On quantum and classical Poisson algebras

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Abstract

Results on derivations and automorphisms of some quantum and classical Poisson algebras, as well as characterizations of manifolds by the Lie structure of such algebras, are revisited and extended. We prove in particular somehow unexpected fact that the algebras of linear differential operators acting on smooth sections of two real vector bundles of rank 1 are isomorphic as Lie algebras if and only if the base manifolds are diffeomorphic, independently whether the line bundles themselves are isomorphic or not.

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1 Introduction

Let us start with an overview of relevant literature on isomorphisms and derivations of infinite-dimensional Lie algebras.

In [PS54], Pursell and Shanks proved the well-known result stating that the Lie algebra of all smooth compactly supported vector fields of a smooth manifold characterizes the differentiable structure of the variety. Similar upshots were obtained in numerous subsequent papers dealing with different Lie algebras of vector fields and related algebras (see e.g. [Abe82, Ame75, AG90, Gra78, Gra93, HM93, Omo76, Skr87]).

Derivations of certain infinite-dimensional Lie algebras arising in Geometry were also studied in different situations (note that in infinite dimension there is no such a clear correspondence between derivations and one-parameter groups of automorphisms as in the finite-dimensional case). Let us mention a result of L. S. Wollenberg [Wol69] who described all derivations of the Lie algebra of polynomial functions on the canonical symplectic space $\mathbb{R}^2$ with respect to the Poisson bracket. It turned out that there are outer derivations of this algebra in contrast to the corresponding Weyl algebra. This can be viewed as a variant of a "no-go" theorem (see [Jos70]) stating that the Dirac quantization problem [Dir58] cannot be solved satisfactorily because the classical and the corresponding quantum algebras are not isomorphic as Lie algebras. An algebraic generalization of the latter fact, known as the *algebraic "no-go" theorem,* has been proved in [GG01] by different methods. Derivations of the Poisson bracket of all smooth functions on a symplectic manifold have been determined in [ADML74] (for the real-analytic case, see [Gra86]). Another important result is the one by F. Takens [Tak73] stating that all derivations of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on a manifold M are inner. The same turned out to be valid for analytic cases [Gra81]. Some cases of the Lie algebras

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of vector fields associated with different geometric structures were studied in a series of papers by Y. Kanie [Kan75]–[Kan81].

Our work [GP03] contains Shanks-Pursell type results for the Lie algebra $\mathcal{D}(M)$ of all linear differential operators of a smooth manifold M, for its Lie subalgebra $\mathcal{D}^1(M)$ of all linear first-order differential operators of M, and for the Poisson algebra $\mathcal{S}(M) = \mathrm{Pol}(T^*M)$ of all polynomial functions on T^*M, the symbols of the operators in $\mathcal{D}(M)$. Furthermore, we computed all the automorphisms of these algebras and showed that the Lie algebras $\mathcal{D}(M)$ and $\mathcal{S}(M)$ are not integrable. The paper [GP05b] provides their derivations, so it is a natural continuation of this previous work and can be considered as a generalization of the results of Wollenberg and Takens. It is also shown which derivations generate one-parameter groups of automorphisms and the explicit form of such one-parameter groups is given.

The first part of the present text is an intuitive description of the major facts explained in [GP05b]. Moreover, experience of different approaches to the quantization problem and the geometric study of differential equations incite to substitute differential operators acting on tensor densities for differential operators on functions. In the frame of our previous works, this substitution requires investigations on a possible characterization of a manifold M by the Lie algebra of differential operators acting on densities on M of arbitrary fixed weight, or more generally, on the potential characterization, by the canonical Lie algebra structure of the space of linear differential operators on smooth sections of an arbitrary $\mathbb{R}$-line bundle L, of the base manifold M or even of the bundle L itself. These problems are solved in the second part of this paper.

2 Derivations of some quantum and classical Poisson algebras

2.1 Locality and weight

In this section we depict the derivations of the algebras $\mathcal{D}^1(M)$, $\mathcal{S}(M)$, and $\mathcal{D}(M)$. Let $(\mathcal{D}, [.,.])$ be one of these three filtered Lie algebras and let C be a derivation of $(\mathcal{D}, [.,.])$. We speak of operators when referring to elements of $\mathcal{D}$ and denote the algebra $C^\infty(M)$ of smooth functions of M by $\mathcal{A}$. The adjoint action of a smooth function f of M, regarded as a differential operator of order 0, on an operator $D \in \mathcal{D}^i$ lowers the filtration degree by 1. This provides a tool for proofs by induction. The idea is really fruitful if derivations have weight 0. Indeed, a bracket such as $[CD, f]$, which involves the chosen derivation, is then also a member of $\mathcal{D}^{i-1}$.

We first prove that any derivation C has a bounded weight, i.e. that there is a positive integer d, such that $C\mathcal{D}^i \subset \mathcal{D}^{i+d}, \forall i \in \mathbb{N}$. The proof uses the derivation property on functions, the characterization of filters "à la Vinogradov", and the result that the $\mathcal{A}$-module $\Omega^1(M)$ of differential 1-forms is spanned by the differentials of a finite number of functions. This last upshot is a consequence of the Whitney's embedding theorem.

