Derivations of the Lie algebras of differential operators *

J. Grabowski, N. Poncin

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Abstract

This paper encloses a complete and explicit description of the derivations of the Lie algebra $\mathcal{D}(M)$ of all linear differential operators of a smooth manifold M, of its Lie subalgebra $\mathcal{D}^1(M)$ of all linear first-order differential operators of M, and of the Poisson algebra $\mathcal{S}(M) = \mathrm{Pol}(T^*M)$ of all polynomial functions on T^*M, the symbols of the operators in $\mathcal{D}(M)$. It turns out that, in terms of the Chevalley cohomology, $H^1(\mathcal{D}(M), \mathcal{D}(M)) = H^1_{\mathrm{DR}}(M)$, $H^1(\mathcal{D}^1(M), \mathcal{D}^1(M)) = H^1_{\mathrm{DR}}(M) \oplus \mathbf{R}^2$, and $H^1(\mathcal{S}(M), \mathcal{S}(M)) = H^1_{\mathrm{DR}}(M) \oplus \mathbf{R}$. The problem of distinguishing those derivations that generate one-parameter groups of automorphisms and describing these one-parameter groups is also solved.

1 Introduction

In [PS54], Pursell and Shanks proved the well-known result stating that the Lie algebra of all smooth compactly supported vector fields of a smooth manifold characterizes the differentiable structure of the variety. Similar upshots were obtained in numerous subsequent papers dealing with different Lie algebras of vector fields and related algebras (see e.g. [Abe82, Ame75, AG90, Gra78, Gra93, HM93, Omo76, Skr87]).

Derivations of certain infinite-dimensional Lie algebras arising in Geometry were also studied in different situations (note that in infinite dimension there is no such a clear correspondence between derivations and one-parameter groups of automorphisms as in the finite-dimensional case). Let us mention a result of L. S. Wollenberg [Wol69] who described all derivations of the Lie algebra of polynomial functions on the canonical symplectic space $\mathbf{R}^2$ with respect to the Poisson bracket. It turned out that there are outer derivations of this algebra in contrast to the corresponding Weyl algebra. This can be viewed as a variant of a "no-go" theorem (see [Jos70]) stating that the Dirac quantization problem [Dir58] cannot be solved satisfactorily because the classical and the corresponding quantum algebras are not isomorphic as Lie algebras. An algebraic generalization of the latter fact, known as the *algebraic "no-go" theorem,* has been proved in [GG01] by different methods. Derivations of the Poisson bracket of all smooth functions on a symplectic manifold have been determined in [ADML74] (for the real-analytic case, see [Gra86]). Another important result is the one by F. Takens [Tak73] stating that all derivations of the Lie algebra $\mathcal{X}(\mathcal{M})$ of smooth vector fields on a manifold M are inner. The same turned out to be valid for analytic cases [Gra81]. Some cases of the Lie algebras of vector fields associated with different geometric structures were studied in a series of papers by Y. Kanie [Kan75]–[Kan81].

Our work [GP03] contains Shanks-Pursell type results for the Lie algebra $\mathcal{D}(M)$ of all linear differential operators of a smooth manifold M, for its Lie subalgebra $\mathcal{D}^1(M)$ of all linear first-order differential operators of M, and for the Poisson algebra $\mathcal{S}(M) = \mathrm{Pol}(T^*M)$ of all polynomial functions on T^*M, the symbols of the operators in $\mathcal{D}(M)$. Furthermore, we computed all the automorphisms of these algebras and showed that $\mathcal{D}(M)$ and $\mathcal{S}(M)$ are not integrable. The current paper contains a description of their derivations, so it is a natural continuation of this previous work and can be considered as a generalization of the results of Wollenberg and Takens. It is also shown which derivations

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generate one-parameter groups of automorphisms and the explicit form of such one-parameter groups is provided.

2 Notations and definitions

Throughout this paper, M is as usually assumed to be a smooth, Hausdorff, second countable, connected manifold of dimension n.

Recall that the space $\mathcal{D}(M)$ (or $\mathcal{D}$ for short) of linear differential operators on $C^\infty(M)$ (or $\mathcal{A}$ for short) is filtered by the order of differentiation, $\mathcal{D}^i$ being the space of at most i-th order operators (for $i \geq 0$; $\mathcal{D}^i = \{0\}$ for $i < 0$), and is equipped with an associative and so a Lie algebra structure, $\circ$ and $[.,.]$ respectively, such that $\mathcal{D}^i \circ \mathcal{D}^j \subset \mathcal{D}^{i+j}$ and $[\mathcal{D}^i, \mathcal{D}^j] \subset \mathcal{D}^{i+j-1}$. Obviously, $\mathcal{D}^0 = \mathcal{A}$ is an associative commutative subalgebra and $\mathcal{D}^1$ is a Lie subalgebra of $\mathcal{D}$. We denote by $\mathcal{D}_c$ (respectively $\mathcal{D}^i_c$) the algebra of differential operators (respectively the space of (at most) i-th order operators) that vanish on constants. For instance $\mathcal{D}^1_c$ is the Lie algebra $\mathcal{X}(M) = \mathrm{Vect}(M)$ (or $\mathcal{X}$ for short) of vector fields of M, i.e. the Lie algebra $\mathrm{Der}\,\mathcal{A}$ of derivations of the algebra of functions. Observe also that we have the canonical splittings $\mathcal{D} = \mathcal{A} \oplus \mathcal{D}_c$, $\mathcal{D}^i = \mathcal{A} \oplus \mathcal{D}^i_c$.

The classical counterpart of $\mathcal{D}$, the space $\mathcal{S}(M)$ (or $\mathcal{S}$ for short) of symmetric contravariant tensor fields on M, is of course naturally graded, $\mathcal{S}_i$ being the space of i-tensor fields (for $i \geq 0$; $\mathcal{S}_i = \{0\}$ for $i < 0$). This counterpart $\mathcal{S}$ is isomorphic— even as a $\mathcal{D}^1_c$-module—to the space $\mathrm{Pol}(T^*M)$ of smooth functions on T^*M that are polynomial on the fibers. Furthermore, it is a commutative associative and a Poisson algebra. These structures $\cdot$ and $\{.,.\}$ verify $\mathcal{S}_i \cdot \mathcal{S}_j \subset \mathcal{S}_{i+j}$ and $\{\mathcal{S}_i, \mathcal{S}_j\} \subset \mathcal{S}_{i+j-1}$ respectively. The Poisson bracket can be viewed as the symmetric Schouten bracket or the standard symplectic bracket. Note that $\mathcal{S}_0 = \mathcal{A}$ is an associative and Lie-commutative subalgebra of $\mathcal{S}$. Clearly, $\mathcal{S}$ is filtered by $\mathcal{S}^i = \oplus_{j \leq i} \mathcal{S}_j$ and $\mathcal{S}^1$ is a Lie subalgebra of $\mathcal{S}$ isomorphic to $\mathcal{D}^1$ and $\mathcal{A} \oplus \mathcal{X}$.

The algebras $\mathcal{D}$ and $\mathcal{S}$ are models of a quantum and a classical Poisson algebra in the sense of [GP03]. All the results of this paper apply to these algebras. It is well known that $\mathcal{D}^i$, $\mathcal{S}_i$, and $\mathcal{S}^i$ are algebraically characterized in the following way:

$$\{D \in \mathcal{D} : [D, \mathcal{A}] \subset \mathcal{D}^i\} = \mathcal{D}^{i+1} \ (i \geq -1), \tag{1}$$

$$\{S \in \mathcal{S} : \{S, \mathcal{A}\} \subset \mathcal{S}_i\} = \mathcal{A} + \mathcal{S}_{i+1} \ (i \geq -1), \tag{2}$$

and

$$\{S \in \mathcal{S} : \{S, \mathcal{A}\} \subset \mathcal{S}^i\} = \mathcal{S}^{i+1} \ (i \geq -1). \tag{3}$$

Moreover, $\mathcal{S}$ is the classical algebra induced by the quantum algebra $\mathcal{D}$. Thus, $\mathcal{S}_i = \mathcal{D}^i/\mathcal{D}^{i-1}$. For any non-zero $D \in \mathcal{D}$, the degree $\deg(D)$ of D is the lowest i, such that $D \in \mathcal{D}^i \backslash \mathcal{D}^{i-1}$. If cl_j is the class in the quotient $\mathcal{S}_j$, the (principal) symbol $\sigma(D)$ of D is

$$\sigma(D) = \mathrm{cl}_{\deg(D)}(D)$$

and the symbol $\sigma_i(D)$ of order $i \geq \deg(D)$ is defined by

$$\sigma_i(D) = \mathrm{cl}_i(D) = \begin{cases} 0, \text{ if } i > \deg(D), \\ \sigma(D), \text{ if } i = \deg(D). \end{cases}$$

