Formal Poisson cohomology of twisted r-matrix induced structures*

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Abstract

Quadratic Poisson tensors of the Dufour-Haraki classification read as a sum of an r-matrix induced structure twisted by a (small) compatible exact quadratic tensor. An appropriate bigrading of the space of formal Poisson cochains then leads to a vertically positive double complex. The associated spectral sequence allows to compute the Poisson-Lichnerowicz cohomology of the considered tensors. We depict this modus operandi, apply our technique to concrete examples of twisted Poisson structures, and obtain a complete description of their cohomology. As richness of Poisson cohomology entails computation through the whole spectral sequence, we detail an entire model of this sequence. Finally, the paper provides practical insight into the operating mode of spectral sequences.

Key-words: Poisson-Lichnerowicz cohomology, r-matrix induced Poisson tensor, exact quadratic structure, vertically positive double complex, spectral sequence

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1 Introduction

It is easily seen that any quadratic Poisson tensor of the Dufour-Haraki classification (DHC), [DH91], reads

$$\Lambda = \Lambda_I + \Lambda_{II} = aY_{23} + bY_{31} + cY_{12} + \Lambda_{II}, \tag{1}$$

where $a, b, c \in \mathbb{R}$, where the Y_i are linear, mutually commuting vector fields ($Y_{ij} = Y_i \wedge Y_j$), and where Λ_{II} is—as Λ_I—a quadratic Poisson structure. This entails of course that Λ_I and Λ_{II} are compatible, i.e. that $[\Lambda_I, \Lambda_{II}] = 0$, where $[.,.]$ is the Schouten bracket. Except for structure 10 of the DHC, where $\Lambda_{II} = (3b+1)(y^2 - 2xz)\partial_{23}$ ($\partial_{23} = \partial_{x_2}\partial_{x_3} = \partial_y\partial_z$), the second Poisson structure is always Koszul-exact, i.e.

$$\Lambda_{II} = \Pi_\phi := (\partial_1\phi)\partial_{23} + (\partial_2\phi)\partial_{31} + (\partial_3\phi)\partial_{12}, \quad \phi \in \mathcal{S}^3\mathbb{R}^{3*}.$$

In [Xu92], P. Xu has proved that any quadratic Poisson tensor of $\mathbb{R}^3$ reads

$$\Lambda = \frac{1}{3}K \wedge \mathcal{E} + \Pi_f, \tag{2}$$

where K is the curl of Λ, $\mathcal{E}$ the Euler field, and $f \in \mathcal{S}^3\mathbb{R}^{3*}$.

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In most cases (only cases 9 and 10 of the DHC are exceptional), term Λ_I of Equation (1), which is twisted by the exact term Λ_{II} and is—as easily seen—implemented by an r-matrix in the stabilizer $\mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda$, $\mathfrak{g}_\Lambda = \{A \in \mathrm{gl}(3,\mathbb{R}) : [A,\Lambda] = 0\}$, is given by

$$\Lambda_I = \frac{1}{3} K \wedge \mathcal{E} + \Pi_{\lambda D},$$

where $\lambda \in \mathbb{R}^*$ and $D = \det(Y_1, Y_2, Y_3)$, whereas

$$\Lambda_{II} = \Pi_\phi = \Pi_{f-\lambda D}.$$

Hence, the difference between decompositions (1) and (2) is that in (1) the biggest possible part of Λ is incorporated into the r-matrix induced structure, whereas in (2) it is incorporated into the exact structure.

We privilege decomposition (1), since a general computing technique allows to deal with the cohomology of Λ_I, [MP06], and Λ_{II} vanishes in many cases. In most of the cases where the small exact tensor Λ_{II} does not vanish, the decomposition

$$\partial_\Lambda := [\Lambda, .] = [\Lambda_I, .] + [\Lambda_{II}, .] =: \partial_{\Lambda_I} + \partial_{\Lambda_{II}}, \quad \partial^2_{\Lambda_I} = \partial^2_{\Lambda_{II}} = \partial_{\Lambda_I}\partial_{\Lambda_{II}} + \partial_{\Lambda_{II}}\partial_{\Lambda_I} = 0$$

leads to a vertically positive double complex and the corresponding spectral sequence allows to deduce bit by bit the cohomology of Λ from that of Λ_I.

In Section 2, we show how twisted r-matrix induced tensors generate vertically positive double complexes. As richness of Poisson cohomology entails computation through the whole associated spectral sequence, we detail a complete model of the sequence in Section 3. Section 4 contains the computation of the cohomology of tensor Λ_4 of the Dufour-Haraki classification. More precisely, Subsection 4.1 provides the second term of the spectral sequence, i.e. the cohomology of the r-matrix induced part $\Lambda_{4,I}$ of Λ_4, which is accessible to the general cohomological technique developed in [MP06]. After some preliminary work in Subsections 4.2 and 4.3, we are prepared to compute, in Subsection 4.4, through the entire spectral sequence, see Theorem 1. As we aim at the extraction of "true results", we are obliged to detail all the isomorphisms involved in the theory of spectral sequences and to read our upshots through these isomorphisms. Hence, in particular, a study of the limiting process in the sequence and of the reconstruction of the cohomology, precedes, in Subsection 4.5.1, the concrete description of the cohomology of twisted structure Λ_4, see Theorem 2 in Subsection 4.5.2, and of twisted tensor Λ_8, Theorem 3 in Subsection 5.

The description of the main features of the cohomology of r-matrix induced Poisson structures has been given in [MP06]. The tight relation between Casimir functions and Koszul-exactness of these Poisson tensors is recalled in Subsection 4.1, see Equation (10) (a generalization can be found in Subsection 4.3, see Equation(11)). Since our r-matrix induced Poisson structures are built with infinitesimal Poisson automorphisms Y_i, see Equation (1), the wedge products of the Y_i constitute a priori "privileged" cocycles. The associative graded commutative algebra structure of the Poisson cohomology space now explains part of the cohomology classes. The second and third term of this cohomology space contain, in addition to the just mentioned wedge products of Casimir functions and infinitesimal automorphisms Y_i, non-bounding cocycles the coefficients of which are—in a broad sense—polynomials on the singular locus of the considered Poisson tensor. The "weight in cohomology" of the singularities increases with closeness of the Poisson structure to Koszul-exactness. The appearance of some "accidental Casimir-like" non bounding cocycles completes the depiction of the main characteristics of the cohomology.

If the r-matrix induced structure is twisted by an exact quadratic tensor, the aforementioned spectral sequence constructs little by little the cohomology of Λ from that of Λ_I. In the examined cases, the basic Casimir C_I of Λ_I is the first term of the expansion by Newton's binomial theorem of the basic Casimir C of Λ. Beyond the emergence of systematic conditions on the coefficients of the powers C^i, $i \in \mathbb{N}$, and the methodic disappearance of monomials on the singular locus of Λ_I, the main impact on Poisson cohomology of twist Λ_{II} is the (partial) passage from first term C_I to complete expansion C, a change that takes place gradually for all powers of these Casimirs, as we compute through the spectral sequence.

2 Vertically positive double complex

2.1 Definition

Let (K,d) be a complex, i.e. a differential space, made up by a graded vector space $K=\oplus_{n\in\mathbb{N}}K^n$ and a differential $d:K^n\to K^{n+1}$ that has weight 1 with respect to this grading. Assume that each term K^n is itself graded,

$$K^n=\oplus_{r,s\in\mathbb{N},r+s=n}K^{rs},$$

so that $K=\oplus_{r,s\in\mathbb{N}}K^{rs}$ is bigraded. We will refer to grading $K=\oplus_{n\in\mathbb{N}}K^n$ as the diagonal grading. Let $p,q\in\mathbb{N},p+q=n$. Differential $d:K^{pq}\to\oplus_{r,s\in\mathbb{N},r+s=n+1}K^{rs}$ induces linear maps

$$d_{ab}:K^{pq}\to K^{p+a,q+b}\quad(a,b\in\mathbb{Z},a+b=1),$$

such that

$$d=\sum_{a,b\in Z,a+b=1}d_{ab}.$$

If $d_{ab}=0,\forall b<0$ (resp. $d_{ab}=0,\forall a<0$), the preceding complex is a *vertically positive double complex* (VPDC) (resp. a *horizontally positive double complex* (HPDC)). Vertically positive and horizontally positive double complexes are *semi-positive double complexes*. A complex that is simultaneously a VPDC and a HPDC is a *double complex* (DC) in the usual sense.

We filter a VPDC (resp. a HPDC) using the *horizontal filtration* (resp. *vertical filtration*)

$${}^hK_p=\oplus_{r\in\mathbb{N},s\geq p}K^{rs}\quad(\text{resp. }{}^vK_p=\oplus_{r\geq p,s\in\mathbb{N}}K^{rs}).$$

These filtrations are compatible (in the usual sense) with the diagonal grading and differential d. Moreover, they are regular, i.e. $K_p\cap K^n=0,\forall p>n$ (as well for $K_p={}^hK_p$ as for $K_p={}^vK_p$), and verify $K_0=K$ and $K_{+\infty}=0$.

The (convergent) spectral sequence (SpecSeq) associated with this graded filtered differential space is extensively studied below. Let us stress that in the following we prove several general results on spectral sequences, which we could not find in literature. In order to increase the reader-friendliness of our paper and to avoid scrolling, we chose to give these upshots in separate subsections that directly precede those where the results are needed.

2.2 Application to twisted r-matrix induced Poisson structures

We will now associate a VPDC to twisted r-matrix induced Poisson tensors. Let

$$\Lambda=\Lambda_I+\Lambda_{II}=aY_{23}+bY_{31}+cY_{12}+\Pi_\phi$$

be as in Equation (1).

Set $Y_i=\ell_{ij}\partial_j$, $\ell_{ij}\in\mathbb{R}^{3*}$ (we use the Einstein summation convention) and $D=\det\ell=\det(\ell_{ij})\in\mathcal{S}^3\mathbb{R}^{3*}$. If $L\in\mathrm{gl}(3,\mathcal{S}^2\mathbb{R}^{3*})$ is the matrix of algebraic (2×2)-minors of ℓ, we have $\partial_i=\frac{L_{ji}}{D}Y_j$. The formal Poisson cochain space $\mathcal{P}$ is made up by the $0-$, $1-$, $2-$, and $3-$cochains

$$C^0=\frac{\sigma}{D},C^1=\frac{\sigma_1}{D}Y_1+\frac{\sigma_2}{D}Y_2+\frac{\sigma_3}{D}Y_3,C^2=\frac{\sigma_1}{D}Y_{23}+\frac{\sigma_2}{D}Y_{31}+\frac{\sigma_3}{D}Y_{12},C^3=\frac{\sigma}{D}Y_{123},\tag{3}$$

where $\sigma,\sigma_1,\sigma_2,\sigma_3\in\mathbb{R}[[x_1,x_2,x_3]]$ and where σ, $\ell_{ij}\sigma_i$, $L_{ij}\sigma_i$ are divisible by D (for any j; 3-cochains do not generate any divisibility condition). In order to understand these results, note first that, if $\mathcal{L}\in\mathrm{gl}(3,\mathcal{S}^4\mathbb{R}^{3*})$ denotes the matrix of algebraic (2×2)-minors of L, we have $\mathcal{L}=(\det L)\tilde{L}^{-1}$ and $L=(\det\ell)\tilde{\ell}^{-1}$. The last equation entails that $\det L=(\det\ell)^2$ and that $L^{-1}=\frac{1}{\det\ell}\tilde{\ell}$. Hence, it follows from the first equation that $\mathcal{L} = (\det \ell)\ell = D\ell$. Let now $C^2 = \sigma_1\partial_{23} + \sigma_2\partial_{31} + \sigma_3\partial_{12}$ be an arbitrary 2-cochain. Since its first term reads

$$\sigma_1\partial_{23} = \frac{\sigma_1}{D^2} L_{j2}L_{k3}Y_{jk} = \frac{\sigma_1}{D^2}\left(\mathcal{L}_{11}Y_{23} + \mathcal{L}_{21}Y_{31} + \mathcal{L}_{31}Y_{12}\right) = \frac{\sigma_1}{D}\left(\ell_{11}Y_{23} + \ell_{21}Y_{31} + \ell_{31}Y_{12}\right), \tag{4}$$

its is clear that any 2-cochain can be written as announced. Conversely, the first term of any 2-vector $C^2 = \frac{\sigma_1}{D}Y_{23} + \frac{\sigma_2}{D}Y_{31} + \frac{\sigma_3}{D}Y_{12}$ reads

$$\frac{\sigma_1}{D}Y_{23} = \frac{\sigma_1}{D}\ell_{2j}\ell_{3k}\partial_{jk} = \frac{\sigma_1}{D}\left(L_{11}\partial_{23} + L_{12}\partial_{31} + L_{13}\partial_{12}\right).$$

Thus, such a 2-vector C^2 is a formal Poisson 2-cochain if and only if $L_{ij}\sigma_i$ is divisible by D for any j. The proofs of the statements concerning 0-, 1-, and 3-cochains are similar.