In order to verify that investigation by local computations is possible, we have to check if any derivation can be restricted to a domain of local coordinates. This means that we must prove that a derivation is always a local operator. We obtain locality using a general technique worked out by De Wilde and Lecomte, see [DWL81]: if an operator $D \in \mathcal{D}$ vanishes in a neighborhood U of a point $x \in M$, it reads $D = \sum_k [X_k, D_k]$, where the sum is finite and the vector fields X_k and operators D_k vanish in some neighborhood $V \subset U$ of x. The derivation property $CD = \sum_k ([CX_k, D_k] + [X_k, CD_k])$ then allows to conclude.

Let us emphasize that significant information on automorphisms and derivations is encoded in the automorphism and derivation properties written for two functions. If $(x^1, \ldots, x^n)$ are local coordinates in an open subset $U \subset M$, we get

$$0 = C[x^i, x^j] = [Cx^i, x^j] + [x^i, Cx^j].$$

The values Cx^i are differential operators over U or polynomials of T^*U. In the first case, we symbolically write the derivatives in these operators Cx^i as monomials in the corresponding components $(\xi_1, \ldots, \xi_n)$ of some linear form $\xi \in (\mathbb{R}^n)^*$. Hence $Cx^i \simeq P^i$, where the P^i are polynomials of T^*U. In this polynomial language, the above result reads

$$\partial_{\xi_j} P^i = \partial_{\xi_i} P^j.$$

Integration furnishes us with a polynomial P such that $\partial_{\xi_i} P = P^i$, i.e. $[P, x^i] = Cx^i$. So derivation C coincides on coordinate functions with an interior derivation. It is easily seen that for an arbitrary function f, the derivations C and $\operatorname{ad} P$ differ by a function, $Cf - [P, f] \in \mathcal{A}$, i.e. locally $C - \operatorname{ad} P$ respects the lowest filter. After a gluing process and a generalization to higher order filters, we conclude that any derivation can be corrected by an interior derivation in such a way that the filtration is respected. We will refer to this property as “property P1”. Let us stress that $P \in \mathcal{D}^{d+1}$ since the weight of C is d. Operator P is not unique, the set of all convenient P is $P + \mathcal{D}^1$.

2.2 Restriction to functions

We prove that for any derivation C that respects the filtration there is a unique vector field Y, such that the derivation $C - \operatorname{ad} Y$ respects the filtration and reduces on functions to a multiple of identity, i.e. $(C - \operatorname{ad} Y)|_{\mathcal{A}} = \kappa \operatorname{id}$, where $\kappa \in \mathbb{R}$ is uniquely determined by C.

In the following we refer to this result as “property P2”. Unless differently stated, we assume that all derivations examined below have been corrected and have acquired properties P1 and P2.

The proof of the preceding upshot is based upon a technique similar to that used in [GP05a, Sect. 2.3.2] and will not be described here.

2.3 Derivations of first order linear differential operators

The Lie algebra of first order differential operators has a canonical splitting, $\mathcal{D}^1(M) = \mathcal{A} \oplus \mathcal{X}(M)$, where $\mathcal{X}(M)$ is the Lie algebra of vector fields of M. In the following we simply write $\mathcal{D}^1$ and $\mathcal{X}$, if no misunderstanding is possible. In view of property P2, we have $Cf = \kappa f$ ($f \in \mathcal{A}, \kappa \in \mathbb{R}$). The derivation property shows that $C|_{\mathcal{X}}$ is a 1-cocycle of the canonical representation of the Lie algebra $\mathcal{X}$ on the space $\mathcal{A}$. These cocycles are well-known (see [Fuc87], [DWL83]): $CX = \lambda \operatorname{div} X + \omega(X)$ ($X \in \mathcal{X}, \lambda \in \mathbb{R}, \omega \in \Omega^1(M) \cap \ker \mathrm{d}$). So we know C on any first-order operator. If we wish to recover the initial, not yet corrected (see 2.2) derivation (we denote it also by C), we have to add again the corrections. Finally,

$$C(f + X) = [Y, f + X] + \kappa f + \lambda \operatorname{div} X + \omega(X), \forall f \in \mathcal{A}, \forall X \in \mathcal{X},$$

where $Y \in \mathcal{X}, \kappa, \lambda \in \mathbb{R}, \omega \in \Omega^1(M) \cap \ker \mathrm{d}$ are uniquely defined by C. The cohomological translation of this result is

$$H^1(\mathcal{D}^1, \mathcal{D}^1) = \mathbb{R}^2 \oplus H^1_{DR}(M).$$

Here $H^1(\mathcal{D}^1, \mathcal{D}^1)$ is the first cohomology space of the Lie algebra $\mathcal{D}^1$ and $H^1_{DR}(M)$ is the first de Rham cohomology group of the underlying manifold M.

