On a general approach to the formal cohomology of quadratic Poisson structures*

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Abstract

We propose a general approach to the formal Poisson cohomology of r-matrix induced quadratic structures, we apply this device to compute the cohomology of structure 2 of the Dufour-Haraki classification, and provide complete results also for the cohomology of structure 7.

Key-words: Poisson cohomology, formal cochain, quadratic Poisson tensor, r-matrix
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1 Introduction

The Poisson cohomology was defined by Lichnerowicz in [Lic77]. It plays an important role in Poisson Geometry, gives several information on the geometry of the manifold, is closely related to the classification of singularities of Poisson structures, naturally appears in infinitesimal deformations of Poisson tensors [second cohomology group (non-trivial infinitesimal deformations), third cohomology group (obstructions to extending a first order deformation to a formal deformation)], and is exploited in the study and the classification of star-products.

The case of regular Poisson manifolds is discussed in [Xu92] and [Vai94], some upshots regarding the Poisson cohomology of Poisson-Lie groups can be found in [GW92]. Many results have been proven only in two dimensions. Nakanishi [Nak97] has computed, using an earlier idea of Vaisman, the cohomology for plane quadratic structures. But also more recent papers by Monnier [Mon01] and Roger and Vanhaecke [RV02] are confined to dimension 2. Ginzburg [Gin99] has studied a spectral sequence, Poisson analog of the Leray spectral sequence of a fibration, which converges to the Poisson cohomology of the manifold. Related cohomologies, the Nambu-Poisson cohomology (which generalizes in a certain sense the Poisson cohomology in dimension 2) [Mon02,1] or the tangential Poisson cohomology (which governs tangential star products and is involved in the Poisson cohomology) [Gam02] have (partially) been computed. Roytenberg has computed the cohomology on the 2-sphere for special covariant structures [Roy02] and Pichereau has taken an interest in Poisson (co)homology and isolated singularities [Pic05].

In Deformation Quantization we are interested in the formal Poisson cohomology, where "formal" means that cochains are multi-vector fields with coefficients in formal series. Let us emphasize that the formal Poisson cohomology associated to a Poisson manifold (M, P), where the Poisson tensor P gives Kontsevich's star-product $*_K$, is linked with the Hochschild cohomology of the associative algebra $(C^\infty(M)[[\hbar]], *_K)$ of formal series in $\hbar$ with coefficients in $C^\infty(M)$.

In [Mon02,2], Monnier has computed the formal cohomology of diagonal Poisson structures.

The aim of this work is to provide a general approach to the formal Poisson cohomology of a broad set of isomorphism classes of quadratic structures and to illustrate this modus operandi through its

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application to two of the most demanding classes of the Dufour-Haraki classification (DHC, see [DH91]).

Our procedure applies to the quadratic Poisson tensors Λ that read as linear combinations of wedge products of mutually commuting linear vector fields. In the three-dimensional Euclidean setting this means that

$$\Lambda = aY_2 \wedge Y_3 + bY_3 \wedge Y_1 + cY_1 \wedge Y_2 \quad a, b, c \in \mathbf{R},$$

where the vector fields Y_1, Y_2, Y_3 have linear coefficients and meet the condition $[Y_i, Y_j] = 0$. Let us stress that most of the structures of the DHC have this *admissible* form. The advantage of the just defined family of admissible tensors is readily understood. If we substitute the Y_i for the standard basic vector fields $\partial_i = \partial/\partial_{x_i}$, the cochains assume—broadly speaking—the shape $\sum f\mathbf{Y}$, where f is a function and $\mathbf{Y}$ is a wedge product of basic fields Y_i. Then the Lichnerowicz-Poisson (LP) coboundary operator $[\Lambda, \cdot]$ is just

$$[\Lambda, f\mathbf{Y}] = [\Lambda, f] \wedge \mathbf{Y}.$$

More precisely,

$$[\Lambda, f] = \sum_i X_i(f)Y_i,$$

where $X_1 = aY_2 - cY_3, X_2 = bY_3 - aY_1, X_3 = cY_1 - bY_2$. So the coboundary of a 0-cochain is a kind of gradient, and, as easily checked, the coboundaries of a 1- and a 2-cochain, decomposed in the Y_i-basis are a sort of curl and of divergence respectively. As in the above "gradient" the operators X_i act in these "curl" and "divergence" as substitutes for the usual partial derivatives. The preceding simplification of the Lichnerowicz-Poisson coboundary operator is of course not restricted to the three-dimensional context.

Let us emphasize that, if cochains are decomposed in the new Y_i-induced bases, their coefficients are rational with fixed denominator. Hence a natural injection of the *real cochain space* $\mathcal{R}$ in a larger *potential cochain space* $\mathcal{P}$. As a matter of fact, such a potential cochain is implemented by a real cochain if and only if specific divisibility conditions are satisfied. This observation directly leads to a *supplementary cochain space* $\mathcal{S}$ of $\mathcal{R}$ in $\mathcal{P}$. It is possible to heave space $\mathcal{S}$ into the category of differential spaces. Since $\mathcal{P}$ is as $\mathcal{R}$ a complex for the LP coboundary operator, we end up with a short exact sequence of differential spaces and an exact triangle in cohomology. It turns out that $\mathcal{S}$-cohomology and $\mathcal{P}$-cohomology are less intricate than $\mathcal{R}$-cohomology, but are important stages on the way to $\mathcal{R}$-cohomology, i.e. to the cohomology of the considered admissible quadratic Poisson structure.

In this work we provide explicit results—obtained by the just depicted method—for the cohomologies of class 2 and class 7 of the DHC. Observe that the representative

$$\Lambda_7 = b(x_1^2 + x_2^2)\partial_1 \wedge \partial_2 + ((2b + c)x_1 - ax_2)x_3\partial_2 \wedge \partial_3 + (ax_1 + (2b + c)x_2)x_3\partial_3 \wedge \partial_1$$

of class 7 reduces to the representative of class 2 for parameter value $c = 0$. Actually computations are quite similar in both cases, hence we refrained from publishing those pertaining to case 7.

We now detail the Lichnerowicz-Poisson cohomology of structure class 2. Remark first that

$$\Lambda_2 = 2bY_2 \wedge Y_3 + aY_3 \wedge Y_1 + bY_1 \wedge Y_2,$$

where $Y_1 = x_1\partial_1 + x_2\partial_2$, $Y_2 = x_1\partial_2 - x_2\partial_1$, and $Y_3 = x_3\partial_3$. As case $b = 0$ has been studied in [Mon02,2], we assume that $b \neq 0$. We denote the determinant $(x_1^2 + x_2^2)x_3$ of the basic fields Y_i by D. The results of this article will entail that the abundance of cocycles that do not bound is tightly related with closeness of the considered Poisson tensor to Koszul-exactness. If $a = 0$, structure Λ_2 is exact and induced by bD. The algebra of Casimir functions is generated by 1 if $a \neq 0$ and by D if $a = 0$. When rewording this statement and writing Λ instead of Λ_2, we get

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \begin{cases} \mathbf{R}, & \text{if} \quad a \neq 0, \\ \bigoplus_{m=0}^{\infty} \mathbf{R}D^m, & \text{if} \quad a = 0. \end{cases}$$

Remind now that our Poisson tensor is built with linear infinitesimal Poisson automorphisms Y_i. It follows that the wedge products of the Y_i constitute "a priori" privileged cocycles. Of course, 2-cocycle

Λ_2 itself, is a linear combination of such privileged cocycles. Moreover, the curl or modular vector field reads here $K(\Lambda) = a(2Y_3 - Y_1)$ and is thus also a combination of privileged cocycles. As the LP cohomology is an associative graded commutative algebra, the first cohomology group of Λ_2 is easy to conjecture:

$$H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda) Y_i.$$

It is well-known that the singularities of the investigated Poisson structure appear in the second and third cohomology spaces. Observe that the singular points of structure Λ_2 are the annihilators of $D' = x_1^2 + x_2^2$. Note also that any homogeneous polynomial $P \in \mathbf{R}[x_1, x_2, x_3]$ of order m reads

$$P = \sum_{\ell=0}^{m} x_3^\ell P_\ell(x_1, x_2) = \sum_{\ell=0}^{m} x_3^\ell \left(D' \cdot Q_\ell + A_\ell x_1^{m-\ell} + B_\ell x_1^{m-\ell-1} x_2 \right),$$

with self-explaining notations. Hence, the algebra of "polynomials" of the affine algebraic variety of singularities is

$$\bigoplus_{m=0}^{\infty} \bigoplus_{\ell=0}^{m} x_3^\ell \left(\mathbf{R} x_1^{m-\ell} + \mathbf{R} x_1^{m-\ell-1} x_2 \right).$$

The third cohomology group contains a part of this formal series. More precisely,

$$H^3(\Lambda) = \mathrm{Cas}(\Lambda)\ Y_{123} \oplus \begin{cases} \mathbf{R}\ \partial_{123}, & \text{if} \quad a \neq 0, \\ \bigoplus_{m=0}^{\infty} \mathbf{R}\ x_3^m \partial_{123} \oplus \bigoplus_{m=0}^{\infty} x_1^m \left(\mathbf{R} x_1 + \mathbf{R} x_2 \right) \partial_{123}, & \text{if} \quad a = 0, \end{cases}$$

where Y_{123} (resp. ∂_{123}) means $Y_1 \wedge Y_2 \wedge Y_3$ (resp. $\partial_1 \wedge \partial_2 \wedge \partial_3$). The reader might object that the "mother-structure" Λ is symmetric in x_1, x_2 and that there should therefore exist a symmetric "twin-cocycle" $\bigoplus_{m=0}^{\infty} x_2^m (\mathbf{R} x_1 + \mathbf{R} x_2) \partial_{123}$. This cocycle actually exists, but is—as easily checked—cohomologous to the visible representative. Finally, the second cohomology space reads

$$\begin{aligned} H^2(\Lambda) = \ & \mathrm{Cas}(\Lambda) Y_{23} \oplus \mathrm{Cas}(\Lambda) Y_{31} \oplus \mathrm{Cas}(\Lambda) Y_{12} \\ & \oplus \begin{cases} \ / \,, & \text{if} \quad a \neq 0, \\ \bigoplus_{m=0}^{\infty} \mathbf{R}\ x_3^m \partial_{12} \oplus \bigoplus_{m=0}^{\infty} x_1^{m-1} \left(\mathbf{R} x_1 \partial_{23} + \mathbf{R} (x_1 \partial_{31} + m x_2 \partial_{23}) \right), & \text{if} \quad a = 0. \end{cases} \end{aligned}$$

For $m \geq 1$, the last cocycle has the form

$$\left(\mathbf{R} x_1^m + \mathbf{R} x_1^{m-1} x_2 \right) \partial_{23} + \left(\int \partial_{x_2} \left(\mathbf{R} x_1^m + \mathbf{R} x_1^{m-1} x_2 \right) dx_1 \right) \partial_{31}$$

and is thus also induced by singularities.