Then, the commutative multiplication and the Poisson bracket of $\mathcal{S}$ verify

$$\sigma(D_1) \cdot \sigma(D_2) = \sigma_{\deg(D_1)+\deg(D_2)}(D_1 \circ D_2) \ \ (D_1, D_2 \in \mathcal{D}) \tag{4}$$

and

$$\{\sigma(D_1), \sigma(D_2)\} = \sigma_{\deg(D_1)+\deg(D_2)-1}([D_1, D_2]) \ \ (D_1, D_2 \in \mathcal{D}). \tag{5}$$

3 Locality and weight

The characterizations (1), (2), and (3) of the filters $\mathcal{D}^{i+1}$ of $\mathcal{D}$ and the terms $\mathcal{S}_{i+1}$ and filters $\mathcal{S}^{i+1}$ of $\mathcal{S}$ $(i \geq -1)$, can be "extended" in the following way:

Lemma 1 *For any $i \geq -1$ and any $k \geq 1$, we have*

$$\{D \in \mathcal{D} : [D, \mathcal{D}^k] \subset \mathcal{D}^i\} = \mathbf{R} \cdot 1 + \mathcal{D}^{i-k+1}, \tag{6}$$

$$\{S \in \mathcal{S} : \{S, \mathcal{S}_k\} \subset \mathcal{S}_i\} = \mathbf{R} \cdot 1 + \mathcal{S}_{i-k+1}, \tag{7}$$

and

$$\{S \in \mathcal{S} : \{S, \mathcal{S}_k\} \subset \mathcal{S}^i\} = \mathbf{R} \cdot 1 + \mathcal{S}^{i-k+1}. \tag{8}$$

Proof. (i) Note first that $\{S \in \mathcal{S} : \{S, \mathcal{S}_k\} = 0\} = \mathbf{R} \cdot 1$. Of course, we need only show that the commutation of S with $\mathcal{S}_k$ implies $S \in \mathbf{R} \cdot 1$. But this is obvious: on a connected Darboux chart domain U, take for instance the polynomials $S_k \in \mathcal{S}_k$ defined by $S_k(x; \xi) = (\xi_i)^k$ and $S_k(x; \xi) = x^i(\xi_i)^k$ $(x \in U$, $\xi \in (\mathbf{R}^n)^*$, $\{\xi_j, x^i\} = \delta^i_j$, $i, j \in \{1, \ldots, n\})$.

More generally, we have $\{S \in \mathcal{S} : \{S, \mathcal{S}_k\} \subset \mathcal{S}_i\} = \mathbf{R} \cdot 1 + \mathcal{S}_{i-k+1}$ for all $i \geq -1$. Take $i \geq 0$. Writing $S = S_{i-k+1} + S'$ with $S_{i-k+1} \in \mathcal{S}_{i-k+1}$ and $S' \in \mathcal{S} \ominus \mathcal{S}_{i-k+1}$, we get $\{S', \mathcal{S}_k\} \subset \mathcal{S}_i \cap (\mathcal{S} \ominus \mathcal{S}_i)$, so $\{S', \mathcal{S}_k\} = 0$ and $S' \in \mathbf{R} \cdot 1$. Hence the conclusion.

(ii) In order to prove (8), observe that it is enough to consider the case $i \geq 0$. If $\{S, \mathcal{S}_k\} \subset \mathcal{S}^i$ and $S = \sum_j S_j$, $S_j \in \mathcal{S}_j$, we have $\{S_j, \mathcal{S}_k\} = 0$ for all $j > i - k + 1$. So $S_j \in \mathbf{R} \cdot 1$ and $S \in \mathbf{R} \cdot 1 + \mathcal{S}^{i-k+1}$.

(iii) Assume $[D, \mathcal{D}^k] \subset \mathcal{D}^i$ and $D \in \mathcal{D} \backslash \mathcal{D}^{i-k+1}$, so that $\deg(D) > i - k + 1$. Clearly, $\sigma_k : \mathcal{D}^k \to \mathcal{S}_k$ is surjective, so any $S_k \in \mathcal{S}_k \backslash \{0\}$ reads $S_k = \sigma(\Delta)$, $\deg(\Delta) = k$. Thus,

$$\{\sigma(D), S_k\} = \{\sigma(D), \sigma(\Delta)\} = \sigma_{\deg(D)+k-1}([D, \Delta]) = 0.$$

So $\sigma(D) \in \mathbf{R} \cdot 1$ and $D \in \mathbf{R} \cdot 1$. Eventually, $D \in \mathbf{R} \cdot 1 + \mathcal{D}^{i-k+1}$. ∎

Let $(\mathcal{P}, [.,.])$ be either the Lie algebra $(\mathcal{D}, [.,.])$, its Lie subalgebra $(\mathcal{D}^1, [.,.])$, or the Poisson algebra $(\mathcal{S}, \{.,.\})$. The sign "$\cdot$" stands for the multiplication "$\circ$" of differential operators and the multiplication "$\cdot$" of polynomials of T^*M. We denote by $\operatorname{Der} \mathcal{P}$ the Lie algebra of all derivations of the Lie algebra $(\mathcal{P}, [.,.])$.

Proposition 1 *Any derivation of the Lie algebra $\mathcal{P}$ is a local operator.*

Proof. If $P \in \mathcal{P}^i$ vanishes on an open $U \subset M$ and if $x_0 \in U$, we have

$$P = \sum_k [X_k, P_k]$$

for certain $X_k \in \mathcal{X}$, $P_k \in \mathcal{P}^i$ with $X_k|_V = P_k|_V = 0$ for some neighborhood $V \subset U$ of x_0. In the quantum case, this follows for instance from [Pon02]. In the classical case, a straightforward adaptation of [DWL81, Ex. 12] shows that the set $\{L_X : \Gamma(\mathcal{S}^{\leq i}TM) \to \Gamma(\mathcal{S}^{\leq i}TM) \mid X \in \mathcal{X}\}$, where $\Gamma(\mathcal{S}^{\leq i}TM)$ is the space of smooth sections of the tensor bundle $\oplus_{j \leq i} \mathcal{S}^j TM$, is locally transitive. Since it is obviously stable under locally finite sums, the announced result is a direct consequence of [DWL81, Prop. 3, Def. 2].

For any $C \in \operatorname{Der} \mathcal{P}$, the preceding decomposition of P and the derivation property then imply that $(CP)(x_0) = 0$. ∎

Lemma 2 *There is a finite set $\mathcal{F} = \{f_1, \ldots, f_m\} \subset C^\infty(M)$ $(m \leq 2n + 1)$, such that*
(j) the $C^\infty(M)$-module $\Omega^1(M)$ of differential 1-forms on M is spanned by $d\mathcal{F} = \{df_1, \ldots, df_m\}$,
(jj) if $P \in \mathcal{P}$ verifies $[P, \mathcal{F}] \subset \mathcal{P}^i$ then $P \in \mathcal{P}^{i+1}$, for any $i \geq -1$.

Proof. Assertion (j) is a consequence of Whitney's embedding theorem (see [AG90, Prop. 2.6], [Whi36]). It suffices to prove (jj) for $i = -1$. Indeed, by induction, if (jj) is verified for i ($i \geq -1$) then it is for $i+1$:

$$\begin{aligned}[P,\mathcal{F}] \subset \mathcal{P}^{i+1} \quad &\Rightarrow [[P,\mathcal{F}],\mathcal{A}] \subset \mathcal{P}^i \\ &\Rightarrow [[P,\mathcal{A}],\mathcal{F}] \subset \mathcal{P}^i \\ &\Rightarrow [P,\mathcal{A}] \subset \mathcal{P}^{i+1} \\ &\Rightarrow P \in \mathcal{P}^{i+2}.\end{aligned}$$

As

$$[D,\mathcal{F}] = 0 \Rightarrow \{\sigma(D),\mathcal{F}\} = 0, \forall D \in \mathcal{D},$$

it is enough to consider the classical case, which is obvious in view of (j). Indeed, if $f \in \mathcal{A}$ we have $df = \sum_{s=1}^m g_s df_s$ ($g_s \in \mathcal{A}$) and if $S \in \mathcal{S} = \mathcal{S}(M) = \mathrm{Pol}(T^*M)$ and Λ denotes the canonical Poisson tensor of T^*M, then

$$\{S,f\} = \Lambda(dS,df) = \sum_{s=1}^m g_s\{S,f_s\} = 0.$$

Hence the result. ∎

Proposition 2 *Any derivation C of the Lie algebra $\mathcal{P}$ has a bounded weight, i.e. there is $d \in \mathbf{N}$, such that*

$$C(\mathcal{P}^i) \subset \mathcal{P}^{i+d}, \forall i \in \mathbf{N}.$$

Proof. Set $d = \max\{\deg(Cf_s), s \in \{1,\ldots,m\}\}$, where deg is the degree in the filtered algebra $\mathcal{P}$ and where the set $\mathcal{F} = \{f_1,\ldots,f_m\}$ is that of Lemma 2. Then C maps all functions into $\mathcal{P}^d$. Indeed, if $f \in \mathcal{A}$ we have for any $f_s \in \mathcal{F}$,

$$0 = C[f,f_s] = [Cf,f_s] + [f,Cf_s]$$

and $[Cf,f_s] \in \mathcal{P}^{d-1}$, so that $Cf \in \mathcal{P}^d$. The announced result can then once more be obtained by induction. Take $P \in \mathcal{P}^{i+1}$ ($i \geq 0$) and apply again the derivation property:

$$C[P,f_s] = [CP,f_s] + [P,Cf_s], \forall s \in \{1,\ldots,m\}.$$

Hence the conclusion. ∎

Remark: Evidently, for $C \in \mathrm{Der}\,\mathcal{D}^1$, we have $C(\mathcal{D}^0) \subset \mathcal{D}^1$ and $C(\mathcal{D}^1) \subset \mathcal{D}^1 \subset \mathcal{D}^2$.