Explanations regarding the divergence can be found in [GP05a, Sect. 2.5.1]. Let us recall that any nowhere vanishing 1-density ρ_0 defines a vector space isomorphism τ_0 between the space of 1-densities and the space of functions. Nevertheless these spaces are not isomorphic as modules over the Lie algebra of vector fields. Indeed, if $\mathcal{L}_X$ and L_X denote the Lie derivatives with respect to a vector field X, of 1-densities and functions respectively, the difference $\tau_0 \circ \mathcal{L}_X \circ \tau_0^{-1} - L_X$ is the value at X of a 1-cocycle of $\mathcal{X}$ with coefficients in $\mathcal{A}$. This cocycle is the divergence implemented by ρ_0. There is no canonical divergence. Nevertheless all divergences induced by nowhere vanishing 1-densities are cohomologous. So these divergences define a privileged cohomology class. The divergence above and below is a fixed divergence of this class.

2.4 Derivations of polynomials on the cotangent bundle

Any (corrected) derivation C of $\mathcal{S}(M) = \mathrm{Pol}(T^*M)$ (in the following we simply note $\mathcal{S}$) restricts to a derivation (still denoted by C) of the Lie algebra $\mathcal{S}^1$ of polynomials of degree 1 at most. Since this algebra is isomorphic to $\mathcal{D}^1$, derivation C reads

$$C(f + X) = \kappa f + \lambda \operatorname{div} X + \omega(X), \forall f \in \mathcal{A}, \forall X \in \mathcal{X}. \tag{1}$$

If we impose the derivation condition, not only for elements of $\mathcal{S}^1$ but for all polynomials in $\mathcal{S}$, the terms of the r.h.s. of Equation (1) either cancel or turn out to be the traces on the $\mathcal{S}^1$-level of derivations of the whole algebra $\mathcal{S}$. In this intuitive approach we confine ourselves to trying to extend these terms as derivations of $\mathcal{S}$.

Since $\mathcal{S}$ is a graded algebra, one of its derivations is the so-called *degree derivation*,

$$\mathrm{Deg} : \mathcal{S}_i \ni P \to (i-1)P \in \mathcal{S}_i,$$

which just multiplies by the (shifted) degree of the argument. Of course $\mathcal{S}_i$ is the space $\mathrm{Pol}^i(T^*M)$ of homogeneous polynomials of degree i. Visibly $-\kappa\,\mathrm{Deg}$ is a derivation of $\mathcal{S}$ that extends the first term of the r.h.s. of Equation (1).

It should be clear that such an extension does not exist for the second term $\lambda\,\mathrm{div}$.

Since ω is locally exact, its value at X locally reads

$$\omega(X) = (\mathrm{d}\,f)(X) = \{X, f\},$$

where f is a local function and where $\{.,.\}$ is the standard Poisson bracket of T^*M. So, if, for any polynomial $P \in \mathcal{S}$, we locally define

$$\overline{\omega}(P) := \{P, f\} = \Lambda(\mathrm{d}\,P, \mathrm{d}\,f),$$

where Λ is the corresponding Poisson tensor, we see that $\overline{\omega}$ is a well and globally defined derivation of $\mathcal{S}$ that extends our third term. It is obvious from the preceding equation that $\overline{\omega}$ is the (vertical) vector field ω^v of T^*M induced by (the pullback of) ω (to the cotangent bundle).

Finally we understand that any derivation C of $\mathcal{S}$ is of the type

$$C(P) = \{Q, P\} - \kappa\,\mathrm{Deg}\,P + \omega^v(P), \forall P \in \mathcal{S},$$

where $Q \in \mathcal{S}, \kappa \in \mathbb{R}, \omega \in \Omega^1(M) \cap \ker \mathrm{d}$. Let us still mention that κ is unique, whereas the set of appropriate (Q, ω) is $\{(Q + h, \omega + \mathrm{d}\,h), h \in \mathcal{A}\}$. The cohomological version of this second result is

$$H^1(\mathcal{S}, \mathcal{S}) = \mathbb{R} \oplus H^1_{\mathrm{DR}}(M),$$

with self-explaining notations.

2.5 Derivations of linear differential operators

The intuitive approach is as in Section 2.4, conclusions are similar. Note nevertheless that the degree derivation, which extends the first term of Equation (1), is tightly connected with the grading of the classical Poisson algebra $\mathcal{S}$. Since the quantum algebra $\mathcal{D}(M)$ ($\mathcal{D}$ for short) is only filtered, we guess that such an extension is no longer possible on the quantum level.

We now understand that any derivation C of $\mathcal{D}$ has the form

$$C(D) = [\Delta, D] + \overline{\omega}(D), \forall D \in \mathcal{D},$$

where $\Delta \in \mathcal{D}, \omega \in \Omega^1(M) \cap \ker \mathrm{d}$. The lowering (its weight with respect to the filtration degree is -1) derivation $\overline{\omega}$ is defined as in Section 2.4. The convenient (Δ, ω) are again $\{(\Delta + h, \omega + \mathrm{d}\,h), h \in \mathcal{A}\}$. Moreover,

$$H^1(\mathcal{D}, \mathcal{D}) = H^1_{\mathrm{DR}}(M).$$

3 Canonical and equivariant quantizations

Needless to say that the above depicted intuitive presentation is merely a kind of "story". Proofs are completely different and comparatively technical. So the quest for equations that annihilate both constants κ and λ (see Eq. (1)) is, in the case of differential operators, a little bit as the hunt for the proverbial needle in a haystack. Computations can be reduced to the search for some operators that intertwine the canonical action of vector fields. These intertwining operators had been discovered in

[Pon02] and in [BHMP02].

