \supseteq \mathcal{I}\}$, is equal to $\{\ker_{\mathcal{A}} \ell\pi_j \mid \ell \in L\}$ for a certain $j \in J$. Then $\mathcal{I} = \bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$.*

Proof. Clearly, $\mathcal{I}$ is contained in $\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$. Fix, for the moment, $i \in J$, $i \neq j$. By the definition of $h(\mathcal{I})$, for each $\ell \in L$ there exists $g \in \mathcal{I}$ with $(\ell\pi_i)(g) \neq 0$. Again, consider the ideal $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i = \mathcal{I} \cap \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$ in $\mathcal{A}_i = \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$.

As above one finds an element $f \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $\langle(\ell\pi_i)(f)\xi,\xi\rangle \geq \frac{1}{2} < \xi,\xi >$ for all $\ell \in L$, $\xi \in \mathfrak{F}_i$, from which one concludes again that f is invertible in $\mathcal{A}_i$, whence $\mathfrak{p}_i \in \mathcal{I}$.

Therefore, the G–invariant ideal $\mathcal{P} := \left\{\sum_{\substack{i\in J\\ i\neq j}} \mathcal{A} * \mathfrak{p}_i * \mathcal{A}\right\}^-$ is contained in $\mathcal{I}$. Consider the closure $\mathcal{P}'$ of the image of $\mathcal{P}$ in $C^*(\mathcal{A}) = C^*(K, \mathcal{C}(L/L_\omega))$, which is also G–invariant, and invariant under the involution. Since all closed subsets of the primitive spectrum of any C^*–algebra are "sets of synthesis" we conclude that

$$\mathcal{P}' = \bigcap_{\ell\in L} \ker_{C^*(\mathcal{A})} \ell\pi_j.$$

We have the dense G–equivariant injections

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathcal{P}', \text{ and} \\ \bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j' &\longrightarrow \mathcal{P}'. \end{aligned}$$

Again, by (1.10) and the Lemma in the introduction, the collections of their respective G–finite vectors coincide. In particular, $\mathcal{P}^{(G)} = \{\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j\}^{(G)}$. But $\{\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j\}^{(G)}$ is dense in $\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$, whence $\mathcal{P} = \bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j \subseteq \mathcal{I}$. $\square$

For illustration and later applications we consider the special case that L is a direct product $L = S \times K$, where S is a compact abelian Lie group, not necessarily connected. As we will see, in this case the collection of groups L^j, $j \in J$, is finite, and a little more can be said on the structure of the C^*–algebras $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$.

One has two projections $\mathrm{pr}_1 : L \to S$, $\mathrm{pr}_2 : L \to K$. With the subgroup L_ω of L there are associated four canonical subgroups, namely

(3.23)
$$\begin{aligned} S_\omega &= L_\omega \cap S, K_\omega = K \cap L_\omega, S' = \mathrm{pr}_1(L_\omega) = S \cap L_\omega K, \\ K' &= \mathrm{pr}_2(L_\omega) = K \cap SL_\omega. \end{aligned}$$

The subgroup K_ω is normal in K', hence $S_\omega \times K_\omega$ is normal in $S' \times K'$. The image of L_ω in $S'/S_\omega \times K'/K_\omega$ under the canonical map $\mu : S' \times K' \to S'/S_\omega \times K'/K_\omega$ is the graph of a certain isomorphism

(3.24)
$$\kappa : S'/S_\omega \to K'/K_\omega, \text{ and } L_\omega \text{ is its pre–image, } L_\omega = \mu^{-1}\{(s,\kappa(s)) \mid s \in S'/S_\omega\}.$$

In particular, also K'/K_ω is a compact abelian Lie group. The connected component $(S'/S_\omega)_\circ$ of S'/S_ω will be written as S°/S_ω with a certain subgroup S° of S' of finite index. Likewise, we have $(K'/K_\omega)_\circ = K^\circ/K_\omega$, and clearly κ map S°/S_ω onto K°/K_ω.

If $j \in J$ is given, the subgroup L^j is of finite index in L_ω, and L^j contains $S_\omega \times L_\omega$, hence

$$L^j = \mu^{-1}\{(s, \kappa(s)) \mid s \in S^j/S_\omega\} \tag{3.25}$$

for a certain group S^j, $S_\omega \subset S^j \subset S'$. As L^j is of finite index in L_ω, the group S^j has to contain S°. Thus, we see that $\{L^i \mid i \in J\}$ is a finite collection. Moreover, we define K^j by $\kappa(S^j/S_\omega) = K^j/K_\omega$. The group K^j can also be described as the stabilizer in K' of $\alpha_j \in K_\omega^\wedge$. Next we choose a measurable projective extension $\widetilde{\alpha}_j$ of α_j to K^j with a measurable cocycle $m_j : K^j/K_\omega \times K^j/K_\omega \to \mathbb{T}$,

$$\widetilde{\alpha}_j : K^j \longrightarrow \mathcal{B}(V_j), \widetilde{\alpha}_j(x)\widetilde{\alpha}_j(y) = m_j(x, y)\widetilde{\alpha}_j(x\,y). \tag{3.26}$$

The cocycles on compact abelian Lie groups are very well known; replacing m_j by a cohomologous one (and modifying $\widetilde{\alpha}_j$ accordingly), we may assume that m_j lives on the finite group K^j/K°. In particular, $\widetilde{\alpha}_j$ is then continuous.

The representation $\widetilde{\alpha}_j$ delivers an m_j–projective representation Z_j of K^j/K_ω in $\mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K, V_j)$ by

$$(Z_j(a)\xi)(k) = \widetilde{\alpha}_j(a)\xi(ka) \tag{3.27}$$

for $k \in K$ and $a \in K^j$ (or $a \in K^j/K_\omega$, clearly Z_j is constant on K_ω–cosets), and hence also a representation A_j of the isomorphic copy S^j/S_ω,

$$A_j(s) = \chi(s)Z_j(\kappa(s)) \tag{3.28}$$

for $s \in S^j/S_\omega$ with a certain unitary character χ on S^j/S_ω to be determined later. Also A_j is projective, the cocycle m'_j on S^j/S_ω being given by $m'_j(s,t) = m_j(\kappa(s), \kappa(t))$.

The intertwining operators $X_j(\ell)$, $Y_j(\ell)$, $\ell \in L^j$ in V_j resp. $\mathfrak{H}_j$, see (2.2), (2.18), can now be specified to be

$$X_j(\ell) = \widetilde{\alpha}_j(b)\chi(s) \;\text{ if }\; \ell = (s, b) \in L^j, (Y_j(\ell)\xi)(k) = X_j(\ell)\xi(\ell^{-1}k\,\ell). \tag{3.29}$$

The crucial properties of the representation A_j are the following.

3.30. *For any $s \in S^j/S_\omega$ and any $k \in K$ the operators $A_j(s)$ and $\pi'_j(k)$ commute. If $\ell = (s, b) \in L^j$ and $a \in K$, whence $\ell\,a = (s, b\,a)$, then*

$$Y_j(\ell)\pi'_j(a) = A_j(s)\pi'_j(b\,a).$$

The straightforward computations are omitted. This implies that the functions $f \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$, see (2.19), can alternatively be described as those functions which are constant on S_ω–cosets, and which satisfy

$$f(\ell(s,k)) = \pi_j'(k)^* A_j(s)^* f(\ell) A_j(s) \pi_j'(k) \text{ for } \ell \in L, k \in K, s \in S^j. \tag{3.31}$$

Finally, we choose, as above, a finite–dimensional $\mathfrak{U}_j$–invariant, see (3.1), subspace $\mathfrak{F}_j$ of $\mathfrak{H}_j$, which we now assume to be irreducible. $\mathfrak{U}_j$-invariant means, by (3.30), to be invariant under $A_j(S^j/S_\omega)$ and under $\pi_j'(K)$. As these two groups commute, $\mathfrak{F}_j$ decomposes into a tensor product,

3.32. *$\mathfrak{F}_j = \mathfrak{P}_j \otimes \mathfrak{Q}_j$, where $\mathfrak{P}_j$ is an irreducible S^j/S_ω–space, and $\mathfrak{Q}_j$ is an irreducible K–space.*

Since the cocycle m_j' lives on S^j/S°, the operators $A_j(s)$, $s \in S^\circ/S_\omega$, are scalar on $\mathfrak{P}_j$. Adapting the above χ to the chosen subspace $\mathfrak{F}_j$ we can arrange that $A_j(S^\circ/S_\omega)$ is trivial on $\mathfrak{P}_j$.

As we have seen earlier, (3.9), the C^*–hull of $\mathfrak{p}_j * \mathcal{L}^1(K,Q) * \mathfrak{p}_j$ is $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$, which is given analogous to (3.31), compare (2.19), (3.6). The functions in the latter algebra are determined by their restriction to S, i.e., $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$ can be identified with

$$\begin{aligned} &\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j \otimes \mathfrak{Q}_j)) \\ &\quad = \{f : S \to \mathcal{B}(\mathfrak{P}_j \otimes \mathfrak{Q}_j) \mid f \text{ is continuous, } f(st) = \\ &\qquad A_j(t)^* \otimes \mathrm{Id}_{\mathfrak{Q}_j} \circ f(s) \circ A_j(t) \otimes \mathrm{Id}_{\mathfrak{Q}_j} \text{ for } s \in S, t \in S^j\} \\ &\quad \cong \mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j), \end{aligned} \tag{3.33}$$

where $\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j))$ has the obvious meaning. A tensor $\varphi \otimes B$ is mapped to the function $s \longmapsto \varphi(s) \otimes B$. Actually, the functions in $\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j))$ are constant on S°–cosets, i.e., this space may be written as $\mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j))$. One should note that

3.34. *the S–isotypical components in $\mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j), S$ acts by left translations, are clearly finite–dimensional, whence they coincide with the S–isotypical components in $\mathfrak{p}_j * \mathcal{L}^1(K,Q) * \mathfrak{p}_j$.*

Also, the L–action on $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$ can easily be transferred into the new picture. An $\ell = (t,k) \in S \times K$ acts on the tensor $\varphi \otimes B \in \mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j)$ by $\ell(\varphi \otimes B) = \varphi' \otimes B'$, where $\varphi'(s) = \varphi(t^{-1}s)$, $B' = \widetilde{\pi}_j(k) \circ B \circ \widetilde{\pi}_j(k)^*$.