Our paper is organized as follows. Sections 2-8 deal with the explicit computation of the cohomology of tensor Λ_2. In Section 9 we completely describe the cohomology of structure Λ_7. Finally, the last section is devoted to explanations regarding the connection of the Poisson structures that are accessible to our method with r-matrix induced Poisson structures. Upshots relating to the admissible quadratic tensors different from Λ_2 and Λ_7 are being published separately.

2 Simplified differential

In this and the following sections, we compute the formal cohomology of the second quadratic Poisson structure of the Dufour-Haraki classification. This structure reads

$$\Lambda = b(x_1^2 + x_2^2) \partial_1 \wedge \partial_2 + x_3(2b x_1 - a x_2) \partial_2 \wedge \partial_3 + x_3(a x_1 + 2b x_2) \partial_3 \wedge \partial_1,$$

where $b \neq 0$ (if $b = 0$, we recover the case studied in [Mon02,2]).

As mentioned above, we get

$$\Lambda = b Y_1 \wedge Y_2 + 2b Y_2 \wedge Y_3 + a Y_3 \wedge Y_1, \tag{1}$$

if we set

$$Y_1 = x_1\partial_1 + x_2\partial_2, Y_2 = x_1\partial_2 - x_2\partial_1, Y_3 = x_3\partial_3. \tag{2}$$

We also know that it is interesting—in order to simplify the coboundary operator—to write the cochains in terms of these fields. Let us recall that in the formal setting, the cochains are the multi-vector fields with coefficients in $\mathbf{R}[[x]]$, the space of formal series in $x = (x_1, x_2, x_3)$ with coefficients in $\mathbf{R}$.

Note first that if $D = (x_1^2 + x_2^2)x_3$ and if $x_{ij} = x_i x_j$,

$$\partial_1 = \frac{1}{D}(x_{13}Y_1 - x_{23}Y_2), \partial_2 = \frac{1}{D}(x_{23}Y_1 + x_{13}Y_2), \partial_3 = \frac{1}{D}(x_1^2 + x_2^2)Y_3.$$

An arbitrary 0-cochain $C^0 = \sum_{I\in\mathbf{N}^3} c_I x^I$ ($I = (i_1, i_2, i_3), c_I \in \mathbf{R}, x^I = x^{i_1,i_2,i_3} = x_1^{i_1}x_2^{i_2}x_3^{i_3}$) can be written

$$C^0 = \sum_{J\in\mathbf{N}^3} \frac{x^J}{D}\alpha_J =: \frac{\sigma}{D} =: \varsigma,$$

with self-explaining notations. Similarly, 1-, 2-, and 3-cochains $C^1 = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3$, $C^2 = \sigma_1\partial_{23} + \sigma_2\partial_{31} + \sigma_3\partial_{12}$, $C^3 = \sigma\partial_{123}$, where $\sigma_i, \sigma \in \mathbf{R}[[x]]$ ($i \in \{1,2,3\}$) and where for instance $\partial_{23} = \partial_2 \wedge \partial_3$, read in terms of the Y_i ($i \in \{1,2,3\}$), $C^1 = \varsigma_1 Y_1 + \varsigma_2 Y_2 + \varsigma_3 Y_3$, $C^2 = \varsigma_1 Y_{23} + \varsigma_2 Y_{31} + \varsigma_3 Y_{12}$, and $C^3 = \varsigma Y_{123}$, again with obvious notations. If $[\cdot,\cdot]$ denotes the Schouten-bracket, the coboundary of C^0 is given by

$$[\Lambda, C^0] = X_1(\varsigma)Y_1 + X_2(\varsigma)Y_2 + X_3(\varsigma)Y_3,$$

where

$$X_1 = bY_2 - aY_3, X_2 = -bY_1 + 2bY_3, X_3 = aY_1 - 2bY_2.$$

Set $\nabla = \sum_i X_i(.)Y_i$. A short computation then shows that

$$[\Lambda, C^0] = \nabla C^0, [\Lambda, C^1] = \nabla \wedge C^1, [\Lambda, C^2] = \nabla.C^2, \text{ and } [\Lambda, C^3] = 0,$$

where the r.h.s. have to be viewed as notations that give the coefficients of the coboundaries in the Y_i-basis. For instance, $[\Lambda, C^2] = (\sum_i X_i(\varsigma_i))Y_{123}$. We easily verifie that

$$X_1\left(\frac{x^J}{D}\right) = [b\left(j_2\frac{x_1}{x_2} - j_1\frac{x_2}{x_1}\right) - a(j_3 - 1)]\frac{x^J}{D},$$

$$X_2(\frac{x^J}{D}) = b[2j_3 - (j_1 + j_2)]\frac{x^J}{D},$$

and

$$X_3(\frac{x^J}{D}) = [a(j_1 + j_2 - 2) - 2b\left(j_2\frac{x_1}{x_2} - j_1\frac{x_2}{x_1}\right)]\frac{x^J}{D}.$$

When writing the quotients ς in the cochains in the form

$$\varsigma = \sum_{r\in\mathbf{N}}\sum_{k=0}^{r}\sum_{\ell=0}^{k}\alpha_{r,k,\ell}\frac{x^{\ell,k-\ell,r-k}}{D},$$

we see that cochains are graded not only with respect to the cochain-degree $d \in \{0,1,2,3\}$, but also with respect to the total degree r in x and the partial degree k. The preceding results regarding $X_i(\frac{x^J}{D})$ allow to see that the coboundary operator is compatible with both degrees, k and r, so that the cohomology can be computed part by part.

3 Fundamental operators

Denote by P_{kr} the space of homogeneous polynomials of partial degree k and total degree r, and by Q_{kr} the space $\frac{1}{D}P_{kr}$. We now study the fundamental operators X_i as endomorphisms of Q_{kr}.

It is easy to verify that

$$X_1\left(\sum_{\ell=0}^k \alpha_\ell \frac{x^{\ell,k-\ell,r-k}}{D}\right) = \sum_{\ell=0}^k [b(k-\ell+1)\alpha_{\ell-1} - a(r-k-1)\alpha_\ell - b(\ell+1)\alpha_{\ell+1}]\frac{x^{\ell,k-\ell,r-k}}{D}, \tag{3}$$

$$X_2 = (2r-3k)\, b\, id_{Q_{kr}}, \tag{4}$$

and

$$X_3\left(\sum_{\ell=0}^k \alpha_\ell \frac{x^{\ell,k-\ell,r-k}}{D}\right) = \sum_{\ell=0}^k [-2b(k-\ell+1)\alpha_{\ell-1} + a(k-2)\alpha_\ell + 2b(\ell+1)\alpha_{\ell+1}]\frac{x^{\ell,k-\ell,r-k}}{D}. \tag{5}$$

Note also that if $D' = x_1^2 + x_2^2$, we have $X_1(D') = 0$, $X_1(x_3) = -ax_3$, $X_1(D) = -aD$, $X_2(D') = -2bD'$, $X_2(x_3) = 2bx_3$, $X_2(D) = 0$, $X_3(D') = 2aD'$, $X_3(x_3) = 0$, $X_3(D) = 2aD$. Hence $X_1(D^\ell) = -a\ell D^\ell$, $X_3(D^\ell) = 2a\ell D^\ell$, for all $\ell \in \mathbf{Z}$. In particular, $D^{\frac{k}{2}-1} \in Q_{k,\frac{3}{2}k}$ is an eigenvector of X_1 and X_3 associated to the eigenvalue $\frac{1}{2}a(2-k)$ and $a(k-2)$ respectively, for all $k \in 2\mathbf{N}$.

Remark If $\mathbf{Y}$ is α, $\alpha Y_1 + \beta Y_2 + \gamma Y_3$, $\alpha Y_{23} + \beta Y_{31} + \gamma Y_{12}$, or αY_{123} $(\alpha, \beta, \gamma \in \mathbf{R})$, the cochains $D^\ell \mathbf{Y}$ are cocycles for all ℓ if $a = 0$ and for $\ell = 0$ otherwise. Indeed, the coboundary of these cochains vanishes if $[\Lambda, D^\ell] = \nabla D^\ell$ does.

In order to compute the spectrum of the endomorphisms X_1 and X_3 of Q_{kr}, note that their matrix in the canonical basis of Q_{kr} is

$$M_0 = \begin{pmatrix} A & B & 0 & \dots & 0 & 0 & 0 \\ -kB & A & 2B & \dots & 0 & 0 & 0 \\ 0 & -(k-1)B & A & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & -2B & A & kB \\ 0 & 0 & 0 & \dots & 0 & -B & A \end{pmatrix},$$

where $(A, B) = (a(k-r+1), -b)$ and $(A, B) = (a(k-2), 2b)$ respectively. A straightforward induction shows that for odd k the determinant of $M_0 - \lambda I$ is

$$((A-\lambda)^2 + B^2)((A-\lambda)^2 + (3B)^2)\dots((A-\lambda)^2 + (kB)^2),$$

whereas for even k its value is

$$(A-\lambda)((A-\lambda)^2 + (2B)^2)((A-\lambda)^2 + (4B)^2)\dots((A-\lambda)^2 + (kB)^2).$$

We thus have the

Proposition 1. *(i) For any $k \in 2\mathbf{N}+1$, the operator X_1 (resp. X_3) has no eigenvalue.*
(ii) For any $k \in 2\mathbf{N}$, the unique eigenvalue of X_1 (resp. X_3) is

$$\lambda = a(k-r+1) \ \ (\textit{resp. } a(k-2)).$$

The vector

$$\frac{1}{D}(x_1^2 + x_2^2)^{\frac{k}{2}} x_3^{r-k} \in Q_{kr}$$

is a basis of eigenvectors.

Note that this result is an extension of the above remark regarding eigenvectors of X_1 and X_3 in the space $Q_{k,\frac{3}{2}k}$, i.e. for $2r - 3k = 0$.