4 Corrections by inner derivations

Proposition 3 *Let $C \in \mathrm{Der}\,\mathcal{P}$. There is (a non-unique) $P \in \mathcal{P}$, such that $C - \mathrm{ad}\,P \in \mathrm{Der}\,\mathcal{P}$ respects the filtration. The set of all elements of $\mathcal{P}$ that have this property is then $P + \mathcal{P}^1$.*

Proof. Take an arbitrary derivation C of the Lie algebra $\mathcal{P}$. Let $(U_\iota,\varphi_\iota)_{\iota \in I}$ be an atlas of M and $\mathcal{U} = (U,\varphi = (x^1,\ldots,x^n))$ any chart of this atlas. The restriction $C|_U$ of the local operator C to the domain U is of course a derivation of the Lie algebra $\mathcal{P}_U$, similar to $\mathcal{P}$ but defined on U instead of M.

Set now

$$P_C^{\mathcal{U},i} = C|_U\ (x^i) \in \mathcal{P}_U^d.$$

This element $P_C^{\mathcal{U},i}$ is equal to or symbolically represented by a polynomial of T^*U of type

$$P_C^{\mathcal{U},i}(x;\xi) = \sum_{|\alpha| \leq d} \gamma_\alpha^i(x)\,\xi^\alpha,$$

where we used standard notations, $\gamma_\alpha^i \in C^\infty(U)$ and $\xi \in (\mathbf{R}^n)^*$.

Since it follows from $C|_U\ [x^i,x^j] = 0$ that $[P_C^{\mathcal{U},i},x^j] = [P_C^{\mathcal{U},j},x^i]$, we get

$$\partial_{\xi_j} P_C^{\mathcal{U},i}(x;\xi) = \partial_{\xi_i} P_C^{\mathcal{U},j}(x;\xi).$$

Thus, there is a polynomial of T^*U,

$$P_C^{\mathcal{U}}(x;\xi) = \sum_{|\alpha|\le d+1} \gamma_\alpha(x)\xi^\alpha$$

(polynomial character in ξ and smooth dependence on x easily checked), such that

$$\partial_{\xi_i} P_C^{\mathcal{U}}(x;\xi) = P_C^{\mathcal{U},i}(x;\xi), \forall i \in \{1,\ldots,n\}.$$

Finally, $P_C^{\mathcal{U}} \in \mathcal{P}_U^{d+1}$ (interpret—if necessary—the polynomial as differential operator) and

$$C|_U\,(x^i) = [P_C^{\mathcal{U}}, x^i], \forall i.$$

For any function $f \in \mathcal{A}$ and any $i \in \{1,\ldots,n\}$, we then have

$$\begin{aligned} 0 = C|_U\,[f|_U, x^i] &= [(Cf)|_U, x^i] + [f|_U, [P_C^{\mathcal{U}}, x^i]] \\ &= [(Cf)|_U - [P_C^{\mathcal{U}}, f|_U], x^i]. \end{aligned}$$

In view of Lemma 2, this entails that

$$(Cf)|_U - [P_C^{\mathcal{U}}, f|_U] \in C^\infty(U). \tag{9}$$

Now we will glue together the elements $P_C^{\mathcal{U}} \in \mathcal{P}_U^{d+1}$. Let $(U_\iota, \varphi_\iota, \psi_\iota)_{\iota\in I}$ be a partition of unity subordinated to the considered atlas and set

$$P_C = \sum_\iota \psi_\iota P_C^{\mathcal{U}_\iota}.$$

Clearly $P_C \in \mathcal{P}^{c+1}$. Furthermore, $C - \mathrm{ad}\ P_C$ is a derivation of $\mathcal{P} = \mathcal{D}$ and of $\mathcal{P} = \mathcal{S}$. Let us emphasize that for $\mathcal{P} = \mathcal{D}^1$, this map $C - \mathrm{ad}\ P_C$ verifies the derivation property in $\mathcal{D}^1$, but is a priori only linear from $\mathcal{D}^1$ into $\mathcal{D}$. For any $\mathcal{P}$, it respects the filtration. Indeed, for any $f \in \mathcal{A}$ and any open $V \subset M$, we have

$$\begin{aligned} (Cf - [P_C, f])|_V &= (Cf)|_V - [\textstyle\sum_i \psi_i|_V\ P_C^{\mathcal{U}_i}|_{U_i\cap V}, f|_V] \\ &= \textstyle\sum_i \psi_i|_V \Big((Cf)|_{U_i} - [P_C^{\mathcal{U}_i}, f|_{U_i}]\Big)|_{U_i\cap V}\ \in C^\infty(V), \end{aligned} \tag{10}$$

in view of (9). We can now proceed by induction. So assume that $CP - [P_C, P] \in \mathcal{P}^i, \forall P \in \mathcal{P}^i$ $(i \ge 0)$. Then, if $P \in \mathcal{P}^{i+1}$ and $f \in \mathcal{A}$,

$$\begin{aligned} [CP - [P_C, P], f] &= C[P,f] - [P, Cf] - [P_C, [P,f]] + [P, [P_C, f]] \\ &= (C[P,f] - [P_C, [P,f]]) - [P, Cf - [P_C, f]] \in \mathcal{P}^i. \end{aligned} \tag{11}$$

Hence the result for $\mathcal{P} = \mathcal{D}$ and $\mathcal{P} = \mathcal{S}$. For $\mathcal{P} = \mathcal{D}^1$, Equation (11) shows that $[P_C, \mathcal{D}^1] \subset \mathcal{D}^1$. In view of Lemma 1, this means that $P_C \in \mathcal{D}^1$. Finally, Equation (10) allows to see that $C(\mathcal{D}^0) \subset \mathcal{D}^0$, so that C respects the filtration. ∎

Remark: Thus the inner derivation of Proposition 3 can be taken equal to 0, in the case $\mathcal{P} = \mathcal{D}^1$.

Proposition 4 *If $C \in \mathrm{Der}\,\mathcal{P}$ respects the filtration, there is a unique vector field $Y \in \mathrm{Der}\,\mathcal{A} \subset \mathcal{P}$ such that $C - \mathrm{ad}\ Y \in \mathrm{Der}\,\mathcal{P}$ respects the filtration and*

$$(C - \mathrm{ad}\ Y)|_{\mathcal{A}} = \kappa\ \mathrm{id},$$

where $\kappa \in \mathbf{R}$ is uniquely determined by C.

Proof. Consider a derivation C of $\mathcal{P}$ that respects the filtration and denote by $\mathcal{C}(\mathcal{P})$ the centralizer of ad $\mathcal{A}$ in the Lie algebra $\mathcal{E} = \mathrm{End}\,\mathcal{P}$ of endomorphisms of $\mathcal{P}$, i.e. the Lie subalgebra $\mathcal{C}(\mathcal{P}) = \{\psi \in \mathcal{E} : [\psi, \mathrm{ad}\ \mathcal{A}]_{\mathcal{E}} = 0\}$, where $[.,.]_{\mathcal{E}}$ is the commutator of endomorphisms of $\mathcal{P}$. The derivation $C \in \mathrm{Der}\,\mathcal{P}$ induces a derivation $\mathrm{ad}_{\mathcal{E}}\, C \in \mathrm{Der}\,\mathcal{E}$, which respects the centralizer. Indeed, for any $\psi \in \mathcal{C}(\mathcal{P})$, we have

$$\begin{aligned} [(\mathrm{ad}_{\mathcal{E}}\, C)\,(\psi), \mathrm{ad}\ \mathcal{A}]_{\mathcal{E}} &= [[C,\psi]_{\mathcal{E}}, \mathrm{ad}\ \mathcal{A}]_{\mathcal{E}} \\ &= -[[\psi, \mathrm{ad}\ \mathcal{A}]_{\mathcal{E}}, C]_{\mathcal{E}} - [\psi, [C, \mathrm{ad}\ \mathcal{A}]_{\mathcal{E}}]_{\mathcal{E}} = 0, \end{aligned}$$

as $[C, \mathrm{ad}\ f]_{\mathcal{E}} = \mathrm{ad}(Cf) \in \mathrm{ad}\,\mathcal{A}$ for each $f \in \mathcal{A}$, since C is a derivation that respects the filtration.