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Propriété de Kazhdan et sous-groupes discrets de covolume infini

J.-F. Quint

Résumé

Soient G un groupe de Lie réel presque simple, de centre fini, de rang réel ≥ 2 et $G = K(\exp \mathfrak{a}^+)K$ une décomposition de Cartan de G. Si Γ est un sous-groupe discret Zariski dense de G, son indicateur de croissance a été défini dans [7] : c'est une fonction homogène concave d'un cône convexe de $\mathfrak{a}^+$ dans $\mathbb{R}_+$. Si Γ est un réseau de G, ψ_Γ est la restriction à $\mathfrak{a}^+$ de la somme ρ des racines positives multipliées par la dimension de leurs espaces poids. Nous montrons ici qu'il existe une forme linéaire θ de $\mathfrak{a}$, strictement positive sur $\mathfrak{a}^+ \setminus \{0\}$, ne dépendant que de G et telle que, pour tout sous-groupe discret Γ de G qui ne soit pas un réseau de G, $\psi_\Gamma \leq \rho - \theta$.

Introduction

Soit G un groupe de Lie réel presque simple de centre fini.

Soient X l'espace symétrique de G, muni d'une métrique riemannienne G-invariante, x un point de X et K son stabilisateur dans G. Si Γ est un sous-groupe discret Zariski dense de G, on note τ_Γ l'exposant de convergence de la série de Dirichlet

$$\sum_{\gamma \in \Gamma} e^{-td(x,\gamma x)} \quad (t \in \mathbb{R}).$$

C'est un réel > 0. L'exposant de convergence des réseaux de G ne dépend que de G. On le note τ_G.

Généralisant un phénomène observé par K. Corlette dans [2] pour les groupes de rang réel 1 ayant la propriété (T) de Kazhdan, E. Leuzinger a montré dans [5] que, si G avait la propriété (T), il existait un réel $\varepsilon > 0$ tel que, pour tout sous-groupe Γ discret de G qui ne soit pas un réseau, on ait $\tau_\Gamma \leq \tau_G - \varepsilon$.

Soit $\mathfrak{a}$ un sous-espace de Cartan de l'algèbre de Lie de G tel que le plat maximal de X stable par $\exp \mathfrak{a}$ contienne x. Soit $\mathfrak{a}^+ \subset \mathfrak{a}$ une chambre de Weyl. Si Γ est un sous-groupe discret Zariski dense de G, nous avons introduit dans [7] son indicateur de croissance : c'est une fonction homogène concave positive, définie sur un cône convexe fermé de $\mathfrak{a}^+$. Elle permet, par exemple, de décrire les exposants de convergence de Γ par rapport à toutes les métriques finsleriennes G-invariantes de X. Si Γ est un réseau, son indicateur de croissance est la restriction à $\mathfrak{a}^+$ d'une forme linéaire ρ qui ne dépend que de G.

Nous démontrons ici :

Théorème. *Supposons G de rang réel ≥ 2. Il existe alors une forme linéaire θ, strictement positive sur $\mathfrak{a}^+ \setminus \{0\}$, telle que, pour tout sous-groupe discret Γ de G qui ne soit pas un réseau, on ait $\psi_\Gamma \leq \rho - \theta$.*

La démonstration de ce résultat repose sur une version forte de la propriété (T), établie par H. Oh dans [6] : il s'agit d'une estimation de la vitesse de décroissance dans G des coefficients matriciels associés aux vecteurs K-finis dans les représentations unitaires de G ne contenant pas de vecteurs G-invariants.

Partant de ce résultat, on utilisera le plan et les méthodes de [3], où A. Eskin et C. McMullen démontrent un résultat de comptage pour les réseaux de G. Précisément, étant donné un réseau Γ dans G, dans [3], on commence par établir, à partir du théorème de Howe-Moore un résultat d'équidistribution des translatés des K-orbites dans $\Gamma \backslash G$, puis on utilise cet énoncé pour estimer le nombre de points de Γ dans les parties K-invariantes à droite de G.

Ici, nous nous donnerons un sous-groupe discret Γ de G qui ne soit pas un réseau. Le théorème de Oh nous permettra de démontrer un résultat de disparition des translatés des K-orbites, la proposition 3.2. Nous en déduirons une majoration pour le comptage des points de Γ dans les parties K-invariantes à droite qui mènera à notre théorème.

Nous démontrerons aussi un analogue de ce résultat pour un groupe de Lie presque simple défini sur n'importe quel corps local $\mathbb{K}$ de caractéristique $\neq 2$.

Je tiens à remercier Y. Benoist qui a attiré mon attention sur ces questions et E. Leuzinger qui m'a communiqué son article [5].

1 Notations

Soit $\mathbb{K}$ un corps local de caractéristique différente de 2.

Si $\mathbb{K}$ est $\mathbb{R}$ ou $\mathbb{C}$, on le munit de la valeur absolue usuelle et on pose $q = e$ et, pour tout $x \neq 0$ dans $\mathbb{K}$, $\omega(x) = -\log |x|$.

Si $\mathbb{K}$ est non-archimédien, on note $\mathcal{O}$ l'anneau de valuation de $\mathbb{K}$, $\mathfrak{m}$ l'idéal maximal de $\mathcal{O}$ et q le cardinal du corps résiduel $\mathcal{O}/\mathfrak{m}$ de $\mathbb{K}$; on note ω la valuation de $\mathbb{K}$ telle que $\omega(\mathfrak{m} \setminus \mathfrak{m}^2) = 1$ et on munit $\mathbb{K}$ de la valeur absolue $x \mapsto q^{-\omega(x)}$.

Soient $\mathbf{G}$ un $\mathbb{K}$-groupe presque simple de $\mathbb{K}$-rang ≥ 2 et G le groupe de ses $\mathbb{K}$-points.

On choisit dans $\mathbf{G}$ un tore $\mathbb{K}$-déployé maximal $\mathbf{A}$. On note A le groupe de ses $\mathbb{K}$-points, $\mathbf{Z}$ le centralisateur de $\mathbf{A}$ dans $\mathbf{G}$ et Z le groupe des $\mathbb{K}$-points de $\mathbf{Z}$.

Soit X le groupe des caractères de $\mathbf{A}$. On note E l'espace vectoriel réel dual de $\mathbb{R} \otimes_{\mathbb{Z}} X$ et, pour χ dans X, on note χ^ω la forme linéaire associée sur E. On note ν l'unique homomorphisme continu de Z dans E tel que, pour tous a dans A et χ dans X, $\chi^\omega(\nu(a)) = -\omega(\chi(a))$.

Dorénavant, on identifie les éléments de X et les formes linéaires associées de E. Soit Σ l'ensemble des racines de $\mathbf{A}$ dans l'algèbre de Lie de $\mathbf{G}$. Alors Σ est un système de racines dans E^*. On choisit dans Σ un système de racines positives Σ^+. On note $E^+ \subset E$ la chambre de Weyl positive de Σ^+ et $Z^+ = \nu^{-1}(E^+)$. On note $\iota : E \to E$ l'involution d'opposition de E^+.

Enfin, on choisit dans G un bon sous-groupe compact maximal relativement à A, qu'on note K. On a la décomposition de Cartan $G = KZ^+K$ et, pour z_1,z_2 dans Z^+, z_1 appartient à Kz_2K si et seulement si $\nu(z_1) = \nu(z_2)$. On note μ l'unique application bi-K-invariante de G dans E^+ dont la restriction à Z^+ vaut ν. Pour g dans G, on a $\mu(g^{-1}) = \iota(\mu(g))$. On a :

Proposition 1.1 (Benoist, [1, 5.1]). *Pour toute partie compacte L de G, il existe une partie compacte M de E telle que, pour tout g dans G, on ait $\mu(LgL) \subset \mu(g) + M$.*

Nous allons appliquer à des problèmes de comptage dans G le résultat suivant de H. Oh. Il s'agit d'une quantification du fait que G possède la propriété (T) de Kazhdan :

Théorème 1.2 (Oh, [6]). *Il existe une fonction $\xi : E^+ \to \mathbb{R}_+$, invariante par ι, telle que*

(i) pour toute représentation de G dans un espace de Hilbert H sans vecteurs G-invariants, on ait, pour v,w K-invariants dans H et g dans G,

$$|\langle gv,w\rangle| \le \xi(\mu(g))\,\|v\|\,\|w\|\,.$$

(ii) il existe une forme linéaire θ de E, strictement positive sur $E^+\setminus\{0\}$, et telle que, pour tout $\varepsilon > 0$, il existe $0 < c < d$ avec

$$\forall x \in E^+ \quad cq^{-\theta(x)} \le \xi(x) \le dq^{-(1-\varepsilon)\theta(x)}.$$

Dorénavant, on fixe de tels ξ et θ.

2 Divergence exponentielle des sous-groupes discrets

Nous rappelons ici les résultats de [7] sur le comportement de l'image par μ d'un sous-groupe discret de G.

Soient ν une mesure de Radon et N une norme sur E. Pour tout cône ouvert $\mathcal{C}$ de E, on note $\tau_{\mathcal{C}}$ l'exposant de convergence de l'intégrale de Dirichlet :

$$\int_{\mathcal{C}} e^{-tN(x)}\mathrm{d}\nu(x) \quad (t \in \mathbb{R})$$

et, pour x dans E, on pose :

$$\psi_\nu(x) = N(x) \inf \tau_{\mathcal{C}},$$

la borne inférieure étant prise sur l'ensemble des cônes ouverts $\mathcal{C}$ de E qui contiennent x. La fonction ψ_ν ne dépend pas de la norme choisie. On l'appelle indicateur de croissance de ν.

Les deux résultats élémentaires suivants sont prouvés dans [7, 5.2] :

Lemme 2.1. *Soit $\theta : E \to \mathbb{R}$ une fonction homogène et continue. On a $\psi_{e^\theta \nu} = \psi_\nu + \theta$.*

Lemme 2.2. *Soient ν et ν' des mesures de Radon sur E. S'il existe une partie compacte M de E et un réel $\omega \geq 0$ tels que, pour tout borélien B de E, $\nu'(B) \leq \omega\nu(B + M)$, alors $\psi_{\nu'} \leq \psi_\nu$.*

Si Γ est un sous-groupe fermé de G, soit ν_Γ l'image par μ d'une mesure de Haar de Γ ; on note ψ_Γ la fonction $\frac{1}{\log q}\psi_{\nu_\Gamma}$. Soit ρ la forme linéaire sur E qui est la somme des racines positives multipliées par la dimension de leurs espaces poids. De la formule d'intégration pour la décomposition de Cartan [4, I.5.2], on déduit tout de suite :

Lemme 2.3. *On a $\psi_G = \rho$.*

Le résultat principal de [7] s'énonce :

Théorème 2.4. *Soit Γ un sous-groupe discret Zariski dense de G. La fonction ψ_Γ est majorée par ρ. Elle est concave, semi-continue supérieurement et positive partout où elle est $> -\infty$.*

3 Disparition des K-orbites

On fixe des mesures de Haar dg sur G et dk sur K. On suppose que $\int_K \mathrm{d}k = 1$.