Corollary 1. *(i) If $k \in 2\mathbf{N}+1$ and if $k \in 2\mathbf{N}, a \neq 0, k \neq r-1$ (resp. $k \in 2\mathbf{N}, a \neq 0, k \neq 2$), the operator X_1 (resp. X_3) is invertible.*
(ii) If $k \in 2\mathbf{N}$ and $a = 0$ or $k = r-1$ (resp. $k \in 2\mathbf{N}$ and $a = 0$ or $k = 2$), the operator X_1 (resp. X_3) is degenerated and the eigenvector

$$\frac{1}{D}(x_1^2 + x_2^2)^{\frac{k}{2}} x_3^{r-k} \in Q_{kr}$$

is a basis of the kernel $kerX_1$ (resp. $kerX_3$). Moreover,

$$kerX_1 \oplus imX_1 = Q_{kr} \text{ (resp. } kerX_3 \oplus imX_3 = Q_{kr}).$$

Proof. Only the last result requires an explanation. It suffices to show that $\ker X_i \cap \mathrm{im} X_i = 0$ ($i \in \{1,3\}$). Let $X_i(q) \in \ker X_i$ ($q \in Q_{kr}$), i.e. $q \in \ker X_i^2$. Since $\ker X_i \subset \ker X_i^2$, dim $\ker X_i^2 \geq 1$. We prove by induction that the eigenvalues of X_i^2 are $-((i+1)\ell b)^2$ ($\ell \in \{0, \ldots, \frac{k}{2}\}$), all of them having multiplicity 2, except 0 that has multiplicity 1. So dim $\ker X_i^2 = 1$ and $\ker X_i = \ker X_i^2 \ni q$. ∎

The following proposition is obvious.

Proposition 2. *(i) If $2r - 3k \neq 0$, operator X_2 is invertible.*
(ii) If $2r - 3k = 0$, we have $X_2 = 0$ and $X_3 = -2X_1$.

The next proposition is an immediate consequence of the commutativity of the Y_i ($i \in \{1,2,3\}$).

Proposition 3. *(i) All commutators of X_i-operators vanish:*

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0.$$

(ii) If X_i is invertible,

$$[X_i^{-1}, X_j] = 0, \forall i, j \in \{1,2,3\}.$$

4 Injection and short exact sequence

Any 1-cochain C_{kr}^1 of degrees k and r can be written in the form $\frac{1}{D}(p_1 Y_1 + p_2 Y_2 + p_3 Y_3)$, with $p_i \in P_{kr}$. Conversely, any element of this type reads $\frac{1}{D}[(p_1 x_1 - p_2 x_2)\partial_1 + (p_1 x_2 + p_2 x_1)\partial_2 + p_3 x_3 \partial_3]$, and is induced by a cochain if and only if $p_1 x_1 - p_2 x_2$ is divisible by D and p_3 by $D' = x_1^2 + x_2^2$. Indeed, the first condition implies that $p_1 x_2 + p_2 x_1$ is also a multiple of D. This cochain is then an element of $P_{k-1,r-2}\partial_1 + P_{k-1,r-2}\partial_2 + P_{k-2,r-2}\partial_3$, of course provided that $k \geq 2, r \geq 2, k \leq r-1$. If $p_3 = 0$, this condition reduces to $k \geq 1, r \geq 2, k \leq r-1$, and if $p_1 = p_2 = 0$, it is replaced by $k \geq 2, r \geq 2, k \leq r$. Of course, the space

$$\mathcal{P}_{kr}^1 = \Delta_{kr} Q_{kr} Y_1 + \Delta_{kr} Q_{kr} Y_2 + \Delta_{k1} Q_{kr} Y_3$$

of *potential* 1-*cochains* of degrees k, r (where $\Delta_{ij} = 1 - \delta_{ij}$, defined by means of Kronecker's symbol, is used in order to group the mentioned cases) has to be taken into account only if the injected space of *real* 1-*cochains* of degrees k, r,

$$\mathcal{R}_{kr}^1 = \Delta_{kr} P_{k-1,r-2}\partial_1 + \Delta_{kr} P_{k-1,r-2}\partial_2 + \Delta_{k1} P_{k-2,r-2}\partial_3,$$

is not vanishing, i.e. if $k \geq 1, r \geq 2, k \leq r$. In the following, we often write x, y, z instead of x_1, x_2, x_3. It is easily checked that the space

$$\mathcal{S}_{kr}^1 = \{ \tfrac{x^{k-1} z^{r-k}}{D} [\Delta_{kr} e x Y_1 + \Delta_{kr} f x Y_2 + \Delta_{k1}(gx + hy) Y_3] : c, d, e, f \in \mathbf{R} \}$$

is supplementary to $\mathcal{R}_{kr}^1$ in $\mathcal{P}_{kr}^1$. Similar spaces $\mathcal{P}_{kr}^d, \mathcal{R}_{kr}^d, \mathcal{S}_{kr}^d$ can be defined for $d = 0, k \geq 2, r \geq 3, k \leq r-1$; $d = 2, k \geq 0, r \geq 1, k \leq r$ and $d = 3, k \geq 0, r \geq 0, k \leq r$. These spaces are described at the end of this section. Hence the whole space of potential cochains $\mathcal{P} = \oplus_{d,k,r} \mathcal{P}_{kr}^d$ is the direct sum of the whole space of real cochains $\mathcal{R} = \oplus_{d,k,r} \mathcal{R}_{kr}^d$ and the supplementary space $\mathcal{S} = \oplus_{d,k,r} \mathcal{S}_{kr}^d$ (we can view k and

r as subscripts running through $\mathbf{N}$ ($k \leq r$), provided that the spaces associated to forbidden values are considered as vanishing).

The spaces $(\mathcal{P}, \partial_{\mathcal{P}})$ and $(\mathcal{R}, \partial_{\mathcal{R}})$, where $\partial_{\mathcal{P}} = \partial_{\mathcal{R}} = [\Lambda, \cdot]$, are differential spaces. Denote by $p_{\mathcal{R}}$ and $p_{\mathcal{S}}$ the projections of $\mathcal{P}$ onto $\mathcal{R}$ and $\mathcal{S}$ respectively and set for any $s \in \mathcal{S}$,

$$\phi s = p_{\mathcal{R}} \partial_{\mathcal{P}} s, \partial_{\mathcal{S}} s = p_{\mathcal{S}} \partial_{\mathcal{P}} s.$$

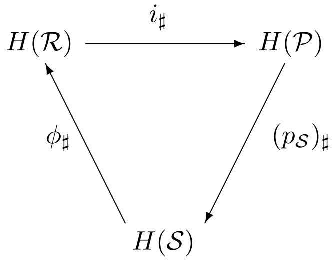
Proposition 4. *(i) The endomorphism $\partial_{\mathcal{S}} \in End\mathcal{S}$ is a differential on $\mathcal{S}$.*
(ii) The linear map $\phi \in \mathcal{L}(\mathcal{S}, \mathcal{R})$ is an anti-homomorphism of differential spaces from $(\mathcal{S}, \partial_{\mathcal{S}})$ into $(\mathcal{R}, \partial_{\mathcal{R}})$.

Proof. Direct consequence of $\partial_{\mathcal{P}}^2 = 0$. ∎

Proposition 5. *If i denotes the injection of $\mathcal{R}$ into $\mathcal{P}$, the sequence*

$$0 \to \mathcal{R} \stackrel{i}{\to} \mathcal{P} \stackrel{p_{\mathcal{S}}}{\to} \mathcal{S} \to 0$$

is a short exact sequence of homomorphisms of differential spaces. Hence the triangle



is exact.

Proof. We only need check that the linear map $\phi_\sharp$ induced by ϕ coincides with the connecting homomorphism Δ. It suffices to remember the definition of Δ. ∎

Remark now that the exact triangle induces for any $(k, r) \in \mathbf{N}^2, k \leq r$, a "long" exact sequence of linear maps:

$$\begin{aligned} 0 \to H^0_{kr}(\mathcal{R}) &\stackrel{i_\sharp}{\to} \dots \\ &\stackrel{\phi_\sharp}{\to} H^d_{kr}(\mathcal{R}) \stackrel{i_\sharp}{\to} H^d_{kr}(\mathcal{P}) \stackrel{(p_{\mathcal{S}})_\sharp}{\to} H^d_{kr}(\mathcal{S}) \stackrel{\phi_\sharp}{\to} H^{d+1}_{kr}(\mathcal{R}) \stackrel{i_\sharp}{\to} \dots \stackrel{(p_{\mathcal{S}})_\sharp}{\to} H^3_{kr}(\mathcal{S}) \to 0. \end{aligned}$$

We denote the kernel and the image of the restricted map $\phi_\sharp \in \mathcal{L}(H^d_{kr}(\mathcal{S}), H^{d+1}_{kr}(\mathcal{R}))$ by $\mathrm{ker}^d_{kr}\phi_\sharp \subset H^d_{kr}(\mathcal{S})$ and $\mathrm{im}^{d+1}_{kr}\phi_\sharp \subset H^{d+1}_{kr}(\mathcal{R})$. Similar notations are used if $i_\sharp$ or $(p_{\mathcal{S}})_\sharp$ are viewed as restricted maps.

Corollary 2. *For any $d \in \{0, 1, 2, 3\}, k \in \mathbf{N}, r \in \mathbf{N}, k \leq r$, we have*

$$\begin{aligned} H^d_{kr}(\mathcal{R}) &\simeq ker^d_{kr} i_\sharp \oplus im^d_{kr} i_\sharp \\ &= im^d_{kr}\phi_\sharp \oplus ker^d_{kr}(p_{\mathcal{S}})_\sharp \\ &\simeq H^{d-1}_{kr}(\mathcal{S})/ker^{d-1}_{kr}\phi_\sharp \oplus H^d_{kr}(\mathcal{P})/ker^d_{kr}\phi_\sharp. \end{aligned}$$

Proof. Apply exactness of the long sequence. ∎

We will compute the $\mathcal{R}$-cohomology by computing the simpler $\mathcal{S}$- and $\mathcal{P}$-cohomology (and in some cases the anti-homomorphism ϕ).

The preceding result is easily understood. The space $Z_{\mathcal{R}}$ of $\mathcal{R}$-cocycles is a subset of $Z_{\mathcal{P}}$. Among the $\mathcal{P}$-classes there may be classes without representatives in $Z_{\mathcal{R}}$. Take now the classes with a non-empty intersection with $Z_{\mathcal{R}}$. Two cocycles in different intersections can not be $\mathcal{R}$-cohomologous. Two cocycles in the same intersection are or not $\mathcal{R}$-cohomologous. Hence the picture of the $\mathcal{P}$- and $\mathcal{R}$-cohomologies. Remember now that the isomorphism $H(\mathcal{R}) \simeq keri_\sharp \oplus imi_\sharp$ means that $H(\mathcal{R}) = keri_\sharp \oplus \chi(imi_\sharp)$, where χ is a linear right inverse of $i_\sharp$. The space $imi_\sharp = \{[\rho]_{\mathcal{P}} : \rho \in Z_{\mathcal{R}}\}$, where $[\cdot]_{\mathcal{P}}$ denotes a class in $\mathcal{P}$, is the space of $\mathcal{P}$-classes with intersection. The space $\chi(imi_\sharp) \simeq imi_\sharp$ is made up of one of the source classes of each $\mathcal{P}$-class with intersection. We obtain the missing $\mathcal{R}$-classes when adding the kernel.