It follows from the description of the centralizer, see [GP03, Theo. 3], that if $\psi \in \mathcal{C}(\mathcal{P})$, then ψ respects the filtration and there is $\psi_1 \in \mathcal{C}(\mathcal{P})$, such that $\psi_1(\mathcal{P}^i) \subset \mathcal{P}^{i-1}$ and $\psi = \ell_{\psi(1)} + \psi_1$. Obviously, the left multiplication $\ell_f : \mathcal{P} \ni P \to f \cdot P \in \mathcal{P}$ by an arbitrary $f \in \mathcal{A}$ belongs to the centralizer $\mathcal{C}(\mathcal{P})$. So $(\mathrm{ad}_{\mathcal{E}}\, C)(\ell_f) \in \mathcal{C}(\mathcal{P})$ and for any $g \in \mathcal{A}$, $[C, \ell_f]_{\mathcal{E}}(g) = [C, \ell_f]_{\mathcal{E}}(1) \cdot g$, i.e. $(C - C(1))(f \cdot g) = (C - C(1))(f) \cdot g + f \cdot (C - C(1))(g)$. As constants are the only central elements and as derivations map central elements to central elements, $C(1) = \kappa$ ($\kappa \in \mathbf{R}$), and the preceding result means that $(C - \kappa\ \mathrm{id})|_{\mathcal{A}}$ is a vector field Y. Finally, $C - \mathrm{ad}\ Y \in \mathrm{Der}\,\mathcal{P}$ respects the filtration and $(C - \mathrm{ad}\ Y)|_{\mathcal{A}} = \kappa\ \mathrm{id}|_{\mathcal{A}}$. Uniqueness of Y is readily obtained. Indeed, if Y is a suitable vector field, then the corresponding constant is necessarily $\kappa = C(1)$ and Y is unique. ∎

5 Characterization of the derivations for the Lie algebra $\mathcal{D}^1$(M)

Let $|\ \eta\ |$ be a fixed smooth nowhere zero 1-density. The associated divergence $\mathrm{div}_{|\eta|}$ (or simply div) is defined for any vector field X as the unique function $\mathrm{div}_{|\eta|} X$ that verifies $L_X |\ \eta\ | = \left(\mathrm{div}_{|\eta|} X\right) |\ \eta\ |$, where $L_X |\ \eta\ |$ is the Lie derivative of $|\ \eta\ |$ in the direction of X. In any local coordinate system in which $|\ \eta\ |$ is a constant multiple of the standard density, this divergence reads $\mathrm{div}_{|\eta|} X = \sum_i \partial_{x^i} X^i$, with self-explaining notations. For details regarding the origin of the class of the divergence, we refer the reader to [Lec02].

Theorem 1 *A map $C : \mathcal{D}^1(M) \to \mathcal{D}^1(M)$ is a derivation of the Lie algebra $\mathcal{D}^1(M) = C^\infty(M) \oplus \mathrm{Vect}(M)$ of first order differential operators on $C^\infty(M)$, if and only if it can be written in the form*

$$C_{Y,\kappa,\lambda,\omega}(X + f) = [Y, X + f] + \kappa\, f + \lambda\ \mathrm{div}\, X + \omega(X), \tag{12}$$

where $Y \in \mathrm{Vect}(M)$, $\kappa, \lambda \in \mathbf{R}$, and $\omega \in \Omega^1(M) \cap \ker d$. All these objects $Y, \kappa, \lambda, \omega$ are uniquely determined by C.

Corollary 1 *The first group of the Chevalley-Eilenberg cohomology of the Lie algebra $\mathcal{D}^1(M)$ of first order differential operators on $C^\infty(M)$ with coefficients in the adjoint representation, is given by*

$$H^1(\mathcal{D}^1(M), \mathcal{D}^1(M)) \simeq \mathbf{R}^2 \oplus H^1_{\mathrm{DR}}(M),$$

where $H^1_{\mathrm{DR}}(M)$ stands for the first space of the de Rham cohomology of M.

Proof. Let C_1 be a derivation of $\mathcal{P}$ that respects the filtration and reduces to κ id ($\kappa \in \mathbf{R}$) on functions. The derivation property, written for $f \in \mathcal{A}$ and $X \in \mathcal{X}$, shows that $C_1(\mathcal{X}) \subset \mathcal{A}$, and written for $X, Y \in \mathcal{X}$, it means that $C_1|_{\mathcal{X}}$ is a 1-cocycle of the Lie algebra $\mathcal{X}$ canonically represented upon $\mathcal{A}$. The cohomology $H(\mathcal{X}, \mathcal{A})$ is known (see e.g. [Fuc87] or [DWL83]). Having fixed a divergence on $\mathcal{X}$, we get

$$C_1|_{\mathcal{X}} = \lambda\ \mathrm{div} + \omega,$$

with $\lambda \in \mathbf{R}$ and $\omega \in \Omega^1(M) \cap \ker d$. Finally,

$$C_1(X + f) = \kappa\, f + \lambda\ \mathrm{div}\, X + \omega(X), \forall X \in \mathcal{X}, \forall f \in \mathcal{A}.$$

To prove uniqueness of $Y, \omega, \kappa, \lambda$, it suffices to write Equation (12) successively for $1 \in \mathcal{A}$, $f \in \mathcal{A}$, and $X \in \mathcal{X}$. Hence Theorem 1. Corollary 1 is clear. Indeed, if C and $C' = C + \mathrm{ad}\,(Z + h)$ ($Z \in \mathcal{X}$, $h \in \mathcal{A}$) are two cohomologous 1-cocycles and if we denote by $(Y, \omega, \kappa, \lambda)$ and $(Y', \omega', \kappa', \lambda')$ the respective unique quadruples, then necessarily $\kappa' = \kappa$, $\lambda' = \lambda$, and $\omega' = \omega - dh$, so that the map

$$H^1(\mathcal{D}^1, \mathcal{D}^1) \ni [C] \to \kappa + \lambda + [\omega] \in \mathbf{R}^2 \oplus H^1_{\mathrm{DR}}(M)$$

is a well-defined vector space isomorphism. ∎

Remarks: 1. Observe that the preceding proof is valid for the generic algebra $\mathcal{P}$, so not only for $\mathcal{D}^1$ but also for $\mathcal{S}$ and $\mathcal{D}$.

2. Note that, for $\lambda \neq 0$ and $\omega = df$ ($f \in \mathcal{A}$), $\lambda \operatorname{div} X + \omega(X) = \lambda \operatorname{div}' X$, where $\operatorname{div}' X = \operatorname{div} X + \lambda^{-1}\omega(X)$ is another divergence.

3. Denote $C_{Y,0,0,0} = C_Y$, $C_{0,1,0,0} = C_{\mathcal{A}}$, $C_{0,0,1,0} = C_{\operatorname{div}}$, $C_{0,0,0,\omega} = C_\omega$. The Lie algebra structure of $\operatorname{Der} \mathcal{D}^1$ is determined by the following commutation relations (the commutators we miss are just 0):

$$[C_Y, C_{Y'}] = C_{[Y,Y']}, [C_Y, C_{\operatorname{div}}] = C_{d(\operatorname{div} Y)}, [C_Y, C_\omega] = C_{d(\omega(Y))}, [C_{\mathcal{A}}, C_{\operatorname{div}}] = C_{\operatorname{div}}, [C_{\mathcal{A}}, C_\omega] = C_\omega. \tag{13}$$

6 Characterization of the derivations for the Lie algebra $\mathcal{S}$(M)

Remark: Let us recall that we mentioned in [GP03, Sect. 4] two specific types of derivations: the canonical derivation $\operatorname{Deg} \in \operatorname{Der} \mathcal{S}$, $\operatorname{Deg} : \mathcal{S}_i \ni S \to (i-1)S \in \mathcal{S}_i$ and the derivation $\overline{\omega} \in \operatorname{Der} \mathcal{P}$ implemented by a closed 1-form ω of M. If U is an arbitrary open subset of M and if $\omega|_U = df_U$, $f_U \in C^\infty(U)$, this cocycle $\overline{\omega}$ is defined by

$$\overline{\omega}(P)|_U = [P|_U, f_U], \forall P \in \mathcal{P}.$$

Remark that, if $\mathcal{P} = \mathcal{D}^1$, $\overline{\omega}$ coincides with the derivation ω. In the case $\mathcal{P} = \mathcal{S}$ the lowering derivation $\overline{\omega}$ can be interpreted as the action of the vertical vector field ω^v (the vertical lift of the section ω of T^*M) on polynomial functions on T^*M.