Rappelons une propriété usuelle des mesures de Haar :

Lemme 3.1. *Soient H un groupe topologique localement compact unimodulaire et K un sous-groupe fermé unimodulaire de H. Alors $K\backslash H$ possède une unique mesure de Radon H-invariante non triviale, à multiplication par un réel > 0 près. Munissons H et K de mesures de Haar à droite* dh *et* dk. *La mesure H-invariante m de $K\backslash H$ peut-être normalisée de façon à ce que, si ψ est une fonction mesurable sur H, ψ soit intégrable si et seulement si la fonction $\hat{\psi} : Kh \mapsto \int_K \psi(kh)\mathrm{d}k$ est définie presque partout et intégrable sur $K\backslash H$ et que, alors, on ait :*

$$\int_{K\backslash H} \hat{\psi}\mathrm{d}m = \int_H \psi(h)\mathrm{d}h.$$

Soit Γ un sous-groupe discret de G. On note m la mesure G-invariante sur $\Gamma\backslash G$ associée à la mesure de Haar de G et à la mesure de comptage sur Γ. On suppose que Γ n'est pas un réseau de G, c'est-à-dire que $m(\Gamma\backslash G) = \infty$. On note p_0 l'image de e dans $\Gamma\backslash G$.

Dans cette section, nous allons prouver :

Proposition 3.2. *Soit* $\varphi : \Gamma\backslash G \to \mathbb{C}$, *une fonction continue à support compact. Il existe un réel* $c > 0$ *tel que, pour tout* g *dans* G, *on ait :*

$$\left|\int_K \varphi(p_0 k g)\mathrm{d}k\right| \leq c\xi(\mu(g)).$$

Nous commencerons par nous ramener au cas où φ est K-invariante, grâce à un raisonnement simple de topologie :

Lemme 3.3. *Soit* X *un espace topologique localement compact et* K *un groupe compact agissant continûment sur* X. *Si* φ *est une fonction continue à support compact sur* X, *il existe une fonction continue* K*-invariante, positive et à support compact* ψ *telle que* $|\varphi| \leq \psi$.

Nous pouvons d'ores et déjà conclure, dans le cas où $\mathbb{K}$ est non-archimédien :

Démonstration de la proposition 3.2 quand $\mathbb{K}$ *est non-archimédien.* D'après le lemme 3.3, on peut supposer que φ est K-invariante. Par ailleurs comme K est un sous-groupe ouvert de G, la mesure de Haar de K est la restriction de la mesure de Haar de G, et, d'après le lemme 3.1, pour tout g dans G, on a :

$$\int_K \varphi(p_0 k g)\mathrm{d}k = \mathrm{card}(\Gamma \cap K) \int_{p_0 K} \varphi(pg)\mathrm{d}m(p).$$

Or, comme Γ n'est pas un réseau de G, la représentation naturelle de G dans $\mathrm{L}^2(\Gamma\backslash G)$ ne possède pas de vecteurs G-invariants ; le résultat est alors une conséquence du théorème 1.2.□

Intéressons-nous à présent au cas où $\mathbb{K}$ est archimédien. Nous utiliserons une décomposition de la mesure de Haar de G :

Lemme 3.4. *Il existe une base de voisinages de* e *dans* G *constituée de voisinages* V *pour lesquels il existe une mesure de Radon* τ *sur* V *telle que, pour toute fonction* φ *continue à support compact dans* KV, *on ait :*

$$\int_{KV} \varphi(g)\mathrm{d}g = \int_{K\times V} \varphi(kv)\mathrm{d}k\mathrm{d}\tau(v).$$

Démonstration. C'est une conséquences de la formule d'intégration pour les décompositions d'Iwasawa de G (*cf.* [4, I.5.1]). □

Enfin, nous utiliserons un cas particulier du théorème du front d'onde, de A. Eskin et C. McMullen :

Lemme 3.5 (Eskin, McMullen, [3]). *Pour tout voisinage* U *de* e *dans* G, *il existe un voisinage* V *de* e *tel que, pour tout* g *dans* G, $KVg \subset KgU$.

Démonstration. Remarquons que, par décomposition de Cartan, quitte à remplacer par la suite V par un voisinage plus petit normalisé par K, il suffit de démontrer ce résultat pour des g dans Z^+K.

Soient $\mathbf{P}$ le $\mathbb{K}$-sous-groupe parabolique minimal de $\mathbf{G}$ associé au choix de $\mathbf{A}$ et de E^+ et P le groupe de ses $\mathbb{K}$-points. Le groupe P possède une base de voisinages de l'élément neutre stables par l'action adjointe des éléments de $(Z^+)^{-1}$. Or, on a la décomposition d'Iwasawa $G = KP$. Soient alors U un voisinage de e dans G normalisé par K et $V \subset U$ un voisinage de e dans P tel que, pour tout z dans Z^+, $z^{-1}Vz \subset V$. L'ensemble KV est un voisinage de e dans G et, pour z dans Z^+ et k dans K, on a :

$$KVzk = Kz(z^{-1}Vz)k \subset KzVk \subset KzUk = K(zk)U,$$

ce qu'il fallait démontrer.□

Nous pouvons à présent démontrer notre proposition :

Démonstration de la proposition 3.2 quand $\mathbb{K}$ *est* $\mathbb{R}$ *ou* $\mathbb{C}$. Soit U un voisinage compact de e dans G. D'après le lemme 3.3, appliqué à la fonction $p \mapsto \sup_{u \in U} |\varphi(pu^{-1})|$, il existe une fonction ψ K-invariante, positive, continue et à support compact sur $\Gamma\backslash G$ telle que, pour p dans $\Gamma\backslash G$ et u dans U, on ait $|\varphi(p)| \leq \psi(pu)$.

Soit alors, comme dans le lemme 3.5, un voisinage V borné de e dans G tel que, pour g dans G, $KVg \subset KgU$. On suppose que V vérifie la propriété du lemme 3.4 et on choisit une mesure τ sur V comme dans cet énoncé. Enfin, soit W un voisinage borné de e tel que $V \subset W$ et que W soit normalisé par K. Comme Γ n'est pas un réseau de G, la représentation naturelle de G dans $\mathrm{L}^2(\Gamma\backslash G)$ ne possède pas de vecteurs G-invariants et, donc, le théorème 1.2 s'applique aux vecteurs K-invariants que sont la fonction ψ et la fonction caractéristique de p_0KW. En d'autres termes, il existe $c > 0$ tel que, pour tout g dans G,

$$\int_{p_0KV} \psi(pg)\mathrm{d}m(p) \leq \int_{p_0KW} \psi(pg)\mathrm{d}m(p) \leq c\xi(\mu(g)).$$

Soit g dans G. D'une part, on a, d'après le lemme 3.1 :

$$\begin{aligned}\int_{KV} \psi(p_0hg)\mathrm{d}h &= \int_{p_0KV} \mathrm{card}(p \cap KV)\psi(pg)\mathrm{d}m(p)\\ &\leq \mathrm{card}(\Gamma \cap KVV^{-1}K) \int_{p_0KV} \psi(pg)\mathrm{d}m(p)\end{aligned}$$

(où l'on a employé le même symbole pour un point de $\Gamma\backslash G$ et pour la classe à droite associée).

D'autre part, pour v dans V, il existe u dans U tel que $Kvg = Kgu$; par conséquent, on a :

$$\int_K \psi(p_0kvg)\mathrm{d}k \geq \int_K \psi(p_0kgu)\mathrm{d}k \geq \int_K |\varphi(p_0kg)|\,\mathrm{d}k$$

et, donc,

$$\int_{KV} \psi(p_0 g)\mathrm{d}g = \int_V \left(\int_K \psi(p_0 kvg)\mathrm{d}k\right) \mathrm{d}\tau(v) \geq \tau(V) \int_K |\varphi(p_0 kg)| \,\mathrm{d}k.$$

Il vient :

$$\int_K |\varphi(p_0 kg)| \,\mathrm{d}k \leq \frac{c\,\mathrm{card}(\Gamma \cap KVV^{-1}K)}{\tau(V)} \xi(\mu(g)),$$

ce qu'il fallait démontrer. □

4 Majoration des exposants de convergence

Dans cette section, nous utilisons la majoration précédente pour démontrer :

Théorème 4.1. *Soit Γ un sous-groupe discret de G. Si Γ n'est par un réseau de G, on a $\psi_\Gamma \leq \rho - \theta$.*

Établissons un résultat préliminaire. Pour tout borélien B de G/K, notons χ_B la fonction indicatrice de l'ensemble $B^{-1} \subset K\backslash G$ et F_B la fonction sur $\Gamma\backslash G$ telle que, pour tout g dans G, $F_B(\Gamma g) = \mathrm{card}(\Gamma \cap gB)$. Munissons $K\backslash G$ de sa mesure G-invariante n associée au choix des mesures de Haar de G et de K. On a :

Lemme 4.2. *Soit φ une fonction continue à support compact dans $\Gamma\backslash G$. Il existe un réel $c > 0$ tel que, pour tout borélien B de mesure finie dans G/K, on ait :*

$$\left|\int_{\Gamma\backslash G} \varphi F_B \mathrm{d}m\right| \leq c \int_{B^{-1}} (\xi \circ \mu)\mathrm{d}n.$$

Démonstration. Soit B un borélien de mesure finie dans G/K. La fonction $g \mapsto \chi_B(Kg)$ est intégrable sur G. Or, pour g dans G, on a :

$$F_B(\Gamma g) = \sum_{\gamma \in \Gamma} \chi_B(\gamma g).$$

D'après le lemme 3.1, la fonction F_B est intégrable sur $\Gamma\backslash G$ et son intégrale est égale à $n(B^{-1})$. De même, en appliquant le lemme 3.1 à $\Gamma\backslash G$ et à $K\backslash G$, pour toute fonction continue φ à support compact dans $\Gamma\backslash G$, on a :

$$\int_{\Gamma\backslash G} \varphi F_B \mathrm{d}m = \int_G \varphi(\Gamma g)\chi_B(Kg)\mathrm{d}g = \int_{B^{-1}} \psi \mathrm{d}n,$$

où, pour g dans G, on a $\psi(Kg) = \int_K \varphi(\Gamma kg)\mathrm{d}k$.