The first of these papers as well as our work [GP03] are based upon formal calculus. In the main, this symbolism consists in the substitution of monomials $\xi_1^{\alpha^1}\ldots\xi_n^{\alpha^n}$ in the components of a linear form $\xi\in(\mathbb{R}^n)^*$ to the derivatives $\partial_{x^1}^{\alpha^1}\ldots\partial_{x^n}^{\alpha^n}f$ of a function f. In other words, we exploit the so-called affine symbol map,

$$\begin{aligned}\sigma_{\mathrm{aff}}:\mathcal{D}(\mathbb{R}^n)\ni D=\textstyle\sum D^{i_1\ldots i_j}(x)\ \partial_{x^{i_1}}\ldots\partial_{x^{i_j}}&\\ \to\sigma_{\mathrm{aff}}(D)=\textstyle\sum D^{i_1\ldots i_j}(x)\xi_{i_1}\ldots\xi_{i_j}&\in\mathcal{S}(\mathbb{R}^n)=\mathrm{Pol}(T^*\mathbb{R}^n),\end{aligned}$$

where the differential operator D is decomposed in some coordinate system and where the coefficients $D^{i_1\ldots i_j}(x)$ are smooth functions. Note that in order to increase readability, we have explicitly written the variables of the involved functions, thus identifying a function with its value at a generic point. This affine symbol method is known in Mechanics as the "normal ordering" or "canonical symbolization/quantization". Its systematic use in Differential Geometry is originated in papers by M. Flato and A. Lichnerowicz [FL80] as well as M. De Wilde and P. Lecomte [DWL83], dealing with the Chevalley-Eilenberg cohomology of the Lie algebra of vector fields associated with the Lie derivative of differential forms. This polynomial modus operandi matured during the last twenty years and developed into a quite powerful and universal computing technique that has been successfully applied in numerous works (see e.g. [LMT96] or [Pon99]). A more detailed explanation of the key features of this non standard procedure can be found in [Pon02].

The reader might ask if tensor field $\sigma_{\mathrm{aff}}(D)$ is independent on the chosen coordinates, i.e. if the vector space isomorphism σ_{aff} commutes with diffeomorphisms. An easy computation allows to convince oneself that only the highest order terms, the principal symbol of D, have an intrinsic meaning.

This problem is tightly connected with equivariant quantization. Let us first replace, as already mentioned, differential operators acting on functions by differential operators acting on densities of arbitrary weight λ. Remember that 0-densities are just functions and that the space of 1/2-densities is pre-Hilbert and deserves particular attention. It seems natural to wonder whether there is an isomorphism $\sigma_\chi\neq\sigma_{\mathrm{aff}}$ between the space $\mathcal{D}_\lambda(\mathbb{R}^n)$ of differential operators on densities of weight λ and the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of polynomials on the cotangent bundle, which commutes with diffeomorphisms or—on the infinitesimal level—with the Lie derivatives with respect to vector fields. The existence of such a total symbol map would imply that the spaces of differential operators on λ- and μ-densities are isomorphic as modules over the Lie algebra of vector fields. The classification of these modules has been obtained in [LMT96]. It shows that there is no $\mathcal{X}(\mathbb{R}^n)$-equivariant symbol map, that we have to relax the invariance condition.

Note however that the canonical action of $\mathrm{SL}(n+1,\mathbb{R})$ on $\mathbb{R}^{n+1}$ induces a "local" action on $\mathbb{R}^n$, which in turn induces an action of the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ on $\mathbb{R}^n$. This tangent action realizes $\mathrm{sl}(n+1,\mathbb{R})$ as a maximal Lie subalgebra sl_{n+1} of the algebra $\mathcal{X}_*(\mathbb{R}^n)$ of polynomial vector fields on $\mathbb{R}^n$. So the aforementioned relaxed equivariance condition can be possibly changed into the equivariance with respect to the action of this maximal algebra sl_{n+1} of infinitesimal projective transformations. More precisely, we are looking for a vector space isomorphism

$$\sigma_{\mathrm{sl}}:\mathcal{D}_\lambda(\mathbb{R}^n)\to\mathrm{Pol}(T^*\mathbb{R}^n),$$

such that

$$\sigma_{\mathrm{sl}}\circ\mathcal{L}_X=L_X\circ\sigma_{\mathrm{sl}},\ \forall X\in\mathrm{sl}_{n+1}$$

and

$$\sigma_{\mathrm{sl}}(D)-\sigma(D)\in\mathrm{Pol}^{\leq k-1}(T^*\mathbb{R}^n),\ \forall D\in\mathcal{D}_\lambda^k(\mathbb{R}^n),$$

where $\mathcal{L}_X$ and L_X are the standard actions of a vector field X on differential operators and tensor fields respectively, where σ denotes the principal symbol and where superscripts $\leq k-1$ and k are the filtration degrees. The last condition is a normalization condition that assures uniqueness of the projectively equivariant symbol and quantization maps σ_{sl} and $\sigma_{\mathrm{sl}}^{-1}$. Existence has also been proven, see [LO99]. Note that σ_{aff} is called the affine symbol map since it intertwines the Lie derivatives with respect to the affine vector fields.