Theorem 2 *A map $C : \mathcal{S}(M) \to \mathcal{S}(M)$ is a derivation of the Lie algebra $\mathcal{S}(M)$ of all infinitely differentiable functions of T^*M that are polynomial along the fibers, if and only if it is of the form*

$$C_{P,\kappa,\omega}(S) = \{P, S\} + \kappa \operatorname{Deg}(S) + \omega^v(S), \tag{14}$$

where $P \in \mathcal{S}(M)$, $\kappa \in \mathbf{R}$, and $\omega \in \Omega^1(M) \cap \ker d$. Here κ is uniquely determined by C, but P and ω are not. The set of all fitting pairs is $\{(P + h, \omega + dh) : h \in C^\infty(M)\}$, so we get uniqueness if we impose that the polynomial function P vanishes on the 0-section of T^*M.

Corollary 2 *The first group of the Chevalley-Eilenberg cohomology of the Lie algebra $\mathcal{S}(M)$ of all polynomial functions on T^*M with coefficients in the adjoint representation, is given by*

$$H^1(\mathcal{S}(M), \mathcal{S}(M)) \simeq \mathbf{R} \oplus H^1_{\mathrm{DR}}(M).$$

Proof. If $C_1 \in \operatorname{Der} \mathcal{S}$ respects the filtration and coincides with κ id ($\kappa \in \mathbf{R}$) on $\mathcal{A}$, the proof of Theorem 1 yields

$$C_1(X + f) = \kappa f + \lambda \operatorname{div} X + \omega(X), \forall X \in \mathcal{X}, \forall f \in \mathcal{A}$$

($\lambda \in \mathbf{R}, \omega \in \Omega^1(M) \cap \ker d$). This outcome is apparent since a derivation of $\mathcal{S}$ that respects the filtration, restricts to a derivation of $\mathcal{S}^1 \simeq \mathcal{D}^1$. It is easy to check that $C_2 = C_1 + \kappa \operatorname{Deg} - \overline{\omega}$ has all the properties of C_1, but verifies in addition

$$C_2(X + f) = \lambda \operatorname{div} X, \forall X \in \mathcal{X}, \forall f \in \mathcal{A}.$$

The derivation property, written for $S \in \mathcal{S}$ and $f \in \mathcal{A}$, shows inductively that C_2 is lowering. It is easily seen that $\lambda = 0$, i.e. that $C_2|_{\mathcal{S}^1} = 0$. Indeed, for any $X \in \mathcal{X}$ and $f \in \mathcal{A}$, we have

$$\{C_2(X^2), f\} = C_2\{X^2, f\} = 2\lambda \left(X^2(f) + X(f) \operatorname{div} X\right). \tag{15}$$

The left hand side of this identity is, with respect to f, a differential operator of order 1, and the right hand side is of order 2, if $\lambda \neq 0$. Hence $\lambda = 0$.

Now, we need only check that any derivation C of $\mathcal{S}$, which vanishes on $\mathcal{S}^1 \simeq \mathcal{D}^1$, is identically zero. Suppose by induction that C vanishes on $\mathcal{S}^k$, $k \geq 1$. We will show that then C also vanishes on $\mathcal{S}_{k+1}$.

For $f \in \mathcal{A}$, $X \in \mathcal{X}$, and $S \in \mathcal{S}_{k+1}$, we have $0 = C(\{S, f\}) = \{C(S), f\}$ and $C(\{X, S\}) = \{X, C(S)\}$, so C maps $\mathcal{S}_{k+1}$ into $\mathcal{A}$ and intertwines the adjoint action of $\mathcal{X}$. Hence $C(\{fX, X^{k+1}\}) = fX\left(C(X^{k+1})\right)$ and the map $D_X : \mathcal{A} \ni f \to C(\{fX, X^{k+1}\}) \in \mathcal{A}$ is a differential operator of order 0. On the other hand, $D_X(f) = -(k+1)C\left(X(f)X^{k+1}\right)$ is of order 0 only if it is just 0. Thus, $C\left(X(f)X^{k+1}\right) = 0$, for all $f \in \mathcal{A}$ and all $X \in \mathcal{X}$. Let us now work locally. Homogeneous polynomials of degree $k+1$ on $(\mathbf{R}^n)^*$, $n = \dim M$ are spanned by $(k+1)$-th powers X^{k+1}, $X \in \mathbf{R}^n$, and any function reads $X(f)$ for any non-vanishing $X \in \mathbf{R}^n$. So polynomials of the form $X(f)X^{k+1}$ locally span $\mathcal{S}_{k+1}$ and $C = 0$. This completes the proof of Theorem 2, except for uniqueness of κ and the convenient pairs (P, ω). Equation (14), written for $S = 1 \in \mathcal{A}$, shows that κ necessarily equals $-C(1)$. Setting $S = f \in \mathcal{A}$, then $S = X \in \mathcal{X}$ in this same equation, we get $(C - C(1)\,\mathrm{id})|_{\mathcal{A}} = (\mathrm{ad}\ P)|_{\mathcal{A}}$ resp. $(C - \mathrm{ad}\ P)|_{\mathcal{X}} = \omega$. So, if (P', ω') is another suitable pair we have $\{P' - P, \mathcal{A}\} = 0$, so that $P' = P + h$, $h \in \mathcal{A}$. But then, $\omega' - \omega = \mathrm{ad}\,(P - P') = -\,\mathrm{ad}\ h = dh$ on $\mathcal{X}$. Corollary 2 is now obvious. ∎

Remark: Denote $C_{P,0,0} = C_P$, $C_{0,1,0} = \mathrm{Deg}$, $C_{0,0,\omega} = \omega^v$. The Lie algebra structure of $\mathrm{Der}\,\mathcal{S}$ is determined by the following commutation relations (the missing commutators are just 0):

$$[C_P, C_{P'}] = C_{\{P,P'\}}\,, [\mathrm{Deg}, C_P] = C_{\mathrm{Deg}(P)}\,, [\omega^v, C_P] = C_{\omega^v(P)}\,, [\omega^v, \mathrm{Deg}] = \omega^v.$$

7 Characterization of the derivations for the Lie algebra $\mathcal{D}$(M)

Lemma 3 *Any derivation $C \in \mathrm{Der}\,\mathcal{D}(M)$ that respects the filtration induces a derivation $\tilde{C} \in \mathrm{Der}\,\mathcal{S}(M)$, which respects the graduation:*

$$\tilde{C} : \mathcal{S}_i(M) \ni S \to \sigma_i(C(\sigma_i^{-1}(S))) \in \mathcal{S}_i(M),$$

for all $i \in \mathbf{N}$.

Theorem 3 *A map $C : \mathcal{D}(M) \to \mathcal{D}(M)$ is a derivation of the Lie algebra $\mathcal{D}(M)$ of all linear differential operators on $C^\infty(M)$ if and only if it can be written in the form*

$$C_{P,\omega}(D) = [P, D] + \overline{\omega}(D), \tag{16}$$

where $P \in \mathcal{D}(M)$ and $\omega \in \Omega^1(M) \cap \ker\, d$ are not unique. Again the appropriate pairs are $(P+h, \omega+dh)$, $h \in C^\infty(M)$, so we get uniqueness if we impose that P is vanishing on constants.

Corollary 3 *The first cohomology group of the Lie algebra $\mathcal{D}(M)$ of all linear differential operators on $C^\infty(M)$ with coefficients in the adjoint representation is isomorphic to the first space of the de Rham cohomology of M:*

$$H^1(\mathcal{D}(M), \mathcal{D}(M)) \simeq H^1_{\mathrm{DR}}(M).$$

Proof. Lemma 3 is a consequence of the surjective character of $\sigma_i : \mathcal{D}^i \to \mathcal{S}_i$ and Equation (5) that links the Poisson and the Lie brackets.

Propositions 3,4 and Theorem 1 show that in order to establish Theorem 3, we can start with $C_1 \in \mathrm{Der}\,\mathcal{D}$, such that $C_1(\mathcal{D}^i) \subset \mathcal{D}^i, \forall i \in \mathbf{N}$ and $C_1(X + f) = \kappa\, f + \lambda\, \mathrm{div}\, X + \omega(X)$, with the usual notations. When correcting by $\overline{\omega}$, we get a filtration-respecting derivation $C_2 = C_1 - \overline{\omega}$, which maps $X + f$ to $C_2(X + f) = \kappa\, f + \lambda\, \mathrm{div}\, X$. The derivation $\tilde{C}_2$ induced on the classical level then verifies $\tilde{C}_2(X + f) = \kappa\, f$. Theorem 2 now implies that

$$\tilde{C}_2(S) = -\kappa\ \mathrm{Deg}(S), \tag{17}$$

for all $S \in \mathcal{S}$.