As for the meaning of ϕ, since $\partial_{\mathcal{S}} s = \partial_{\mathcal{P}} s - \phi s$, this anti-homomorphism is nothing but the correction that turns the Poisson-coboundary in a coboundary on $\mathcal{S}$. If ϕ were 0, we would of course have $H(\mathcal{R}) = H(\mathcal{P})/H(\mathcal{S})$. We recover this result as special case of the preceding corollary.

Corollary 3. *For any* $d \in \{0,1,2,3\}, k \in \mathbf{N}, r \in \mathbf{N}, k \leq r$,

(i) if

$$H^{d-1}_{kr}(\mathcal{S}) = 0 \quad (resp.\ H^{d-1}_{kr}(\mathcal{S}) = H^d_{kr}(\mathcal{S}) = 0),$$

then

$$i_\sharp \in Isom(H^d_{kr}(\mathcal{R}), im^d_{kr} i_\sharp) \quad (resp.\ i_\sharp \in Isom(H^d_{kr}(\mathcal{R}), H^d_{kr}(\mathcal{P}))),$$

(ii) if

$$H^d_{kr}(\mathcal{P}) = 0 \quad (resp.\ H^{d-1}_{kr}(\mathcal{P}) = H^d_{kr}(\mathcal{P}) = 0),$$

then

$$\phi_\sharp \in Isom(H^{d-1}_{kr}(\mathcal{S})/ker^{d-1}_{kr}\phi_\sharp, H^d_{kr}(\mathcal{R})) \quad (resp.\ \phi_\sharp \in Isom(H^{d-1}_{kr}(\mathcal{S}), H^d_{kr}(\mathcal{R}))).$$

Proof. Obvious. ∎

The below basic formulas are obtained by straightforward computations. For instance, a potential 0-cochain of degree (k,r), $\pi = \frac{p}{D} \in \mathcal{P}^0_{kr} = Q_{kr}$, is a member of the corresponding real cochain space $\mathcal{R}^0_{kr} = P_{k-2,r-3}$ $(k \geq 2, r \geq 3, k \leq r-1)$, if and only if $p \in P_{kr}$ is divisible by $D' = x_1^2 + x_2^2 = x^2 + y^2$. If $p = \sum_{\ell=0}^k \alpha_\ell x^{\ell,k-\ell,r-k}$, this divisibility condition means that $\alpha_0 - \alpha_2 + \alpha_4 - \ldots = 0$ and $\alpha_1 - \alpha_3 + \alpha_5 - \ldots = 0$. Hence, a potential cochain can be made a real cochain by changing the coefficients α_{k-1} and α_k, so that any potential cochain can be written in a unique way as the sum of a real cochain and an element of $\mathcal{S}^0_{kr} = \{\frac{x^{k-1}z^{r-k}}{D}(cx+dy) : c,d \in \mathbf{R}\}$.

Formulary 1

1. 0-cochains

$k \geq 2, r \geq 3, k \leq r-1$:

$\mathcal{P}^0_{kr} = Q_{kr}$
$\mathcal{R}^0_{kr} = P_{k-2,r-3}$
$\mathcal{S}^0_{kr} = \{\frac{x^{k-1}z^{r-k}}{D}(cx+dy) : c,d \in \mathbf{R}\}$

A potential cochain $\pi = \frac{p}{D}$ is a real cochain if and only if p is divisible by D'

2. 1-cochains

$k \geq 1, r \geq 2, k \leq r$:

$\mathcal{P}^1_{kr} = \Delta_{kr} Q_{kr} Y_1 + \Delta_{kr} Q_{kr} Y_2 + \Delta_{k1} Q_{kr} Y_3$

$\mathcal{R}^1_{kr} = \Delta_{kr} P_{k-1,r-2} \partial_1 + \Delta_{kr} P_{k-1,r-2} \partial_2 + \Delta_{k1} P_{k-2,r-2} \partial_3$

$\mathcal{S}^1_{kr} = \{\frac{x^{k-1}z^{r-k}}{D}[\Delta_{kr} exY_1 + \Delta_{kr} fxY_2 + \Delta_{k1}(gx+hy)Y_3] :$
$e,f,g,h \in \mathbf{R}\}$

A potential cochain $\pi = \frac{1}{D}[\Delta_{kr}p_1Y_1 + \Delta_{kr}p_2Y_2 + \Delta_{k1}p_3Y_3]$ is a real cochain if and only if $\Delta_{kr}[p_1x_1 - p_2x_2]$ and $\Delta_{k1}p_3$ are divisible by D'

3. 2-cochains

$k \geq 0, r \geq 1, k \leq r$:

$\mathcal{P}^2_{kr} = \Delta_{k0}Q_{kr}Y_{23} + \Delta_{k0}Q_{kr}Y_{31} + \Delta_{kr}Q_{kr}Y_{12}$

$\mathcal{R}^2_{kr} = \Delta_{k0}P_{k-1,r-1}\partial_{23} + \Delta_{k0}P_{k-1,r-1}\partial_{31} + \Delta_{kr}P_{k,r-1}\partial_{12}$

$\mathcal{S}^2_{kr} = \{\frac{x^k z^{r-k}}{D}[\Delta_{k0}iY_{23} + \Delta_{k0}jY_{31}] : i, j \in \mathbf{R}\}$

A potential cochain $\pi = \frac{1}{D}[\Delta_{k0}p_1Y_{23} + \Delta_{k0}p_2Y_{31} + \Delta_{kr}p_3Y_{12}]$ is a real cochain if and only if $\Delta_{k0}[p_1x_1 - p_2x_2]$ is divisible by D'

4. 3-cochains

$k \geq 0, r \geq 0, k \leq r$:

$\mathcal{P}^3_{kr} = Q_{kr}Y_{123}$

$\mathcal{R}^3_{kr} = P_{kr}\partial_{123}$

$\mathcal{S}^3_{kr} = 0$

The spaces of potential and real cochains coincide

We write the coefficients (e, f, g, h) (resp. (i, j)) of the coboundary $\partial_{\mathcal{S}}\sigma^0_{kr}$ (resp. $\partial_{\mathcal{S}}\sigma^1_{kr}$) in terms of the coefficients (c, d) (resp. $(e\Delta_{kr}, f\Delta_{kr}, g\Delta_{k1}, h\Delta_{k1})$) of the supplementary cochain $\sigma^0_{kr} \in \mathcal{S}^0_{kr}$, $k \geq 2, r \geq 3, k \leq r-1$ (resp. $\sigma^1_{kr} \in \mathcal{S}^1_{kr}$, $k \geq 1, r \geq 2, k \leq r$) and of the Pauli type matrices

$$\mu_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mu_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mu_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The expressions of $\phi(\sigma^d_{kr})$ are not all indispensable.

Formulary 2

1. 0-cochains

Coefficients of $\partial_{\mathcal{S}}\sigma^0_{kr}$ in terms of the coefficients of σ^0_{kr}, $k \geq 2, r \geq 3, k \leq r-1$:

$$\begin{pmatrix} e \\ f \\ \hline g \\ h \end{pmatrix} = \begin{pmatrix} 2b(r-k)\mu_1 - a(r-k-1)\mu_3 \\ \hline a(k-2)\mu_0 - 2bk\mu_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

2. 1-cochains

Coefficients of $\partial_{\mathcal{S}}\sigma^1_{kr}$ in terms of the coefficients of σ^1_{kr}, $k \geq 1, r \geq 2, k \leq r$:

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 2bk\mu_0 - a(k-2)\mu_2 \mid a(r-k-1)\mu_1 + 2b(r-k)\mu_3 \end{pmatrix} \begin{pmatrix} e\Delta_{kr} \\ f\Delta_{kr} \\ \hline g\Delta_{k1} \\ h\Delta_{k1} \end{pmatrix}$$

Value of $\phi(\sigma^1_{rr})$ in terms of the coefficients of σ^1_{rr}, $r \geq 2, a = 0$:

$$\begin{array}{rcl} \phi(\sigma^1_{rr}) & = & \frac{bx^{r-2}}{D}[-rx(gx+hy)Y_{23} + (-h\,x^2 + rg\,xy + (r-1)h\,y^2)Y_{31}] \\ & = & -bx^{r-2}[(rgx + (r-1)hy)\partial_{23} + hx\partial_{31}] \end{array}$$

3. 2-cochains

Coboundary $\partial_{\mathcal{S}}\sigma^2_{kr}$, $k \geq 0, r \geq 1, k \leq r$:

$$\partial_{\mathcal{S}}\sigma^2_{kr} = 0$$

Value of $\phi(\sigma^2_{rr})$ in terms of the coefficients of σ^2_{rr}, $r \geq 1$:

$$\phi(\sigma^2_{rr}) = \frac{x^{r-1}}{D}[(ai - brj)x - briy]Y_{123} \tag{6}$$

4. 3-cochains

Coboundary $\partial_{\mathcal{S}}\sigma^3_{kr}$, $k \geq 0, r \geq 0, k \leq r$:

$$\partial_{\mathcal{S}}\sigma^3_{kr} = 0$$

5 0 - cohomology spaces

5.1 $\mathcal{S}$ - cohomology

Proposition 6. *The* 0 *- cohomology space of* $\mathcal{S}$ *vanishes:* $H^0(\mathcal{S}) = 0$.

Proof. If (c, d) are the coefficients of an arbitrary cochain σ^0_{kr} $(k \geq 2, r \geq 3, k \leq r-1)$, we have for instance

$$M_0 \begin{pmatrix} c \\ d \end{pmatrix} = 0, \quad M_0 = a(k-2)\mu_0 - 2bk\mu_2.$$

Since $det\, M_0 = a^2(k-2)^2 + 4b^2k^2 > 0$, it follows that $\sigma^0_{kr} = 0$. ∎

Corollary 4. *For any* $k \geq 2, r \geq 3, k \leq r-1$, *we have* $dim\ im^1_{kr}\ \partial_{\mathcal{S}} = dim\ \mathcal{S}^0_{kr} = 2$.

5.2 $\mathcal{P}$ - cohomology and $\mathcal{R}$ - cohomology

Theorem 1. *The* 0 *- cohomology spaces of* $\mathcal{P}$ *and* $\mathcal{R}$ *coincide:*
(i) if $a \neq 0$, $H^0(\mathcal{P}) = H^0(\mathcal{R}) = H^0_{23}(\mathcal{P}) = H^0_{23}(\mathcal{R}) = \mathbf{R}$,
(ii) if $a = 0$, $H^0(\mathcal{P}) = H^0(\mathcal{R}) = \bigoplus_{k=1}^{\infty} H^0_{2k,3k}(\mathcal{P}) = \bigoplus_{k=1}^{\infty} H^0_{2k,3k}(\mathcal{R}) = \bigoplus_{m=0}^{\infty} \mathbf{R}\, D^m$.