Or, d'après la proposition 3.2, si φ est une fonction continue à support compact dans $\Gamma\backslash G$, il existe un réel $c > 0$ tel que, pour tout g dans G, on ait $|\int_K \varphi(\Gamma kg)\mathrm{d}k| \leq c\xi(\mu(g))$. Il vient bien, pour tout borélien B de mesure finie dans G/K,

$$\left|\int_{\Gamma\backslash G} \varphi F_B \mathrm{d}m\right| \leq c \int_{B^{-1}} (\xi \circ \mu)\mathrm{d}n.$$

□

Terminons à présent :

Démonstration du théorème 4.1. Soit V un voisinage compact de e dans G. D'après le proposition 1.1, il existe une partie compacte M de E telle que, pour tout g dans G, $\mu(V^{-1}g) \subset \mu(g) + M$. Alors, pour g dans V, on a, pour tout borélien B de E, $(\Gamma \cap \mu^{-1}(B)) \subset (\Gamma \cap g\mu^{-1}(B+M))$ et, donc, $F_{\mu^{-1}(B+M)}(\Gamma g) \geq \mathrm{card}(\Gamma \cap \mu^{-1}(B))$.

Donnons-nous une fonction φ positive, continue, à support compact dans p_0V, avec $\int_{\Gamma\backslash G} \varphi \mathrm{d}m = 1$. Soit $c > 0$, comme dans le lemme 4.2. On a, pour tout borélien B de E,

$$\mathrm{card}(\Gamma \cap \mu^{-1}(B)) \leq \int_{\Gamma\backslash G} \varphi F_{\mu^{-1}(B+M)} \mathrm{d}m \leq c \int_{(\mu^{-1}(B+M))^{-1}} (\xi \circ \mu) \mathrm{d}n.$$

Or $(\mu^{-1}(B+M))^{-1} = \mu^{-1}(\iota(B+M))$ et, donc, en notant ν_G l'image par μ de la mesure de Haar de G,

$$\int_{(\mu^{-1}(B+M))^{-1}} (\xi \circ \mu) \mathrm{d}n = \int_{\iota(B+M)} \xi \mathrm{d}\nu_G = \int_{B+M} \xi \mathrm{d}\nu_G,$$

d'où le résultat, par définition de θ et d'après les lemmes 2.2 et 2.3. □

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Singular masas of von Neumann algebras: examples from the geometry of spaces of nonpositive curvature[1]

Guyan Robertson

Abstract

If Γ is a group, then the von Neumann algebra $VN(\Gamma)$ is the convolution algebra of functions $f \in \ell_2(\Gamma)$ which act by convolution on $\ell_2(\Gamma)$ as bounded operators. Maximal abelian $\star$-subalgebras (masas) of von Neumann algebras have been studied from the early days.

If Γ is a torsion free cocompact lattice in a semisimple Lie group G of rank r with no centre and no compact factors then the geometry of the symmetric space $X = G/K$ may be used to define and study masas of $VN(\Gamma)$. These masas are of the form $\mathrm{VN}(\Gamma_0)$, where Γ_0 is the period group of some Γ-periodic maximal flat in X. There is a similar construction if Γ is a lattice in a p-adic Lie group G, acting on its Bruhat-Tits building.

Consider the compact locally symmetric space $M = \Gamma\backslash X$. Assume that T^r is a totally geodesic flat torus in M and let $\Gamma_0 \cong \mathbb{Z}^r$ be the image of the fundamental group $\pi(T^r)$ under the natural monomorphism from $\pi(T^r)$ into $\Gamma = \pi(M)$. Then $\mathrm{VN}(\Gamma_0)$ is a masa of $\mathrm{VN}(\Gamma)$. If in addition $\mathrm{diam}(T^r)$ is less than the length of a shortest closed geodesic in M then $\mathrm{VN}(\Gamma_0)$ is a *singular masa* : its unitary normalizer is as small as possible. This last result is joint work with A. M. Sinclair and R. R. Smith [RSS].

1 Background

Let Γ be an ICC group: each element in Γ other than the identity has infinite conjugacy class. The group von Neumann algebra is the convolution algebra

$$\mathrm{VN}(\Gamma) = \{f \in \ell^2(\Gamma) : g \mapsto f \star g \text{ is in } B(\ell^2(\Gamma))\}.$$

It is well known that $\mathrm{VN}(\Gamma)$ is a **factor of type** II_1. This means

(a) $\mathrm{VN}(\Gamma)$ is a strongly closed $\star$-subalgebra of $B(\ell^2(\Gamma))$, with trivial centre;

(b) there is a faithful trace on $\mathrm{VN}(\Gamma)$ defined by $\mathrm{tr}(f) := f(1)$.

The group Γ may be embedded as a subgroup of the unitary group of $\mathrm{VN}(\Gamma)$ by identifying an element $\gamma \in \Gamma$ with the corresponding delta function δ_γ. A major result of A. Connes [Co, Corollary 3] implies:

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Theorem. (A. Connes) If Γ_1,Γ_2 are countable amenable ICC groups then $\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2)$, the algebra being isomorphic to the hyperfinite II_1 factor.

At the opposite extreme from amenable groups there is the
Rigidity Conjecture (A. Connes) If ICC groups Γ_1,Γ_2 have Property (T) of Kazhdan, then

$$\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2) \Rightarrow \Gamma_1 \cong \Gamma_2.$$

Compare this with the

Rigidity Theorem (Mostow-Margulis-Prasad) For $i = 1, 2$, let Γ_i be a lattice in G_i, a connected non-compact simple Lie group with trivial centre, $G_1 \neq \mathrm{PSL}_2(\mathbb{R})$. Then

$$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2.$$

In Mostow's proof of rigidity ([Mo]: the cocompact, higher rank case), maximal flats of symmetric spaces play an important role. There is some reason to hope that masas of von Neumann algebras might play a similar role for Connes' conjecture.

2 Maximal abelian $\star$-subalgebras

Let $\mathcal{A}$ be a maximal abelian $\star$-subalgebra (masa) of $\mathrm{VN}(\Gamma)$. Say that $\mathcal{A}$ is a **singular masa** if :

$$u \in \mathrm{VN}(\Gamma),\ u \text{ unitary},\ u\mathcal{A}u^* = \mathcal{A} \Rightarrow u \in \mathcal{A}.$$

Singular masas [2] always exist [P1], but are hard to construct explicitly.

If $\mathcal{A} = \mathrm{VN}(\Gamma_0)$, where Γ_0 is a subgroup of Γ, then
$\mathrm{VN}(\Gamma_0)$ embeds as a subalgebra of $\mathrm{VN}(\Gamma)$ via $f \mapsto \overline{f}$, where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let $\Gamma_1 < \Gamma_0 < \Gamma$, with Γ_0 abelian. Define the commutant $\mathrm{VN}(\Gamma_1)'$ to be the centralizer of $\mathrm{VN}(\Gamma_1)$ in $\mathrm{VN}(\Gamma_0)$. Suppose that, for all $x \notin \Gamma_0$, the set*

$$A_x = \{x_1^{-1} x x_1 : x_1 \in \Gamma_1\}$$

is infinite. Then $\mathrm{VN}(\Gamma_1)' = \mathrm{VN}(\Gamma_0)$. In particular, $\mathrm{VN}(\Gamma_0)$ is a masa of $\mathrm{VN}(\Gamma)$.

(This result is contained in [Di], in the case $\Gamma_1 = \Gamma_0$.)

[2] If the unitary normalizer of $\mathcal{A}$ generates $\mathrm{VN}(\Gamma)$ then $\mathcal{A}$ is a *Cartan* masa. $\mathrm{VN}(\Gamma)$ may not contain a Cartan masa: e.g. $\Gamma = \mathbb{F}_2$. S. Popa [P2] has recently used Cartan masas to construct isomorphism invariants for certain II_1 factors.

Proof. Let $f \in \mathrm{VN}(\Gamma_1)'$ and $x \notin \Gamma_0$.
Then $\delta_{x_1^{-1}} * f * \delta_{x_1} = f$ (for all $x_1 \in \Gamma_1$)
$\Rightarrow f$ is constant on A_x
$\Rightarrow f = 0$ on A_x (since $f \in \ell^2(\Gamma)$ and $\#A_x = \infty$)
$\Rightarrow f(x) = 0$ (for all $x \notin \Gamma_0$)
$\Rightarrow f \in \mathrm{VN}(\Gamma_0)$. □

There is a **conditional expectation** $\mathbb{E}_{\mathcal{A}} : \mathrm{VN}(\Gamma) \to \mathcal{A}$ onto any masa $\mathcal{A}$, which extends to an orthogonal projection on $\ell^2(\Gamma)$. If $\mathcal{A} = \mathrm{VN}(\Gamma_0)$, where Γ_0 is an abelian subgroup of Γ and if $f \in \mathrm{VN}(\Gamma)$, then

$$\mathbb{E}_{\mathcal{A}} f(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition.[SS] Say that $\mathcal{A}$ is a **strongly singular masa** of $\mathrm{VN}(\Gamma)$ if

$$\|\mathbb{E}_{u\mathcal{A}u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \geq \|u - \mathbb{E}_{\mathcal{A}}(u)\|_2$$

for all unitaries $u \in \mathrm{VN}(\Gamma)$. [Here $\| \cdot \|_{\infty,2}$ means: operator norm on domain, ℓ^2 norm on range.]

This condition implies that any unitary $u \in \mathrm{VN}(\Gamma)$ which normalizes $\mathcal{A}$ necessarily lies in $\mathcal{A}$. Therefore $\mathcal{A}$ is a singular masa.

2.1 Construction of masas

Let Γ be an ICC group and let Γ_0 be an abelian subgroup. Here is a condition that ensures that $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

(SS) If $x_1, \dots, x_m, y_1, \dots, y_n \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 y_j \,, \tag{2.1}$$

then some $x_i \in \Gamma_0$.