Let us emphasize that Theorem 3 is based upon Theorem 1 and Theorem 2, itself built upon Theorem 1. The point is that the degree-derivation is not generated by any canonical quantum derivation. Therefore the proof of Theorem 3 is a little bit more complicated than that of Theorem 2.

Observe first that Equation (17) means that, for each $i \in \mathbf{N}$,

$$C_2|_{\mathcal{D}^i} = \kappa(1-i)\ \mathrm{id} + \chi_i, \tag{18}$$

where $\chi_i \in \mathrm{Hom}_{\mathbf{R}}(\mathcal{D}^i, \mathcal{D}^{i-1})$. Indeed, it entails that, if $D \in \mathcal{D}^i$, $i \in \mathbf{N}$, the operator $C_2 D - \kappa(1-i)D$ has a vanishing i-th order symbol. As easily checked, $\chi_0 = 0$, $\chi_1(X+f) = \kappa\, f + \lambda\ \mathrm{div}\, X$, and $\chi_i f = i\kappa\, f$ ($X \in \mathcal{X}, f \in \mathcal{A}, i \in \mathbf{N}$). Injecting now this structure into the derivation property, written for $D^i \in \mathcal{D}^i$ and $\Delta^j \in \mathcal{D}^j$, $i, j \in \mathbf{N}$, we obtain

$$\chi_{i+j-1}[D^i, \Delta^j] = [\chi_i D^i, \Delta^j] + [D^i, \chi_j \Delta^j]. \tag{19}$$

When using the decomposition $\mathcal{D} = \mathcal{A} \oplus \mathcal{D}_c$, we denote by π_0 and π_c the projections onto $\mathcal{A}$ and $\mathcal{D}_c$ respectively. Furthermore, if $D \in \mathcal{D}$, we set $D_0 = \pi_0 D = D(1)$ and $D_c = \pi_c D = D - D(1)$, and if $C \in \mathrm{End}\ \mathcal{D}$, we set $C_0 = \pi_0 \circ C \in \mathrm{Hom}_{\mathbf{R}}(\mathcal{D}, \mathcal{A})$ and $C_c = \pi_c \circ C \in \mathrm{Hom}_{\mathbf{R}}(\mathcal{D}, \mathcal{D}_c)$.

The projections on $\mathcal{A}$ of Equation (19), written for $D_c^i \in \mathcal{D}_c^i$ ($i \geq 2$) and $f \in \mathcal{A}$, then for $D_c^i \in \mathcal{D}_c^i$ and $\Delta_c^j \in \mathcal{D}_c^j$ ($i + j \geq 3$), read

$$(\chi_{i,c} D_c^i)(f) = (i-1)\kappa\, D_c^i(f) + \chi_{i-1,0}[D_c^i, f]_c \tag{20}$$

and

$$\chi_{i+j-1,0}[D_c^i, \Delta_c^j] = D_c^i(\chi_{j,0}\Delta_c^j) - \Delta_c^j(\chi_{i,0} D_c^i) \tag{21}$$

respectively. Exploiting first Equation (20) with $i = 2$ and $D_c^i = Y^2$, $Y \in \mathcal{X}$, we get the upshot

$$\chi_{2,c} Y^2 = (2\lambda + \kappa) Y^2 + 2\lambda\, (\mathrm{div}\, Y)\, Y, \tag{22}$$

for all $Y \in \mathcal{X}$. If we apply the second order symbol σ_2 to both sides of this equation, we see that

$$2\lambda + \kappa = 0. \tag{23}$$

There is an atlas of M such that in each chart $(U, x^1, \ldots, x^n)$ the divergence takes the classical form, $\mathrm{div}\left(\sum_i X^i \partial_{x^i}\right) = \sum_i \partial_{x^i} X^i$. We work in such a chart and write ∂ (resp. f' and $B(f)$, $f \in C^\infty(U)$) instead of ∂_{x^1} (resp. ∂f and $\chi_{2,0}(f\partial^2)$). For $i = 1, D_c^i = g\partial$, $j = 2, \Delta_c^j = f\partial^2$, $f, g \in C^\infty(U)$, Equation (21) yields

$$B(gf' - 2fg') = \lambda f' g'' + g\, (B(f))'. \tag{24}$$

In particular, $B(f') = (B(f))'$ and $B(g') = 0$. But then $B = 0$, $\lambda = 0$ (see Equation (24)), $\kappa = 0$ (see Equation (23)), and $C_2|_{\mathcal{D}^1} = 0$.

We now proceed by induction and show that $C_2|_{\mathcal{D}^{k+1}} = 0$, if $C_2|_{\mathcal{D}^k} = 0$, $k \geq 1$. As $C_2|_{\mathcal{D}^{k+1}}$ only depends on the $(k+1)$-th order symbol, as $[C_2(D), f] = 0$, and $C_2([X, D]) = [X, C_2(D)]$, for all $D \in \mathcal{D}^{k+1}$, $X \in \mathcal{X}$, and $f \in \mathcal{A}$, C_2 defines a map $\tilde{C}_2 : \mathcal{S}_{k+1} \to \mathcal{A}$ that intertwines the adjoint action of $\mathcal{X}$. We have shown in the proof of Theorem 2 that such a map necessarily vanishes. Hence $C_2 = 0$. This completes the proof of Theorem 3. Indeed, the statement regarding the appropriate pairs (P, ω) is obvious. The same is true for Corollary 3. ∎

Remarks: 1. Denote $C_{P,0} = C_P$, $C_{0,\omega} = \overline{\omega}$. The Lie algebra structure of $\mathrm{Der}\, \mathcal{D}$ is determined by the following commutation relations (the missing commutator is 0):

$$[C_P, C_{P'}] = C_{[P,P']}\,,\ [\overline{\omega}, C_P] = C_{\overline{\omega}(P)}\ .$$

2. Corollary 1, Corollary 2, and Corollary 3 imply that the first adjoint cohomology spaces of the Lie algebras $\mathcal{D}^1$, $\mathcal{S}$, and $\mathcal{D}$ are independent of the smooth structure of M, provided that the topology of M remains unchanged.

3. It is worth comparing our cohomological results with those obtained in other recent papers. Let $D_M = (\mathrm{End}\ \mathcal{A})_{\mathrm{loc},c}$ be the Lie algebra of local endomorphisms of $\mathcal{A}$ that vanish on constants. A well-known theorem of Peetre, [Pee60], guarantees that these operators are locally differential. The main theorem of [Pon99] asserts that the first three local cohomology groups $H^p(D_M, \mathcal{A})_{\mathrm{loc}}$ ($p \in \{1, 2, 3\}$)

of D_M canonically represented upon $\mathcal{A}$ are isomorphic to the corresponding groups $H^p_{\rm DR}(M)$ of the de Rham cohomology of M. In particular,

$$H^1(D_M, C^\infty(M))_{\rm loc} \simeq H^1_{\rm DR}(M).$$

Let us quote from [AAL02] the outcome

$$H^1(\mathrm{Vect}(M), \mathcal{D}(M)) \simeq H^1(\mathrm{Vect}(M), \mathcal{D}^i(M)) \simeq \mathbf{R} \oplus H^1_{\rm DR}(M),$$

for all $i \in \mathbf{N}$.

8 Integrability of derivations

In this section we distinguish those derivations that generate (smooth) one-parameter groups of automorphisms of the Lie algebra $\mathcal{P}$ (we will call such derivations *integrable*) and we find explicit forms of these one-parameter groups of automorphisms. The smoothness of a curve in $\mathrm{Aut}\,\mathcal{P}$ is defined in the obvious way with relation to the smooth structure on M. For instance, Φ_t is smooth in $\mathrm{Aut}\,\mathcal{D}$ if for any $D \in \mathcal{D}$ and any $f \in C^\infty(M)$ the induced map $(t, x) \mapsto \Phi_t(D)(f)(x)$ is a smooth function on $\mathbf{R} \times M$, a curve Φ_t in $\mathrm{Aut}\,\mathcal{S}$ is smooth if for any $S \in \mathcal{S}$ the induced map $(t, y) \mapsto \Phi_t(S)(y)$ is a smooth function on $\mathbf{R} \times T^*M$, etc. In the following all one-parameter groups will be assumed to be smooth.

Since the group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M is embedded in $\mathrm{Aut}\,\mathcal{P}$ (see [GP03]), a partial problem is the determination of one-parameter groups of diffeomorphisms. This, however, is well known and the one-parameter groups of diffeomorphisms are just flows $\mathrm{Exp}(tY)$ of complete vector fields Y. Note that in general it is hard to decide if a given diffeomorphism is implemented by a vector field, since neighbourhoods of identity in the connected component of the group $\mathrm{Diff}(M)$ are far from being filled up by flows (even in the case when M is compact and all vector fields are complete (see [Gra88, Kop70, Pal73])); that differs $\mathrm{Diff}(M)$ from finite-dimensional Lie groups.

