Lie algebraic characterization of manifolds*

Janusz Grabowski†, Norbert Poncin‡

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Abstract

Results on characterization of manifolds in terms of certain Lie algebras growing on them, especially Lie algebras of differential operators, are reviewed and extended. In particular, we prove that a smooth (real-analytic, Stein) manifold is characterized by the corresponding Lie algebra of linear differential operators, i.e. isomorphisms of such Lie algebras are induced by the appropriate class of diffeomorphisms of the underlying manifolds.

Keywords: algebraic characterization; smooth, real-analytic, and Stein manifolds; automorphisms; Lie algebras; differential operators; principal symbols

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1 Algebraic characterizations of manifold structures

Algebraic characterizations of topological spaces and manifolds can be traced back to the work of I. Gel'fand and A. Kolmogoroff [GK37] in which compact topological spaces K are characterized by the algebras $\mathcal{A} = C(K)$ of continuous functions on them. In particular, points p of these spaces are identified with maximal ideals p^* in these algebras consisting of functions vanishing at p. This identification easily implies that isomorphisms of the algebras $C(K_1)$ and $C(K_2)$ are induced by homeomorphisms between K_1 and K_2, so that the algebraic structure of $C(K)$ characterizes K uniquely up to homeomorphism. Here $C(K)$ may consist of complex or real functions as well.

All above can be carried over when we replace K with a compact smooth manifold M and $C(K)$ with the algebra $\mathcal{A} = C^\infty(M)$ of all real smooth functions on M. In this case algebraic isomorphisms between $C^\infty(M_1)$ and $C^\infty(M_2)$ are induced by diffeomorphisms $\phi : M_2 \to M_1$, i.e. they are of the form $\phi^*(f) = f \circ \phi$. If our manifold is non-compact then no longer maximal ideals must be of the form p^*. However, there is still an algebraic characterization of ideals p^* as one-codimensional (or just maximal finite-codimensional) ones, so that the

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†E-mail: jagrab@impan.gov.pl

‡E-mail: norbert.poncin@uni.lu

algebraic structure of $C^\infty(M)$ still characterizes M, for M being Hausdorff and second countable. A similar result is true in the real-analytic (respectively the holomorphic) case, i.e. when we assume that M is a real-analytic (respectively a Stein) manifold and that $\mathcal{A}$ is the algebra of all real-analytic (respectively all holomorphic) functions on M (see [Gra78]). As it was recently pointed out to us by Alan Weinstein, the assumption that M is second countable is crucial for the standard proofs that one-codimensional ideals are of the form p^*. This remark resulted in alternative proofs [Mrč03, Gra03], which are valid without additional assumptions about manifolds.

There are other algebraic structures canonically associated with a smooth manifold M, for example the Lie algebra $\mathcal{X}(M)$ of all smooth vector fields on M or the associative (or Lie) algebra $\mathcal{D}(M)$ of linear differential operators acting on $C^\infty(M)$. Characterization of manifolds by associated Lie algebras is a topic initiated in 1954 with the well-known paper [PS54] of L.E. Pursell and M.E. Shanks that appeared in the Proceedings of the American Mathematical Society.

The main theorem of this article states that, if the Lie algebras $\mathcal{X}_c(M_i)$ of smooth compactly supported vector fields of two smooth manifolds M_i ($i \in \{1,2\}$) are isomorphic Lie algebras, then the underlying manifolds are diffeomorphic, and—of course—vice versa. The central idea of the proof of this result is the following. If M is a smooth variety and p a point of M, denote by $\mathcal{I}(\mathcal{X}_c(M))$ the set of maximal ideals of $\mathcal{X}_c(M)$ and by p^∞ the maximal ideal

$$p^\infty = \{X \in \mathcal{X}_c(M) : X \text{ is flat at } p\}.$$

The map $p \in M \to p^\infty \in \mathcal{I}(\mathcal{X}_c(M))$ being a bijection and the property "maximal ideal" being an algebra-isomorphism invariant, the correspondence

$$\begin{array}{ccc} p_1^\infty \in \mathcal{I}(\mathcal{X}_c(M_1)) & \to & p_2^\infty \in \mathcal{I}(\mathcal{X}_c(M_2)) \\ \uparrow & & \downarrow \\ p_1 \in M_1 & & p_2 \in M_2 \end{array}$$

is a bijection and even a diffeomorphism.

Similar upshots exist in the analytic cases. Note that here "flat" means zero and that the Lie algebra of all $\mathbf{R}$-analytic vector fields of an $\mathbf{R}$-analytic compact connected manifold is simple, so has no proper ideals. Hence, the above maximal ideals in particular, and ideals in general, are of no use in these cases. Maximal finite-codimensional subalgebras turned out to be an efficient substitute for maximal ideals. This idea appeared around 1975 in several papers, e.g. in [Ame75], and is the basis of the general algebraic framework developed in [Gra78] and containing the smooth, the $\mathbf{R}$-analytic and the holomorphic cases. Here, if $\mathcal{M}(\mathcal{X}_\bullet(M))$ is the set of all maximal finite-codimensional subalgebras of $\mathcal{X}_\bullet(M)$, where subscript $\bullet$ means smooth, $\mathbf{R}$-analytic or holomorphic, the fundamental bijection is $p \in M \to p^0 \in \mathcal{M}(\mathcal{X}_\bullet(M))$ with

$$p^0 = \{X \in \mathcal{X}_\bullet(M) : (Xf)(p) = 0, \forall f \in C^\bullet(M)\}.$$