Hence, if we substitute the Y_i for the standard basic vector fields ∂_i, the cochains assume—roughly speaking—the shape $\sum f\mathbf{Y}$, where f is a function and $\mathbf{Y}$ is a wedge product of basic fields Y_i. Then the Lichnerowicz-Poisson coboundary operator $\partial_{\Lambda_I} = [\Lambda_I, \cdot]$ is just

$$\partial_{\Lambda_I}(f\mathbf{Y}) = [\Lambda_I, f\mathbf{Y}] = [\Lambda_I, f] \wedge \mathbf{Y}. \tag{5}$$

More precisely, the coboundary operator associated with Λ_I is given by

$$[\Lambda_I, C^0] = \nabla C^0, [\Lambda_I, C^1] = \nabla \wedge C^1, [\Lambda_I, C^2] = \nabla . C^2, \text{ and } [\Lambda_I, C^3] = 0, \tag{6}$$

where $\nabla = \sum_i X_i(\cdot)Y_i$, $X_1 = cY_2 - bY_3, X_2 = aY_3 - cY_1, X_3 = bY_1 - aY_2$, and where the RHS have to be viewed as notations that give the coefficients of the coboundaries in the Y_i-basis. For instance, $[\Lambda_I, C^2] = (\sum_i X_i(\frac{\sigma_i}{D}))Y_{123}$.

Of course the formal power series $\sigma, \sigma_1, \sigma_2, \sigma_3$ in Equation (3) read

$$\sum_{J\in\mathbb{N}^3} c_J X^J = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} c_{j_1j_2j_3} x_1^{j_1}x_2^{j_2}x_3^{j_3} \quad (c_{j_1j_2j_3} \in \mathbb{R}).$$

The degrees $j_1, j_2, j_3 \in \mathbb{N}$ and the cochain degree $c \in \{0, 1, 2, 3\}$ induce a 4-grading of the formal Poisson cochain space $\mathcal{P}$ of polyvector fields with coefficients in formal power series. Let us emphasize that the degrees j_i are read in the numerators σ of the decomposition $C = \sum \frac{\sigma}{D}\mathbf{Y}$. They are tightly related with the r-matrix induced nature of Λ_I and were basic in the method developed in [MP06]. In the following we use the degrees $r = j_1 + j_2 + c$ and $s = j_3$ (depending on the considered Poisson tensor, other degrees could be used, but the preceding ones encompass the majority of twisted structures) that generate a bigrading of $\mathcal{P}$, $\mathcal{P} = \oplus_{r,s\in\mathbb{N}}\mathcal{P}^{rs}$. When defining the diagonal degree $n = r + s$, we get a graded space

$$\mathcal{P} = \oplus_{n\in\mathbb{N}}\mathcal{P}^n, \mathcal{P}^n = \oplus_{r,s\in\mathbb{N},r+s=n}\mathcal{P}^{rs}.$$

These degrees differ from those defined in [Vai05] on foliated manifolds (M, F), for arbitrary smooth coefficients, by means of a normal bundle H of the foliation, such that $TM = H \oplus F$.

We now determine the weights of the coboundary operators ∂_{Λ_I} and $\partial_{\Lambda_{II}}$ with respect to r and s. Actually D is an eigenvector of the basic fields Y_i, hence of the fundamental fields X_i, $Y_iD = \lambda_iD$, $X_iD = \mu_iD$, $\lambda_i, \mu_i \in \mathbb{R}$. Indeed, since $\pi_{\lambda D} = \lambda(\partial_1 D\,\partial_{23} + \partial_2 D\,\partial_{31} + \partial_3 D\,\partial_{12})$, it follows from Equation (4) (take $\sigma_j = \lambda\partial_j D$) (and its cyclic permutations) that

$$\pi_{\lambda D} = \frac{\lambda}{D}\left(Y_1D\,Y_{23} + Y_2D\,Y_{31} + Y_3D\,Y_{12}\right).$$

But $\pi_{\lambda D}$ is part of Λ_I and is—more precisely—of type (1), i.e. reads

$$\pi_{\lambda D} = \mathfrak{l}_1Y_{23} + \mathfrak{l}_2Y_{31} + \mathfrak{l}_3Y_{12}$$

$(\mathfrak{l}_1, \mathfrak{l}_2, \mathfrak{l}_3 \in \mathbb{R})$. Hence,

$$Y_iD = \frac{\mathfrak{l}_i}{\lambda}D =: \lambda_iD, \forall i \in \{1, 2, 3\}.$$

In view of Equations (3) and (6), the degrees j_1, j_2, j_3 of the Λ_I-coboundary $\partial_{\Lambda_I} C$ of any cochain C only depend on the values $X_i\left(\frac{\sigma}{D}\right)$ of the fundamental linear fields X_i for an arbitrary formal power series $\sigma = \sum_J c_J X^J$. Since

$$X_i\left(\frac{\sigma}{D}\right) = \sum_J c_J \frac{1}{D}\left(X_i - \mu_i \,\mathrm{id}\right) X^J,$$

it is clear that ∂_{Λ_I} preserves the total degree $\mathfrak{t} = j_1 + j_2 + j_3$.

In the following, we focus on the first twisted quadratic Poisson structures that appear in the DHC, i.e. on classes 4, 8, and 11, see [DH91]. Let us recall that

$$\Lambda_4 = ayz\partial_{23} + axz\partial_{31} + \left(bxy + z^2\right)\partial_{12} = aY_{23} + aY_{31} + bY_{12} + \frac{z^3}{D}Y_{12} = \Lambda_{4,I} + \Lambda_{4,II},$$

$$a \neq 0, b \neq 0, Y_1 = x\partial_1, Y_2 = y\partial_2, Y_3 = z\partial_3, D = xyz,$$

$$\Lambda_8 = \left(\frac{a+b}{2}(x^2+y^2) \pm z^2\right)\partial_{12} + axz\partial_{23} + ayz\partial_{31} = aY_{23} + \frac{a+b}{2}Y_{12} \pm \frac{z^3}{D}Y_{12} = \Lambda_{8,I} + \Lambda_{8,II},$$

$$a \neq 0, b \neq 0, Y_1 = x_1\partial_1 + x_2\partial_2, Y_2 = x_1\partial_2 - x_2\partial_1, Y_3 = x_3\partial_3, D = (x^2+y^2)z,$$

$$\Lambda_{11} = \left(ax^2 + bz^2\right)\partial_{12} + (2a+1)xz\partial_{23} = Y_{23} + aY_{12} + b\frac{z^3}{D}\left((3a+1)Y_{12} + Y_{23}\right) = \Lambda_{11,I} + \Lambda_{11,II},$$

$$a \neq \frac{-1}{3}, b \neq 0, Y_1 = \mathcal{E}, Y_2 = x\partial_2, Y_3 = (3a+1)z\partial_3, D = (3a+1)x^2 z.$$

Owing to the above remarks, it is obvious that $\partial_{\Lambda_{i,I}}, i \in \{4,8,11\}$, preserves the partial degree $\mathfrak{p} = j_1 + j_2$ (and, as aforementioned, the total degree $\mathfrak{t}$). Hence, its weight with respect to (r,s) is $(1,0)$:

$$d' := d_{10} := \partial_{\Lambda_{i,I}} : \mathcal{P}^{rs} \to \mathcal{P}^{r+1,s} \quad (i \in \{4,8,11\})$$

(dependence on i omitted in d' and d_{10}).

As for the weight of $\partial_{\Lambda_{i,II}}, i \in \{4,8,11\}$, with respect to (r,s), let us first recall that, if f and g are some functions, and if $\mathbf{X}$ and $\mathbf{Y}$ denote wedge products of Y_1, Y_2, Y_3 with (non-shifted) degrees α and β respectively, we have

$$[f\mathbf{X}, g\mathbf{Y}] = f[\mathbf{X}, g] \wedge \mathbf{Y} + (-1)^{\alpha\beta-\alpha-\beta} g[\mathbf{Y}, f] \wedge \mathbf{X}. \tag{7}$$

Of course, the RHS of the preceding equation is a linear combination of terms of the type $fY_i(g)\mathbf{Z}$ or $gY_i(f)\mathbf{Z}$, where $\mathbf{Z}$ is a wedge product of Y_1, Y_2, Y_3 of degree $\alpha + \beta - 1$. It follows that $\partial_{\Lambda_{i,II}} C^c$, $i \in \{4,8,11\}$, $C^c \in \mathcal{P}$, is a formal series of terms of the type

$$[\frac{z^3}{D}\mathbf{X}, \frac{X^J}{D}\mathbf{Y}].$$

Any such term is a linear combination of terms of the type

$$\frac{z^3}{D}Y_i\left(\frac{X^J}{D}\right)\mathbf{Z} \quad \text{and} \quad \frac{X^J}{D}Y_i\left(\frac{z^3}{D}\right)\mathbf{Z}.$$

As D is an eigenvector of Y_i, this entails that coboundary $\partial_{\Lambda_{i,II}} C^c$ has the form

$$\partial_{\Lambda_{i,II}} C^c = \sum \frac{\sum_K c_K X^K}{D^2}\mathbf{Z},$$

where in each term $k_1 + k_2 = j_1 + j_2$ and $k_3 = j_3 + 3$, and where the degree of wedge product $\mathbf{Z}$ is $\alpha + \beta - 1 = c + 1$. When dividing the preceding numerators by D (see above), we find that the weight of $\partial_{\Lambda_{i,II}}$ with respect to (r,s) is $(-1,2)$:

$$d'' := d_{-12} := \partial_{\Lambda_{i,II}} : \mathcal{P}^{rs} \to \mathcal{P}^{r-1,s+2} \quad (i \in \{4,8,11\})$$

(dependence on i omitted in d'' and d_{-12}).

Finally, $(\mathcal{P}, \partial_{\Lambda_i})$, $i \in \{4, 8, 11\}$, endowed with the previously mentioned gradings

$$\mathcal{P} = \oplus_{n\in\mathbb{N}}\mathcal{P}^n, \mathcal{P}^n = \oplus_{r,s\in\mathbb{N},r+s=n}\mathcal{P}^{rs}$$

and the differential

$$d := \partial_{\Lambda_i} = \partial_{\Lambda_{i,I}} + \partial_{\Lambda_{i,II}} = d' + d'' = d_{10} + d_{-12},$$

is a VPDC. We will compute the cohomology $H(\Lambda_i) = H(\mathcal{P}, d)$ using the SpecSeq associated with this VPDC (see above).

3 Model of the spectral sequence associated with a VPDC

As mentioned above, a VPDC, a HPDC, and a DC can canonically be viewed as regular filtered graded differential spaces. Hence, a SpecSeq (two, for any DC) is associated with each one of these complexes.