Before we start the investigation into one-parameter subgroups in $\mathrm{Aut}\,\mathcal{P}$ we have to define the group-analogue of the divergence, which is important for the case $\mathcal{P} = \mathcal{D}^1(M)$. Let us stress that in this paper the divergence is not an arbitrary 1-cocycle of vector fields with coefficients into functions, but a cocycle obtained as described in [GP03] from a nowhere vanishing 1-density or as depicted in [GMM03] from an odd volume form. These cocycles or divergences form some privileged cohomology class. We will integrate any such divergence $\mathrm{div} : \mathcal{X}(M) \to C^\infty(M)$ to a group 1-cocycle $J : \mathrm{Diff}(M) \to C^\infty(M)$, which is a sort of Jacobian. Indeed, if $|\eta|$ is the odd volume form inducing the divergence and if $\phi \in \mathrm{Diff}(M)$, we have $\phi^*|\eta| = J(\phi)|\eta|$ for a unique positive smooth function $J(\phi)$. It is easily verified that if ϕ is a diffeomorphism between two domains of local coordinates and if f and g are the component functions of $|\eta|$ in the corresponding bases, then locally

$$J(\phi)(x) = \frac{g(\phi(x))}{f(x)} |\det\, \partial_x \Phi|,$$

where Φ is the local form of ϕ. For any $\phi, \psi \in \mathrm{Diff}(M)$, we clearly have

$$J(\phi \circ \psi) = \psi^*(J(\phi)) \cdot J(\psi). \tag{25}$$

In particular,

$$J(\phi^{-1}) = \frac{1}{J(\phi) \circ \phi^{-1}}\ .$$

A similar concept may be found under the name of Jacobi determinant in [AMR88, Def. 6.5.12]. Let us put $\mathrm{Div}(\phi) = \ln J(\phi)$.

Proposition 5 *For any $X \in \mathcal{X}(M)$ and $\phi \in \mathrm{Diff}(M)$, we have*

(a)

$$\phi^*(\mathrm{div}\ \phi_*(X)) = \mathrm{div}\, X + X(\mathrm{Div}(\phi)) \tag{26}$$

and, if X is complete,

(b)

$$\mathrm{Div}(\mathrm{Exp}(tX)) = \int_0^t (\mathrm{div}\, X) \circ \mathrm{Exp}(sX) ds. \tag{27}$$

Proof. (a) By definition of the action of ϕ on vector fields and differential forms, $\phi^*(i_{\phi_*(X)}|\eta|) = i_X(\phi^*|\eta|)$, so that

$$\phi^*(\mathrm{div}_{|\eta|}\, \phi_*(X)) = \mathrm{div}_{\phi^*|\eta|}(X).$$

Since $\phi^*|\eta| = J(\phi)|\eta|$, (26) follows.

(b) Let us put $F_t = \mathrm{Div}(\mathrm{Exp}(tX))$. It is easy to see that $F_{t+s} = F_t + F_s \circ \mathrm{Exp}(tX)$, which implies the differential equation

$$\dot{F}_t = X(F_t) + \dot{F}_0. \tag{28}$$

Additionally, we have the initial conditions $F_0 = 0$ and, due to

$$\dot{F}_0|\eta| = \frac{d}{dt}|_{t=0}(\mathrm{Exp}(tX))^*|\eta| = (\mathrm{div}\, X)|\eta|,$$

$\dot{F}_0 = \mathrm{div}\, X$. Applying formally the variation of constant method, we find

$$F_t = (\mathrm{Exp}(tX))^*(\int_0^t \dot{F}_0 \circ \mathrm{Exp}(-sX)ds) = \int_0^t (\mathrm{div}\, X) \circ \mathrm{Exp}(sX)ds.$$

It is easily verified that this integral is really a solution. Equation (28) is in fact a PDE of first order, which can be written in the form

$$L_{\hat{X}} F = \dot{F}_0,$$

with $\hat{X} = \partial_t - X \in \mathcal{X}(\mathbf{R} \times M)$. A well-known consequence of the theorem of Frobenius allows to see that this equation, completed by the boundary condition $F|_M = 0$, has locally one unique solution. Hence,

$$F_t = \int_0^t (\mathrm{div}\, X) \circ \mathrm{Exp}(sX) ds.$$

8.1 The case $\mathcal{D}^1(M)$

Theorem 8 of [GP03] states that an endomorphism Φ of $\mathcal{D}^1$ is an automorphism of the Lie algebra $\mathcal{D}^1$ if and only if it reads

$$\Phi_{\phi,K,\Lambda,\Omega}(X + f) = \phi_*(X) + (K\, f + \Lambda\, \mathrm{div}\, X + \Omega(X)) \circ \phi^{-1}, \tag{29}$$

where ϕ is a diffeomorphism of M, K, Λ are constants, $K \neq 0$, Ω is a closed 1-form on M, and ϕ_* is the push-forward

$$(\phi_*(X))(f) = (X(f \circ \phi)) \circ \phi^{-1}, \tag{30}$$

all the objects ϕ, K, Λ, Ω being uniquely determined by Φ. The one-parameter group condition

$$\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t} \circ \Phi_{\phi_s,K_s,\Lambda_s,\Omega_s} = \Phi_{\phi_{t+s},K_{t+s},\Lambda_{t+s},\Omega_{t+s}}$$

gives immediately

$$\phi_{t+s} = \phi_t \circ \phi_s, \quad K_{t+s} = K_t \cdot K_s, \quad \Lambda_{t+s} = \Lambda_t + K_t \cdot \Lambda_s,$$

and, in view of (26),

$$\Omega_{t+s} = K_t \Omega_s + \phi_s^* \Omega_t + \Lambda_t\, d(\mathrm{Div}(\phi_s)), \tag{31}$$

with the initial conditions $\phi_0 = \mathrm{id}_M$, $K_0 = 1$, $\Lambda_0 = 0$, $\Omega_0 = 0$. One solves easily: $\phi_t = \mathrm{Exp}(tY)$, $K_t = e^{\kappa t}$, $\Lambda_t = \lambda \frac{e^{\kappa t}-1}{\kappa}$ (with $\frac{e^{\kappa t}-1}{\kappa} = t$ if $\kappa = 0$), for some unique complete vector field Y and some unique real numbers κ, λ. To solve (31) we derive the differential equation

$$\dot{\Omega}_t = \kappa \Omega_t + \lambda\, d(\mathrm{Div}(\phi_t)) + \phi_t^* \omega,$$

where $\omega = \dot{\Omega}_0$. This is an inhomogeneous linear equation, which can be solved by the method of variation of the constant. We get

$$\Omega_t = e^{\kappa t}\int_0^t e^{-\kappa s}(\lambda\, d(\mathrm{Div}(\phi_s)) + \phi_s^*\omega)ds$$

and, in view of (27),

$$\Omega_t = \int_0^t e^{\kappa(t-s)}\left(\lambda\, d\left(\int_0^s \mathrm{div}\, Y\circ \mathrm{Exp}(uY)du\right) + (\mathrm{Exp}(sY))^*\omega\right)ds.$$

Since $\dot{\phi}_0 = Y$, $\dot{K}_0 = \kappa$, $\dot{\Lambda}_0 = \lambda$, $\dot{\Omega}_0 = \omega$, we get the following:

Theorem 4 *A derivation*

$$C_{Y,\kappa,\lambda,\omega}(X+f) = [Y, X+f] + \kappa\, f + \lambda\ \mathrm{div}\, X + \omega(X)$$

of $\mathcal{D}^1(M)$ induces a one-parameter group Φ_t of automorphisms of $\mathcal{D}^1(M)$ if and only if the vector field Y is complete. In this case the group is of the form

$$\Phi_t(X+f) = (\mathrm{Exp}(tY))_*(X) + \left(e^{\kappa t} f + \lambda \tfrac{e^{\kappa t}-1}{\kappa}\,\mathrm{div}\, X\right)\circ \mathrm{Exp}(-tY) +$$
$$\left(\int_0^t e^{\kappa(t-s)}\left(\lambda\int_0^s X(\mathrm{div}\, Y\circ \mathrm{Exp}(uY))du + ((\mathrm{Exp}(sY))^*\omega)\,(X)\right)ds\right)\circ \mathrm{Exp}(-tY).$$

8.2 The case $\mathcal{S}(M)$

We know from [GP03, Theorem 9] that an endomorphism Φ of $\mathcal{S}$ is an automorphism of the Lie algebra $\mathcal{S}$ if and only if it has the form

$$\Phi = \overline{\phi}\circ \mathcal{U}_K \circ e^{\overline{\Omega}}.$$

Here $\phi \in \mathrm{Diff}(M)$ and if $\mathcal{S}$ is interpreted as the algebra $\mathrm{Pol}(T^*M)$ of polynomial functions on T^*M, the automorphism $\overline{\phi}$ is implemented by the phase lift ϕ^* of ϕ to the cotangent bundle T^*M, a symplectomorphism of T^*M. If, on the other hand, $\mathcal{S}$ is viewed as the algebra $\Gamma(\mathcal{S}TM)$ of symmetric contravariant tensor fields on M, the automorphism $\overline{\phi}$ is the standard action of ϕ on such tensor fields.