Proof. The equality of both cohomologies is a consequence of Corollary 3. So let $k \geq 2, r \geq 3, k \leq r-1$ and $\pi^0_{kr} = q \in \mathcal{P}^0_{kr} \cap ker\partial_{\mathcal{P}}$. We have $X_1(q) = X_2(q) = 0$. Apply now Proposition 2 and Corollary 1. If $3k - 2r \neq 0$ and if $3k - 2r = 0$, $a \neq 0$ and $k \neq r-1$, the cocycle vanishes. If $3k - 2r = 0$ and $a = 0$ or $k = r-1$, $q = \alpha D^{\frac{k}{2}-1}$ $(\alpha \in \mathbf{R})$. ∎

6 1 - cohomology spaces

6.1 $\mathcal{S}$ - cohomology

Proposition 7. *If* $a \neq 0$, $H^1(\mathcal{S}) = 0$, *and if* $a = 0$, $H^1(\mathcal{S}) = \bigoplus_{m=2}^{\infty} H^1_{mm}(\mathcal{S}) = \bigoplus_{m=2}^{\infty} \frac{x_1^{m-1}}{D}(\mathbf{R}x_1 + \mathbf{R}x_2)Y_3$.

Proof. Set $M_1 = 2bk\mu_0 - a(k-2)\mu_2$ and $N_1 = a(r-k-1)\mu_1 + 2b(r-k)\mu_3$ $(k \geq 1, r \geq 2, k \leq r)$. The coefficients of any 1-cocycle of degrees k, r verify

$$\Delta_{kr} \begin{pmatrix} e \\ f \end{pmatrix} = -\Delta_{k1} M_1^{-1} N_1 \begin{pmatrix} g \\ h \end{pmatrix}.$$

If $k = 1$, the cocycle vanishes. If $k = r$ the cocycle equation reads $N_1 \begin{pmatrix} g \\ h \end{pmatrix} = 0$, with $N_1 = -a\mu_1$. For $a \neq 0$, the cocycle vanishes again, and for $a = 0$, the cohomology space is $H^1_{rr}(\mathcal{S}) = \frac{x^{r-1}}{D}(\mathbf{R}x + \mathbf{R}y)Y_3$, since coboundaries do not exist for $k = r$. Finally, if $2 \leq k \leq r-1, r \geq 3$, Corollary 4 shows that there is no cohomology. (A direct proof of this result, based upon the multiplication table $\mu_i\mu_j = (-1)^{ij+1}[\delta_{ij}\mu_0 + \varepsilon_{ijk}\mu_k]$ $(i, j \in \{1, 2, 3\})$, where ε_{ijk} is the Levi-Civita symbol, is also easy.) ∎

6.2 $\mathcal{P}$ - cohomology and $\mathcal{R}$ - cohomology

Theorem 2. *The first cohomology groups of $\mathcal{P}$ and $\mathcal{R}$ are isomorphic:*
(i) $H^1(\mathcal{P}) = H^1(\mathcal{R}) = H^1_{23}(\mathcal{P}) = H^1_{23}(\mathcal{R}) = \mathbf{R}Y_1 + \mathbf{R}Y_2 + \mathbf{R}Y_3$*, if* $a \neq 0$,
(ii) $H^1(\mathcal{P}) = H^1(\mathcal{R}) = \bigoplus_{k=1}^{\infty} H^1_{2k,3r}(\mathcal{P}) = \bigoplus_{k=1}^{\infty} H^1_{2k,3r}(\mathcal{R}) = \bigoplus_{m=0}^{\infty} D^m(\mathbf{R}Y_1 + \mathbf{R}Y_2 + \mathbf{R}Y_3)$*, if* $a = 0$.

Proof. Let $\pi^1_{kr} = \Delta_{kr}q_1Y_1 + \Delta_{kr}q_2Y_2 + \Delta_{k1}q_3Y_3$ $(k \geq 1, r \geq 2, k \leq r)$ be a member of $\mathcal{P}^1_{kr} \cap ker\partial_{\mathcal{P}}$. The cocycle equation reads

$$X_2(\Delta_{k1}q_3) - X_3(\Delta_{kr}q_2) = 0,$$
$$X_3(\Delta_{kr}q_1) - X_1(\Delta_{k1}q_3) = 0,$$
$$X_1(\Delta_{kr}q_2) - X_2(\Delta_{kr}q_1) = 0.$$

Moreover, for $k \geq 2, r \geq 3, k \leq r-1$, this cocycle can be a coboundary, i.e. there is $\pi^0_{kr} = q$ in $\mathcal{P}^0_{kr}$ such that

$$X_1(q) = q_1,\ X_2(q) = q_2,\ X_3(q) = q_3.$$

1. $\mathbf{k = 1}$ or $\mathbf{k = r}$

We treat the first case (resp. the second case). The cocycle equation implies $X_3(q_1) = X_3(q_2) = 0$ (resp. $X_2(q_3) = 0$). In view of Corollary 1 (resp. Proposition 2) X_3 (resp. X_2) is invertible and cocycles vanish.

2. $\mathbf{2 \leq k \leq r-1}$ (then $r \geq 3$)

2.a. $\mathbf{2r - 3k \neq 0}$

Set $q = X_2^{-1}(q_2)$. Due to Proposition 3 and the cocycle relations, we then have $X_1(q) = q_1, X_3(q) = q_3$, so that all cocycles are coboundaries.

2.b. $\mathbf{2r - 3k = 0}$ (then $k \in 2\mathbf{N}$)

Here $X_2 = 0$ and $X_3 = -2X_1$. The cocycle condition is $X_1(q_2) = 0$ and $X_3(q_1) = X_1(q_3)$, i.e. $X_1(2q_1 + q_3) = 0$.

2.b.I $\mathbf{a \neq 0}$ and $\mathbf{k \neq r-1}$ (i.e. $a \neq 0$ and $(k, r) \neq (2, 3)$)

In this case $q_2 = 0$ and we choose $q = X_1^{-1}(q_1)$. This entails that $X_3(q) = q_3$.

2.b.II $\mathbf{a = 0}$ or $\mathbf{k = r-1}$ (i.e. $a = 0$ or $(k, r) = (2, 3)$)

Corollary 1 allows to decompose q_1 in the form $q_1 = \alpha_1 D^{\frac{k}{2}-1} + X_1(q)$ $(\alpha_1 \in \mathbf{R}, q \in Q_{kr})$. It is clear that $q_2 = \alpha_2 D^{\frac{k}{2}-1} + X_2(q)$ $(\alpha_2 \in \mathbf{R})$ and that $q_3 = \alpha'_3 D^{\frac{k}{2}-1} - 2q_1 = \alpha_3 D^{\frac{k}{2}-1} + X_3(q)$ $(\alpha'_3, \alpha_3 \in \mathbf{R})$. Hence, the considered cocycle is cohomologous to $D^{\frac{k}{2}-1}(\alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3)$. As the kernel and the image of

X_1 and X_3 are supplementary and X_2 vanishes, such cocycles—with different α-coefficients—can not be cohomologous.

As for the isomorphism between the $\mathcal{P}$- and $\mathcal{R}$-cohomology, remember that $H^0(\mathcal{S}) = 0$. Corollary 3 then implies that $i_\sharp \in Isom(H^1_{kr}(\mathcal{R}), H^1_{kr}(\mathcal{P}))$, for all k, r. Indeed, if $H^1_{kr}(\mathcal{S}) \neq 0$, we have $k = r \geq 2$ and $H^1_{kr}(\mathcal{P}) = 0 = im^1_{kr} i_\sharp$. ∎

7 2 - cohomology spaces

7.1 $\mathcal{S}$ - cohomology

Proposition 8. *If $a \neq 0$, $H^2(\mathcal{S}) = H^2_{11}(\mathcal{S}) = \frac{x_1}{D}(\mathbf{R}Y_{23} + \mathbf{R}Y_{31})$, and if $a = 0$, $H^2(\mathcal{S}) = \bigoplus_{m=1}^\infty H^2_{mm}(\mathcal{S}) = \bigoplus_{m=1}^\infty \frac{x_1^m}{D}(\mathbf{R}Y_{23} + \mathbf{R}Y_{31})$.*

Proof. As $\mathcal{S}^3 = 0$, any cochain is a cocycle. Let σ^2_{kr} $(k \geq 0, r \geq 1, k \leq r)$ be a cocycle with coefficients $(\Delta_{k0}i, \Delta_{k0}j)$. If $k \geq 1, r \geq 2, k \leq r$, it is a coboundary if there are coefficients $(\Delta_{kr}e, \Delta_{kr}f, \Delta_{k1}g, \Delta_{k1}h)$, such that

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} 2bk\mu_0 - a(k-2)\mu_2 \mid a(r-k-1)\mu_1 + 2b(r-k)\mu_3 \end{pmatrix} \begin{pmatrix} \Delta_{kr}e \\ \Delta_{kr}f \\ \hline \Delta_{k1}g \\ \Delta_{k1}h \end{pmatrix}.$$

If $r \geq 1, k = 0$, all cochains vanish. If $r = 1, k = 1$, there are no coboundaries. Consider now the case $r \geq 2, k \geq 1$. If $k \neq r$ and if $k = r$ and $a \neq 0$, any cocycle is a coboundary. If $k = r$ and $a = 0$, the unique coboundary is 0. ∎

7.2 $\mathcal{P}$ - cohomology

Proposition 9. *The second cohomology group of the complex $(\mathcal{P}, \partial_\mathcal{P})$ is,*
(i) if $a \neq 0$,

$$\begin{aligned} H^2(\mathcal{P}) &= H^2_{23}(\mathcal{P}) \oplus H^2_{11}(\mathcal{P}) \\ &= \mathbf{R}Y_{23} + \mathbf{R}Y_{31} + \mathbf{R}Y_{12} \\ &\quad \oplus \tfrac{1}{D}\left[\mathbf{R}\left((D - \tfrac{a}{b}x_1x_2x_3)\partial_{23} + \tfrac{a}{b}x_1^2x_3\partial_{31}\right) + \mathbf{R}\left((D + \tfrac{a}{b}x_1x_2x_3)\partial_{31} - \tfrac{a}{b}x_2^2x_3\partial_{23}\right)\right], \end{aligned}$$