In order to introduce notations, let us recall that, if (K, d, K_p, K^n) is any (regular, i.e. $K_p \cap K^n = 0, \forall p > n$) filtered (subscripts) graded (superscripts) differential space (in our work p and n can be regarded as positive integers), the associated SpecSeq (E_r, d_r) $(r \in \mathbb{N})$ is defined by

$$E_r^{pq} = Z_r^{pq}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{pq}),$$

where $Z_r^{pq} = K_p \cap d^{-1}K_{p+r} \cap K^{p+q}$ and $B_r^{pq} = K_p \cap dK_{p-r} \cap K^{p+q}$ are the spaces of "weak cocycles" and "strong coboundaries" of order r in $K_p \cap K^{p+q}$, and

$$d_r : E_r^{pq} \ni [\mathfrak{z}_r^{pq}]_{E_r^{pq}} \to [d\mathfrak{z}_r^{pq}]_{E_r^{p+r,q+1-r}} \in E_r^{p+r,q+1-r}.$$

In the following, we also use the vector space isomorphism

$$\sigma_r : E_{r+1}^{pq} \to H^{pq}(E_r, d_r),$$

which assigns to each $[\mathfrak{z}_{r+1}^{pq}]_{E_{r+1}^{pq}}$, $\mathfrak{z}_{r+1}^{pq} \in Z_{r+1}^{pq} \subset Z_r^{pq}$, the d_r-cohomology class $[[\mathfrak{z}_{r+1}^{pq}]_{E_r^{pq}}]_{d_r}$, $[\mathfrak{z}_{r+1}^{pq}]_{E_r^{pq}} \in E_r^{pq} \cap \ker d_r$. For more detailed results on spectral sequences, we refer the reader to [Cle85], [God52], [CE56], [Vai73], ... In these monographs, a model for the SpecSeq associated with a (HP)DC is partially depicted up to $r = 2$. It is well-known that spectral sequences are particularly easy to use, if many spaces E_2^{pq} (or E_r^{pq} $(r > 2)$) vanish. Due to richness of Poisson cohomology, this lacunary phenomenon is less pronounced in our setting. Since we have thus to compute through the whole SpecSeq, we need the complete description of the entire model of the SpecSeq (E_r, d_r) $(r \in \mathbb{N})$ associated with a VPDC.

So consider an arbitrary VPDC and let $G^{pq}(K)$ $(p, q \in \mathbb{N})$ be the term of degree (p, q) of the bigraded space associated with the filtered graded space K. It is clear that the mapping

$$I_0 : E_0^{pq} = K_p \cap K^{p+q}/K_{p+1} \cap K^{p+q} = G^{pq}(K) \ni [\mathfrak{z}_0^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_0^{pq}} \to z^{qp} \in K^{qp},$$

where z^{rs} (as well as—in the following—all Latin characters with double superscript) is an element of K^{rs} (whereas German Fraktur characters with double superscript, such as $\mathfrak{z}_0^{pq}$, do not refer to the bigrading of K), is an isomorphism of bigraded vector spaces (i.e. a vector space isomorphism that respects the bigrading). It is easily seen that, when reading d_0 through this isomorphism, we get the compound map

$$\overline{d}_0 = I_0 d_0 I_0^{-1} = d_{10}.$$

Thus $I_0 : (E_0, d_0) \to (K, \overline{d}_0)$ is an isomorphism between bigraded differential spaces, and induces an isomorphism

$$I_{0\sharp} : H^{pq}(E_0, d_0) \ni [[\mathfrak{z}_0^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_0^{pq}}]_{d_0} \to [z^{qp}]_{\overline{d}_0} \in H^{pq}(K, \overline{d}_0) =: {}^0H^{pq}(K) = {}^0H^q(K^{*p})$$

of bigraded vector spaces, where the last space is the q-term of the cohomology space of $(K^{*p}, \bar{d}_0 = d_{10})$. Hence the bigraded vector space isomorphism

$$I_1 = I_{0\sharp}\sigma_0 : E_1^{pq} \ni [\mathfrak{z}_1^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_1^{pq}} \to [[\mathfrak{z}_1^{pq}]_{E_0^{pq}}]_{d_0} \to [z^{qp}]_{\bar{d}_0} \in {}^0H^q(K^{*p}).$$

We now again verify straightforwardly that differential d_1 read on model ${}^0H(K)$ is induced by d_{01}, i.e. that

$$\bar{d}_1 = I_1 d_1 I_1^{-1} = d_{01\sharp}.$$

Finally,

$$I_2 = I_{1\sharp}\sigma_1 : E_2^{pq} \ni [\mathfrak{z}_2^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_2^{pq}} \to [[\mathfrak{z}_2^{pq}]_{E_1^{pq}}]_{d_1} \to [[z^{qp}]_{\bar{d}_0}]_{\bar{d}_1} \in {}^1H^p({}^0H^q(K))$$

is an isomorphism of bigraded vector spaces. As for the sense of the last space, note that $({}^0H^q(K) = \oplus_p {}^0H^q(K^{*p}), \bar{d}_1)$ is a complex. Observe now that the inverse I_2^{-1} is less straightforward than I_0^{-1} and I_1^{-1}. Indeed, if $[[z^{qp}]_{\bar{d}_0}]_{\bar{d}_1} \in {}^1H^p({}^0H^q(K))$, representative z^{qp} is generally not a member of Z_2^{pq}. However, since the considered class makes sense,

$$\begin{array}{l} d_{10} z^{qp} = 0 \\ d_{01} z^{qp} + d_{10} z^{q-1,p+1} = 0, \end{array}$$

where $z^{q-1,p+1} \in K^{q-1,p+1}$. Thus, $\mathfrak{z}_2^{pq} := z^{qp} + z^{q-1,p+1} \in Z_2^{pq}$ and

$$I_2^{-1}[[z^{qp}]_{\bar{d}_0}]_{\bar{d}_1} = [\mathfrak{z}_2^{pq}]_{E_2^{pq}}.$$

So

$$\bar{d}_2[[z^{qp}]_{\bar{d}_0}]_{\bar{d}_1} = I_2[d\mathfrak{z}_2^{pq}]_{E_2^{p+2,q-1}} = [[d_{-12} z^{qp} + d_{01} z^{q-1,p+1}]_{\bar{d}_0}]_{\bar{d}_1}.$$

The preceding results extend those given in [Vai73] (for a HPDC). They can easily be adapted to the most frequently encountered situations where only some terms d_{ab} of d do not vanish.

In the following, we complete the description of the SpecSeq associated with a VPDC, assuming that $d = d_{10} + d_{-12} := d' + d''$. This hypothesis entails that $d'^2 = d''^2 = d'd'' + d''d' = 0$, i.e. that d' and d'' are two anticommuting differentials. Hereafter, we denote by ${}^rH(.)$ $(r \in \mathbb{N})$ the cohomology of differential $\bar{d}_{2r}$ and by $[.]_r$ the corresponding classes. Moreover, we will deal with strongly triangular systems of type

$$\begin{array}{ll} d' z^{qp} = 0 & (\mathfrak{E}_0) \\ d'' z^{qp} + d' z^{q-2,p+2} = 0 & (\mathfrak{E}_1) \\ \ldots & \\ d'' z^{q-2(k-2),p+2(k-2)} + d' z^{q-2(k-1),p+2(k-1)} = 0. & (\mathfrak{E}_{k-1}) \end{array}$$

Note that when solving such a system, we prove at each stage that some d'-cocycle is actually a d'-coboundary. We refer to this kind of system using the notation $S(z^{qp}; k)$ or $S(k; z^{q-2(k-1),p+2(k-1)})$ depending on the necessity to emphasize the first or the last unknown or entry of an ordered solution.

Proposition 1. *The spectral sequence associated to a VPDC with differential $d = d_{10} + d_{-12} = d' + d''$ admits the following model. The model of E_0, isomorphisms I_0 and I_0^{-1}, and differential $\bar{d}_0$ are the same as above. For any $r \in \{1, 2, \ldots\}$,*

(i) The map

$$I_{2r-1} : E_{2r-1}^{pq} \ni [\mathfrak{z}_{2r-1}^{pq} = \sum_{i=0}^{q} z^{q-i,p+i}]_{E_{2r-1}^{pq}} \to [[[z^{qp}]_0]_1 \ldots]_{r-1} \in {}^{r-1}H^{pq}({}^{r-2}H(\ldots({}^0H(K))))$$

is a bigraded vector space isomorphism. Its inverse I_{2r-1}^{-1} *associates to any RHS-class the LHS-class with representative* $\mathfrak{z}_{2r-1}^{pq} = \sum_{i=0}^{r-1} z^{q-2i,p+2i}$, *where* $(z^{qp}, \ldots, z^{q-2(r-1),p+2(r-1)})$ *is any solution of system* $S(z^{qp}; r)$. *Furthermore,* $\overline{d}_{2r-1} = 0$.

(ii) The model of E_{2r}^{pq} *and the corresponding isomorphisms* I_{2r} *and* I_{2r}^{-1} *coincide with those pertaining to* E_{2r-1}^{pq}. *Moreover,*

$$\overline{d}_{2r}[[[z^{qp}]_0]_1 \ldots]_{r-1} = [[[d'' z^{q-2(r-1),p+2(r-1)}]_0]_1 \ldots]_{r-1}, \tag{8}$$

where $z^{q-2(r-1),p+2(r-1)}$ *is the last entry of an arbitrary solution of* $S(z^{qp}; r)$.

Proof. It is easier to prove an extended version of Proposition 1. Indeed, let us complete assertions (i) and (ii) by item

(iii) Existence (resp. vanishing) of a class $[[[z^{qp}]_0]_1 \ldots]_{r-1}$ is equivalent with existence of at least one solution of system $S(z^{qp}; r)$ (resp. with existence of $z^{q-1,p}$ and of $z_i^{q+1,p-2}$, $i \in \{1, \ldots, r-1\}$, which induce systems $S(i; z_i^{q+1,p-2})$ with solution, such that

$$z^{qp} + d' z^{q-1,p} + d'' \sum_{i=1}^{r-1} z_i^{q+1,p-2} = 0.)$$

The proof is by induction on r. Observe first that the assertions are valid for $r = 1$ (see above). Assume now that all items hold for $r \in \{1, \ldots, \ell - 1\}$. Proceeding as above, we easily show that $I_{2\ell-1} := I_{2(\ell-1)\sharp}\sigma_{2(\ell-1)}$ is the appropriate bigraded vector space isomorphism. In order to determine $I_{2\ell-1}^{-1}$, take any RHS-class $[[[z^{qp}]_0]_1 \ldots]_{\ell-1}$.

Let us first prove assertion (iii). Existence of class $[[[z^{qp}]_0]_1 \ldots]_{\ell-1}$ is equivalent with existence of class $[[[z^{qp}]_0]_1 \ldots]_{\ell-2}$ (itself equivalent to existence of at least one solution

$$z^{q-2j,p+2j} \quad (0 \leq j \leq \ell - 2)$$

for $S(z^{qp}; \ell - 1)$, by induction) and condition

$$\overline{d}_{2(\ell-1)}[[[z^{qp}]_0]_1 \ldots]_{\ell-2} = 0.$$

Using the induction assumptions, we see that the last condition is equivalent, first with

$$[[[d'' z^{q-2(\ell-2),p+2(\ell-2)}]_0]_1 \ldots]_{\ell-2} = 0,$$

then with existence of

$$z^{q-2(\ell-1),p+2(\ell-1)}$$

and $z_i^{q-2(\ell-2),p+2(\ell-2)}$ $(1 \leq i \leq \ell - 2)$, which implement systems $S(i; z_i^{q-2(\ell-2),p+2(\ell-2)})$ with solution, say

$$z_i^{q-2j,p+2j} \quad (1 \leq \ell - i - 1 \leq j \leq \ell - 2),$$

such that

$$d'' \left(z^{q-2(\ell-2),p+2(\ell-2)} + \sum_{i=1}^{\ell-2} z_i^{q-2(\ell-2),p+2(\ell-2)} \right) + d' z^{q-2(\ell-1),p+2(\ell-1)} = 0. \tag{9}$$

Assume now that all this holds and define new $z^{q-2j,p+2j}$ $(0 \leq j \leq \ell - 1)$. For each j, take just the sum of the old $z^{q-2j,p+2j}$ and of all existing $z_i^{q-2j,p+2j}$. These new $z^{q-2j,p+2j}$ form a solution of $S(z^{qp}; \ell)$. Note first that for $j \in \{0, \ell - 1\}$, the old and new $z^{q-2j,p+2j}$ coincide. Hence, the last equation $(\mathfrak{E}_{\ell-1})$ of $S(z^{qp}; \ell)$ is nothing but Equation (9). Moreover, it is easily checked that Equations $(\mathfrak{E}_{\ell-2}), \ldots, (\mathfrak{E}_0)$ are also verified. Conversely, if $S(z^{qp}; \ell)$ has a solution, the successive classes $[z^{qp}]_0, [[z^{qp}]_0]_1, \ldots, [[[z^{qp}]_0]_1 \ldots]_{\ell-1}$ are actually defined. It suffices to note that $\overline{d}_0 z^{qp} = 0$ and that, by induction,

$$\overline{d}_{2r}[[[z^{qp}]_0]_1 \ldots]_{r-1} = [[[d'' z^{q-2(r-1),p+2(r-1)}]_0]_1 \ldots]_{r-1} = -[[[d' z^{q-2r,p+2r}]_0]_1 \ldots]_{r-1} = 0,$$

for any $r \in \{1, \ldots, \ell-1\}$.