Further, $K \in \mathbf{R}^*$, Ω is a closed 1-form on M and the automorphism $\mathcal{U}_K \in \mathrm{Aut}\,\mathcal{S}$, $\mathcal{U}_K : \mathcal{S}_i \ni S \to K^{i-1}S \in \mathcal{S}_i$, is, for $K>0$, the exponential of the derivation $\ln K$ Deg, whereas the automorphism $e^{\overline{\Omega}}$ induced by the lowering derivation $\overline{\Omega}$, i.e. the action of the vertical vector field Ω^v, is the composition with the translation $\mathcal{T}_\Omega$ by Ω in T^*M. Note that since the homothety h_K of T^*M by K acts on homogeneous polynomials of degree i by multiplication by K^i, the automorphism $\mathcal{U}_K$ can be written also in the form $\mathcal{U}_K(S) = K^{-1}\, S\circ h_K$. Hence, every one-parameter group of automorphisms of the Lie algebra $\mathcal{S}$ has the form

$$\Phi_{\phi_t,K_t,\Omega_t}(S) = K_t^{-1}\, S\circ \mathcal{T}_{\Omega_t}\circ h_{K_t}\circ (\phi_t^{-1})^*.$$

It is easy to prove the following commutation relations.

Proposition 6

$$\begin{aligned} h_K\circ\phi^* &= \phi^*\circ h_K\ ,\\ \mathcal{T}_\Omega\circ h_K &= h_K\circ \mathcal{T}_{K^{-1}\Omega}\ ,\\ (\phi^{-1})^*\circ\mathcal{T}_\Omega &= \mathcal{T}_{\phi^*\Omega}\circ(\phi^{-1})^*. \end{aligned}$$

These relations together with the one-parameter group property yield

$$\begin{aligned} \phi_{t+s} &= \phi_t\circ\phi_s\ ,\\ K_{t+s} &= K_t\cdot K_s\ ,\\ \Omega_{t+s} &= \Omega_t + K_t\cdot\phi_t^*\Omega_s\ , \end{aligned}$$

with the initial conditions $\phi_0 = \mathrm{id}_M$, $K_0 = 1$, $\Omega_0 = 0$. The obvious unique solutions are $\phi_t = \mathrm{Exp}(tY)$ for a certain complete vector field Y, $K_t = e^{\kappa t}$ for a certain $\kappa \in \mathbf{R}$, and

$$\Omega_t = \int_0^t e^{\kappa s}\,(\mathrm{Exp}(sY))^*\omega\,ds$$

for a certain closed 1-form ω on M.

Let us systematically characterize derivations by the unique triplet with first member vanishing on the 0-section. As well-known there is a Lie algebra isomorphism between $\mathcal{S}_1(M) = \mathrm{Pol}^1(T^*M)$ and $\mathcal{X}(M)$. We denote a homogeneous first order polynomial P on T^*M by P_* when it is viewed as vector field of M. Note that the hamiltonian vector field X_P of P is nothing but the phase lift $(P_*)^*$ of P_*. We then have the following theorem.

Theorem 5 *A derivation*

$$C_{P,\kappa,\omega}(S) = \{P, S\} + \kappa\,\mathrm{Deg}(S) + \omega^v(S) \tag{32}$$

of the Lie algebra $\mathcal{S}(M)$ of all infinitely differentiable functions of T^*M that are polynomial along the fibers, where P is vanishing on the 0-section, is integrable if and only if the polynomial function P belongs to $\mathcal{S}_1(M)$ and is complete, i.e. the hamiltonian vector field X_P of P is complete, i.e. the basis vector field P_* is complete. In this case the one-parameter group of automorphisms Φ_t generated by $C_{P,\kappa,\omega}$ reads

$$\Phi_t(S) = e^{-\kappa t}\,S \circ \mathcal{T}_{\int_0^t e^{\kappa s}\,(\mathrm{Exp}(sP_*))^*\omega\,ds} \circ h_{e^{\kappa t}} \circ \mathrm{Exp}(-tX_P).$$

8.3 The case $\mathcal{D}(M)$

Let us eventually recall that Theorem 10 of [GP03] asserts that automorphisms of the Lie algebra $\mathcal{D}$ have the form

$$\Phi = \phi_* \circ \mathcal{C}^a \circ e^{\overline{\Omega}}, \tag{33}$$

where $e^{\overline{\Omega}}$ ($\Omega \in \Omega^1(M) \cap \ker d$) is the formerly mentioned automorphism of $\mathcal{D}$ and where ϕ_* ($\phi \in \mathrm{Diff}(M)$) is the automorphism of $\mathcal{D}$ defined by $\phi_*(D) = \phi \circ D \circ \phi^{-1}$, $\phi(f)$ being of course $f \circ \phi^{-1}$ ($D \in \mathcal{D}, f \in \mathcal{A}$). Moreover, superscript a is 0 or 1, so that $\mathcal{C}^a$ is $\mathcal{C}^0 = \mathrm{id}$ or $\mathcal{C}^1 = \mathcal{C}$, $\mathcal{C}$ being the opposite of the conjugation operator $*$. Remember that for an oriented manifold M with volume form η, the conjugate $D^* \in \mathcal{D}$ of a differential operator $D \in \mathcal{D}$ is defined by

$$\int_M D(f) \cdot g \mid \eta \mid = \int_M f \cdot D^*(g) \mid \eta \mid,$$

for any compactly supported $f, g \in \mathcal{A}$. Since $(D \circ \Delta)^* = \Delta^* \circ D^*$ ($D, \Delta \in \mathcal{D}$), the operator $\mathcal{C} := -*$ verifies $\mathcal{C}(D \circ \Delta) = -\mathcal{C}(\Delta) \circ \mathcal{C}(D)$ and is thus an automorphism of $\mathcal{D}$. Formal calculus allows to show that this automorphism exists for any manifold, orientable or not. Clearly, the automorphism $\mathcal{C}$ is not implemented by a derivation and (33) belongs to the connected component of identity only if $a = 0$. Thus we can consider one-parameter groups of automorphisms of the form

$$\Phi_{\phi_t,\Omega_t} = (\phi_t)_* \circ e^{\overline{\Omega_t}}.$$

It is easy to prove that

$$e^{\overline{\Omega}} \circ \phi_* = \phi_* \circ e^{\overline{\phi^*\Omega}},$$

so that the one-parameter group property yields

$$\begin{aligned} \phi_{t+s} &= \phi_t \circ \phi_s\ , \\ \Omega_{t+s} &= \Omega_t + \phi_t^*\Omega_s\ , \end{aligned}$$

with initial conditions $\phi_0 = \mathrm{id}_M$, $\Omega_0 = 0$. The obvious general solutions are $\phi_t = \mathrm{Exp}(tY)$ for a complete vector field Y, and $\Omega_t = \int_0^t (\mathrm{Exp}(sY))^*\omega\,ds$ for a certain closed 1-form ω. Thus we get the following.

Theorem 6 *A derivation*

$$C_{P,\omega}(D) = [P, D] + \overline{\omega}(D) \tag{34}$$

of the Lie algebra $\mathcal{D}(M)$ of all differential operators is integrable if and only if $P \in \mathcal{X}(M)$ and P is complete. In this case the one-parameter group of automorphisms Φ_t generated by $C_{P,\omega}$ reads

$$\Phi_t = (\mathrm{Exp}(tP))_* \circ e^{\overline{\int_0^t (\mathrm{Exp}(sP))^* \omega\, ds}}.$$

Remark: The results of this section describing commutations rules for automorphisms easily imply that $\mathrm{Aut}\,\mathcal{P}$ is an infinite-dimensional regular Lie group in the sense of A. Kriegl and P. Michor (see [KM97a] or [KM97b, Ch. 8]). The integrable derivations (in fact, those with compact supports) form the Lie algebra of $\mathrm{Aut}\,\mathcal{P}$.

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Janusz GRABOWSKI
Polish Academy of Sciences
Institute of Mathematics
Śniadeckich 8
P.O. Box 21
00-956 Warsaw, Poland
Email: jagrab@impan.gov.pl

Norbert PONCIN
University of Luxembourg
Mathematics Laboratory
avenue de la Faïencerie, 162A
L-1511 Luxembourg City, Grand-Duchy of Luxembourg
Email: norbert.poncin@uni.lu