This method works well for a large class of the Lie algebras of vector fields which are simultaneously modules over the algebra of functions $\mathcal{A}$. A pure algebraic framework in this direction has been developed by S. M. Skryabin [Skr87] who has proven a very general "algebraic Pursell-Shanks theorem":

Theorem 1 *If $\mathcal{L}_i$ is a Lie subalgebra in the Lie algebra $Der\mathcal{A}_i$ of derivations of a commutative associative unital algebra $\mathcal{A}_i$ over a field of characteristic 0 which is simultaneously an $\mathcal{A}_i$-submodule of $Der\mathcal{A}_i$ and is non-singular in the sense that $\mathcal{L}_i(\mathcal{A}_i) = \mathcal{A}_i$, $i = 1, 2$, then any Lie algebra isomorphism $\Phi : \mathcal{L}_1 \to \mathcal{L}_2$ is induced by an associative algebra isomorphism $\phi^* : \mathcal{A}_2 \to \mathcal{A}_1$, i.e.*

$$\Phi(X)(f) = (\phi^*)^{-1}(X(\phi^*(f))).$$

Note that Skryabin's proof does not refer to the structure of maximal ideals in $\mathcal{A}$ but uses the $\mathcal{A}$-module structure on $\mathcal{L}$.

Other types of Lie algebras of vector fields (see e.g. [KMO77]) have also been considered but the corresponding methods have been developed for each case separately. Let us mention the Lie algebras of vector fields preserving a given submanifold [Kor74], a given (generalized) foliation [Gra93], a symplectic or contact form [Omo76], the Lie algebras of Hamiltonian vector fields or Poisson brackets of functions on a symplectic manifold [AG90] and Jacobi brackets in general [Gra00], Lie algebras of vector fields on orbit spaces and G-manifolds [Abe82], Lie algebras of vector fields on affine and toric varieties [HM93, CGM99, Sie96], Lie algebroids [GG01], and many others.

In our work [GP03], we have examined the Lie algebra $\mathcal{D}(M)$ of all linear differential operators on the space $C^\infty(M)$ of smooth functions of M, its Lie subalgebra $\mathcal{D}^1(M)$ of all first-order differential operators and the Poisson algebra $\mathcal{S}(M) = Pol(T^*M)$ of all polynomial functions on the cotangent bundle, the symbols of the operators in $\mathcal{D}(M)$. We have obtained in each case Pursell-Shanks type results in a purely algebraic way. Furthermore, we have provided an explicit description of all the automorphisms of any of these Lie algebras.

In this notes we depict this last paper assuming a philosophical and pedagogic point of view, we prove a general algebraic Pursell-Shanks type result, and, with the help of some topology, we extend our smooth Pursell-Shanks type result from [GP03] to the real-analytic and holomorphic cases.

2 Lie algebras of differential operators

2.1 Abstract definitions

The goal being a work on the algebraic level, we must define some general algebra, modelled on $\mathcal{D}(M)$, call it a quantum Poisson algebra. A quantum Poisson algebra is an associative *filtered* algebra $\mathcal{D} = \cup_i \mathcal{D}^i$ ($\mathcal{D}^i = \{0\}$, for $i < 0$) with unit 1,

$$\mathcal{D}^i \cdot \mathcal{D}^j \subset \mathcal{D}^{i+j},$$

such that the canonical *Lie bracket* verifies

$$[\mathcal{D}^i, \mathcal{D}^j] \subset \mathcal{D}^{i+j-1}. \tag{1}$$

Note that $\mathcal{A} = \mathcal{D}^0$ is an associative commutative subalgebra of $\mathcal{D}$ and that K, the underlying field, is naturally imbedded in $\mathcal{A}$.

Similarly we heave the algebra $\mathcal{S}(M) = Pol(T^*M)$ (or $\mathcal{S}(M) = \Gamma(\mathcal{S}TM)$) of smooth functions on T^*M that are polynomial along the fibers (respectively, of symmetric contravariant tensor fields of M), classical counterpart of $\mathcal{D}(M)$, on the algebraic level and define a classical Poisson algebra as a *commutative* associative *graded* algebra $\mathcal{S} = \oplus_i \mathcal{S}_i$ ($\mathcal{S}_i = \{0\}$, for $i < 0$) with unit 1,

$$\mathcal{S}_i \mathcal{S}_j \subset \mathcal{S}_{i+j},$$

which is equipped with a *Poisson* bracket $\{.,.\}$ such that

$$\{\mathcal{S}_i, \mathcal{S}_j\} \subset \mathcal{S}_{i+j-1}.$$

Here $\mathcal{A} = \mathcal{S}_0$ is obviously an associative and Lie-commutative subalgebra of $\mathcal{S}$.

Let us point out that quantum Poisson algebras canonically induce classical Poisson algebras. Indeed, starting from $\mathcal{D} = \cup_i \mathcal{D}^i$, we get a graded vector space when setting $\mathcal{S}_i = \mathcal{D}^i / \mathcal{D}^{i-1}$. If the degree $deg(D)$ of an arbitrary non-zero $D \in \mathcal{D}$ is given by the lowest filter that contains D and if cl_i denotes the class in the quotient $\mathcal{S}_i$, we define the principal symbol $\sigma(D)$ of D by

$$\sigma(D) = \mathrm{cl}_{deg(D)}(D)$$

and the symbol $\sigma_i(D)$ of order $i \geq deg(D)$ by

$$\sigma_i(D) = \mathrm{cl}_i(D) = \begin{cases} 0, \text{ if } i > deg(D), \\ \sigma(D), \text{ if } i = deg(D). \end{cases}$$