As for the second part of (iii), note that a class $[[[z^{qp}]_0]_1 \ldots]_{\ell-1}$ vanishes if and only if there is $z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)}$ that generates a system $S(z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)}; \ell-1)$ with solution, say

$$z_{\ell-1}^{q+2(\ell-j-1)-1,p-2(\ell-j-1)} \quad (0 \leq j \leq \ell-2),$$

such that

$$[[[z^{qp}]_0]_1 \ldots]_{\ell-2} = -\overline{d}_{2(\ell-1)}[[[z_{\ell-1}^{q+2(\ell-1)-1,p-2(\ell-1)}]_0]_1 \ldots]_{\ell-2} = -[[[d'' z_{\ell-1}^{q+1,p-2}]_0]_1 \ldots]_{\ell-2}.$$

But, by induction, $[[[z^{qp} + d'' z_{\ell-1}^{q+1,p-2}]_0]_1 \ldots]_{\ell-2} = 0$ if and only if there are $z^{q-1,p}$ and $z_i^{q+1,p-2}$ $(1 \leq i \leq \ell-2)$, which induce systems $S(i; z_i^{q+1,p-2})$ with solution, such that

$$z^{qp} + d'' z_{\ell-1}^{q+1,p-2} + d'' \sum_{i=1}^{\ell-2} z_i^{q+1,p-2} + d' z^{q-1,p} = 0.$$

Hence the conclusion.

We now revert to items (i) and (ii). For any RHS-class $[[[z^{qp}]_0]_1 \ldots]_{\ell-1} \in {}^{\ell-1}H^{pq}({}^{\ell-2}H(\ldots({}^0H(K))))$, the corresponding system $S(z^{qp}; \ell)$ admits, as just explained, at least one solution $z^{q-2j,p+2j}$ $(0 \leq j \leq \ell-1)$. Set

$$\mathfrak{z}_{2\ell-1}^{pq} := \sum_{j=0}^{\ell-1} z^{q-2j,p+2j}.$$

As $d\mathfrak{z}_{2\ell-1}^{pq} = d'' z^{q-2(\ell-1),p+2(\ell-1)} \in K^{q-2\ell+1,p+2\ell}$, we see that $\mathfrak{z}_{2\ell-1}^{pq} \in Z_{2\ell-1}^{pq} = K_p \cap d^{-1} K_{p+2\ell-1} \cap K^{p+q}$. Hence $I_{2\ell-1}^{-1}$. As $d_{2\ell-1}[\mathfrak{z}_{2\ell-1}^{pq}]_{E_{2\ell-1}^{pq}} \in E_{2\ell-1}^{p+2\ell-1,q-2\ell+2}$, it is clear that $\overline{d}_{2\ell-1} = 0$. Thus, the statement concerning the model of $E_{2\ell}^{pq}$ and the isomorphisms $I_{2\ell}$ and $I_{2\ell}^{-1}$ is obvious. Finally, as $d_{2\ell}[\mathfrak{z}_{2\ell}^{pq}]_{E_{2\ell}^{pq}} \in E_{2\ell}^{p+2\ell,q-2\ell+1}$, we get

$$\overline{d}_{2\ell}[[[z^{qp}]_0]_1 \ldots]_{\ell-1} = [[[d'' z^{q-2(\ell-1),p+2(\ell-1)}]_0]_1 \ldots]_{\ell-1}. \quad \blacksquare$$

Remark. Result (8) can be rephrased as $\overline{d}_{2r} = \left((-1)^{r-1} d''(d'^{-1} d'')^{r-1}\right)_\sharp$, for any $r \in \{1, 2, \ldots\}$.

4 Formal cohomology of Poisson tensor Λ_4

As aforementioned, we use the just depicted SpecSeq associated with the above detailed VPDC implemented by the twisted r-matrix induced Poisson structure Λ_4.

4.1 Computation of the second term of the SpecSeq

In this section, we give the second term $E_2 \simeq {}^0H(\mathcal{P})$ of the SpecSeq. Note that ${}^0H(\mathcal{P})$ is the formal Poisson cohomology of $\overline{d}_0 = d' = d_{10} = \partial_{\Lambda_{4,I}}$. As already elucidated in the Introduction, we came up with decomposition (1), since the cohomology of ∂_{Λ_I} is always accessible by the technique proposed in [MP06]. Hence, cohomology space ${}^0H(\mathcal{P})$ can be obtained (quite straightforwardly) by this modus operandi. Let us emphasize that our results are in accordance, as well with similar upshots in [Mon02,2], as with our comments in [MP06], regarding the tight relation between Casimir functions and Koszul-exactness or "quasi-exactness", the appearance of "accidental Casimir-like" non bounding cocycles, and the increase of the "weight in cohomology" of the singularities, with closeness of the considered Poisson structure to Koszul-exactness.

If $\frac{b}{a} \in \mathbb{Q}^*$, we denote by $(\beta, \alpha) \sim (b, a)$, $\alpha \in \mathbb{N}^*$, the irreducible representative of $\frac{b}{a}$. Remember that, see [MP06], for $\frac{b}{a} \in \mathbb{Q}^*_+$, a quasi-exact structure

$$\Lambda = a\partial_1(pq)\partial_{23} + a\partial_2(pq)\partial_{31} + b\partial_3(pq)\partial_{12}, \tag{10}$$

$p = p(x, y)$, $q = q(z)$, exhibits the basic Casimir $p^\alpha q^\beta$. Furthermore, we set $D = xyz$, $D' = xy$, and write $\mathcal{A}_\alpha Y_3$, $\alpha \in \mathbb{N}^*$, instead of $D'^\alpha z^{-1} Y_3 = D'^\alpha \partial_3$, and $\oplus_{ij} \ldots Y_{ij}$ instead of $\ldots Y_{23} + \ldots Y_{31} + \ldots Y_{12}$. Remark also that the algebra of polynomials of the algebraic variety of singularities of $\Lambda_{4,I}$ is $\mathbb{R}[[x]] \oplus \mathbb{R}[[y]] \oplus \mathbb{R}[[z]]$, where it is understood that term $\mathbb{R}$ is considered only once.

The following proposition is now almost obvious.

Proposition 2.

1. *If $\frac{b}{a} \in \mathbb{Q}^*_+$, the algebra of $\Lambda_{4,I}$-Casimirs is $\mathrm{Cas}(\Lambda_{4,I}) = \oplus_{i\in\mathbb{N}} \mathbb{R} D'^{\alpha i} z^{\beta i}$ and the cohomology space ${}^0H(\mathcal{P})$ is given by*

$$\begin{aligned} E_2 \simeq {}^0H(\mathcal{P}) \quad = \quad & \mathrm{Cas}(\Lambda_{4,I}) \oplus \bigoplus_i \mathrm{Cas}(\Lambda_{4,I}) Y_i \oplus \bigoplus_{ij} \mathrm{Cas}(\Lambda_{4,I}) Y_{ij} \oplus \mathrm{Cas}(\Lambda_{4,I}) Y_{123} \\ & \oplus \mathbb{R}[[z]]\partial_{12} \oplus \mathbb{R}[[z]]\partial_{123} \oplus \begin{cases} \mathbb{R}[[x]]\partial_{23} \oplus \mathbb{R}[[y]]\partial_{31} \oplus (\mathbb{R}[[x]] \oplus \mathbb{R}[[y]])\partial_{123}, \textit{if } b = a \\ 0, \textit{otherwise} \end{cases} \end{aligned}$$

2. *If $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}^*_+$, we have $\mathrm{Cas}(\Lambda_{4,I}) = \mathbb{R}$ and*

$$\begin{aligned} E_2 \simeq {}^0H(\mathcal{P}) \quad = \quad & \mathrm{Cas}(\Lambda_{4,I}) \oplus \bigoplus_i \mathrm{Cas}(\Lambda_{4,I}) Y_i \oplus \bigoplus_{ij} \mathrm{Cas}(\Lambda_{4,I}) Y_{ij} \oplus \mathrm{Cas}(\Lambda_{4,I}) Y_{123} \\ & \oplus \begin{cases} \mathbb{R}\mathcal{A}_\alpha Y_3 \oplus \mathcal{A}_\alpha(\mathbb{R}Y_{23} + \mathbb{R}Y_{31}) \oplus \mathbb{R}\mathcal{A}_\alpha Y_{123}, \ \textit{if } (-1, \alpha) \sim (b, a) \\ 0, \textit{otherwise} \end{cases} \\ & \oplus \mathbb{R}[[z]]\partial_{12} \oplus \mathbb{R}[[z]]\partial_{123} \end{aligned}$$

Remark. Due to the properties—used below—of the preceding (non bounding) $\Lambda_{4,I}$-cocycles, we classify these representatives as follows:

1. Representatives of type 1: All cocycles with cochain degree 0, the $1-$ and $2-$cocycles that contain a Casimir (maybe the accidental Casimir $\mathcal{A}_\alpha$), except cocycles $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$

2. Representatives of type 2: All $3-$cocycles, all cocycles with singularities, and cocycles $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$

4.2 Prolongable systems $\mathbf{S(z^{qp}; r)}$

Since computation through the whole SpecSeq will shape up as inescapable, we need the below corollary of Proposition (1). It allows to short-circuit the process of computing the successive terms of the sequence. Let us specify that in the following a system of representatives of a space of classes is made up by representatives that are in 1-to-1 correspondence with the considered classes.

Corollary 1. *If, for some fixed $r \in \mathbb{N}^*$, all the classes $[[[z^{qp}]_0]_1 \ldots]_{r-1}$ in model space ${}^{r-1}H({}^{r-2}H(\ldots {}^0H(K)))$, appendant on a SpecSeq associated to a VPDC with differential $d = d_{10} + d_{-12} = d' + d''$, give rise to an enlarged system $S(z^{qp}; s)$ with solution, for some fixed $s \geq r$, the following upshots hold:*

1. *All the differentials $\overline{d}_{2r-1}, \overline{d}_{2r}, \ldots, \overline{d}_{2s-1}$ vanish*

2. *Differential $\overline{d}_{2s}$ is defined by $\overline{d}_{2s}[[[z^{qp}]_0]_1 \ldots]_{r-1} = [[[d'' z^{q-2(s-1), p+2(s-1)}]_0]_1 \ldots]_{r-1}$*

3. *Any system (z^{qp}) of representatives of ${}^{r-1}H({}^{r-2}H(\ldots {}^0H(K)))$ is in 1-to-1 correspondence with the system $(\mathfrak{z}^{pq}_{2s} := \sum_{k=0}^{s-1} z^{q-2k, p+2k})$ of representatives of E_{2s}*

Proof. Induction on s. ∎

4.3 Forecast

In order to increase readability of our paper, some intuitive advisements are necessary.