Theorem. *Condition* (SS) *implies that* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

The proof of this result is contained in [RSS]. It can be used to construct strongly singular masas of $\mathrm{VN}(\Gamma)$, for certain geometrically defined groups Γ, acting on spaces of nonpositive curvature.

Let G be a semisimple Lie group of rank r with no centre and no compact factors. Let Γ be a torsion free cocompact lattice in G. Then Γ acts freely on the symmetric space $X = G/K$ and the quotient manifold $M = \Gamma\backslash X$ is a compact locally symmetric space, with fundamental group $\pi(M) = \Gamma$.

Let $T^r \subset M$ be a totally geodesic embedding of a flat r-torus in M. The inclusion $i : T^r \to M$ induces an injective homomorphism $i_* : \pi(T^r) \to \pi(M)$. (Reason: no geodesic loop in M can be null-homotopic.)

Let $\Gamma_0 = i_*\pi(T^r) \cong \mathbb{Z}^r < \Gamma$. Under these assumptions, the following results hold.

Theorem A. $\mathrm{VN}(\Gamma_0)$ *is a masa of* $\mathrm{VN}(\Gamma)$.

Theorem B. [RSS] *Let σ be the length of a shortest closed geodesic in M. If* $\mathrm{diam}(T^r) < \sigma$ *then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

3 Proofs

Theorem A is a consequence of a stronger result. Recall that a geodesic L in X is *regular* if it lies in only one maximal flat. See the appendix below for further details. A regular geodesic in $M = \Gamma\backslash X$ is, by definition, the image of a regular geodesic in X under the canonical projection. It follows from [Mo, §11] that T^r contains a closed regular geodesic.

Theorem A′. Let $x_1 \in \Gamma_0$ be the class of a *regular* closed geodesic c in T^r, and

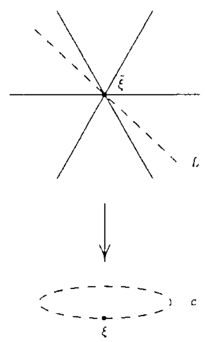
$$\Gamma_1 = \langle x_1 \rangle \cong \mathbb{Z} < \Gamma_0.$$

Then $\mathrm{VN}(\Gamma_1)' = \mathrm{VN}(\Gamma_0)$.

Consequently $\mathrm{VN}(\Gamma_0)$ is the unique masa containing $\mathrm{VN}(\Gamma_1)$.

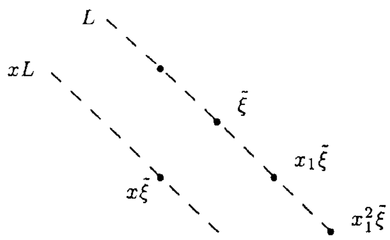
Proof of Theorem A′. (Using Lemma 2.1.)

Lift c to a geodesic L in X through $\tilde{\xi}$, where $p(\tilde{\xi}) = \xi$.



Regularity of the geodesic c means that L lies in a *unique* maximal flat F_0 and $p(F_0) = T^r$.

Now x_1 acts on L by translation.



Suppose that $A_x = \{x_1^{-n} x x_1^n : n \in \mathbb{Z}\}$ is finite, and let

$$\delta = \sup\{d(\eta, x_1^{-n} x x_1^n \eta) : \eta \in [\tilde{\xi}, x_1\tilde{\xi}], n \in \mathbb{Z}\}.$$

Then

$$d(x_1^n \eta, x x_1^n \eta) \leq \delta \qquad (\eta \in [\tilde{\xi}, x_1\tilde{\xi}],\ n \in \mathbb{Z}).$$

Therefore

$$d(\zeta, x\zeta) \leq \delta \text{ for all } \zeta \in L.$$

In other words, L is a parallel translate of xL. This implies that L and xL lie in a common maximal flat, namely F_0. In particular $x\tilde{\xi} \in F_0$. It follows that $p[\tilde{\xi}, x\tilde{\xi}]$ is a closed geodesic in T^r. Consequently $x \in \Gamma_0$. □

Rather than proving Theorem B in complete generality, we prove a special case of it, which contains all the essential ideas of the general proof [RSS].

Corollary. *Let* $\Gamma = \pi(M_g)$*, the fundamental group of a compact Riemann surface* M_g *of genus* $g \geq 2$*. Let* c *be a closed geodesic of minimal length* σ *in* M_g*. Let* $\gamma_0 = [c] \in \Gamma$*, and let* $\Gamma_0 \cong \mathbb{Z}$ *be the subgroup of* Γ *generated by* γ_0*. Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of the* II_1 *factor* $\mathrm{VN}(\Gamma)$.

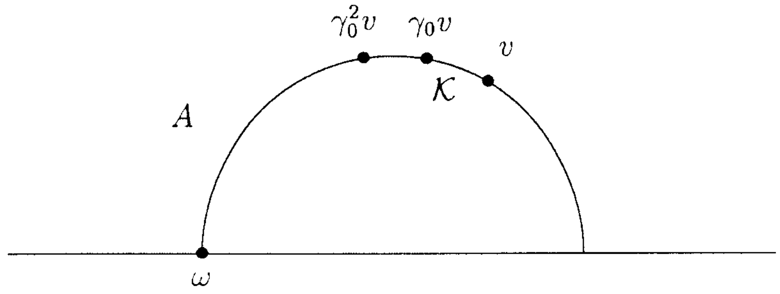
Proof. This uses condition (SS). The universal covering of M_g is the Poincaré upper half-plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

The boundary of $\mathfrak{H}$ is $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\}$. Also Γ acts isometrically on $\mathfrak{H}$.

The minimal closed geodesic c lifts to a geodesic A in $\mathfrak{H}$. Fix $v \in A$, and let $\mathcal{K} = [v, \gamma_0 v]$. Then

$$A = \bigcup_{n \in \mathbb{Z}} \gamma_0^n \mathcal{K} = \Gamma_0 \mathcal{K}. \tag{3.1}$$



Suppose that $x_1, \ldots, x_m, y_1, \ldots, y_n \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 y_j \,. \tag{3.2}$$

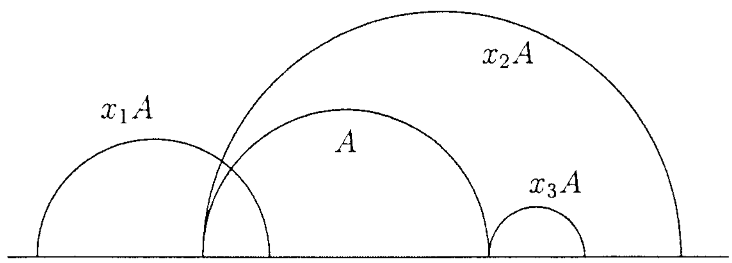
Let $\delta = \max\{d(y_j\kappa, \kappa); 1 \le j \le n,\, \kappa \in \mathcal{K}\}$. Then

$$\begin{aligned} y_j\mathcal{K} \underset{\delta}{\subset} \mathcal{K} \quad &\Rightarrow \quad \Gamma_0 y_j \mathcal{K} \underset{\delta}{\subset} \Gamma_0\mathcal{K} = A \\ &\Rightarrow \quad x_i\Gamma_0 y_j\mathcal{K} \underset{\delta}{\subset} x_i A \end{aligned}$$

[Here the notation $P \underset{\delta}{\subset} Q$ means that $d(p, Q) \le \delta$, for all $p \in P$.]
Hence

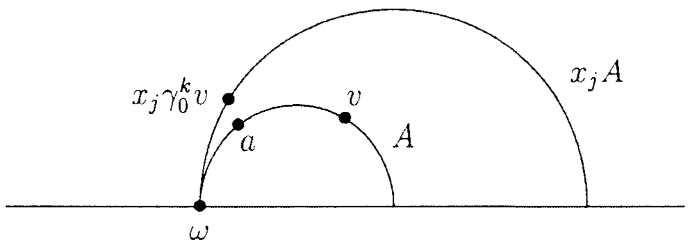
$$A = \Gamma_0\mathcal{K} \underset{\delta}{\subset} x_1 A \cup x_2 A \cup \cdots \cup x_m A \,. \tag{3.3}$$

This implies that each boundary point of A is a boundary point of some x_jA.

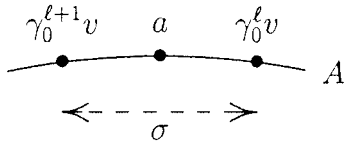


Now $\omega = \gamma_0^\infty v$ is a boundary point of some x_jA. We show that this implies $x_j \in \Gamma_0$. Choose $k \in \mathbb{Z}$, $a \in A$ such that

$$d(x_j\gamma_0^k v, a) < \frac{\sigma}{2} \,.$$



Choose $\ell \in \mathbb{Z}$ such that $d(a, \gamma_0^\ell v) \leq \frac{\sigma}{2}$:



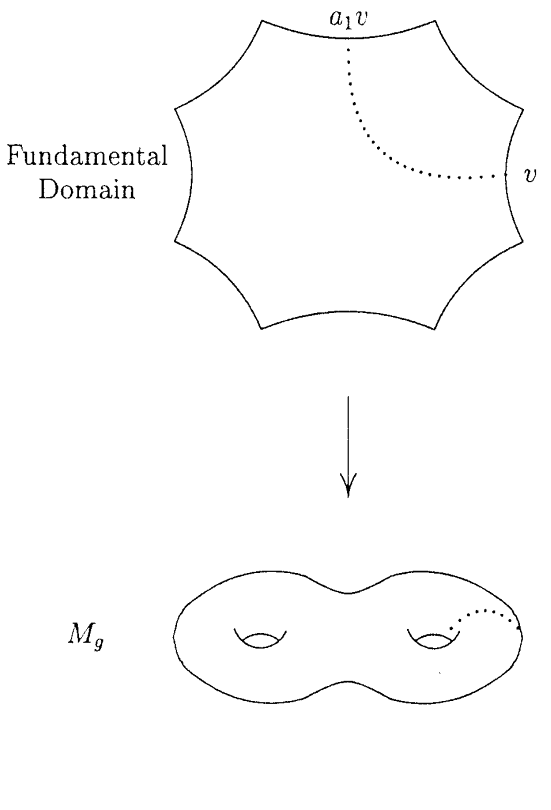
Then $d(\gamma_0^{-\ell} x_j \gamma_0^k v, v) = d(x_j\gamma_0^k v, \gamma_0^\ell v) \leq d(x_j\gamma_0^k v, a) + d(a, \gamma_0^\ell v) < \sigma$.
This implies $\gamma_0^{-\ell} x_j \gamma_0^k = 1$. For otherwise $[v, \gamma_0^{-\ell} x_j \gamma_0^k v]$ projects to a closed geodesic in M_g of length $< \sigma$.