If M is endowed with a projective structure, the projectively equivariant quantization map $Q = \sigma_{\mathrm{sl}}^{-1}$, which exists in any chart of any projective atlas, is of course well-defined on M. This quantization procedure $Q : \mathrm{Pol}(T^*M) \to \mathcal{D}_\lambda(M)$ defines a family of invariant $*$-products on T^*M. Indeed, it is easily checked that if for any formal parameter $\hbar$ and any $P \in \mathrm{Pol}^k(T^*M)$, we set

$$Q_\hbar P = \hbar^k QP,$$

then

$$F *_\hbar G = Q_\hbar^{-1}(Q_\hbar F \circ Q_\hbar G), \;\; \forall F, G \in \mathrm{Pol}(T^*M)$$

is such a 1-parameter family.

As mentioned, the upshots of Sections 2.4 and 2.5 can be deduced from some invariant operators obtained in [Pon02] and [BHMP02]—although independent proofs have also been found recently. The paper [Pon02] fits into the frame of a series of works, specially by P. Lecomte, P. Mathonet, and E. Tousset [LMT96], H. Gargoubi and V. Ovsienko [GO96], P. Cohen, Yu. Manin, and D. Zagier [CMZ97], C. Duval and V. Ovsienko [DO97]. A small dimensional hypothesis in [Pon02], which was believed to be inherent in the used canonical symbolization technique, was the starting point of [BHMP02]. Here, the authors prove—using a conceptual method—existence and uniqueness of a projectively equivariant symbol between the space $\mathcal{D}_p^k(M)$ ($\mathcal{D}_p^k$ for short) of kth order differential operators transforming differential p-forms into functions, and the space $\mathcal{S}_p^k(M)$ ($\mathcal{S}_p^k$ for short) of the corresponding symbols, the underlying manifold M being endowed with a flat projective structure. This invariant symbol map, which is explicitly known in terms of a "divergence operator" and the "Koszul differentials", can be used as a substitute for the affine or canonical symbol method and allows to get rid of the dimensional assumption. Roughly speaking, the search for $\mathcal{X}$-equivariant operators between $\mathcal{D}_p^k$ and $\mathcal{D}_q^\ell$ reduces to investigations on sl-equivariant operators on the classical level between $\mathcal{S}_p^k$ and $\mathcal{S}_q^\ell$, and the subsequent exploitation of the maximality of the projective algebra.

4 Integrability of derivations

Let us come back to the derivations of the algebras $\mathcal{D}^1(M)$, $\mathcal{S}(M)$, and $\mathcal{D}(M)$. For these non-integrable infinite-dimensional Lie algebras, there is no such clear correspondence between derivations and 1-parameter groups of automorphisms as in the finite-dimensional setting. Our goal is to find in each of these cases the most general form of a 1-parameter group of automorphisms. Moreover, computations should unmask a derivation that can be viewed as the generator of the chosen group of automorphisms. Finally, we wonder if it is possible to characterize those derivations that induce 1-parameter groups of automorphisms.

First remark that any diffeomorphism ϕ of M canonically induces an automorphism ϕ_* of the considered algebra $\mathcal{D}$. If $\mathcal{D} = \mathcal{D}^1(M)$ or $\mathcal{D} = \mathcal{D}(M)$, this automorphism is defined by

$$(\phi_* D)f = D(f \circ \phi) \circ \phi^{-1}, \;\; \forall D \in \mathcal{D}, \forall f \in \mathcal{A}.$$

If $\mathcal{D} = \mathcal{S}(M) = \mathrm{Pol}(T^*M)$, we set

$$\phi_* P = P \circ (\phi^\sharp)^{-1}, \;\; \forall P \in \mathcal{D},$$

where $\phi^\sharp$ is the phase lift of ϕ. So 1-parameter groups of diffeomorphisms are special 1-parameter groups of automorphisms, known a priori, since they are just flows of complete vector fields.