The basic idea of the theory of spectral sequences is that computation of the successive terms $E_r \simeq H(E_{r-1}, d_{r-1})$ $(r \in \mathbb{N}^*)$ allows to detect their inductive limit E_∞, which—for a convergent sequence—is isomorphic with the graded space $G(H)$ associated to the sought-after filtered cohomology space H. We then hope to be able to reconstruct this filtered space H from the corresponding graded space $G(H)$. Let us recall that space H is of course the cohomology of the filtered graded differential space associated with the SpecSeq. Hence, in our case, $H = H(\Lambda_4)$. It is clear that the successive cohomology computations take place on the concrete model side. To determine H, we have to pull our results back to the theoretical side, and more precisely to read them through the numerous isomorphisms involved.

Actually the application of spectral sequences presented in this work, provides a beautiful insight into the operating mode of a SpecSeq. Since—roughly spoken—the "weak cocycle condition" in the definition of Z_r^{pq} (resp. the "strong coboundary condition" in the definition of B_r^{pq}) converges to the usual cocycle condition (resp. the usual coboundary condition), we understand that, when passing from one estimate E_{r-1} of H to the next approximation E_r, we obtain an increasing number of conditions on our initial weak non bounding cocycles of E_2 and an increasing number of bounding cocycles. Moreover, when we compute through the SpecSeq, the aforementioned pullbacks, see Proposition (1), add up solutions of crescive systems,

$$\mathfrak{z}_{2r}^{pq} = z^{qp} + \sum_{k=1}^{r-1} z^{q-2k,p+2k}.$$

The next remarks aim at anticipation of these systems. The reader is already familiar with Casimirs of exact and quasi-exact structures. When taking an interest in slightly more general quasi-exact tensors,

$$\Lambda = a\partial_1((p+r)q)\partial_{23} + a\partial_2((p+r)q)\partial_{31} + b\partial_3((p+r)q)\partial_{12}, \tag{11}$$

$a, b \in \mathbb{R}^*$, $p = p(x,y)$, $q = q(z)$, $r = r(z)$, it is natural to ask which polynomials of the type $(p+c\,r)^n q^m$, $c \in \mathbb{R}$, $n, m \in \mathbb{N}$, $(n,m) \neq (0,0)$, are Casimir functions. It is easily checked that structure Λ_4 has this form and that the Casimir conditions read $am = bn$ and $3bn = ca(2n+m)$. So, for $\frac{b}{a} \in \mathbb{Q}_+^*$, the basic Casimir C of Λ_4 and its powers C^i, $i \in \mathbb{N}$, are given by

$$C^i = (p + \frac{3b}{2a+b} r)^{\alpha\, i} q^{\beta\, i} = (D' + \frac{z^2}{2a+b})^{\alpha\, i} z^{\beta\, i} = D'^{\alpha\, i} z^{\beta\, i} + \sum_{k=1}^{\alpha\, i} \frac{\complement_{\alpha\, i}^k}{(2a+b)^k} D'^{\alpha\, i-k} z^{\beta\, i+2k}.$$

These powers C^i (non bounding cocycles of $H = H(\Lambda_4)$) will be obtained—while we compute through the SpecSeq—from those, $D'^{\alpha\, i} z^{\beta\, i}$, of the Casimir of $\Lambda_{4,I}$ (non bounding cocycles of $E_2 \simeq {}^0H(\mathcal{P})$). Hence, the above-quoted solutions and corresponding systems $S(D'^{\alpha\, i} z^{\beta\, i}, \alpha\, i + 1)$.

4.4 Computation through the SpecSeq

In view of the preceding awareness, it is natural to set

$$Z^{q_{ic}-2k, p_i+2k} = \frac{\complement_{\alpha i}^k}{(2a+b)^k} D'^{\alpha i-k} z^{\beta i+2k} \begin{cases} A_{ik} \\ B_{ik}Y_1 + C_{ik}Y_2 + D_{ik}Y_3 \\ E_{ik}Y_{23} + F_{ik}Y_{31} \end{cases},$$

where $k \in \{0, 1, \ldots, \alpha\, i\}$ and $A_{ik}, B_{ik}, C_{ik}, D_{ik}, E_{ik}, F_{ik} \in \mathbb{R}$. More precisely, if $\frac{b}{a} \in \mathbb{Q}_+^*$, we have $(b,a) \sim (\beta, \alpha)$, $\alpha, \beta \in \mathbb{N}^*$, and we ask that $i \in \mathbb{N}$, if $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}_+^*$, we choose $i = 0$, and if moreover $(b,a) \sim (\beta, \alpha) = (-1, \alpha)$, $\alpha \in \mathbb{N}^*$, we also accept the value $i = 1$, but add the conditions $A_{10} = B_{10} = C_{10} = 0$. We define $(q_{ic}, p_i) := (2\alpha\, i + 2 + c, \beta\, i + 1)$, where $c \in \{0, 1, 2\}$ denotes the cochain degree, so that the double superscript in the LHS is the bidegree $(r,s) = (j_1 + j_2 + c, j_3)$ of the RHS.

Observe that the Z^{q_{ic}, p_i} are exactly the representatives of type 1 of the classes of $E_2 \simeq {}^0H(\mathcal{P})$.

Lemma 1. *For any admissible exponent i and any cochain degree $c \in \{0, 1, 2\}$, the cochains $Z^{q_{ic}-2k, p_i+2k}$, $k \in \{0, 1, \ldots, \alpha\, i\}$, constitute a solution of system $S(Z^{q_{ic}, p_i}; \alpha\, i+1)$, if and only if, for any $k \in \{0, 1, \ldots, \alpha\, i-1\}$,*

$$A_{i,k+1} = A_{ik}, \text{ if } c=0, \quad (C_0)$$

$$B_{i,k+1} + C_{i,k+1} = \frac{(\alpha i - k + 1)\,(B_{ik} + C_{ik}) - 2D_{ik}}{\alpha i - k} \quad \text{and} \quad D_{i,k+1} = D_{ik}, \text{ if } c=1, \quad (C_1)$$

$$E_{i,k+1} - F_{i,k+1} = \frac{\alpha i - k + 1}{\alpha i - k}\,(E_{ik} - F_{ik}),\ \text{if } c=2. \quad (C_2)$$

Furthermore,

$$d'' Z^{q_{ic}-2\alpha i, p_i+2\alpha i} = d'' Z^{2+c, p_i+2\alpha i} = \begin{cases} 0, \text{ for } c=0, \\ (2a+b)^{-\alpha i}\,(B_{i,\alpha i} + C_{i,\alpha i} - 2D_{i,\alpha i})\, z^{2+i(2\alpha+\beta)} \partial_{12}, \text{ for } c=1, \\ (2a+b)^{-\alpha i}\,(E_{i,\alpha i} - F_{i,\alpha i})\, z^{3+i(2\alpha+\beta)} \partial_{123}, \text{ for } c=2, \end{cases}$$

is a d'-coboundary if and only if the coefficient vanishes.

Proof. We must compute the differentials $d' = \partial_{\Lambda_{4,I}} = [\Lambda_{4,I}, .]$ and $d'' = \partial_{\Lambda_{4,II}} = [D^{-1} z^3 Y_{12}, .] =: [f\mathbf{X}, .]$ on the $Z^{q_{ic}-2k, p_i+2k}$. These cochains have the form $g\mathcal{Y} := D^{-1} X^J \mathcal{Y} := D^{-1} D'^n z^m \sum_j r_j \mathbf{Y}_j$, $n, m \in \mathbb{N}, r_j \in \mathbb{R}$, where the degree c of wedge product $\mathbf{Y}_j$ is independent of j. Hence, Equation (7) gives

$$d''(g\mathcal{Y}) = [f\mathbf{X}, g\mathcal{Y}] = f[\mathbf{X}, g] \wedge \mathcal{Y} + (-1)^c g[\mathcal{Y}, f] \wedge \mathbf{X}.$$

On the other hand, Equations (5) and (6) entail $d'(g\mathcal{Y}) = [\Lambda_{4,I}, g\mathcal{Y}] = [\Lambda_{4,I}, g] \wedge \mathcal{Y} = \sum_\ell X_\ell(g)\, Y_\ell \wedge \mathcal{Y}$, where $X_1 = bY_2 - aY_3, X_2 = aY_3 - bY_1, X_3 = a(Y_1 - Y_2)$. Since

$$Y_\ell \left(\frac{X^J}{D}\right) = (j_\ell - 1)\frac{X^J}{D}$$

(same notations as above), we get

$$d'(g\mathcal{Y}) = g\,(b(n-1) - a(m-1))\,(Y_1 - Y_2) \wedge \mathcal{Y}.$$

In particular, we recover the result $d' Z^{q_{ic}, p_i} = ig(b\alpha - a\beta)(Y_1 - Y_2) \wedge \mathcal{Y} = 0$, and, when setting $a = 0, b = 1, \mathcal{Y} = 1$, we find

$$[\mathbf{X}, g] = g(n-1)(Y_1 - Y_2).$$

We now compute $d'' Z^{q_{ic}-2k, p_i+2k}$, $k \in \{0, 1, \ldots, \alpha\, i\}$, and $d' Z^{q_{ic}-2(k+1), p_i+2(k+1)}$, $k \in \{0, 1, \ldots, \alpha\, i-1\}$.

1. $c = 0$

   It follows from the preceding equations that

$$d'' Z^{q_{i0}-2k, p_i+2k} = \mathtt{C}_{\alpha i}^{k} (\alpha\, i - k) A_{ik} (2a+b)^{-k} D^{-1} D'^{\alpha i - k} z^{3+\beta i + 2k} (Y_1 - Y_2)$$

   and that

$$d' Z^{q_{i0}-2(k+1), p_i+2(k+1)} = -\mathtt{C}_{\alpha i}^{k+1} (k+1) A_{i,k+1} (2a+b)^{-k} D^{-1} D'^{\alpha i - k} z^{3+\beta i + 2k} (Y_1 - Y_2).$$

   Since for any $p, n \in \mathbb{N}, p < n$, we have $\mathtt{C}_n^p (n-p) = \mathtt{C}_n^{p+1}(p+1)$, the sum of these coboundaries vanishes if and only if $A_{i,k+1} = A_{ik}$, for any $k \in \{0, 1, \ldots, \alpha\, i-1\}$. For $k = \alpha\, i$, we get

$$d'' Z^{q_{i0}-2\alpha i, p_i+2\alpha i} = d'' Z^{2, p_i+2\alpha i} = 0.$$

2. $c=1$

A short computation shows that

$$\begin{aligned} d''Z^{q_{i1}-2k,p_i+2k} = \; & \complement_{\alpha i}^{k}(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}[-D_{ik}(\alpha\, i-k)Y_{23}-D_{ik}(\alpha\, i-k)Y_{31} \\ & +((B_{ik}+C_{ik})(\alpha\, i-k+1)-2D_{ik})\,Y_{12}] \end{aligned}$$

and that

$$\begin{aligned} d'Z^{q_{i1}-2(k+1),p_i+2(k+1)} = \; & -\complement_{\alpha i}^{k+1}(k+1)(2a+b)^{-k}D^{-1}D'^{\alpha\, i-k}z^{3+\beta\, i+2k}[-D_{i,k+1}Y_{23} \\ & -D_{i,k+1}Y_{31}+(B_{i,k+1}+C_{i,k+1})Y_{12}]. \end{aligned}$$

If $k\in\{0,1,\ldots,\alpha\, i-1\}$, the sum of these coboundaries vanishes if and only if

$$B_{i,k+1}+C_{i,k+1}=\frac{(\alpha i-k+1)\,(B_{ik}+C_{ik})-2D_{ik}}{\alpha i-k}\quad\text{and}\quad D_{i,k+1}=D_{ik}.$$

Furthermore, for $k=\alpha\, i$, the first of the preceding "coboundary equations" provides the announced result for $d''Z^{q_{i1}-2\alpha i,p_i+2\alpha i}$. As $\mathbb{R}[[z]]\partial_{12}$ is part of the cohomology of $\overline{d}_0=d'$, this d''-coboundary is a d'-coboundary if and only if its coefficient vanishes.