Therefore $x_j = \gamma_0^{\ell-k} \in \Gamma_0$. $\square$

In the usual presentation of $\pi(M_g)$,

$$\Gamma = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

we can take $\gamma_0 \in \{a_i^{\pm 1}, b_j^{\pm 1}\}$.



3.1 The ICC Property

Recall that $VN(\Gamma)$ is a II_1 factor if and only if the group Γ is *ICC*. If Γ were a lattice in a semisimple Lie group then the argument of [GHJ, Lemma 3.3.1] (which uses the Borel density theorem) proves that Γ is *ICC*. However not all the groups of interest to us are embedded in a natural way as subgroups of linear groups. We therefore show how to use a geometric argument to verify the *ICC* property of Γ. This argument applies much more generally; in particular to the group actions on buildings which we consider later.

Proposition. A group Γ of isometries of $\mathfrak{H}$ which acts cocompactly on $\mathfrak{H}$ is ICC.

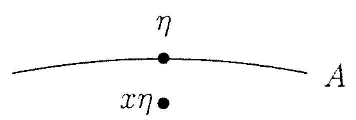
Proof. By assumption, $\Gamma\mathcal{K} = \mathfrak{H}$ where $\mathcal{K} \subset \mathfrak{H}$ is compact.
Let $x \in \Gamma - \{1\}$. Suppose that $C = \{y^{-1}xy : y \in \Gamma\}$ is finite.
Let $\delta = \max\{d(\kappa, y^{-1}xy\kappa) : \kappa \in \mathcal{K}, y \in \Gamma\}$. Then

$$d(y\kappa, xy\kappa) = d(\kappa, y^{-1}xy\kappa) \leq \delta, \qquad y \in \Gamma, \kappa \in \mathcal{K}.$$

Therefore, for all $\xi \in \mathfrak{H}$

$$d(\xi, x\xi) \leq \delta. \tag{3.4}$$

Choose $\eta \in \mathfrak{H}$ such that $x\eta \neq \eta$
Choose a geodesic A in $\mathfrak{H}$ with $\eta \in A$, $x\eta \notin A$.



Now it follows from (3.4) that $A \underset{\delta}{\subset} xA$. This implies that $A = xA$. In particular $x\eta \in A$, a contradiction. $\square$

3.2 A Free Group Analogue

If X is a finite connected graph with fundamental group $\Gamma = \pi(X)$ then Γ is a finitely generated free group. Also Γ acts freely and cocompactly on the universal covering tree $\tilde{X}$ with boundary $\partial\tilde{X}$. Let $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of Γ generated by one of the free generators of Γ.

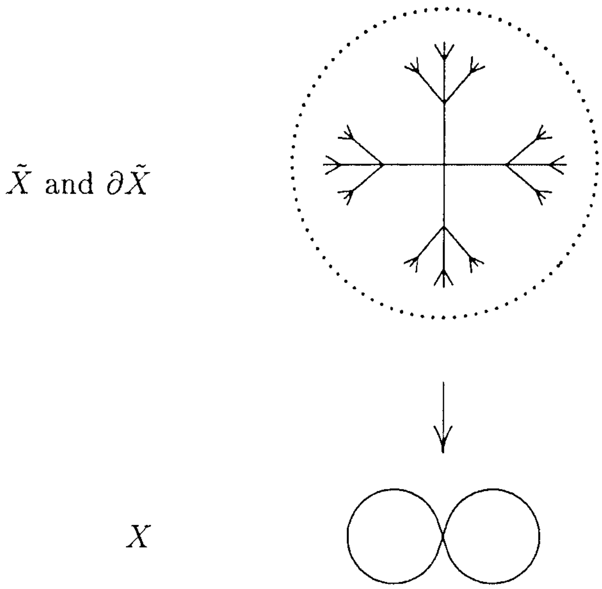
This setup is a combinatorial analogue of the Corollary above, where the fundamental group Γ of a Riemann surface acts on the Poincaré upper half plane. Exactly the same proof shows that Γ, Γ_0, satisfy condition **(SS)**.

In the figure below, $\Gamma = \mathbb{Z} \star \mathbb{Z} = \langle a, b\rangle$, the free group on two generators and $\Gamma_0 = \langle a\rangle \cong \mathbb{Z}$. Thus $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

3.3 Euclidean Buildings

More generally, suppose Γ acts freely and transitively on the vertex set of a euclidean building Δ and Γ_0 is an abelian subgroup which acts transitively on the vertex set of an apartment (flat). Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. [The proof is essentially the same as that of Theorem B.]

There exist many examples where $\Gamma < \mathrm{PGL}_3(\mathbb{K})$, $\mathbb{K}$ a nonarchimedean local field [CMSZ].



Example: $\mathbb{K} = \mathbf{F}_4((X))$, the field of Laurent series with coefficients in the field $\mathbf{F}_4$ with four elements. Let Γ be the torsion free group with generators $x_i, 0 \leq i \leq 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \leq j \leq 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \leq j \leq 20. \end{cases}$$

For each j, $0 \leq j \leq 6$,

$$\Gamma_0 = \langle x_j, x_{j+7}, x_{j+14} \rangle \cong \mathbb{Z}^2$$

satisfies the hypotheses.

3.4 A Borel subgroup

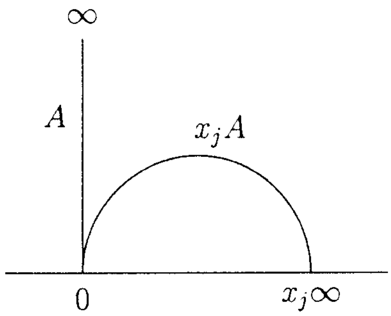
The geometric methods outlined above apply also to other situations. Here is an example.

Proposition. Let Γ be the upper triangular subgroup of $\mathrm{PSL}_n(\mathbb{Q})$, $n \geq 2$, and let Γ_0 be the diagonal subgroup of Γ. Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

We illustrate the proof in the case $n = 2$. Here Γ acts on the Poincaré upper half plane $\mathfrak{H}$.

$$\Gamma = \{g \in \mathrm{PSL}_2(\mathbb{Q}) : g\infty = \infty\} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

$$\Gamma_0 = \{g \in \Gamma : g0 = 0\} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}.$$



Note that Γ_0 stabilizes the geodesic

$$\begin{aligned} A &= \mathbb{R}^+ i \\ &= \Gamma_0 \mathcal{K}, \qquad \text{where } \mathcal{K} = [i, 2i]. \end{aligned}$$

In order to show that condition (SS) holds, proceed as in the proof of the Corollary in Section 3. As in (3.3), suppose that

$$A \underset{\delta}{\subset} x_1 A \cup x_2 A \cup \cdots \cup x_m A,$$

for some $x_1, \ldots, x_m \in \Gamma$, and $\delta > 0$. Then 0 is a boundary point of some $x_j A$. Now since $x_j \in \Gamma$, $x_j\infty = \infty$. Therefore $x_j A = A$ and $x_j 0 = 0$. It follows that $x_j \in \Gamma_0$. □

4 Appendix: Symmetric Spaces

We conclude with a quick summary of some essential facts about symmetric and locally symmetric spaces [BH].

Let G be a semisimple Lie group with no centre and no compact factors.

The corresponding *symmetric space* is $X = G/K$ where K is a maximal compact subgroup.

The *rank* r of X is the dimension of a maximal *flat* in X. That is, the maximal dimension of an isometrically embedded euclidean space in X.

A geodesic L in X is *regular* if it lies in only one maximal flat; it is called *singular* if it is not regular.

Let F be a maximal flat in X and let $x \in F$. Let S_x denote the union of all the singular geodesics through x. A connected component of $F - S_x$ is called a *Weyl chamber* with origin x.

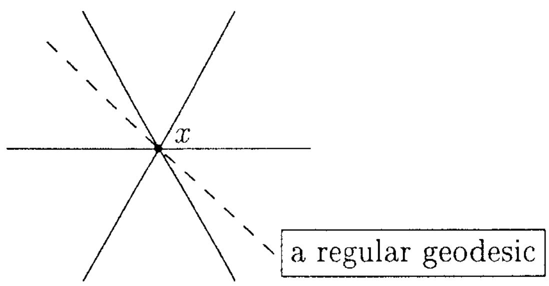
Example Consider a rank 2 example.

$$\begin{aligned} G &= \mathrm{SL}_3(\mathbb{R}) \\ X &= \{x \in \mathrm{SL}_3(\mathbb{R}) : x \text{ is positive definite}\} \end{aligned}$$

G acts transitively on X by $x \mapsto gxg^t$ and the stabilizer of I is $\mathrm{SO}_3(\mathbb{R})$. Therefore

$$X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$$

A maximal flat F is 2-dimensional. There are six Weyl chambers in F with a given origin $x \in F$.



A flat through I has the form $\exp \mathfrak{a}$, where $\mathfrak{a}$ is a linear subspace of

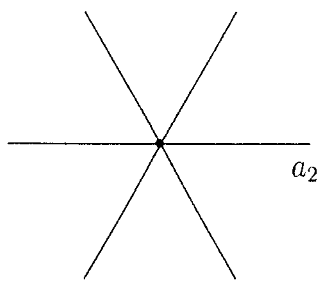
$$S_n(\mathbb{R}) = \{x \in \mathrm{M}_n(\mathbb{R}) : x = x^t, \ \mathrm{trace}(x) = 0\} \qquad \text{(the tangent space at } I)$$

such that $xy = yx$ for all $x, y \in \mathfrak{a}$.

The geodesic $t \mapsto \exp tx$ through I in X is regular if and only if the eigenvalues of $x \in S_n(\mathbb{R})$ are all distinct. To see why this is so, consider

$$\mathfrak{a}_0 = \{\mathrm{diag}(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}.$$

Parametrize elements of $\mathfrak{a}_0$ by points on a plane through the origin in $\mathbb{R}^3$, as in the figure below.



If a_1, a_2, a_3 are all distinct then a matrix in $S_n(\mathbb{R})$ which commutes with $a = \mathrm{diag}(a_1, a_2, a_3)$ is necessarily diagonal and so lies in $\mathfrak{a}_0$. Thus the geodesic $t \mapsto \exp ta$ lies in a unique maximal flat $\exp \mathfrak{a}_0$.