We have shown in [GP03] that any automorphism Φ of $\mathcal{D}^1(M)$ has the form

$$\Phi(f + X) = \phi_*(X) + (K\,f + \Lambda\,\mathrm{div}\,X + \Omega(X)) \circ \phi^{-1}, \;\; \forall f \in \mathcal{A}, \forall X \in \mathcal{X}, \tag{2}$$

where $\phi \in \mathrm{Diff}(M)$, $K \in \mathbb{R}\backslash\{0\}$, $\Lambda \in \mathbb{R}$, and $\Omega \in \Omega^1(M) \cap \ker \mathrm{d}$ are uniquely determined by the chosen automorphism. Let $\Phi_t = \Phi_{\phi_t, K_t, \Lambda_t, \Omega_t}$ be an arbitrary 1-parameter group of automorphisms. Smoothness with respect to the differential structure of M is assumed. In other words, we suppose that the map

$$\mathbb{R} \times M \ni (t, x) \to (\Phi_t D)(f)(x) \in \mathbb{R}$$

is smooth for any $D \in \mathcal{D}^1(M)$ and any $f \in \mathcal{A}$. When computing the l.h.s. of the group condition

$$\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t} \circ \Phi_{\phi_s,K_s,\Lambda_s,\Omega_s} = \Phi_{\phi_{t+s},K_{t+s},\Lambda_{t+s},\Omega_{t+s}}, \tag{3}$$

we get terms that can easily be compared with the corresponding terms of the r.h.s., except for one term,

$$\Lambda_t\,(\mathrm{div}\,\phi_{s*}X) \circ \phi_t^{-1}, \tag{4}$$

which is not of one of the four types in the r.h.s. of Equation (2). So this term has to be transformed.

Let us recall that the divergence is implemented by a fixed nowhere vanishing 1-density ρ_0. It is quite obvious that the divergence of the push-forward of a vector field X coincides with the divergence with respect to the pull-back of ρ_0. More precisely,

$$\mathrm{div}_{\rho_0}\,\phi_* X = (\mathrm{div}_{\phi^*\rho_0}\,X) \circ \phi^{-1}, \tag{5}$$

where subscript s has been omitted. It is clear that for any diffeomorphism ϕ there is a unique positive smooth function $J(\phi)$, such that

$$\phi^*\rho_0 = (J(\phi))\,\rho_0. \tag{6}$$

Furthermore, the reader might have guessed that the essential local building block of $J(\phi)(x)$ is $|\det \partial_x\varphi\,|$, where φ is the local form of ϕ. Hence, the following property of J:

$$J(\phi \circ \psi) = \psi^*\,(J(\phi))\,.\,J(\psi), \quad \forall \phi, \psi \in \mathrm{Diff}(M). \tag{7}$$

Remember now that if G is a group and A is a left G-module, a group 1-cocycle is a map $C : G \to A$ such that

$$C(g_1 \cdot g_2) = g_1.\,(C(g_2)) + C(g_1), \quad \forall g_1, g_2 \in G, \tag{8}$$

where "." is the action of G and "$\cdot$" the group multiplication. Note that this cocycle-condition is similar to that of the Hochschild cohomology of an associative algebra. The unique difference between Equation (7) and Equation (8) is the operation in the r.h.s. When applying the logarithm to both sides of Equation (7), we finally get

$$(\ln \circ J)(\phi \circ \psi) = \psi^*\,((\ln \circ J)(\phi)) + (\ln \circ J)(\psi).$$

So

$$\ln \circ J \in \mathcal{Z}^1(\mathrm{Diff}(M), C^\infty(M)) \tag{9}$$

is a 1-cocycle of the group of diffeomorphisms valued in the module of smooth functions. Moreover, it can be proven that

$$(\ln \circ J)(\mathrm{Exp}(tX)) = \int_0^t \mathrm{div}\,X \circ \mathrm{Exp}(sX)\,\mathrm{d}s, \tag{10}$$

for any complete vector field X. Equations (6), (9), and (10) show that

$$\mathrm{Div} = \ln \circ J$$

is the group analogue of the divergence.

This group divergence allows to rewrite term (4) in an appropriate form. Starting from Equation (5), we obtain

$$\begin{aligned}(\mathrm{div}_{\phi^*\rho_0}\,X) \circ \phi^{-1} &= \left(\mathrm{div}_{(J(\phi))\rho_0}\,X\right) \circ \phi^{-1}\\ &= (\mathrm{div}_{\rho_0}\,X + X\,((\ln \circ J)\,(\phi))) \circ \phi^{-1}\\ &= (\mathrm{div}\;X + \mathrm{d}\,(\mathrm{Div}\,\phi)\,(X)) \circ \phi^{-1}.\end{aligned}$$

The terms of group condition (3) can now easily be compared. This comparison leads to the equations

$$\begin{gathered}\phi_t \circ \phi_s = \phi_{t+s}, \phi_0 = \mathrm{id},\\ K_t K_s = K_{t+s}, K_0 = 1,\\ \Lambda_t + K_t\Lambda_s, \Lambda_0 = 0,\\ K_t\Omega_s + \phi_s^*\Omega_t + \Lambda_t\,\mathrm{d}(\mathrm{Div}\,\phi_s) = \Omega_{t+s}, \Omega_0 = 0,\end{gathered}$$

where id is the identity map. The first of these results for instance means that the 1-parameter family of diffeomorphisms is actually a 1-parameter group of diffeomorphisms, i.e. the flow of a complete vector field Y: $\phi_t = \mathrm{Exp}(tY)$. Other equations are a little bit more complicated, but can be solved, so that the explicit form of K_t, Λ_t, and Ω_t, i.e. of $\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t}$ is known. Furthermore, the solutions of the preceding equations involve, in addition to vector field Y, two real numbers κ, λ and a closed 1-form ω. All these objects are uniquely determined by the chosen 1-parameter group of automorphisms and characterize, as explained in Subsection 2.3, a derivation of $\mathcal{D}^1(M)$. We say that this derivation, which is special in the sense that it is associated with a complete vector field, induces the 1-parameter group of automorphisms.