3. $c=2$

We immediately obtain

$$d''Z^{q_{i2}-2k,p_i+2k}=\complement_{\alpha i}^{k}(\alpha\, i-k+1)(2a+b)^{-k}D^{-1}D'^{\alpha i-k}z^{3+\beta i+2k}(E_{ik}-F_{ik})Y_{123}$$

and

$$d'Z^{q_{i2}-2(k+1),p_i+2(k+1)}=-\complement_{\alpha i}^{k+1}(k+1)(2a+b)^{-k}D^{-1}D'^{\alpha\, i-k}z^{3+\beta\, i+2k}(E_{i,k+1}-F_{i,k+1})Y_{123}.$$

Hence the announced upshots. ∎

Let us recall that the admissible values of i (and the potential conventions on coefficients A_{10},B_{10},C_{10}) depend on quotient b/a. Moreover, for $b/a\in\mathbb{R}^*\setminus\mathbb{Q}_+^*,(b,a)\not\sim(-1,\alpha),\alpha\in\mathbb{N}^*$, we set $\alpha=1\in\mathbb{N}^*$. Actually, in this case, α needed not be defined before, as it was systematically multiplied by $i=0$.

The following theorem provides the complete description of the considered SpecSeq.

Theorem 1. *The even terms $E_{2(n-1)\alpha+4}=E_{2(n-1)\alpha+6}=\ldots=E_{2n\alpha+2}$ ($n\in\mathbb{N}$; for $n=0$, this package contains only term E_2) of the above defined SpecSeq are canonically isomorphic (i.e. $d_{2(n-1)\alpha+4}=d_{2(n-1)\alpha+6}=\ldots=d_{2n\alpha}=0$) and admit the below system of representatives:*

1. *All representatives of type 2 of $E_2\sim{}^0H(\mathcal{P})$, except*

$$\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12}\quad\text{and}\quad\mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123},$$

for all admissible $i\in\{0,1,\ldots,n-1\}$.

2. *All representatives of type 1 of $E_2\sim{}^0H(\mathcal{P})$, altered as follows:*

   - *For all admissible $i\in\{n,n+1,\ldots\}$,*

$$Z^{q_{ic},p_i}\rightsquigarrow\sum_{k=0}^{\alpha\, n}Z^{q_{ic}-2k,p_i+2k},$$

where the coefficients $A_{ik},B_{ik},C_{ik},D_{ik},E_{ik},F_{ik}$ incorporated into the terms of the RHS verify conditions $(C_0)-(C_2)$ of Lemma 1 up to $k=\alpha\, n-1$.

- *For all admissible* $i \in \{0, 1, \ldots, n-1\}$,

$$Z^{q_{ic},p_i} \rightsquigarrow \begin{cases} \left(D' + \frac{z^2}{2a+b}\right)^{\alpha i} z^{\beta i} A_{i0}, & \textit{if} \quad c = 0, \\ \left(D' + \frac{z^2}{2a+b}\right)^{\alpha i} z^{\beta i} \left(B_{i0}(Y_1 + \frac{1}{2}Y_3) + C_{i0}(Y_2 + \frac{1}{2}Y_3)\right), & \textit{if} \quad c = 1, \\ D'^{\alpha i} z^{\beta i} E_{i0}(Y_{23} + Y_{31}), & \textit{if} \quad c = 2. \end{cases}$$

Proof. The proof is by induction on n. For $n = 0$, Theorem 1 is obviously valid. Assume now that it holds true for $0, 1, \ldots, n-1$ ($n \in \mathbb{N}^*$). We first transfer the description of $E_{2(n-2)\alpha+4} = \ldots = E_{2(n-1)\alpha+2}$ to the concrete model side, in order to compute $\overline{d}_{2(n-1)\alpha+2}$. When having a look at the packages of terms that are known to be isomorphic, we see that the only differentials (under $\overline{d}_{2(n-1)\alpha+2}$) that do not vanish are $\overline{d}_{2m\alpha+2}$ ($m \in \{0, 1, \ldots, n-2\}$). Hence the target of vector space isomorphism

$$I_{2(n-1)\alpha+2} : E_{2(n-1)\alpha+2} \to {}^{(n-2)\alpha+1}H({}^{(n-3)\alpha+1}H(\ldots {}^1H({}^0H(\mathcal{P})))),$$

which—as it appears from its general description—maps the system of $E_{2(n-1)\alpha+2}$-representatives onto the system evidently made up by:

1. All representatives of type 2 of E_2, except $\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12}$ and $\mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123}$, for all admissible $i \in \{0, 1, \ldots, n-2\}$.

2. All representatives of type 1 of E_2, Z^{q_{ic},p_i}, i admissible, $c \in \{0, 1, 2\}$, with, for all admissible $i \in \{0, 1, \ldots, n-2\}$, $B_{i0} + C_{i0} = 2D_{i0}$, if $c = 1$, and $E_{i0} = F_{i0}$, if $c = 2$.

We now compute the cohomology of space $({}^{(n-2)\alpha+1}H(\ldots {}^0H(\mathcal{P})), \overline{d}_{2(n-1)\alpha+2})$. If z^{qp} is one of the representatives of the preceding system,

$$\overline{d}_{2(n-1)\alpha+2}[[z^{qp}]_0 \ldots]_{(n-2)\alpha+1} = [[d'' z^{q-2\alpha(n-1),p+2\alpha(n-1)}]_0 \ldots]_{(n-2)\alpha+1}, \tag{12}$$

where $z^{q-2\alpha(n-1),p+2\alpha(n-1)}$ is the last entry of an arbitrary solution of $S(z^{qp}; \alpha(n-1)+1)$.

The d''-coboundary of any z^{qp} of type 2 vanishes. This is obvious if z^{qp} is a 3-cochain or has the form $\mathrm{Cas}(\Lambda_{4,I})Y_{12}$ (as $d'' = [\Lambda_{4,II}, .] = [D^{-1}z^3Y_{12}, .]$). If z^{qp} is a 2-cochain with singularities, e.g. $D^{-1}p(x)Y_{23}$, where $p(x)$ is a polynomial in x, we get $d''z^{qp} = [D^{-1}z^3Y_{12}, D^{-1}p(x)Y_{23}] = -z^3p(x)D^{-2}Y_{123} + z^3p(x)D^{-2}Y_{123} = 0$. Hence, for any type 2 representative z^{qp}, system $S(z^{qp}; s)$ admits solution $(z^{qp}, 0, \ldots, 0)$, for any $s \in \mathbb{N}^*$ ($\mathbf{S_1}$, representative extended by 0 [reference needed in the following]), and Coboundary (12) vanishes.

Let now z^{qp} be a representative Z^{q_{ic},p_i} of the first type. We know from Lemma 1 that $Z^{q_{ic}-2k,p_i+2k}$, $k \in \{0, 1, \ldots, \alpha i\}$, with coefficients that verify (C_0)-(C_2), is a solution of $S(Z^{q_{ic},p_i}; \alpha i + 1)$.

1. For any admissible $i \in \{n, n+1, \ldots\}$, this solution can be truncated to a solution of $S(Z^{q_{ic},p_i}; \alpha n + 1)$ ($\mathbf{S_2}$, truncated standard solution). Hence, Coboundary (12) vanishes.

2. If i is admissible in $\{0, 1, \ldots, n-2\}$, we have

$$B_{i0} + C_{i0} = 2D_{i0} \quad \text{and} \quad E_{i0} = F_{i0}.$$

   It then follows from (C_1) and (C_2) that the same relation holds for $k = \alpha i$, i.e. that $B_{i,\alpha i} + C_{i,\alpha i} = 2D_{i,\alpha i}$ and $E_{i,\alpha i} = F_{i,\alpha i}$. This however implies that $d''Z^{q_{ic}-2\alpha i,p_i+2\alpha i} = 0$, so that system $S(Z^{q_{ic},p_i}; \alpha n + 1)$ admits an obvious solution ($\mathbf{S_3}$, standard solution extended by 0) and that Coboundary (12) vanishes again.

3. If $i = n-1$ is admissible,

$$\overline{d}_{2(n-1)\alpha+2}[[Z^{q_{n-1,c},p_{n-1}}]_0 \ldots]_{(n-2)\alpha+1} = [[d''Z^{q_{n-1,c}-2\alpha(n-1),p_{n-1}+2\alpha(n-1)}]_0 \ldots]_{(n-2)\alpha+1}.$$

   In view of Lemma 1, this class vanishes for $c = 0$, and coincides, if $c = 1$ (resp. $c = 2$), up to a coefficient, with class $[[z^{(n-1)(2\alpha+\beta)+2}\partial_{12}]_0 \ldots]_{(n-2)\alpha+1}$ (resp. $[[z^{(n-1)(2\alpha+\beta)+3}\partial_{123}]_0 \ldots]_{(n-2)\alpha+1}$). The above depicted system of representatives of ${}^{(n-2)\alpha+1}H(\ldots {}^0H(\mathcal{P}))$ shows that the preceding two classes do not vanish. Hence, the cocycle-condition is equivalent with the annihilation of the mentioned coefficient, i.e. with

$$B_{n-1,\alpha(n-1)}+C_{n-1,\alpha(n-1)}=2D_{n-1,\alpha(n-1)}\quad(\text{resp.}\quad E_{n-1,\alpha(n-1)}=F_{n-1,\alpha(n-1)}),$$

or, as already explained,

$$B_{n-1,0}+C_{n-1,0}=2D_{n-1,0}\quad(\text{resp.}\quad E_{n-1,0}=F_{n-1,0}). \tag{13}$$

Since it clearly follows from our computations that the space of $\overline{d}_{2(n-1)\alpha+2}$-coboundaries is generated by the two just encountered non-vanishing classes, cohomology space ${}^{(n-1)\alpha+1}H({}^{(n-2)\alpha+1}H(\ldots{}^{0}H(\mathcal{P})))$ has the same system of representatives than its predecessor ${}^{(n-2)\alpha+1}H(\ldots{}^{0}H(\mathcal{P}))$, but with exclusions carried out and conditions on $B_{i0},C_{i0},D_{i0},E_{i0},F_{i0}$ valid for all admissible $i\in\{0,1,\ldots,n-1\}$.

It now suffices to apply Corollary 1 to cohomology space ${}^{(n-1)\alpha+1}H({}^{(n-2)\alpha+1}H(\ldots{}^{0}H(\mathcal{P})))$. Observe first that $(\mathbf{S_1})$-$(\mathbf{S_3})$ entail existence of a solution of $S(z^{qp};\alpha\,n+1)$, for all representatives z^{qp} dissimilar from $Z^{q_{n-1,c},p_{n-1}}$. But, as the coefficients of these last representatives—viewed as representatives of the preceding $\overline{d}_{2(n-1)\alpha+2}$-cohomology space—satisfy Conditions (13), the coboundaries $d''Z^{q_{n-1,c}-2\alpha(n-1),p_{n-1}+2\alpha(n-1)}$ vanish. So the previously met solution of $S(Z^{q_{n-1,c},p_{n-1}};\alpha\,(n-1)+1)$ can be indefinitely extended by 0 ($\mathbf{S_4}$, standard solution extended by 0). Finally, Corollary 1 is applicable for $s=\alpha\,n+1$.