Now the *commutative* associative multiplication and the *Poisson* bracket are obtained as follows:

$$\sigma(D_1)\sigma(D_2) = \sigma_{deg(D_1)+deg(D_2)}(D_1 \cdot D_2) \ \ (D_1, D_2 \in \mathcal{D})$$

and

$$\{\sigma(D_1), \sigma(D_2)\} = \sigma_{deg(D_1)+deg(D_2)-1}([D_1, D_2]) \ \ (D_1, D_2 \in \mathcal{D}).$$

Remark that the commutativity of the associative multiplication is a direct consequence of Equation (1) and that Leibniz's rule simply passes from $[.,.]$ to $\{.,.\}$. So "dequantization" is actually a passage from non-commutativity to commutativity, the trace of non-commutativity on the classical level being the Poisson bracket. Note also that the quantum Poisson algebra $\mathcal{D}$ and the induced classical limit $\mathcal{S}$ of $\mathcal{D}$ have the same basic algebra $\mathcal{A}$. The principal symbol map $\sigma : \mathcal{D} \to \mathcal{S}$ has the following important property:

$$\{\sigma(D_1), \sigma(D_2)\} = \begin{cases} \sigma([D_1, D_2]) \\ \text{or} \\ 0 \end{cases}.$$

There is a canonical quantum Poisson algebra associated with any unital associative commutative algebra $\mathcal{A}$, namely the algebra $\mathcal{D}(\mathcal{A})$ of linear differential operators on $\mathcal{A}$. Note that this algebraic approach to differential operators goes back to some ideas of Grothendieck and that it was extensively developed by A. M. Vinogradov. The filtration $\mathcal{D}^i(\mathcal{A})$ is defined inductively: $\mathcal{D}^0(\mathcal{A}) = \{m_f : f \in \mathcal{A}\}$, where $m_f(g) = f \cdot g$ (so $\mathcal{D}^0(\mathcal{A})$ is canonically isomorphic with $\mathcal{A}$), and

$$\mathcal{D}^{i+1}(\mathcal{A}) = \{D \in Hom(\mathcal{A}) : [D, m_f] \in \mathcal{D}^i(\mathcal{A}), \ \textit{for all } f \in \mathcal{A}\},$$

where $[.,.]$ is the commutator.

It can be seen that, in the fundamental example $\mathcal{A} = C^\infty(M)$, we get $\mathcal{D} = \mathcal{D}(\mathcal{A}) = \mathcal{D}(M), \mathcal{S} = \mathcal{S}(M) = Pol(T^*M) = \Gamma(\mathcal{S}TM)$ and the above algebraically defined Poisson bracket coincides with the canonical Poisson bracket on $\mathcal{S}$, i.e. the standard symplectic bracket in the first interpretation of $\mathcal{S}$, and the symmetric Schouten bracket in the second. The situation is completely analogous in real-analytic and holomorphic cases. Note only that in the holomorphic case the role of T^*M is played by a complex vector bundle $T^*_{(1,0)}M$ over M whose sections are holomorphic 1-forms of the type $(1,0)$ and the Poisson structure on $T^*_{(1,0)}M$ is represented by a holomorphic bivector field being a combination of wedge products of vector fields of type $(1,0)$.

2.2 Key-idea

Remember that the first objective is to establish Pursell-Shanks type results, that is—roughly speaking—to deduce a geometric conclusion from algebraic information. As in previous papers on this topic, functions should play a central role. So our initial concern is to obtain an algebraic characterization of "functions", i.e., in the general algebraic context of an arbitrary quantum Poisson algebra $\mathcal{D}$, of $\mathcal{A} \subset \mathcal{D}$, and more generally of all filters $\mathcal{D}^i \subset \mathcal{D}$.

It is clear that

$$\begin{aligned} \mathcal{A} &\subset \mathrm{Nil}(\mathcal{D}) \\ &:= \{D \in \mathcal{D} : \forall \Delta \in \mathcal{D}, \exists n \in \mathbf{N} : \overbrace{[D,[D,\ldots[D}^{n},\Delta]]] = 0\} \end{aligned}$$

and that

$$\mathcal{D}^{i+1} \subset \{D \in \mathcal{D} : [D,\mathcal{A}] \subset \mathcal{D}^i\} \ (i \geq -1).$$

Our conjecture is that functions (respectively, $(i+1)$th order "differential operators") are the only locally nilpotent operators, i.e. the sole operators whose repeated adjoint action upon any operator ends up by zero (respectively, the only operators for which the commutator with all functions is of order i).

It turns out that both guesses are confirmed if we show that for any $D \in \mathcal{D}$,

$$\forall f \in \mathcal{A}, \exists n \in \mathbf{N} : \overbrace{[D,[D,\ldots[D}^{n},f]]] = 0 \Longrightarrow D \in \mathcal{A}.$$

We call this property, which states that if the repeated adjoint action of an operator upon any function ends up by zero then this operator is a function, the distinguishing property of the Lie bracket.

At this point it is natural to ask if any bracket is distinguishing and—in the negative—if the commutator bracket of $\mathcal{D}(M)$ is. Obviously, the algebra of all linear differential operators acting on the polynomials in a variable $x \in \mathbf{R}$ is a simple example of a non-distinguishing Lie algebra. It suffices to consider the operator d/dx. The answer to the second question is positive. We refrain from describing here the technical constructive proof given in [GP03]. We will present further another topological proof, which can be adapted to real-analytic and holomorphic cases.