(ii) if $a = 0$,

$$\begin{aligned} H^2(\mathcal{P}) &= \bigoplus_{k=1}^\infty H^2_{2k,3r}(\mathcal{P}) \oplus H^2_{11}(\mathcal{P}) \oplus \bigoplus_{m=1}^\infty H^2_{0m}(\mathcal{P}) \\ &= \bigoplus_{m=0}^\infty D^m (\mathbf{R}Y_{23} + \mathbf{R}Y_{31} + \mathbf{R}Y_{12}) \\ &\quad \oplus \mathbf{R}\partial_{23} + \mathbf{R}\partial_{31} \oplus \bigoplus_{m=0}^\infty \mathbf{R}x_3^m\partial_{12}. \end{aligned}$$

Proof. The cocycle equation for a cochain $\pi^2_{kr} = \Delta_{k0}q_1Y_{23} + \Delta_{k0}q_2Y_{31} + \Delta_{kr}q_3Y_{12}$ $(k \geq 0, r \geq 1, k \leq r; q_1, q_2, q_3 \in Q_{kr})$ reads

$$X_1(\Delta_{k0}q_1) + X_2(\Delta_{k0}q_2) + X_3(\Delta_{kr}q_3) = 0.$$

This cocycle can be the coboundary of a cochain $\pi^1_{kr} = \Delta_{kr}\varrho_1Y_1 + \Delta_{kr}\varrho_2Y_2 + \Delta_{k1}\varrho_3Y_3$ $(k \geq 1, r \geq 2, k \leq r; \varrho_1, \varrho_2, \varrho_3 \in Q_{kr})$:

$$\begin{aligned} X_2(\Delta_{k1}\varrho_3) - X_3(\Delta_{kr}\varrho_2) &= \Delta_{k0}q_1, \\ X_3(\Delta_{kr}\varrho_1) - X_1(\Delta_{k1}\varrho_3) &= \Delta_{k0}q_2, \\ X_1(\Delta_{kr}\varrho_2) - X_2(\Delta_{kr}\varrho_1) &= \Delta_{kr}q_3. \end{aligned}$$

1. $\mathbf{k = 0, r \geq 1}$ (then $k \leq r$)

We get $X_3(q_3) = 0$. If $a \neq 0$, the cocycle π^2_{0r} vanishes, and if $a = 0$, it has the form

$$\pi^2_{0r} = \frac{\alpha_0}{D} z^r Y_{12} = \alpha_0 z^{r-1}\partial_{12} \quad (a = 0, r \geq 1, \alpha_0 \in \mathbf{R}).$$

2. $\mathbf{k \geq 1, r \geq 1, k = r}$

Cocycle condition: $q_2 = -X_2^{-1} X_1(q_1)$.

2.a. $\mathbf{k = r = 1}$

It follows immediately from Equation 3, Equation 4 and Equation 2, that

$$\begin{aligned} \pi_{11}^2 &= \tfrac{1}{D}(\alpha_1 x + \alpha_2 y) Y_{23} + \tfrac{1}{D}\left((\alpha_2 + \tfrac{a}{b}\alpha_1)x + (\tfrac{a}{b}\alpha_2 - \alpha_1)y\right) Y_{31} \\ &= \tfrac{1}{D}\left[\alpha_1\left((D - \tfrac{a}{b}xyz)\partial_{23} + \tfrac{a}{b}x^2 z\partial_{31}\right) + \alpha_2\left((D + \tfrac{a}{b}xyz)\partial_{31} - \tfrac{a}{b}y^2 z\partial_{23}\right)\right] \quad (\alpha_1, \alpha_2 \in \mathbf{R}). \end{aligned} \tag{7}$$

2.b. $\mathbf{k = r \geq 2}$

As the coboundary condition reads $X_2(\varrho_3) = q_1, X_1(\varrho_3) = -q_2$, we only need choose $\varrho_3 = X_2^{-1}(q_1)$.

3. $\mathbf{k \geq 1, k \leq r - 1}$ (then $r \geq 2$)

Cocycle condition: $X_1(q_1) + X_2(q_2) + X_3(q_3) = 0$.

3.a. $\mathbf{k = 1}$

Coboundary condition: $X_3(\varrho_2) = -q_1, X_3(\varrho_1) = q_2, X_1(\varrho_2) - X_2(\varrho_1) = q_3$. It suffices to take $\varrho_1 = X_3^{-1}(q_2), \varrho_2 = -X_3^{-1}(q_1)$.

3.b. $\mathbf{k \geq 2}$ (then $r \geq 3$)

All Δ-factors in the cocycle and coboundary conditions disappear.

3.b.I. $\mathbf{3k - 2r \neq 0}$

Set $\varrho_2 = 0, \varrho_3 = X_2^{-1}(q_1), \varrho_1 = -X_2^{-1}(q_3)$.

3.b.II. $\mathbf{3k - 2r = 0}$

Since $X_2 = 0$ and $X_3 = -2X_1$, the equations read $X_1(q_1 - 2q_3) = 0$ and $X_1(\varrho_2) = \frac{1}{2}q_1, X_1(2\varrho_1 + \varrho_3) = -q_2, X_1(\varrho_2) = q_3$ respectively.

3.b.II.α. $\mathbf{a \neq 0, (k, r) \neq (2, 3)}$

As X_1 is invertible, $q_1 = 2q_3$ and we set $\varrho_1 = 0, \varrho_2 = X_1^{-1}(q_3), \varrho_3 = -X_1^{-1}(q_2)$.

3.b.II.β. $\mathbf{a = 0}$ or $\mathbf{(k, r) = (2, 3)}$

Corollary 1 allows to write q_i $(i \in \{1, 2, 3\})$ in a unique way as the sum $q_i = \alpha_{2+i} D^{\frac{k}{2}-1} + q'_i$, $\alpha_{2+i} \in \mathbf{R}, q'_i \in Q_{k, \frac{3}{2}k}$ of an element of $ker\, X_1$ and an element of $im\, X_1$. As $q_1 - 2q_3 - (\alpha_3 - 2\alpha_5)D^{\frac{k}{2}-1} = q'_1 - 2q'_3 \in ker\, X_1 \cap im\, X_1$, we conclude that $q'_1 = 2q'_3$. Hence $q'_1 Y_{23} + q'_2 Y_{31} + q'_3 Y_{12}$ is the coboundary of $\varrho_1 Y_1 + \varrho_2 Y_2 + \varrho_3 Y_3$, where $\varrho_1 = 0, X_1(\varrho_2) = q'_3, X_1(\varrho_3) = -q'_2$. Finally, the original cocycle is cohomologous to

$$\pi_{k, \frac{3}{2}k}^2 \simeq D^{\frac{k}{2}-1}\left(\alpha_3 Y_{23} + \alpha_4 Y_{31} + \alpha_5 Y_{12}\right)$$
$$(a = 0 \text{ and } k \in \{2, 4, 6, \ldots\} \text{ or } a \neq 0 \text{ and } k = 2; \alpha_3, \alpha_4, \alpha_5 \in \mathbf{R}).$$

Due to the supplementary character of the kernel and the image of X_1, two different cocycles of this type can not be cohomologous. ∎

7.3 $\mathcal{R}$ - cohomology

Theorem 3. *(i) If* $a \neq 0$,

$$H^2(\mathcal{R}) = H^2_{23}(\mathcal{R}) = \mathbf{R}Y_{23} + \mathbf{R}Y_{31} + \mathbf{R}Y_{12},$$

(ii) if $a = 0$,

$$\begin{aligned} H^2(\mathcal{R}) &= \bigoplus_{k=1}^{\infty} H^2_{2k,3k}(\mathcal{R}) \oplus \bigoplus_{m=1}^{\infty} H^2_{mm}(\mathcal{R}) \oplus \bigoplus_{m=1}^{\infty} H^2_{0m}(\mathcal{R}) \\ &= \bigoplus_{m=0}^{\infty} D^m \left(\mathbf{R}Y_{23} + \mathbf{R}Y_{31} + \mathbf{R}Y_{12}\right) \\ &\oplus \bigoplus_{m=1}^{\infty} x_1^{m-2} \left(\mathbf{R}x_1\partial_{23} + \mathbf{R}(x_1\partial_{31} + (m-1)x_2\partial_{23})\right) \oplus \bigoplus_{m=0}^{\infty} \mathbf{R}x_3^m\partial_{12}. \end{aligned}$$

Proof. We apply once more Corollary 3.

If $(kr) \neq (mm)$ $(m \geq 1)$, the map $i_\sharp$ is an isomorphism between $H^2_{kr}(\mathcal{R})$ and $H^2_{kr}(\mathcal{P})$.

If $(kr) = (11)$, we get $i_\sharp \in Isom(H^2_{11}(\mathcal{R}), im^2_{11}\, i_\sharp)$. Since no coboundaries exist for this degree, the $\mathcal{R}$-cohomology is made up by the $\mathcal{P}$-cocycles in $\mathcal{R}$. But, $\pi^2_{11} \in \mathcal{R}$, see 7, if and only if $a\alpha_1 = a\alpha_2 = 0$. So, for $a \neq 0$, we have $H^2_{11}(\mathcal{R}) = 0$, whereas for $a = 0$, we obtain $H^2_{11}(\mathcal{R}) = \mathbf{R}\partial_{23} + \mathbf{R}\partial_{31}$.

Consider now $(kr) = (mm)$ $(m \geq 2)$. As $H^1_{mm}(\mathcal{P}) = H^2_{mm}(\mathcal{P}) = 0$, the map $\phi_\sharp$ is an isomorphism between $H^1_{mm}(\mathcal{S})$ and $H^2_{mm}(\mathcal{R})$. So, if $a \neq 0$, both spaces vanish. In the case $a = 0$, we obtain, see Formulary 2, $H^2_{mm}(\mathcal{R}) = x^{m-2}\left(\mathbf{R}x\partial_{23} + \mathbf{R}(x\partial_{31} + (m-1)y\partial_{23})\right)$. ∎

Remark. So, when passing from the $\mathcal{P}$- to the $\mathcal{R}$-cohomology, the $\mathcal{P}$-class of degree $(kr) = (11)$ is lost for $a \neq 0$, whereas for $a = 0$ new $\mathcal{R}$-classes of degree $(kr) = (mm)$, $m \geq 2$ appear. This is coherent with the picture described in section 4, if $\phi(\sigma)$, $\sigma = \frac{x^{m-1}}{D}(gx + hy)Y_3 \in \mathcal{S}^1_{mm} \cap ker\partial_{\mathcal{S}}$, $m \geq 2$, $g, h \in \mathbf{R}$, is a coboundary in $\mathcal{P}$ but not in $\mathcal{R}$. Since $\pi_{\mathcal{S}}\partial_{\mathcal{P}}\sigma = \partial_{\mathcal{S}}\sigma = 0$, we definitely have $\phi(\sigma) = \partial_{\mathcal{P}}\sigma$. If $\phi(\sigma)$ were an $\mathcal{R}$-coboundary, $\phi(\sigma) = \partial_{\mathcal{R}}\rho$, $\rho \in \mathcal{R}^1_{mm}$, the difference $\rho - \sigma$ would be a $\mathcal{P}$-coboundary in view of Theorem 2: $\rho - \sigma = \partial_{\mathcal{P}}p$, $p \in \mathcal{P}^0_{mm} = 0$.