We are now ready to understand the following

Theorem 1. *A derivation*

$$C_{Y,\kappa,\lambda,\omega}(X+f) = [Y, X+f] + \kappa\, f + \lambda \,\mathrm{div}\, X + \omega(X)$$

of $\mathcal{D}^1(M)$ *induces a one-parameter group* Φ_t *of automorphisms of* $\mathcal{D}^1(M)$ *if and only if the vector field* Y *is complete. In this case the group is of the form*

$$\Phi_t(X+f) = (\mathrm{Exp}(tY))_*(X) + \left(e^{\kappa t} f + \lambda \tfrac{e^{\kappa t}-1}{\kappa}\,\mathrm{div}\, X\right) \circ \mathrm{Exp}(-tY) +$$
$$\left(\int_0^t e^{\kappa(t-s)} \left(\lambda \int_0^s X(\mathrm{div}\, Y \circ \mathrm{Exp}(uY))du + ((\mathrm{Exp}(sY))^*\omega)\,(X)\right) ds\right) \circ \mathrm{Exp}(-tY).$$

Similar upshots have been obtained for the algebras $\mathcal{S}(M)$ and $\mathcal{D}(M)$. They will not be described here.

5 Differential operators on real line bundles

In [GP03] we have taken an interest in characterizations of manifold structures, especially by the Lie algebra of linear differential operators acting on the functions of the chosen manifold M. We now extend these results to the Lie algebra of differential operators acting on tensor densities over M of arbitrary weight and even to differential operators acting on the smooth sections of an arbitrary $\mathbb{R}$-line bundle L. Our objectives are to examine if this Lie algebra structure recognizes the base manifold M and maybe even the bundle L itself.

Let $\pi : L \to M$ be a real vector bundle of rank 1 over a smooth, Hausdorff, second countable, and connected manifold. We define the algebra $\mathcal{D}(L) = \cup_{k\in\mathbb{N}}\mathcal{D}^k(L)$ of differential operators on L in the standard way. Note first that the space $\mathrm{Sec}(L)$ of smooth sections of L is an $\mathcal{A}$-module, so that any function $f \in \mathcal{A}$ induces an endomorphism $m_f : \mathrm{Sec}(L) \ni s \to fs \in \mathrm{Sec}(L)$ of the space $\mathrm{Sec}(L)$. Then set

$$\mathcal{D}^0(L) = \{D \in \mathrm{End}(\mathrm{Sec}(L)) : [D, m_f] = 0, \forall f \in \mathcal{A}\},$$
$$\mathcal{D}^{k+1}(L) = \{D \in \mathrm{End}(\mathrm{Sec}(L)) : [D, m_f] \in \mathcal{D}^k(L), \forall f \in \mathcal{A}\} \;\; (k \in \mathbb{N}),$$

where $\mathrm{End}(\mathrm{Sec}(L))$ denotes the algebra of endomorphisms of $\mathrm{Sec}(L)$ and $[.,.]$– the commutator bracket associated with the composition multiplication.

Proposition 1. *Any differential operator on* L *is a local operator.*

Proof. Indeed, if we denote $\mathcal{D}^{-1}(L) = \{0\}$, then we can proceed now by induction and consider a kth-order ($k \geq 0$) differential operator D on L, a section $s \in \mathrm{Sec}(L)$ that vanishes in an open subset $U \subset M$, an arbitrary point $x \in U$, and a function $\alpha \in \mathcal{A}$, which vanishes outside U and has constant value 1 in some neighborhood of x in U. Since $[D, m_\alpha] \in \mathcal{D}^{k-1}(L)$ is a local operator, $D(\alpha s) = \alpha D(s)$ in U, so $(D(s))(x) = D(0) = 0$. ∎

Remark 1. *In the following we write* $\mathcal{D}(L)$ *in the form* $\mathcal{D}(L \to M)$*, if we wish to put emphasis on the base manifold* M*, and in the form* $\mathcal{D}(M)$*, if* L *is the trivial bundle* $M \times \mathbb{R}$*. This algebra* $\mathcal{D}(M)$ *is nothing but the usual algebra of linear differential operators acting on the space of smooth functions of* M*.*

Proposition 2. *The space $\mathcal{D}(L) = \cup_{k\in\mathbb{N}}\mathcal{D}^k(L)$ is a quantum Poisson algebra in the sense of [GP03].*

Proof. In view of the above proposition it is sufficient to check it locally. Let U be an open subset of M such that L is trivial over U. Any local trivialisation of L in U, i.e. any nowhere vanishing section $\varsigma \in \mathrm{Sec}(L_U)$ of L over U, induces a canonical vector space isomorphism

$$\iota_\varsigma : \mathrm{Sec}(L_U) \ni \varphi\varsigma \to \varphi \in C^\infty(U).$$