Hence, spaces $E_{2(n-1)\alpha+4}=\ldots=E_{2n\alpha+2}$ coincide and we build, from the known system z^{qp} of representatives of ${}^{(n-1)\alpha+1}H(\ldots{}^{0}H(\mathcal{P}))$, a system of $E_{2n\alpha+2}$ by just summing-up the entries of any solutions of the systems $S(z^{qp};\alpha\,n+1)$. For any Z^{q_{ic},p_i}, the coefficients of which verify

$$B_{i0}+C_{i0}=2D_{i0}\quad(c=1)\quad\text{and}\quad E_{i0}=F_{i0}\quad(c=2),$$

the standard $Z^{q_{ic}-2k,p_i+2k}$, $k\in\{0,1,\ldots,\alpha\,i\}$, are solution, see Lemma 1, of $S(Z^{q_{ic},p_i};\alpha\,i+1)$, e.g. if we choose

$$A_{ik}=A_{i0}\ (c=0),\quad B_{ik}=B_{i0},C_{ik}=C_{i0},D_{ik}=D_{i0}\ (c=1),\quad\text{and}\quad E_{ik}=F_{ik}=0\ (c=2,k\neq0). \tag{14}$$

If we pull the concrete side representatives back to theoretical side representatives using these solutions, we exactly get, see S_1-S_4, the sought-after system. ∎

Remark.

1. We already observed previously the obvious fact that when pulling RHS-representatives back, using different solutions of the standard system, we obtain equivalent LHS-representatives. These equivalent LHS-representatives would implement cohomologous cocycles in cohomology space $H(\Lambda_4)$. Choice (14) will induce in cohomology the most basic possible cocycles.

2. Note also that in view of Theorem 1 and our conventions on coefficients B_{10},C_{10}, cocycle $\mathbb{R}\mathcal{A}_\alpha Y_3$ disappears from all spaces E_{2r}, $r\geq 2\alpha+4$.

4.5 Limit of the SpecSeq and reconstruction of the cohomology

The limit of the SpecSeq can be guessed from Theorem 1. However, we already stressed the importance of a careful reading of all results through the isomorphisms involved in the theory of spectral sequences. The proof of Theorem 1 shows for instance that the appropriate Casimir functions appear, when we pull the RHS-representatives back to the LHS, i.e. read them through isomorphism $I^{-1}_{2(n-1)\alpha+3}$. Hence, a precise description of the isomorphisms that lead now to the cohomology of Λ_4 is essential.

4.5.1 General results

Let us consider the SpecSeq associated with a (regular) filtered graded differential space (K, d, K_p, K^n) and recall that the limit spaces $E_\infty^{pq}, Z_\infty^{pq}, B_\infty^{pq}$ are defined exactly as spaces $E_r^{pq}, Z_r^{pq}, B_r^{pq}$, see Section 3, so that Z_∞^{pq} and B_∞^{pq} are the spaces of cocycles and coboundaries in $K_p \cap K^{p+q}$ respectively. For any fixed p and q, regularity implies that the target space of the restriction of d_r to E_r^{pq} vanishes, if $r > q+1$. Thus, there is a canonical linear surjective map $\vartheta_r^{pq} : E_r^{pq} \to H^{pq}(E_r, d_r) \to E_{r+1}^{pq}$. For $s \geq r > q+1$, we define $\theta_{rs}^{pq} := \vartheta_{s-1}^{pq} \circ \ldots \circ \vartheta_r^{pq} : E_r^{pq} \to E_s^{pq}$, and for $r > q+1$, we set

$$\theta_r^{pq} : E_r^{pq} \ni [\mathfrak{z}_r^{pq}]_{E_r^{pq}} \to [\mathfrak{z}_r^{pq}]_{E_\infty^{pq}} \in E_\infty^{pq}. \tag{15}$$

Due to regularity, the first two of the well-known inclusions $Z_\infty^{pq} \subset Z_r^{pq}, Z_\infty^{p+1,q-1} \subset Z_{r-1}^{p+1,q-1}$, and $B_{r-1}^{pq} \subset B_\infty^{pq}$ are actually double inclusions, and $Z_\infty^{p+1,q-1} + B_{r-1}^{pq} \subset Z_\infty^{p+1,q-1} + B_\infty^{pq} \subset Z_\infty^{pq}$. Hence, map θ_r^{pq} is canonical, linear and surjective. It is known that space E_∞^{pq} together with the preceding linear surjections θ_r^{pq} is a model of the inductive limit of the inductive system $(E_r^{pq}, \theta_{rs}^{pq})$. Consider now a first quadrant SpecSeq (i.e. $p, q \in \mathbb{N}$) and assume that $K_0 = K$. For any p, q, the SpecSeq collapses at

$$r > \sup(p, q+1),$$

more precisely, $E_r^{pq} = E_\infty^{pq}$ and $\theta_r^{pq} = \mathrm{id}$. Indeed, in this case, in addition to the aforementioned double inclusions ($r > q+1$), we now have also $B_{r-1}^{pq} = K_p \cap dK_{p+1-r} \cap K^{p+q} = K_p \cap dK_0 \cap K^{p+q} = B_\infty^{pq}$ ($r > p$). Hence the announced results.

The SpecSeq associated with any filtered graded differential space is convergent in the sense that limit E_∞^{pq} is known to be isomorphic as a vector space with term G^{pq} of the bigraded space $G(H(K))$, G for short, associated with the filtered graded space $H(K)$. Let us recall that the filtration of $H(K)$ is induced by that of K. More precisely, injection $i : (K_p, d) \to (K, d)$ is a morphism of differential spaces and $H_p := i_\sharp H(K_p) \subset H(K)$ is the mentioned filtration of $H(K)$. In order to reduce notations, we denote the terms of the grading of $H(K)$ simply by H^n. It is a fact that the filtration and the grading of $H(K)$ are compatible and that filtration H_p is regular if its generatrix K_p is. Hence, $H_p = \oplus_{q\in\mathbb{N}} H_p \cap H^{p+q} =: \oplus_{q\in\mathbb{N}} H_p^{p+q}$. Finally, it is a matter of knowledge that the isomorphism, say ι, between $G^{pq} := H_p^{p+q}/H_{p+1}^{p+q}$ and E_∞^{pq} is canonical,

$$\iota : E_\infty^{pq} \ni [\mathfrak{z}_\infty^{pq}]_{E_\infty^{pq}} \to [[\mathfrak{z}_\infty^{pq}]_{H_p^{p+q}}]_{G^{pq}} \in G^{pq}. \tag{16}$$

We now reconstruct $H(K)$ from G. Let us again focus on a first quadrant SpecSeq associated with a (regular) filtered complex (K, d, K_p, K^n) (such that $K_0 = K$). For any $n \in \mathbb{N}$, we denote by $G^{n-j_1,j_1}, G^{n-j_2,j_2}, \ldots, G^{n-j_{k_n},j_{k_n}}$, $n \geq j_1 > j_2 > \ldots > j_{k_n} \geq 0$, the non vanishing $G^{pq} = H_p^{p+q}/H_{p+1}^{p+q}$, $p+q = n$. Since $H_0 = H(K)$ and $H_p^n = H_p \cap H^n = 0, \forall p > n$, it follows that

$$\begin{aligned} H^n = H_0^n = \ldots = H_{n-j_1}^n \supset H_{n-j_1+1}^n = \ldots = H_{n-j_2}^n \\ \supset H_{n-j_2+1}^n \ldots H_{n-j_{k_n}}^n \supset H_{n-j_{k_n}+1}^n = \ldots = H_n^n = 0. \end{aligned}$$

Hence,

$$H^n/H_{n-j_2}^n = G^{n-j_1,j_1}, \ldots, H_{n-j_{k_n-1}}^n/H_{n-j_{k_n}}^n = G^{n-j_{k_n-1},j_{k_n-1}}, H_{n-j_{k_n}}^n = G^{n-j_{k_n},j_{k_n}}.$$

However, if $B/A = C$, A a vector subspace of B, the sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$, is a short exact sequence of vector spaces. A short exact sequence in a category is split if and only if kernel A admits in vector space B a complementary subspace that is a subobject, or—alternatively—if and only if there is a right inverse morphism $\chi : C \to B$ of projection p. Of course, in the category of vector spaces such a sequence is always split. If χ is a linear right inverse of p, we have $B = A \oplus \chi(C)$.

Let us now come back to our circumstances. If $\chi_1, \ldots, \chi_{k_n-1}$ denote splittings of the involved sequences, central extension H^n is given by

$$H^n = \chi_1(G^{n-j_1,j_1}) \oplus \ldots \oplus \chi_{k_n-1}(G^{n-j_{k_n-1},j_{k_n-1}}) \oplus G^{n-j_{k_n},j_{k_n}}. \tag{17}$$

It follows of course from Equation (17) that $H(K)$ is—in this vector space setting—isomorphic with $G = G(H(K))$. It is known that in the case of ring coefficients, extension problems may prevent the reconstruction of $H(K)$ from $G(H(K))$.

4.5.2 Application to Poisson tensor Λ_4

The next proposition provides a system of representatives of the cohomology space of

$$\Lambda_4 = ayz\partial_{23} + axz\partial_{31} + (bxy + z^2)\partial_{12} \quad (a \neq 0, b \neq 0).$$

Remember that $D' = xy$ and $Y_1 = x\partial_1, Y_2 = y\partial_2, Y_3 = z\partial_3$. If $\frac{b}{a} \sim \frac{\beta}{\alpha} \in \mathbb{Q}^*_+$, we define

$$\mathrm{Cas}(\Lambda_4) := \oplus_{i\in\mathbb{N}}\mathbb{R}\left(D' + \frac{z^2}{2a+b}\right)^{\alpha\, i} z^{\beta\, i}$$

and use the above introduced notation $\mathrm{Cas}(\Lambda_{4,I}) = \oplus_{i\in\mathbb{N}}\mathbb{R}D'^{\alpha\, i}z^{\beta\, i}$. If $\frac{b}{a} \in \mathbb{R}^*\backslash\mathbb{Q}^*_+$, we set $\mathrm{Cas}(\Lambda_4) := \mathbb{R}$ and, as aforementioned, $\mathcal{A}_\alpha = D'^{\alpha}z^{-1}$.

Theorem 2. *1. If $\frac{b}{a} \in \mathbb{Q}^*_+$, the cohomology of Λ_4 is given by*

$$\begin{aligned} E_\infty \sim G \sim H(\Lambda_4) \quad = \quad & \mathrm{Cas}(\Lambda_4) \oplus \mathrm{Cas}(\Lambda_4)(Y_1 + \frac{1}{2}Y_3) \oplus \mathrm{Cas}(\Lambda_4)(Y_2 + \frac{1}{2}Y_3) \\ & \oplus \mathrm{Cas}(\Lambda_{4,I})(Y_{23} + Y_{31}) \oplus \mathrm{Cas}(\Lambda_{4,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{4,I})Y_{123} \\ & \oplus \bigoplus_{k\in\mathbb{N}\backslash\mathbb{N}\,(2\alpha+\beta)+2} \mathbb{R}z^k\partial_{12} \oplus \bigoplus_{k\in\mathbb{N}\backslash\mathbb{N}\,(2\alpha+\beta)+3} \mathbb{R}z^k\partial_{123} \\ & \oplus \begin{cases} \mathbb{R}[[x]]\partial_{23} \oplus \mathbb{R}[[y]]\partial_{31} \oplus (\mathbb{R}[[x]] \oplus \mathbb{R}[[y]])\partial_{123}, \textit{ if } b = a \\ 0, \textit{ otherwise} \end{cases} \end{aligned}$$