8 3 - cohomology spaces

8.1 $\mathcal{S}$ - cohomology

Proposition 10. *The third cohomology group of* $\mathcal{S}$ *vanishes:* $H^3(\mathcal{S}) = 0$.

Proof. Obvious. ∎

8.2 $\mathcal{P}$ - cohomology

Proposition 11. *(i) For* $a \neq 0$,

$$\begin{aligned} H^3(\mathcal{P}) &= H^3_{23}(\mathcal{P}) \oplus H^3_{00}(\mathcal{P}) \\ &= \mathbf{R}Y_{123} \oplus \mathbf{R}\partial_{123}, \end{aligned}$$

(ii) for $a = 0$,

$$\begin{aligned} H^3(\mathcal{P}) &= \bigoplus_{k=1}^{\infty} H^3_{2k,3k}(\mathcal{P}) \oplus \bigoplus_{m=0}^{\infty} H^3_{0m}(\mathcal{P}) \\ &= \bigoplus_{m=0}^{\infty} D^m\mathbf{R}Y_{123} \oplus \bigoplus_{m=0}^{\infty} \mathbf{R}x_3^m\partial_{123}. \end{aligned}$$

Proof. Of course any cochain $\pi = qY_{123} \in \mathcal{P}^3_{kr}$, $q \in Q_{kr}$, $k \geq 0, r \geq 0, k \leq r$ is a cocycle. For $k \geq 0, r \geq 1, k \leq r$, this cocycle is a coboundary, if there are $\varrho_1, \varrho_2, \varrho_3 \in Q_{kr}$ such that

$$X_1(\Delta_{k0}\varrho_1) + X_2(\Delta_{k0}\varrho_2) + X_3(\Delta_{kr}\varrho_3) = q.$$

1. r = 0

The cocycle reads

$$\pi^3_{00} = \frac{\alpha_0}{D}Y_{123} = \alpha_0\partial_{123} \quad (\alpha_0 \in \mathbf{R})$$

and no coboundaries do exist.

2. $\mathbf{k=0, r\geq 1}$

The coboundary condition reads $X_3(\varrho_3)=q$. If $a\neq 0$, we set $\varrho_3 = X_3^{-1}(q)$. Otherwise we decompose q in the form $q=\frac{\alpha_1 z^r}{D}+q'$, $\alpha_1\in\mathbf{R}, q'\in im\, X_3$. Then the cocycle

$$\pi^3_{0r} = \frac{\alpha_1 z^r}{D} Y_{123} = \alpha_1 z^r \partial_{123} \quad (a=0, r\geq 1, \alpha_1\in\mathbf{R})$$

is not a coboundary (except of course if $\alpha_1=0$).

3. $\mathbf{k\geq 1, r\geq 1, k=r}$

Coboundary condition: $X_1(\varrho_1)+X_2(\varrho_2)=q$. As $2r-3k=-r\neq 0$, it suffices to take $\varrho_1=0$ and $\varrho_2=X_2^{-1}(q)$.

3. $\mathbf{k\geq 1, r\geq 1, k\leq r-1}$ (then $r\geq 2$)

Condition: $X_1(\varrho_1)+X_2(\varrho_2)+X_3(\varrho_3)=q$.

3.a. $\mathbf{2r-3k\neq 0}$

We only need choose $\varrho_1=\varrho_3=0$ and $\varrho_2=X_2^{-1}(q)$.

3.b. $\mathbf{2r-3k=0}$ (then $(kr)\in\{(23),(46),(69),\ldots\}$)

Here $X_2=0$ and $X_3=-2X_1$ and the condition reads $X_1(\varrho_1-2\varrho_3)=q$. If $a\neq 0$ and $(kr)\neq(23)$, take $\varrho_3=0$ and $\varrho_1=X_1^{-1}(q)$. If $a=0$ or $(kr)=(23)$, set again $q=\alpha_2 D^{\frac{k}{2}-1}+q'$, $\alpha_2\in\mathbf{R}, q'\in im\, X_1$. The cocycle

$$\pi^3_{2m,3m} = D^{m-1}\alpha_2 Y_{123} \quad (a=0, m\in\mathbf{N}^* \text{ or } a\neq 0, m=1; \alpha_2\in\mathbf{R})$$

can not be a coboundary (if $\alpha_2\neq 0$).

8.3 $\mathcal{R}$ - cohomology

Theorem 4. *(i) If $a\neq 0$,*

$$\begin{aligned} H^3(\mathcal{R}) &= H^3_{23}(\mathcal{R})\oplus H^3_{00}(\mathcal{R}) \\ &= \mathbf{R}Y_{123}\oplus\mathbf{R}\partial_{123}, \end{aligned}$$

(ii) if $a=0$,

$$\begin{aligned} H^3(\mathcal{R}) &= \textstyle\bigoplus_{k=1}^{\infty} H^3_{2k,3k}(\mathcal{R})\oplus\bigoplus_{m=0}^{\infty} H^3_{0m}(\mathcal{R})\oplus\bigoplus_{m=1}^{\infty} H^3_{mm}(\mathcal{R}) \\ &= \textstyle\bigoplus_{m=0}^{\infty} D^m\mathbf{R}Y_{123}\oplus\bigoplus_{m=0}^{\infty}\mathbf{R}x_3^m\partial_{123}\oplus\bigoplus_{m=0}^{\infty} x_1^m(\mathbf{R}x_1+\mathbf{R}x_2)\partial_{123}. \end{aligned}$$

Proof. If $a\neq 0, (kr)\neq(11)$ and if $a=0, (kr)\neq(mm), m\geq 1$, then

$$i_\sharp\in Isom(H^3_{kr}(\mathcal{R}), H^3_{kr}(\mathcal{P})) \quad (a\neq 0, (kr)\neq(11) \text{ or } a=0, (kr)\neq(mm), m\geq 1).$$

As $H^3_{mm}(\mathcal{P})=0$, $m\geq 1$, it follows from Corollary 3 that $\phi_\sharp\in Isom(H^2_{mm}(\mathcal{S})/ker^2_{mm}\phi_\sharp, H^3_{mm}(\mathcal{R}))$. If $a=0, m\geq 2$, the group $H^2_{mm}(\mathcal{P})$ also vanishes and

$$\phi_\sharp\in Isom(H^2_{mm}(\mathcal{S}), H^3_{mm}(\mathcal{R})) \quad (a=0, m\geq 2).$$

For $m=1$, we have to compute the kernel $ker^2_{mm}\phi_\sharp$. If $\phi_\sharp[\sigma]_\mathcal{S}=0$, $\sigma\in\mathcal{S}^2_{11}\cap ker\,\partial_\mathcal{S}=\mathcal{S}^2_{11}$, the image $\phi(\sigma)=\partial_\mathcal{P}\sigma$ is an $\mathcal{R}$-coboundary $\partial_\mathcal{R}\rho$, $\rho\in\mathcal{R}^2_{11}$. So, $\rho-\sigma\in\mathcal{P}^2_{11}\cap ker\,\partial_\mathcal{P}$. Equation 7 shows that

$$\rho-\sigma=\pi^2_{11}=\frac{1}{D}\left[\left((\alpha-\frac{a}{b}\beta)x+\beta y\right)Y_{23}+\left(\beta x+(\frac{a}{b}\beta-\alpha)y\right)Y_{31}\right]+\frac{x}{D}\left[\frac{a}{b}\beta Y_{23}+\frac{a}{b}\alpha Y_{31}\right],$$

where the first (resp. second) term of the r.h.s. is a member of $\mathcal{R}^2_{11}$ (resp. $\mathcal{S}^2_{11}$). Hence, if $a = 0$, the cochain σ vanishes and $ker^2_{11}\,\phi_\sharp = 0$:

$$\phi_\sharp \in Isom(H^2_{11}(\mathcal{S}), H^3_{11}(\mathcal{R})) \quad (a = 0).$$

If $a \neq 0$ and $\sigma = \frac{x}{D}(iY_{23} + jY_{31})$, see Formulary 2, we get $\alpha = -\frac{b}{a}j, \beta = -\frac{b}{a}i$, which provides ρ in terms of σ. Equations 3, 4, and 6 then immediately give $ker^2_{11}\,\phi_\sharp = H^2_{11}(\mathcal{S})$, so that

$$H^3_{11}(\mathcal{R}) = 0 \quad (a \neq 0).$$

The announced theorem is then a direct consequence of Equation 6. ∎

9 Further results

In this section, we provide complete results regarding the formal cohomology of structure 7 of the DHC,

$$\Lambda_7 = b(x_1^2 + x_2^2)\partial_1 \wedge \partial_2 + ((2b + c)x_1 - ax_2)x_3\partial_2 \wedge \partial_3 + (ax_1 + (2b + c)x_2)x_3\partial_3 \wedge \partial_1.$$

We obtained these upshots, using the same method as above for structure 2. Computations are quite long and will not be published here. We assume that $c \neq 0$, otherwise we recover structure 2.

In the following theorems, the Y_i ($i \in \{1, 2, 3\}$) denote the same vector fields as above, namely,

$$Y_1 = x_1\partial_1 + x_2\partial_2, Y_2 = x_1\partial_2 - x_2\partial_1, Y_3 = x_3\partial_3.$$

Moreover, we set

$$D' = x_1^2 + x_2^2, D = (x_1^2 + x_2^2)x_3.$$

If $\frac{b}{c} \in \mathbf{Q}, b(2b + c) < 0$, we denote by $(\beta, \gamma) \simeq (b, c)$ the irreducible representative of the rational number $\frac{b}{c}$, with positive denominator, $\beta \in \mathbf{Z}, \gamma \in \mathbf{N}^*$. If $\frac{b}{c} \in \mathbf{Q}, b(2b+c) > 0$, $(\beta, \gamma) \simeq (b, c)$ denotes the irreducible representative with positive numerator, $\beta \in \mathbf{N}^*, \gamma \in \mathbf{Z}^*$. Furthermore, we write Λ instead of Λ_7, $\bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij}$ instead of $\mathrm{Cas}(\Lambda)Y_{23} + \mathrm{Cas}(\Lambda)Y_{31} + \mathrm{Cas}(\Lambda)Y_{12}$, $\mathrm{Sing}(\Lambda) = \bigoplus_{r\geq 0} \mathrm{Sing}^r(\Lambda)$ instead of $\mathbf{R}[[x_3]] = \bigoplus_{r\geq 0} \mathbf{R}x_3^r$, and $C_\gamma Y_3$ ($\gamma \in \{2, 4, 6, \ldots\}$) instead of $\mathbf{R}D'^{\frac{\gamma}{2}-1}x_3^{-1}Y_3 = \mathbf{R}D'^{\frac{\gamma}{2}-1}\partial_3$.