2.3 An algebraic Pursell-Shanks type result

We aspire to give an algebraic proof of the theorem stating that two manifolds M_1 and M_2 are diffeomorphic if the Lie algebras $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$ are isomorphic. So let us consider a Lie algebra isomorphism

$$\Phi : \mathcal{D}_1 \to \mathcal{D}_2$$

between two quantum Poisson algebras $\mathcal{D}_1$ and $\mathcal{D}_2$.

In the following, we discuss two necessary assumptions.

2.3.1 Distinguishing property

The next proposition is a first step towards our aim.

If $\mathcal{D}_1, \mathcal{D}_2$ are *distinguishing* quantum Poisson algebras then Φ respects the filtration.

The proof is by induction on the "order of differentiation" and uses the above algebraic characterizations of functions and filters, hence the distinguishing character of $\mathcal{D}_1$ and $\mathcal{D}_2$. For instance, $\Phi(\mathcal{A}_1) = \Phi(\mathrm{Nil}(\mathcal{D}_1)) = \mathrm{Nil}(\mathcal{D}_2) = \mathcal{A}_2$ (so that in particular $\Phi(\mathcal{D}_1^0) \subset \mathcal{D}_2^0$).

2.3.2 Non-singularity property

Since $\Phi(\mathcal{A}_1) = \mathcal{A}_2$, the Lie algebra isomorphism Φ restricts to a vector space isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$. If this restriction respected the associative multiplication it would be an associative algebra isomorphism, which—as well known—would in the geometric context, $\mathcal{D}_i = \mathcal{D}(M_i)$ $(i \in \{1, 2\})$, be induced by a diffeomorphism between M_1 and M_2.

So the question arises if we are able to deduce from the Lie algebra structure any information regarding the associative algebra structure and in particular the left and right multiplications $\ell_f : \mathcal{D} \ni D \to f \cdot D \in \mathcal{D}$ and $r_f : \mathcal{D} \ni D \to D \cdot f \in \mathcal{D}$ by a function $f \in \mathcal{A}$.

Observe first that ℓ_f and r_f commute with the adjoint action by functions, i.e. are members of the centralizer of $ad\,\mathcal{A}$ in the Lie algebra $End(\mathcal{D})$ of endomorphisms of $\mathcal{D}$, which is the Lie subalgebra $\mathcal{C}(\mathcal{D}) = \{\Psi \in End(\mathcal{D}) : \Psi \circ ad\,\mathcal{A} = ad\,\mathcal{A} \circ \Psi\}$.

On the other hand, it is not possible to extract from the Lie bracket more information than

$$\ell_f - r_f = ad\,f, \tag{2}$$

where the right hand side is of course a *lowering* member of the centralizer, i.e. a mapping in the centralizer which lowers the order of differential operators.

Thus the centralizer might be the brain wave. In particular it should, in view of (2), be possible to describe it as the algebra of those endomorphisms Ψ of $\mathcal{D}$ that respect the filtration and are of the form $\Psi = \ell_f + \Psi_1$, where $f = \Psi(1)$ and $\Psi_1 \in \mathcal{C}(\mathcal{D})$ is lowering. When trying to prove this conjecture, we realize that it holds if $[\mathcal{D}^1, \mathcal{A}] = \mathcal{A}$, in the sense that any function is a finite sum of brackets. In the geometric context this means that $[\mathcal{X}(M), C^\infty(M)] = \mathcal{X}(M)\,(C^\infty(M)) = C^\infty(M)$, a non-singularity assumption that appears in many papers of this type, e.g. [Gra78, Skr87], and is of course verified. Hence, a second proposition:

If $\mathcal{D}$ is a *non-singular* and distinguishing quantum Poisson algebra then any $\Psi \in \mathcal{C}(\mathcal{D})$ respects the filtration and $\Psi = \ell_{\Psi(1)} + \Psi_1$, $\Psi_1 \in \mathcal{C}(\mathcal{D})$ being lowering.

2.3.3 Isomorphisms

Having in view to use the centralizer to show that Φ respects the associative multiplication, we must visibly read the Lie algebra isomorphism $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$ as Lie algebra isomorphism $\Phi_* : End(\mathcal{D}_1) \to End(\mathcal{D}_2)$. We only need set $\Phi_*(\Psi) = \Phi \circ \Psi \circ \Phi^{-1}$, $\Psi \in End(\mathcal{D}_1)$. As $\Phi_*(\mathcal{C}(\mathcal{D}_1)) = \mathcal{C}(\mathcal{D}_2)$, it follows from the above depicted structure of the centralizer that for any $f, g \in \mathcal{A}_1$, $\Phi_*(\ell_f)(\Phi(g)) = (\Phi_*(\ell_f))\,(1) \cdot \Phi(g)$, i.e.

$$\Phi(f \cdot g) = \Phi(f \cdot \Phi^{-1}(1)) \cdot \Phi(g). \tag{3}$$

Remark that $\zeta = \Phi^{-1}(1) \in \mathcal{Z}(\mathcal{D}_1)$, where $\mathcal{Z}(\mathcal{D}_1)$ denotes the center of the Lie algebra $\mathcal{D}_1$. In view of (3),

$$\Phi(f \cdot g) = \Phi(g \cdot f) = \Phi(f) \cdot \Phi(g \cdot \zeta) = \Phi(f) \cdot \Phi(\zeta \cdot g) = \Phi(\zeta^2) \cdot \Phi(f) \cdot \Phi(g), \tag{4}$$

for all $f, g \in \mathcal{A}_1$. This in turn implies that $\Phi(\zeta^2)$ is invertible, say $\Phi(\zeta^2) = \kappa^{-1}$, since the image of Φ, so $\mathcal{A}_2$, is contained in the ideal generated by $\Phi(\zeta^2)$. If we put $A(f) = \kappa^{-1} \cdot \Phi(f)$, then, due to (4), $A(f \cdot g) = A(f) \cdot A(g)$, so A is an associative algebra isomorphism. Thus we get the following algebraic Pursell-Shanks type theorem.