4.1 Locally Symmetric Spaces

Let Γ be a torsion free cocompact lattice in G.

$M = \Gamma \backslash X$ is a compact *locally symmetric space* of nonpositive curvature.

$X = G/K$ is the universal covering space of M and the fundamental group of M is $\pi(M) = \Gamma$.

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Abstracts

On spectral characterizations of amenability

Claire Anantharaman-Delaroche

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Given a measured G-space (X, μ) and a probability measure m on G, we discuss the informations on the G-space X that are provided by the spectral analysis of the operator $\pi_X(m)$, where π_X is the unitary representation of G associated with (X, μ). Emphasis is put on characterizing the amenability of the G-action.

The heat kernel on symmetric spaces, fifteen years later

Jean–Philippe Anker

University of Orléans, France

At a previous conference (Luxembourg, September 1987), we conjectured an upper bound for the heat kernel on noncompact Riemannian symmetric spaces G/K. Our guess was based on explicit expressions available in some particular cases, namely when $\mathbf{rank}(G/K) = 1$, when G is complex or when $G = \mathrm{SU}(p, q)$.

In the meantime, this conjectural upper bound has been established in a rather large range and has proved to be a lower bound too. We shall give an overview of the present state of the subject, including recent joint work with P. Ostellari.

Divisible convex sets and prehomogeneous vector spaces

Yves Benoist

ENS Paris, France

A properly convex open cone in $\mathbf{R}^m$ is called divisible if there exists a discrete subgroup Γ of $\mathrm{GL}(\mathbf{R}^m)$ preserving C such that the quotient $\Gamma\backslash C$ is compact. We describe the Zariski closure G of such a group Γ.

It was known that this group G is reductive. We show that if C is divisible but is neither a product nor a symmetric cone, then Γ is Zariski dense in $\mathrm{GL}(\mathbf{R}^m)$. The main step is to prove that the representation of G in $\mathbf{R}^m$ is prehomogeneous.

Inducing and restricting unitary representations of nilpotent Lie groups

Hidenori Fujiwara

Kinki University, Japan

Let $G = \exp \mathfrak{g}$ be a connected, simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$. Given an analytic subgroup $H = \exp \mathfrak{h}$ of G with Lie algebra $\mathfrak{h}$ and a unitary character χ of H, we construct the induced representation $\tau = \mathrm{ind}_H^G \chi$ of G. On the other hand, given an analytic subgroup $K = \exp \mathfrak{k}$ and an irreducible unitary representation π of G, we restrict π to K.

It is well known that there exists a strong parallelism between these two operations; inducing and restricting. We discuss this, focusing our attention on algebras of differential operators and a Frobenius reciprocity attached to these two procedures. Our study will be done in terms of the celebrated orbit method.

Extending positive definite functions from subgroups of locally compact groups

Eberhard Kaniuth

University of Paderborn, Germany

Let G be a locally compact group and H a closed subgroup of G. We call H an extending subgroup of G if every continuous positive definite function on H extends to some continuous positive definite function on G. Also, G is said to have the extension property when each closed subgroup of G is extending. The talk will first give a survey on what is known for some time regarding these properties and then focus on recent results for nilpotent groups.

Ramanujan complexes

Alex Lubotzky

University of Jerusalem, Israel

Ramanujan graphs are finite k-regular graphs the eigenvalues λ of their adjancency matrix satisfy $|\lambda| \leq 2\sqrt{k-1}$ or $\lambda = \pm k$. Examples were constructed as quotients of the tree associated to $PGL_2(K)$ when K is a non-archimedean local field.

The talk will describe a work in progress (jointly with B. Samuels and U. Vishne) on generalizing this concept from graphs to higher dimensional simplicial complexes, and constructions as quotients of the building associated with $PGL_d(K)$.

Bochner–Riesz means on the Heisenberg group and fractional integration on the dual

Detlef Müller

University of Kiel, Germany

Let L denote the sub–Laplacian on the Heisenberg group $\mathbb{H}_n$ and $T_r^\lambda := (1 - rL)_+^\lambda$ the corresponding Bochner–Riesz operator. Let Q denote the homogeneous dimension and D the Euclidean dimension of $\mathbb{H}_n$.

In joint work with D. Gorges, we prove a.e. convergence of the Bochner–Riesz means $T_r^\lambda f$ as $r \to 0$ for $\lambda > 0$ and for all $f \in L^p(\mathbb{H}_n)$, provided that

$$\frac{Q-1}{Q}\Big(\frac{1}{2} - \frac{\lambda}{D-1}\Big) \quad < \quad 1/p \quad \leq \quad 1/2.$$

Our proof is based on explicit formulas for the operators ∂_{ω^a}, $a \in \mathbb{C}$, defined on the dual of $\mathbb{H}_n$ by $\partial_{\omega^a} \hat{f} := \widehat{\omega^a f}$, which may be of independent interest. Here ω is given by $\omega(z,u) := |z|^2 - 4iu$ for all $(z,u) \in \mathbb{H}_n$.

Root graded Lie groups

Karl-Hermann Neeb

University of Darmstadt, Germany

Root graded Lie groups are (mostly infinite-dimensional) Lie groups whose Lie algebra is root graded in a topological sense. This means that it has a grading like a finite-dimensional simple complex Lie algebra by a finite irreducible reduced root system Δ, it contains the corresponding finite-dimensional simple complex Lie algebra, and it is generated by the root spaces. These Lie algebras are determined up to central extensions by the root system Δ and a coordinate structure. Root graded Lie algebras show up naturally in many geometric situations and also in mathematical physics, so that it is natural to ask for corresponding Lie groups. This problem is discussed in our lecture. We show that under natural assumptions on the coordinate structures there always is a centerfree Lie group corresponding to the centerfree Lie algebra for a given root system and coordinate structure. Then one has to face the problem to construct central extensions of these groups, which leads to interesting period maps relating K-groups and cyclic homology of topological algebras. In particular we obtain Lie group versions of Steinberg groups for algebras whose period maps satisfy a certain discreteness condition.

Hilbert bimodules associated to self-similar group actions

Volodia Nekrashevych
University of Kiev, Ukraine

We will talk about the Hilbert bimodules related to self-similar actions of groups and semigroups on the shifts of finite type. Basic examples of such groups and semigroups are the adding machines, the branch groups (like the Grigorchuk group), the groups related to aperiodic tilings, etc. The respective Cuntz-Pimsner algebras will be considered. Relations with hyperbolic dynamics and Thompson groups will be discussed.

Hierarchomorphisms of trees and combinatorial analogs of the group of diffeomorphisms of the circle

Yuri A. Neretin
University of Moscow, Russia

Let T be an inifinite tree, Abs be its boundary. A homeomorphism $q : Abs \to Abs$ is a hierarchomorphism if there exists a finite subtree $I \subset T$ such that q admits an extension to a map $T \setminus I \to T$. A group $Hier(T)$ of hierarchomorphisms contains the R. Thompson group and the group of locally analytical diffeomorphisms of the p-adic projective line. Properties of groups $Hier(T)$ are similar to properties of the group of diffeomorphisms of the circle. We discuss some constructions of unitary representations of $Hier(T)$.

Synthesis properties of orbits of compact groups

Detlev Poguntke
University of Bielefeld, Germany

The notion of sets of synthesis is best known in the case of $\mathcal{L}^1(G)$, G a locally compact abelian group. With each closed ideal I in $\mathcal{L}^1(G)$ there is associated a closed subset of the structure space $\mathcal{L}^1(G)\hat{} = \hat{G}$, namely the hull $h(I) = \{\chi \in \hat{G} \mid \ker_{\mathcal{L}^1(G)} \chi \supset I\}$. A closed subset A of $\hat{G}$ is called a set of synthesis if there exists exactly one closed ideal I in $\mathcal{L}^1(G)$ with $h(I) = A$; then I is necessarily equal to the kernel $k(A) := \bigcap_{\chi \in A} \ker_{\mathcal{L}^1(G)} \chi$ of A.

This notion (or these notions) can be generalized immediately to arbitrary Banach algebras, as soon as one agrees on the structure space to be considered. After recalling some known results on $\mathcal{L}^1$-algebras of nilpotent Lie groups, mainly where A is an orbit of a compact group acting homomorphically, we consider algebras of the following type:

Let K be a closed normal subgroup of a compact group L, and let Q be a symmetric semisimple involutive commutative Banach algebra, on which L acts. Suppose that L acts transitively on the Gelfand space $\hat{Q}$. Then one may form the generalized $\mathcal{L}^1$–algebra $B := \mathcal{L}^1(K, Q)$, which is endowed with a natural L–action.

As structure space $\hat{B}$ we take the collection of kernels of irreducible involutive representations of B (which coincides with the set $\mathrm{Priv}(B)$ of all primitive ideals) equipped with the Jacobson topology. It is shown that L–orbits in $\hat{B}$ are closed (which is not completely obvious!), and that they are sets of synthesis. Also the empty set is a set of synthesis, i.e., each proper closed ideal in B is contained in the kernel of an irreducible representation. The proof, briefly scetched, is an exercise in representation theory of compact groups.

Singular masas of von Neumann algebras: examples from the geometry of spaces of nonpositive curvature

Guyan Robertson

University of Newcastle, Australia

If Γ is a group, then the von Neumann algebra $VN(\Gamma)$ is the convolution algebra of functions $f \in \ell_2(\Gamma)$ which act by convolution on $\ell_2(\Gamma)$ as bounded operators. Maximal abelian $\star$-subalgebras (masas) of von Neumann algebras have been studied from the early days.

If Γ is a torsion free cocompact lattice in a semisimple Lie group G of rank r with no centre and no compact factors then the geometry of the symmetric space $X = G/K$ may be used to define and study masas of $VN(\Gamma)$. These masas are of the form $VN(\Gamma_0)$, where Γ_0 is the period group of some Γ-periodic maximal flat in X. There is a similar construction if Γ is a lattice in a p-adic Lie group G, acting on its Bruhat-Tits building.

Consider the compact locally symmetric space $M = \Gamma \backslash X$. Assume that T^r is a totally geodesic flat torus in M and let $\Gamma_0 \cong \mathbf{Z}^r$ be the image of the fundamental group $\pi(T^r)$ under the natural monomorphism from $\pi(T^r)$ into $\Gamma = \pi(M)$. Then $VN(\Gamma_0)$ is a masa of $VN(\Gamma)$. If in addition $diam(T^r)$ is less than the length of a shortest closed geodesic in M then $VN(\Gamma_0)$ is a *(strongly) singular masa* : its unitary normalizer is as small as possible. This result is part of joint work with A. M. Sinclair and R. R. Smith.