2. If $\frac{b}{a} \in \mathbb{R}^* \setminus \mathbb{Q}^*_+$, we have

$$\begin{aligned} E_\infty \sim G \sim H(\Lambda_4) \quad = \quad & \mathrm{Cas}(\Lambda_4) \oplus \mathrm{Cas}(\Lambda_4)(Y_1 + \frac{1}{2}Y_3) \oplus \mathrm{Cas}(\Lambda_4)(Y_2 + \frac{1}{2}Y_3) \\ & \oplus \mathrm{Cas}(\Lambda_4)(Y_{23} + Y_{31}) \oplus \mathrm{Cas}(\Lambda_4)Y_{12} \oplus \mathrm{Cas}(\Lambda_4)Y_{123} \\ & \oplus \begin{cases} \oplus\mathbb{R}\mathcal{A}_\alpha(Y_{23} + Y_{31}) \oplus \mathbb{R}\mathcal{A}_\alpha Y_{123}, \ \textit{if } (b,a) \sim (-1,\alpha) \\ 0, \textit{ otherwise} \end{cases} \\ & \oplus \begin{cases} \bigoplus_{k\in\mathbb{N}\backslash\{2,2\alpha+1\}} \mathbb{R}z^k\partial_{12} \oplus \bigoplus_{k\in\mathbb{N}\backslash\{3,2\alpha+2\}} \mathbb{R}z^k\partial_{123}, \ \textit{if } (b,a) \sim (-1,\alpha) \\ \bigoplus_{k\in\mathbb{N}\backslash\{2\}} \mathbb{R}z^k\partial_{12} \oplus \bigoplus_{k\in\mathbb{N}\backslash\{3\}} \mathbb{R}z^k\partial_{123}, \textit{ otherwise} \end{cases} \end{aligned}$$

Proof. Fix $a, b \in \mathbb{R}^*$ and take any representative of E_2. Remember that the representatives of type 1 are exactly the cochains Z^{q_{ic},p_i} (i admissible, $c \in \{0,1,2\}$). Moreover, we say that a representative of type 2 is critical if it has the form $\mathbb{R}z^{i(2\alpha+\beta)+2}\partial_{12}$ or $\mathbb{R}z^{i(2\alpha+\beta)+3}\partial_{123}$ (i admissible). If the considered representative z^{qp} is of type 2 and not critical (resp. of type 2 and critical, of type 1), we choose $n \in \mathbb{N}$ such that $2n\alpha + 2 > \sup(p, q+1)$ (resp. $2n\alpha + 2 > \sup(p, q+1, 2i\alpha + 2)$ [hence, we have $n - 1 \geq i$], $2n\alpha + 2 > \sup(p_i, q_{ic} + 1, 2i\alpha + 2)$). The system of representatives of $E_\infty \sim G$ specified in Theorem 2 arises now from Theorem 1 and from the canonical isomorphisms (15) (condition: $r > \sup(p, q+1)$) and (16). These representatives are representatives of bases of the non vanishing G^{pq}. In order to compute $H(K)$, it suffices to build arbitrary splittings in keeping with Equation (17). Hence, it suffices to choose, for any class of any basis of the concerned G^{pq}, an arbitrary representative, e.g. the aforementioned one. It follows that $H(K)$ admits exactly the same representatives as $E_\infty \sim G$. ∎

Remarks. Hence, the twist makes a threefold impact on cohomology. When applying our computing device, see Theorem 1, we get, at *each* turn of the handle, on the model level, roughly speaking, cocycle-conditions on the coefficients related with an *additional* power of the basic Casimir $C_{\Lambda_{4,I}}$ of $\Lambda_{4,I}$, and we exclude a *supplementary* pair of singularity-induced classes. These conditions appear in cohomology as terms Y_i or Y_{ij} with the same "Casimir-coefficient". Eventually, the cocycle-conditions allow to lift the mentioned *accessory* power of $C_{\Lambda_{4,I}}$ to the real level as power of Casimir C_{Λ_4} of Λ_4 or—depending on cochain degree—as power of Casimir $C_{\Lambda_{4,I}}$. We know that such a lift is not unique and that two different ones are cohomologous. It follows from Theorem 2 (resp. from the proof of Theorem 1) that any term of $\mathrm{Cas}(\Lambda_4)(Y_{23}+Y_{31})$ is a Λ_4-cocycle (resp. can be chosen as lift of the corresponding term in $\mathrm{Cas}(\Lambda_{4,I})(Y_{23}+Y_{31})$, as well as this term itself). So any term of $\mathrm{Cas}(\Lambda_4)(Y_{23}+Y_{31})$ is cohomologous to the analogous term in $\mathrm{Cas}(\Lambda_{4,I})(Y_{23}+Y_{31})$. Finally, the aforementioned proof allows to see that $\Lambda_{4,I}$-cocycle $\mathbb{R}A_\alpha Y_3 = \mathbb{R}D'^\alpha z^{-1}Y_3$, which is not a product of two Λ_4-cocycles, is a Λ_4-cocycle if and only if its coefficient vanishes.

Let us in the end have a look at singularities. The singular locus of $\Lambda_{4,I}$ (resp. Λ_4) is made up by the three coordinate axes (resp. the axis of abscissæ and the axis of ordinates). Comparing the results of Proposition 2 and of Theorem 2, we see that the twist $\Lambda_{4,II}$, which removes the z-axis from the singular locus, cancels only part of the corresponding polynomials in cohomology. We already observed in [MP06] that, for r-matrix induced tensors, some coefficients of non bounding 2- or 3-cocycles can just be interpreted as polynomials on singularities via an extension of the polynomial ring of the singular locus. In the case of twisted r-matrix induced structures, some of these polynomial coefficients are simply not polynomials on singularities.

5 Formal cohomology of Poisson tensor Λ_8

We now describe the cohomology space of the twisted quadratic Poisson structure

$$\Lambda_8 = \left(\frac{\mathfrak{a}+\mathfrak{b}}{2}(x^2+y^2) \pm z^2\right)\partial_{12} + \mathfrak{a}xz\partial_{23} + \mathfrak{a}yz\partial_{31} \quad (\mathfrak{a}\neq 0, \mathfrak{b}\neq 0).$$

If we substitute c (resp. b) for $-\mathfrak{b}$ (resp. $(\mathfrak{a}+\mathfrak{b})/2$), tensor Λ_8 reads

$$\Lambda_8 = b(x^2+y^2)\partial_{12} + (2b+c)xz\partial_{23} + (2b+c)yz\partial_{31} \pm z^2\partial_{12}. \tag{18}$$

Henceforth we use parameters b and c. Assumptions $\mathfrak{a}\neq 0, \mathfrak{b}\neq 0$ are equivalent with $2b+c\neq 0, c\neq 0$. Moreover, the r-matrix induced part $\Lambda_{8,I} = b(x^2+y^2)\partial_{12} + (2b+c)xz\partial_{23} + (2b+c)yz\partial_{31}$ of Λ_8 is nothing but structure Λ_7 with parameter $a=0$, see [MP06, Section 9], so that term $E_2 \simeq H(\Lambda_{8,I})$ of the spectral sequence follows from [MP06, Theorems 6,8,9].

Let us recall that the Y_i stem from $\Lambda_{8,I}$, i.e. from Λ_7. Hence, $Y_1 = x\partial_1 + y\partial_2, Y_2 = x\partial_2 - y\partial_1, Y_3 = z\partial_3$. We set $D' = x^2+y^2$. Moreover, if $\frac{b}{c}\in\mathbb{Q}, b(2b+c) < 0$, we denote by $(\beta,\gamma)\sim(b,c)$ the irreducible representative of the rational number $\frac{b}{c}$, with positive denominator, $\beta\in\mathbb{Z}, \gamma\in\mathbb{N}^*$, and if $\frac{b}{c}\in\mathbb{Q}, b(2b+c) > 0$, $(\beta,\gamma)\sim(b,c)$ denotes the irreducible representative with positive numerator, $\beta\in\mathbb{N}^*, \gamma\in\mathbb{Z}^*$.

Theorem 3. *The terms of the cohomology space of Λ_8 (see (18)) are given by the following equations:*

1. *If $\frac{b}{c}\in\mathbb{Q}, b(2b+c) > 0$,*

$$\begin{aligned}
H^0(\Lambda_8) &= \mathrm{Cas}(\Lambda_8) = \oplus_{i\in\mathbb{N},\gamma i\in 2\mathbb{Z}}\mathbb{R}\left(D' \pm \frac{z^2}{3b+c}\right)^{(\beta+\frac{\gamma}{2})i} z^{\beta\, i}\,,\\
H^1(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3)\,,\\
H^2(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{8,I})Y_{23} \oplus \bigoplus_{k\in\mathbb{N}\setminus\mathbb{N}(3\beta+\gamma)+2}\mathbb{R}z^k\partial_{12}\,,\\
H^3(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{123} \oplus \bigoplus_{k\in\mathbb{N}\setminus\mathbb{N}(3\beta+\gamma)+3}\mathbb{R}z^k\partial_{123}\,,
\end{aligned}$$

where $\mathrm{Cas}(\Lambda_{8,I}) = \oplus_{i\in\mathbb{N},\gamma i\in 2\mathbb{Z}}\mathbb{R}D'^{(\beta+\frac{\gamma}{2})i}z^{\beta i}$.

2. *If* $\frac{b}{c} \notin \mathbb{Q}$ *or* $\frac{b}{c} \in \mathbb{Q}, b(2b+c) < 0$,

$$\begin{aligned}
H^0(\Lambda_8) &= \mathrm{Cas}(\Lambda_8) = \mathbb{R}\,,\\
H^1(\Lambda_8) &= \mathrm{Cas}(\Lambda_8)Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3)\,,\\
H^2(\Lambda_8) &= \mathrm{Cas}(\Lambda_8)Y_{12} \oplus \mathrm{Cas}(\Lambda_8)Y_{23}\\
&\oplus \begin{cases} \bigoplus_{k\in\mathbb{N}\setminus\{2,\gamma-1\}} \mathbb{R}z^k\partial_{12}, \textit{if } (b,c)\sim(-1,\gamma),\ \gamma\in\{4,6,8,...\}\\ \bigoplus_{k\in\mathbb{N}\setminus\{2\}} \mathbb{R}z^k\partial_{12}, \textit{otherwise}\end{cases}\\
&\oplus \begin{cases} \mathbb{R}\mathcal{A}_\gamma Y_{23}, \textit{if } (b,c)\sim(-1,\gamma), \gamma\in\{4,6,8,...\}\\ 0, \textit{otherwise}\ ,\end{cases}\\
H^3(\Lambda_8) &= \mathrm{Cas}(\Lambda_8)Y_{123}\\
&\oplus \begin{cases} \bigoplus_{k\in\mathbb{N}\setminus\{3,\gamma\}} \mathbb{R}z^k\partial_{123}, \textit{if } (b,c)\sim(-1,\gamma), \gamma\in\{4,6,8,...\}\\ \bigoplus_{k\in\mathbb{N}\setminus\{3\}} \mathbb{R}z^k\partial_{123}, \textit{otherwise}\end{cases}\\
&\oplus \begin{cases} \mathbb{R}\mathcal{A}_\gamma Y_{123}, \textit{if } (b,c)\sim(-1,\gamma), \gamma\in\{4,6,8,....\}\\ 0, \textit{otherwise}\ ,\end{cases}
\end{aligned}$$

where $\mathcal{A}_\gamma = D'^{\frac{\gamma}{2}-1}z^{-1}$.

3. *If* $b = 0$,

$$\begin{aligned}
H^0(\Lambda_8) &= \mathrm{Cas}(\Lambda_8) = \oplus_{i\in\mathbb{N}}\mathbb{R}\left(D' \pm \frac{z^2}{c}\right)^i\,,\\
H^1(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_2 \oplus \mathrm{Cas}(\Lambda_8)(Y_1+Y_3)\,,\\
H^2(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{12} \oplus \mathrm{Cas}(\Lambda_{8,I})Y_{23} \oplus \bigoplus_{k\in\mathbb{N}\setminus\{2\mathbb{N}+2\}} \mathbb{R}z^k\partial_{12}\,,\\
H^3(\Lambda_8) &= \mathrm{Cas}(\Lambda_{8,I})Y_{123} \oplus \bigoplus_{k\in\mathbb{N}\setminus\{2\mathbb{N}+3\}} \mathbb{R}z^k\partial_{123}\,,
\end{aligned}$$

where $\mathrm{Cas}(\Lambda_{8,I}) = \oplus_{i\in\mathbb{N}}\mathbb{R}D'^i$.

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