Theorem 2 *Let $\mathcal{D}_i$ ($i \in \{1, 2\}$) be non-singular and distinguishing quantum Poisson algebras. Then every Lie algebra isomorphism $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$ respects the filtration and its restriction $\Phi|_{\mathcal{A}_1}$ to $\mathcal{A}_1$ has the form $\Phi|_{\mathcal{A}_1} = \kappa A$, where $\kappa \in \mathcal{A}_2$ is invertible and central in $\mathcal{D}_2$ and $A : \mathcal{A}_1 \to \mathcal{A}_2$ is an associative algebra isomorphism.*

Remark. The central elements in $\mathcal{D}(M)$ are just constants. This immediately follows from the well-known corresponding property of the symplectic Poisson bracket on $\mathcal{S} = \mathcal{S}(M) = Pol(T^*M)$. If this symplectic property holds good for the classical limit $\mathcal{S}$ of $\mathcal{D}$, we say that $\mathcal{D}$ is *symplectic*. We have assumed this property to obtain a version of Theorem 2 in [GP03]. Now we have proven that this assumption is superfluous.

2.4 Isomorphisms of the Lie algebras of differential operators

Let now M be a finite-dimensional paracompact and second countable smooth (respectively real-analytic, Stein) manifold, let $\mathcal{A} = \mathcal{A}(M)$ be the commutative associative algebra of all real smooth (respectively real-analytic, holomorphic) functions on M, let $\mathcal{D} = \mathcal{D}(M) = \mathcal{D}(\mathcal{A}(M))$ be the corresponding algebra of differential operators, and let $\mathcal{S} = \mathcal{S}(M)$ be the classical limit of $\mathcal{D}(M)$.

Lemma 1 *The quantum Poisson algebra $\mathcal{D}(M)$ is distinguishing and non-singular.*

Proof. Let us work first in the smooth case. Let D be a linear differential operator on $\mathcal{A}(M)$ such that for every $f \in \mathcal{A}(M)$ there is n for which $(ad_D)^n(f) = 0$, where $ad_D(D') = [D, D']$ is the commutator in the algebra of differential operators. It suffices to show that $D \in \mathcal{A}(M)$.

The algebra $\mathcal{A}(M)$ admits a complete metric, which makes it into a topological algebra such that all linear differential operators are continuous (see section 6 of the book [KM97]). It is then easy to see that $Ker((ad_D)^n) = \{f \in \mathcal{A}(M) : (ad_D)^n(f) = 0\}$, $n = 1, 2, \ldots$, is a closed subspace of $\mathcal{A}(M)$. By assumption, $\bigcup_n Ker((ad_D)^n) = \mathcal{A}(M)$, so $Ker((ad_D)^{n_0}) = \mathcal{A}(M)$ for a certain n_0 according to the Baire property of the topology on $\mathcal{A}(M)$. Passing now to principal symbols, we can write $X_D^{n_0}(f) = 0$ for all $f \in \mathcal{A}(M)$, where X_D is the Hamiltonian vector field on T^*M of the principal symbol $\sigma(D)$ of D with respect to the canonical Poisson bracket on T^*M. Here we regard $\mathcal{A}(M)$ as canonically embedded in the algebra of polynomial functions on T^*M. Hence the 0-order operator $(ad_g)^{n_0}(X_D^{n_0})$, which is the multiplication by $n_0!(X_D(g))^{n_0}$, vanishes on $\mathcal{A}(M)$ for all $g \in \mathcal{A}(M)$, so $X_D(g) = \{\sigma(D), g\} = 0$ for all $g \in \mathcal{A}(M)$. This in turn implies that $\sigma(D) \in \mathcal{A}(M)$, so $D \in \mathcal{A}(M)$.

The proof in the holomorphic case is completely analogous (for the topology we refer to section 8 of [KM97]), so let us pass to the real-analytic case. Now, the natural topology is not completely metrizable and we cannot apply the above procedure. However, there is a Stein neighbourhood $\tilde{M}$ of M, i.e. a Stein manifold $\tilde{M}$ of complex dimension equal to the dimension of M, containing M as a real-analytic closed submanifold, whose germ along M is unique [Gt58], so that D can be complexified to a linear holomorphic differential operator $\tilde{D}$. The complexified operator has clearly the analogous property: for every holomorphic function f on $\tilde{M}$ there is n for which $(ad_{\tilde{D}})^n(f) = 0$, so $\tilde{D}$, thus D, is a multiplication by a function.