Some comments on the results given in the theorems hereafter can be found below.

Theorem 5. *If $a \neq 0$, the cohomology spaces are*

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \mathbf{R}, \quad H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda)Y_i,$$

$$H^2(\Lambda) = \bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij}, \quad H^3(\Lambda) = \mathrm{Cas}(\Lambda)Y_{123} \oplus \mathrm{Sing}^0(\Lambda)\partial_{123}$$

Theorem 6. *If $a = 0$ and $b = 0$, the cohomology is*

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \bigoplus_{r\geq 0} \mathbf{R}D'^r, \quad H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda)Y_i,$$

$$H^2(\Lambda) = \bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij} \oplus \mathrm{Sing}(\Lambda)\partial_{12}, \quad H^3(\Lambda) = \mathrm{Cas}(\Lambda)Y_{123} \oplus \mathrm{Sing}(\Lambda)\partial_{123}$$

Theorem 7. *If $a = 0$ and $2b + c = 0$, the cohomology groups are*

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \bigoplus_{r\geq 0} \mathbf{R}x_3^r, \quad H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda)Y_i \oplus C_2Y_3,$$

$$H^2(\Lambda) = \bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij} \oplus \mathrm{Sing}(\Lambda)\partial_{12} \oplus C_2Y_3 \wedge (\mathbf{R}Y_1 + \mathbf{R}Y_2),$$

$$H^3(\Lambda) = \mathrm{Cas}(\Lambda)Y_{123} \oplus \mathrm{Sing}(\Lambda)\partial_{123} \oplus C_2Y_3 \wedge \mathbf{R}Y_{12}$$

Theorem 8. *If $a = 0$ and $\frac{b}{c} \notin \mathbf{Q}$ or $\frac{b}{c} \in \mathbf{Q}, b(2b+c) < 0$,*

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \mathbf{R}, \quad H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda)Y_i \oplus \begin{cases} (b,c) \simeq (-1,\gamma), \gamma \in \{4,6,8,\ldots\} : C_\gamma Y_3 \\ otherwise : 0 \end{cases},$$

$$H^2(\Lambda) = \bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij} \oplus \mathrm{Sing}(\Lambda)\partial_{12} \oplus \begin{cases} (b,c) \simeq (-1,\gamma), \gamma \in \{4,6,8,\ldots\} : \\ C_\gamma Y_3 \wedge (\mathbf{R}Y_1 + \mathbf{R}Y_2) \\ otherwise : \\ 0 \end{cases},$$

$$H^3(\Lambda) = \mathrm{Cas}(\Lambda)Y_{123} \oplus \mathrm{Sing}(\Lambda)\partial_{123} \oplus \begin{cases} (b,c) \simeq (-1,\gamma), \gamma \in \{4,6,8,\ldots\} : C_\gamma Y_3 \wedge \mathbf{R}Y_{12} \\ otherwise : 0 \end{cases}$$

Theorem 9. *If $a = 0$ and $\frac{b}{c} \in \mathbf{Q}, b(2b+c) > 0$,*

$$H^0(\Lambda) = \mathrm{Cas}(\Lambda) = \bigoplus_{n \in \mathbf{N}, n\gamma \in 2\mathbf{Z}} \mathbf{R}D'^{n\beta + \frac{n\gamma}{2}} x_3^{n\beta}, \quad H^1(\Lambda) = \bigoplus_i \mathrm{Cas}(\Lambda)Y_i,$$

$$H^2(\Lambda) = \bigoplus_{ij} \mathrm{Cas}(\Lambda)Y_{ij} \oplus \mathrm{Sing}(\Lambda)\partial_{12}, \quad H^3(\Lambda) = \mathrm{Cas}(\Lambda)Y_{123} \oplus \mathrm{Sing}(\Lambda)\partial_{123}$$

Remark. The preceding results allow to ascertain that Casimir functions are closely related with Koszul-exactness or "quasi-exactness" of the considered structure. Observe that $C_\gamma = \mathbf{R}D'^{-1+\frac{\gamma}{2}} x_3^{-1}$ ($\gamma \in \{2,4,6,\ldots\}$) has the same form as the basic Casimir in Theorem 9 and that the negative superscript in x_3^{-1} can only be compensated via multiplication by $Y_3 = x_3\partial_3$. Clearly, such a compensation is not possible for $\mathbf{R}D'^{-n+\frac{n\gamma}{2}} x_3^{-n}$, $n > 1$. Hence cocycle $C_\gamma Y_3$ is in some sense "Casimir-like" and "accidental". Note eventually that the "weight" of the singularities in cohomology increases with closeness of the considered Poisson structure to Koszul-exactness.

10 Suitable family of quadratic structures

The objective of this final section is to explain that our technique applies to all the quadratic Poisson classes induced by a special type of r-matrices.

It is well-known that the action tangent to the canonical action of the Lie group $GL(n, \mathbf{R})$ on $\mathbf{R}^n$, is the Lie algebra homomorphism

$$J : g := gl(n, \mathbf{R}) \ni a = (a_{ij}) \to a_{ij} x_i \partial_j \in Sec(T\mathbf{R}^n) =: \mathcal{X}^1(\mathbf{R}^n)$$

($Sec(T\mathbf{R}^n)$: Lie algebra of smooth sections of the tangent bundle $T\mathbf{R}^n$, i.e. Lie algebra of vector fields of $\mathbf{R}^n$; $x = (x_1, \ldots, x_n)$: canonical coordinates of $\mathbf{R}^n$; $\partial_1, \ldots, \partial_n$: partial derivatives with respect to these coordinates), which is a Lie isomorphism if valued in the Lie algebra $\mathcal{X}_0^1(\mathbf{R}^n)$ of linear vector fields.

Let us recall that a standard construction allows to associate to any Lie algebroid E a Gerstenhaber algebra, made up by a graded Poisson-Lie algebra structure on the shifted Grassmann algebra $Sec(\bigwedge E)[1]$ of multi-sections of E. We denote this Schouten-Nijenhuis superbracket, which extends the algebroid bracket on $Sec(E)$, by $[.,.]_E$ or simply by $[.,.]$, if no confusion is possible.

The above Lie homomorphism J extends to a Gerstenhaber homomorphism

$$J : \bigwedge g \to \mathcal{X}(\mathbf{R}^n) := Sec(\bigwedge T\mathbf{R}^n),$$

where the Gerstenhaber structures have been obtained as just mentioned. Let

$$\tilde{\bigwedge} \mathbf{R}^n = \bigoplus_k \left(\mathcal{S}^k(\mathbf{R}^n)^* \otimes \bigwedge{}^k \mathbf{R}^n \right)$$

be the Gerstenhaber subalgebra—of the algebra $\mathcal{X}(\mathbf{R}^n)$ of poly-vector fields—made up by k-vectors with coefficients in the corresponding space of homogeneous polynomials in $x \in \mathbf{R}^n$. It is obvious that J viewed as Gerstenhaber homomorphism with target algebra $\tilde{\bigwedge}\mathbf{R}^n$,

$$J : \bigwedge g \to \tilde{\bigwedge}\mathbf{R}^n,$$

is onto. It is also known that the restriction J^k to the space $\bigwedge^k g$ has a non-trivial kernel (provided that $k, n \geq 2$).

Remember that a classical r-matrix is a bi-matrix $r \in g \wedge g$ that verifies the classical Yang-Baxter equation $[r, r] = 0$. The space $\mathcal{S}^2(\mathbf{R}^n)^* \otimes \bigwedge^2 \mathbf{R}^n$ of quadratic bi-vectors coincides with the image $J^2(g \wedge g)$ and

$$J^3[r, r] = [J^2 r, J^2 r],\ r \in g \wedge g.$$

So any bi-vector $\Lambda = J^2 r$ that is the image of an r-matrix is a quadratic Poisson structure of $\mathbf{R}^n$. Conversely, any quadratic Poisson tensor of $\mathbf{R}^n$ is induced by at least one bi-matrix. However, the characterization of those quadratic Poisson structures that are implemented by an r-matrix, is an open problem (see [MMR02]). Quadratic Poisson structures generated by an r-matrix are of importance e.g. in deformation quantization, in particular in view of Drinfeld's method.

One easily understands that the study of r-matrix induced structures involves the orbit O_Λ and stabilizer G_Λ of the considered structure Λ for the canonical action on tensors of the general linear group $GL(n, \mathbf{R})$. Our paper is confined to the three-dimensional Euclidean setting. Let us recall that the isotropy group G_Λ is a Lie subgroup of $G := GL(3, \mathbf{R})$, the Lie algebra of which is the stabilizer g_Λ of Λ for the corresponding infinitesimal action,

$$g_\Lambda = \{a \in gl(3, \mathbf{R}) : [Ja, \Lambda] = 0\}.$$

As already stated, the objective of this paper is to provide a universal approach to the formal Poisson cohomology for a broad family of quadratic structures in the classical physical space. Let us just indicate that the classes of the DHC that are accessible to our modus operandi are exactly those classes that are implemented by r-matrices in $g_\Lambda \wedge g_\Lambda$. The quadratic Poisson tensors can defacto be classified according to their membership in the family of structures induced by an r-matrix in the "stabilizer". It then turns out that the members of this family are those tensors that read as a linear combination of wedge products of mutually commuting linear vector fields, hence those that are accessible to the above detailed and applied technique.

We refer to the classes implemented by an r-matrix in the stabilizer as admissible classes. The DHC gives the quadratic Poisson structures up to linear transformations. Let us finally mention that admissibility is of course effectively independent of the chosen representative. This means that if $\Lambda = J^2 r, r \in g_\Lambda \wedge g_\Lambda, [r, r] = 0$, then any equivalent structure $A_*\Lambda, A \in G$—A_* denotes the natural action of A—has the same property, i.e. $A_*\Lambda$ is also induced by an r-matrix and this matrix can be chosen in the stabilizer of $A_*\Lambda$. Indeed, it is easily seen that the orbit of $J^2 r$ for the canonical G-action, is nothing but the pointwise image of the orbit of r for the adjoint action Ad of G,

$$A_*(J^2 r) = J^2(\mathrm{Ad}(A)r).$$

Since the adjoint action respects the Schouten-Nijenhuis bracket, $\mathrm{Ad}(A)r$ is an r-matrix in the stabilizer of $A_*\Lambda$.

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