Free Group Representations and Their Realizations on the Boundary

Tim Steger

University of Sassari, Italy

Let Γ be a noncommutative free group on finitely many generators. We consider unitary representations of Γ which are weakly contained in the regular representations. Equivalently, these are the "tempered" representations, those whose matrix coefficients are almost in ℓ^2. Let Ω be the natural boundary of Γ. A representation which acts in a certain well-defined natural way on some L^2-space on Ω is called a *boundary representation.* All boundary representations are tempered. Conversely, if π is any tempered representation, there is an inclusion of π into some boundary representation. Such an inclusion is a *boundary realization* of π.

Consideration of examples leads to the *duplicity conjecture*: a given irreducible tempered representation has at most two inequivalent, irreducible boundary realizations. We give the details of this conjecture.

There are lots of representations of Γ, and one's intuition is that a "generic" representation is irreducibile. However, proving the irreducibility of a specific representation is usually difficult. In many cases, an analysis going by way of boundary realizations works. In certain cases one can prove simultaneously that a representation is irreducible and that it has exactly two inequivalent, irreducible boundary realizations; in others that it has exactly one boundary realization.

We sketch the construction of a large class of examples of irreducibile tempered representations of Γ. The construction is based on vector-valued multiplicative functions, and covers in a uniform way many of the previously studied examples.

Finally, we mention *Paschke's Conjecture*: if $f \in \ell^1(\Gamma)$ is of finite support, then there are at most finitely many irreducibile tempered representations π such that $\pi(f)$ has a nonzero kernel.

The Plancherel formula for real almost algebraic groups

Pierre Torasso

University of Poitiers, France

(joint work with M. S. Khalgui, Tunisia.)

In his lectures given at the University of Maryland during the special year held in 1982-83, M. Duflo stated a concrete Plancherel formula for real almost algebraic groups. We give a proof of it in the philosophy of the orbit method and following the lines of the one given in 1987 by M. Duflo and M. Vergne for simply connected semi-simple Lie groups.

Let G be our almost algebraic Lie group and $\mathfrak{g}$ its Lie algebra. The main ingredients of the proof are :

- the Harish-Chandra's descent method which, interpreting Plancherel formula as an equality of semi-invariant generalized functions, allows one to reduce it to such an equality on a neighbourhood of zero in $\mathfrak{g}(s)$, the centralizer in $\mathfrak{g}$ of any elliptic element s in G,

- the character formula near elliptic elements for the representations of the group constructed by M. Duflo, recently proved by the authors : roughly speaking, the character of a representation associated to a coadjoint orbit Ω is given, near an elliptic element s of G, by the Fourier transform of a canonical measure closely related to the Liouville measure on Ω^s, the set of s-fixed points in Ω,

- the Poisson-Plancherel formula near peculiar elliptic elements, the one said to be in good position. If s is such an element, this formula, generalizing the classical Poisson summation formula, states that the Fourier transform of an invariant distribution, which is the sum of a series of Harish-Chandra type orbital integrals of elliptic elements in $\mathfrak{g}(s)$, is a generalized function on $\mathfrak{g}(s)^*$ whose product by a Lebesgue measure is a tempered complex measure supported on the set of G-admissible and strongly regular forms contained in $\mathfrak{g}(s)^*$.

Property (T) and harmonic maps

Alain Valette

University of Neuchâtel, Switzerland

We plan to sketch the proofs of the following two results:

Theorem 1 (Y. Shalom, unpublished). Let G be a simple Lie group with finite centre, and maximal compact subgroup K. If G does not have property (T), then there exists a Hibert space-valued, non constant harmonic map $G/K \to \mathcal{H}$ which is equivariant with respect to an action of G on $\mathcal{H}$ by affine isometries.

Theorem 2. For $G = Sp(n, 1)$ with $n \geq 2$, every harmonic G-equivariant map $G/K \to \mathcal{H}$ is constant.

The proof of Theorem 2 is based on recent ideas of M. Gromov. Altogether, these two results provide a new, geometric proof of property (T) for $Sp(n, 1)$.

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Harish-Chandra decomposition of Banach-Lie groups

Harald Biller

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The Harish-Chandra decomposition, a construction principle for unitary representations of semi-simple Lie groups on Hilbert spaces of holomorphic functions, is generalized to certain linear Lie groups of infinite dimension.

Haagerup property and spaces with walls

Pierre-Alain Chérix

University of Geneva, Switzerland

In the late seventies Haagerup proved that the length function associated with a free generating system of a non abelian free group is conditionnally negative definite. The existence of such a proper conditionnally negative definite function is one possible definition of Haagerup property. In the eighties, many different definitions of that property were introduced and were proved to be equivalent. Later Haaglund and Paulin introduced the notion of space with walls and of groups acting properly on such spaces. With Valette, they proved that a group acting properly on a space with walls has Haagerup property. The result of Haagerup can be translated directly in that context. It seems natural to ask whether the converse is true or not. In a joined work with Martin and Valette, we introduced a generalized notion of mesured space with walls and we proved that for countable groups,the converse is true. Namely, a countable group which has Haagerup property is acting properly on a measured space with walls.

Unitary duality, weak topologies and thin sets in locally compact groups

Jorge Galindo

University Jaume I de Castellón, Spain

Let G be a locally compact group with sufficiently many finite-dimensional representations (i.e. a MAP group). A general (probably exceedingly general) question is to what extent its finite-dimensional representations can be used to understand G.

In this talk we shall discuss some of the well-known cases of groups strongly determined by their finite-dimensional representations, such as Abelian or Moore groups, and some obstructions to a general theory, represented by van der Waerden or Kazhdan groups. The discussion will be based on unitary dualities, Bohr compactifications and thin sets as ways to relate a group to its finite-dimensional representations.

Turning to concrete results, we shall sketch joint work with Salvador Hernández characterizing the existence of Bohr compact subsets in a locally compact group G (that is, sets that are compact in the Bohr compactification bG of G) by means of the existence of I_0-sets in the sense of Hartman and Ryll-Nardzewski (a set $A \subseteq G$ is an I_0-set if every complex-valued function on A can be extended to an almost periodic function on G). This will be essential in proving that a locally compact group has no infinite I_0-sets if and only if it has at most countably many inequivalent irreducible finite-dimensional representations. A similar approach will be used to show that discrete goups always contain infinite weak Sidon sets in the sense of Picardello (a subset A of a locally compact group is a weak Sidon set if every complex-valued function on A can be extended to a function belonging to $B(G)$, the Fourier-Stieltjes algebra of G).

Fourier inversion on rank one compact symmetric spaces

Francisco Gonzalez

University of Lausanne, Switzerland

Conditions for the pointwise Fourier inversion of K-invariant functions using Cesàro means of a given order are established on rank one compact symmetric spaces G/K.

Spectral decomposition and discrete series representations on a p-adic group

Volker Heiermann

Humboldt University, Berlin, Germany

Topic of my talk will be the proof of a conjecture of A. Silberger on infinitesimal characters of discrete series representations of a p-adic group G. More precisely, I'll show the following: a cuspidal representation σ of a Levi subgroup L corresponds to the infinitesimal character of a discrete series representation of G, if and only if σ is a pole of Harish-Chandra's μ-function of order equal to the parabolic rank of L. The proof is by a spectral decomposition starting from a Fourier inversion formula analog to the Plancherel formula. This formula has been established previously in [*Une formule de Plancherel pour les éléments de l'algèbre de Hecke d'un groupe réductif p-adique,* Comm. Math. Helv. **76**, 388-415, 2001]. The results take part of my Habilitation thesis.

Integral Geometry and hypergroups

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(The corresponding results are joint with M.I. Graev).

It is well known that the Radon Transform is closely related to the Fourier transform and harmonic analysis on the group of real numbers (or the additive groups of vectors in a finite dimensional real linear space). Similarly there are relations between some standard problems of Integral Geometry (in the sense of

Gelfand and Graev) and some commutative hypergroups (in the sense of J. Delsarte) and harmonic analysis on these hypergoups. This result can be treated as an answer for an old I.M. Gelfand's question on algebraic foundations of Integral Geometry.

Théorie des représentations et K-Théorie

Nicolas Prudhon

University Louis Pasteur, Strasbourg, France

Dans cet exposé, nous nous intéresserons aux liens entre la K-théorie des C^*-algèbres associées aux groupes de Lie semi-simples et ‡ la théorie des représentations de ces groupes.

Nous rappellerons tout d'abord quelques notions concernant la K-théorie des C^*-algèbres, puis nous expliquerons comment calculer la K-théorie de la C^*-algèbre réduite $C_r^*(G)$ d'un groupe de Lie semi-simple connexe G, via l'application d'indice de Connes-Kasparov, l'induc- tion de Dirac

$$\mu : R(K) \to K_*(C_r^*(G)) \quad ,$$

où K est un sous groupe compact maximal de G. Celle-ci est un ismorphisme, comme l'ont démontré A. Wassermann puis V. Lafforgue. Nous expliciterons sur l'exemple de $SL_2(\mathbf{R})$ le lien avec la théorie des séries discrètes. Ces résultats sont ‡ rapprocher avec ceux de Atiyah-Schmid sur la construction des séries discrètes sur le noyau L^2 d'un opérateur de Dirac.

Pour finir, nous nous intéresserons au calcul de la K-théorie de la C^*-algèbre maximale $C^*(G)$ d'un tel groupe G. Nous verrons que lorsque le groupe possède la propriété T de Kazhdan, la K-théorie de cette C^*-algèbre est différente de celle précédemment étudiée en ce sens que la représentation régulière

$$\lambda : C^*(G) \to C_r^*(G)$$

n'induit pas un ismorphisme en K-théorie (alors que c'est le cas par exemple pour $SL_2(\mathbf{R})$), et nous verrons comment la propriété T est "détectée" par l'induction de Dirac.

Property (T) and exponential growth of discrete subgroups

Jean-François Quint

ENS Paris, France

We recall the definition and properties of the indicator of growth of a discrete subgroup of a semisimple Lie group. In case the ambient Lie group has real ranq greater than 2, we apply results of H. Oh, related to property (T), to give controls on the indicator of growth.