The non-singularity of $\mathcal{D}(M)$ follows directly from Proposition 3.5 of [Gra78]. ∎

The above Lemma shows that we can apply Theorem 2 to $\mathcal{D}(M)$ in all, i.e. smooth, real-analytic, and holomorphic, cases. We obtain in the same way—mutatis mutandis—Pursell-Shanks type results for the Lie algebras $\mathcal{S}(M)$ and $\mathcal{D}^1(M)$:

Theorem 3 *The Lie algebras $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$ (respectively $\mathcal{S}(M_1)$ and $\mathcal{S}(M_2)$, or $\mathcal{D}^1(M_1)$ and $\mathcal{D}^1(M_2)$) of all differential operators (respectively all symmetric contravariant tensors, or all differential operators of order 1) on two smooth (respectively real-analytic, holomorphic) manifolds M_1 and M_2 are isomorphic if and only if the manifolds M_1 and M_2 are smoothly (respectively bianalytically, biholomorphically) diffeomorphic.*

2.5 Automorphisms of the Lie algebras of differential operators

In view of the above Pursell-Shanks type results, the study of the Lie algebra isomorphisms, e.g. between $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$, can be reduced to the examination of the corresponding Lie algebra automorphisms, say $\Phi \in Aut(\mathcal{D}(M), [.,.])$. The standard idea in this kind of problems is the simplification of the considered arbitrary automorphism Φ via multiplication by automorphisms identified a priori. Here, the automorphism $A = \kappa^{-1}\Phi \in Aut(C^\infty(M), \cdot)$ induced by Φ (see Theorem 2) can canonically be extended to an automorphism $A_* \in Aut(\mathcal{D}(M), [.,.])$. It suffices to set $A_*(D) = A \circ D \circ A^{-1}$, for any $D \in \mathcal{D}(M)$. Then, evidently,

$$\Phi_1 = A_*^{-1} \circ \Phi \in Aut(\mathcal{D}(M), [.,.]) \text{ and } \Phi_1|_{C^\infty(M)} = \kappa\, id,$$

where id is the identity map of $C^\infty(M)$.

Some notations are necessary. In the following, we use the canonical splitting $\mathcal{D}(M) = C^\infty(M) \oplus \mathcal{D}_c(M)$, where $\mathcal{D}_c(M)$ is the Lie algebra of differential operators vanishing on constants. Moreover, we denote by $\bullet_0$ and $\bullet_c$ the projections onto $C^\infty(M) = \mathcal{D}^0(M)$ and $\mathcal{D}_c(M)$ respectively.

2.5.1 Automorphisms of $\mathcal{D}^1(\mathbf{M})$

The formerly explained reduction to the problem of the determination of all automorphisms Φ_1, which coincide with $\kappa\, id$ on functions, is still valid. Furthermore, in the case of the Lie algebra $\mathcal{D}^1(M)$ the preceding splitting reads $\mathcal{D}^1(M) = C^\infty(M) \oplus \mathcal{D}^1_c(M) = C^\infty(M) \oplus \mathcal{X}(M)$.

It follows from the automorphism property that

$$(\Phi_1)_c|_{\mathcal{X}(M)} = id \text{ and } (\Phi_1)_0|_{\mathcal{X}(M)} \in \mathcal{Z}^1(\mathcal{X}(M), C^\infty(M)),$$

where $\mathcal{Z}^1(\mathcal{X}(M), C^\infty(M))$ is the space of 1-cocycles of the Lie algebra of vector fields canonically represented upon functions by Lie derivatives. We know that these cocycles are locally given by

$$(\Phi_1)_0|_{\mathcal{X}(M)} = \lambda\, div + df, \tag{5}$$

where $\lambda \in \mathbf{R}$ and f is a smooth function. When trying to globally define the right hand side of Equation (5), we naturally substitute a closed 1-form $\omega \in \Omega^1(M) \cap ker\, d$ to the exact 1-form df. In order to globalize the divergence div, note the following. If M is oriented by a volume Ω, we have $div_\Omega = div_{-\Omega}$. So the divergence can be defined with respect to $\mid \Omega \mid$. This pseudo-volume may be viewed as a pair $\{\Omega, -\Omega\}$ and exists on any manifold, orientable or not. Alternatively, we may interpret $\mid \Omega \mid$ as a nowhere vanishing tensor density of weight 1, i.e. as a section of the vector bundle $\mathbb{F}_1(TM)$ of 1-densities, which is everywhere non-zero. This bundle being of rank 1, the existence of such a section is equivalent to the triviality of the bundle. However, the fiber bundle $\mathbb{F}_1(TM)$ is known to be trivial for any manifold M. Thus, a nowhere vanishing tensor 1-density ρ_0 always exists and the divergence can be defined with respect to this ρ_0. For a more rigorous approach the reader is referred to [GP03]. Eventually, the above cocycles globally read

$$(\Phi_1)_0|_{\mathcal{X}(M)} = \lambda\, div_{\rho_0} + \omega.$$

Let us fix a divergence div. For any $f + X \in \mathcal{D}^1(M) = C^\infty(M) \oplus \mathcal{X}(M)$, we then obtain

$$\Phi_1(f + X) = \kappa f + \lambda\, div\, X + \omega(X) + X.$$

Since the initial arbitrary automorphism $\Phi \in Aut(\mathcal{D}^1(M), [.,.])$ has been decomposed as $\Phi = A_* \circ \Phi_1$, with A_* induced by a diffeomorphism $\varphi \in Diff(M)$ and denoted φ_* below, we finally have the theorem:

Theorem 4 *A linear map $\Phi : \mathcal{D}^1(M) \to \mathcal{D}^1(M)$ is an automorphism of the Lie algebra $\mathcal{D}^1(M) = C^\infty(M) \oplus \mathcal{X}(M)$ of linear first-order differential operators on $C^\infty(M)$ if and only if it can be written in the form*

$$\Phi(f + X) = (\kappa f + \lambda\, div\, X + \omega(X)) \circ \varphi^{-1} + \varphi_*(X),$$

where φ is a diffeomorphism of M, λ, κ are constants ($\kappa \neq 0$), ω is a closed 1-form on M, and φ_* is defined by

$$(\varphi_*(X))(f) = (X(f \circ \varphi)) \circ \varphi^{-1}.$$

All the objects $\varphi, \lambda, \kappa, \omega$ are uniquely determined by Φ.

2.5.2 Automorphisms of $\mathcal{D}(\mathbf{M})$

As previously, we need only seek the automorphisms $\Phi_1 \in Aut(\mathcal{D}(M), [.,.])$, such that $\Phi_1(f) = \kappa f$, $f \in C^\infty(M)$. Such an automorphism visibly restricts to a similar automorphism of the Lie algebra $\mathcal{D}^1(M)$. Hence,

$$\Phi_1(f + X) = \kappa f + \lambda\, div\, X + \omega(X) + X. \tag{6}$$

Since $\omega \in \mathcal{Z}^1(\mathcal{X}(M), C^\infty(M))$, it is reasonable to think that ω might be extended to $\overline{\omega} \in \mathcal{Z}^1(\mathcal{D}(M), \mathcal{D}(M)) = Der(\mathcal{D}(M))$, where $Der(\mathcal{D}(M))$ is the Lie algebra of all derivations of $(\mathcal{D}(M), [.,.])$. If in addition this derivation $\overline{\omega}$ were lowering, it would generate an automorphism $e^{\overline{\omega}}$, which could possibly be used to cancel the term $\omega(X)$ in Equation (6). But ω is locally exact, $\omega|_U = df_U$ (U: open subset of M, $f_U \in C^\infty(U)$). So it suffices to ensure that the inner derivations associated to the functions f_U glue together. Lastly,

$$\overline{\omega}(D)|_U = [D|_U, f_U],$$

$D \in \mathcal{D}(M)$, and $\Phi_2 = \Phi_1 \circ e^{-\kappa^{-1}\overline{\omega}} \in Aut(\mathcal{D}(M), [.,.])$ actually verifies

$$\Phi_2(f + X) = \kappa f + \lambda\, div\, X + X. \tag{7}$$

It is interesting to note that the automorphism $e^{\overline{\omega}}$ is, for $\omega = df$ ($f \in C^\infty(M)$), simply the inner automorphism $e^{\overline{\omega}} : \mathcal{D}(M) \ni D \to e^f \cdot D \cdot e^{-f} \in \mathcal{D}(M)$.

An analogous extension $\overline{div}$ of the cocycle div is at least not canonical.

At this stage a new idea has to be injected. It is easily checked that every quantum automorphism Φ of $\mathcal{D}(M)$ induces a classical automorphism $\tilde{\Phi}$ of $\mathcal{S}(M)$, $\tilde{\Phi}(\sigma(D)) = \sigma(\Phi(D))$. Of course, the converse is not accurate. Moreover, any automorphism of $\mathcal{S}(M)$ restricts to an automorphism of $\mathcal{S}_0(M) \oplus \mathcal{S}_1(M) \simeq \mathcal{D}^1(M)$. So it would be natural to try to benefit from this "algebra hierarchy" and compute, having already obtained the automorphisms of $\mathcal{D}^1(M)$, those of $\mathcal{S}(M)$. This approach however turns out to be rather merely elegant than really necessary. We do not employ it here.

The automorphism property shows that

$$\Phi_2|_{\mathcal{D}^i(M)} = \kappa^{1-i} id + \psi_i, \tag{8}$$

where $i \in \mathbf{N}$ and $\psi_i \in Hom_{\mathbf{R}}(\mathcal{D}^i(M), \mathcal{D}^{i-1}(M))$. This is equivalent to saying that

$$\tilde{\Phi}_2|_{\mathcal{S}_i(M)} = \kappa^{1-i} id.$$

Such a classical automorphism $\mathcal{U}_\kappa : \mathcal{S}_i(M) \ni P \to \kappa^{1-i} P \in \mathcal{S}_i(M)$ really exists. Indeed, the degree $deg : \mathcal{S}_i(M) \ni P \to (i-1)P \in \mathcal{S}_i(M)$ is known to be a derivation of $\mathcal{S}(M)$ and, for $\kappa > 0$, the automorphism of $\mathcal{S}(M)$ generated by $-log(\kappa) \cdot deg$ is precisely $\mathcal{U}_\kappa$. Nevertheless, it is hard to imagine that this classical automorphism $\mathcal{U}_\kappa$ is induced by a quantum automorphism, since no graduation exists on the quantum level. Therefore, the first guess is

$$\kappa \stackrel{?}{=} 1.$$

In order to validate or invalidate this supposition, we project the automorphism property, combined with the newly discovered structure (8) of Φ_2, onto $C^\infty(M)$. It is worth stressing that all information is, here as well as above, enclosed in the 0-order terms.

Our exploitation of this projection is based on formal calculus. In the main, this symbolism consists in the substitution of monomials $\xi_1^{\alpha^1} \ldots \xi_n^{\alpha^n}$ in the components of a linear form $\xi \in (\mathbf{R}^n)^*$ to the derivatives $\partial_{x^1}^{\alpha^1} \ldots \partial_{x^n}^{\alpha^n} f$ of a function f.

This method is known in Mechanics as the *normal ordering* or *canonical symbolization/quantization*. Its systematic use in Differential Geometry is originated in papers by M. Flato and A. Lichnerowicz [FL80] as well as M. De Wilde and P. Lecomte [DWL83], dealing with the Chevalley-Eilenberg cohomology of the Lie algebra of vector fields associated with the Lie derivative of differential forms. This polynomial modus operandi matured during the last twenty years and developed into a powerful computing technique, successfully applied in numerous works (see e.g. [LMT96] or [Pon99]).

A by now standard application of the normal ordering leads to a system of equations in specially the above constants λ and κ. When solving the system, we get two possibilities, $(\lambda, \kappa) = (0, 1)$ and $(\lambda, \kappa) = (1, -1)$. This outcome surprisingly cancels the conjecture $\kappa = 1$.

2.5.2.1 $(\lambda, \kappa) = (0, 1)$

Equation (7), which gives Φ_2 on $\mathcal{D}^1(M)$, suggests that Φ_2 could coincide with *id* on the whole algebra $\mathcal{D}(M)$.

In fact our automorphism equations show that computations reduce to the determination of some intertwining operators between the $\mathcal{X}(M)$-modules of kth-order linear differential operators mapping differential p-forms into functions. These equivariant operators have been obtained in [Pon02] and [BHMP02]. They allow to conclude that we actually have $\Phi_2 = id$.

The paper [Pon02], following works by P. Lecomte, P. Mathonet, and E. Tousset [LMT96], H. Gargoubi and V. Ovsienko [GO96], P. Cohen, Yu. Manin, and D. Zagier [CMZ97], C. Duval and V. Ovsienko [DO97], gives the classification of the preceding modules. Additionally, it provides the complete description of the above-mentioned intertwining operators, thus answering a question by P. Lecomte whether some homotopy operator—which locally coincides with the Koszul differential [Lec94] and is equivariant if restricted to low-order differential operators—is intertwining for all orders of differentiation.

A small dimensional hypothesis in [Pon02], which was believed to be inherent in the used canonical symbolization technique, was the starting point of [BHMP02]. Here, the authors prove the existence and the uniqueness of a projectively equivariant symbol map (in the sense of P. Lecomte and V. Ovsienko [LO99]) for the spaces of differential operators transforming p-forms into functions, the underlying manifold being endowed with a flat projective structure. The substitution of this equivariant symbol to the previously used canonical symbol allowed to get rid of the dimensional assumption and unexpected intertwining operators were discovered in the few supplementary dimensions.

2.5.2.2 $(\lambda, \kappa) = (1, -1)$

Computations being rather technical, we confine ourselves to mention that our quest for automorphisms, by means of the canonical symbolization, leads to a symbol that reminds of the opposite of the conjugation operator. Thus, this operator might be an astonishing automorphism. Remember that for an oriented manifold M with volume Ω, the conjugate $D^* \in \mathcal{D}(M)$ of a differential operator $D \in \mathcal{D}(M)$ is defined by

$$\int_M D(f) \cdot g \mid \Omega \mid = \int_M f \cdot D^*(g) \mid \Omega \mid,$$

for any compactly supported $f, g \in C^\infty(M)$. Since $(D \circ \Delta)^* = \Delta^* \circ D^*$, $D, \Delta \in \mathcal{D}(M)$, the operator $\mathcal{C} := -*$ verifies $\mathcal{C}(D \circ \Delta) = -\mathcal{C}(\Delta) \circ \mathcal{C}(D)$ and is thus an automorphism of $\mathcal{D}(M)$. Formal calculus allows to show that this automorphism exists for any manifold (orientable or not). At last, the computations can again be reduced to the formerly described intertwining operators and we get $\Phi_2 = \mathcal{C}$. Hence the conclusion:

Theorem 5 *A linear map $\Phi : \mathcal{D}(M) \to \mathcal{D}(M)$ is an automorphism of the Lie algebra $\mathcal{D}(M)$ of linear differential operators on $C^\infty(M)$ if and only if it can be written in the form*

$$\Phi = \varphi_* \circ \mathcal{C}^a \circ e^{\overline{\omega}},$$

where φ is a diffeomorphism of M, $a = 0, 1$, $\mathcal{C}^0 = id$ and $\mathcal{C}^1 = \mathcal{C}$, and ω is a closed 1-form on M. All the objects φ, a, ω are uniquely determined by Φ.

2.5.3 Automorphisms of $\mathcal{S}$(M)

The study of the automorphisms of $\mathcal{S}(M)$ is similar to the preceding one regarding $\mathcal{D}(M)$ and even simpler, in view of the existence of the degree-automorphism $\mathcal{U}_\kappa$. We obtain the following upshot.

Theorem 6 *A linear map $\Phi : \mathcal{S}(M) \to \mathcal{S}(M)$ is an automorphism of the Lie algebra $\mathcal{S}(M)$ of polynomial functions on T^*M with respect to the canonical symplectic bracket if and only if it can be written in the form*

$$\Phi(P) = \mathcal{U}_\kappa(P) \circ \varphi^* \circ Exp(\omega^v),$$

where κ is a non-zero constant, $\mathcal{U}_\kappa(P) = \kappa^{1-i}P$ for $P \in \mathcal{S}_i(M)$, φ^* is the phase lift of a diffeomorphism φ of M, and $Exp(\omega^v)$ is the vertical symplectic diffeomorphism of T^*M, which is nothing but the translation by a closed 1-form ω on M. All the objects κ, φ, ω are uniquely determined by Φ.

It is interesting to compare this result with those of [AG90] and [Gra00].

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Janusz GRABOWSKI
Polish Academy of Sciences, Institute of Mathematics
Śniadeckich 8, P.O.Box 21, 00-956 Warsaw, Poland
Email: jagrab@impan.gov.pl

Norbert PONCIN
University of Luxembourg, Mathematics Laboratory
avenue de la Faïencerie, 162 A
L-1511 Luxembourg City, Grand-Duchy of Luxembourg
Email: norbert.poncin@uni.lu