This isomorphism induces itself an isomorphism of quantum Poisson algebras,

$$\mathcal{I}_\varsigma : \mathcal{D}(L_U) \ni \Delta \to \iota_\varsigma \circ \Delta \circ \iota_\varsigma^{-1} \in \mathcal{D}(U). \tag{11}$$

A straightforward induction on the degree of differentiation allows to see that $\mathcal{I}_\varsigma$ respects the filtration. ∎

A gauge change entails a change in the identification of $\mathcal{D}(L_U)$ with $\mathcal{D}(U)$. Indeed, if ς' is another nowhere vanishing section of L_U, we have $\varsigma' = \psi\varsigma$, $\psi \in C^\infty(U)$, and, as easily verified,

$$\mathcal{I}_\varsigma(\Delta) = m_\psi \circ \mathcal{I}_{\varsigma'}(\Delta) \circ m_{\psi^{-1}}, \forall \Delta \in \mathcal{D}(L_U), \tag{12}$$

so that $\mathcal{I}_{\varsigma'}^{-1} \circ \mathcal{I}_\varsigma$ is a Lie algebra automorphism of $\mathcal{D}(L_U)$. Hence the local isomorphisms $\mathcal{I}_\varsigma$ cannot be glued canonically. Note nevertheless that if ψ is a (non-zero) constant, the identifications $\mathcal{I}_\varsigma$ and $\mathcal{I}_{\varsigma'}$ coincide.

Let us recall that quantum Poisson algebras (qPa) canonically induce classical Poisson algebras (cPa) in the sense of [GP03]. We denote by $\mathcal{S}(\mathcal{D})$ the cPa implemented by a qPa $\mathcal{D}$. We also know that $\mathcal{S}(\mathcal{D}(M))$ coincides with the algebra $\mathrm{Sec}(\mathcal{S}TM)$ of symmetric contravariant tensor fields over M and with the algebra $\mathrm{Pol}(T^*M)$ of smooth functions on the cotangent bundle of M that are polynomial along the fibers.

Theorem 2. *The classical Poisson algebras $\mathcal{S}(\mathcal{D}(L))$ and $\mathcal{S}(\mathcal{D}(M))$ induced by $\mathcal{D}(L)$ and $\mathcal{D}(M)$ are canonically isomorphic cPa.*

Proof. In any qPa $\mathcal{D}$ we can define the kth-order symbol $\sigma_k(D)$ of any differential operator $D \in \mathcal{D}$, the degree of which is $\leq k$ (see [GP03]). In the fundamental case $\mathcal{D} = \mathcal{D}(M)$, this algebraically defined symbol coincides with the usual geometric kth-order symbol. It is then clear (see (12)) that

$$\sigma_k(\mathcal{I}_\varsigma(\Delta)) = \sigma_k(\mathcal{I}_{\varsigma'}(\Delta)), \forall \Delta \in \mathcal{D}^k(L_U). \tag{13}$$

It is obvious that $\sigma_k(\mathcal{I}_\varsigma(\Delta)) \in \mathcal{S}_k(\mathcal{D}(U))$, $\Delta \in \mathcal{D}^k(L_U)$ only depends on the kth-order symbol of Δ, i.e. is actually defined on $\mathcal{S}_k(\mathcal{D}(L_U))$. Indeed, for any $P \in \mathcal{S}_k(\mathcal{D}(L_U))$, if $P = \sigma_k(\Delta) = \sigma_k(\Delta')$, we have $\sigma_k(\mathcal{I}_\varsigma(\Delta)) - \sigma_k(\mathcal{I}_\varsigma(\Delta')) = 0$, since $\sigma_k(\Delta - \Delta') = 0$. So

$$\Phi_\varsigma^U : \mathcal{S}_k(\mathcal{D}(L_U)) \ni P \to \sigma_k(\mathcal{I}_\varsigma(\sigma_k^{-1}(P))) \in \mathcal{S}_k(\mathcal{D}(U))$$

is a cPa isomorphism. The morphism properties with respect to the associative commutative and the Poisson-Lie multiplications are direct consequences of the definitions of these operations . and $\{.,.\}$:

$$\sigma(\Delta).\sigma(\Delta') = \sigma_{\deg\Delta+\deg\Delta'}(\Delta \circ \Delta'),$$

$$\{\sigma(\Delta), \sigma(\Delta')\} = \sigma_{\deg\Delta+\deg\Delta'-1}([\Delta, \Delta']),$$

where $\sigma(\Delta)$ and $\deg\Delta$ are the principal symbol and the degree of Δ respectively (see [GP03]). In view of (13) the isomorphisms Φ_ς^U define a global cPa isomorphism

$$\Phi : \mathcal{S}(\mathcal{D}(L)) \to \mathcal{S}(\mathcal{D}(M)),$$

such that

$$(\Phi P)|_U = \Phi_\varsigma^U(P|_U),$$

for any $P \in \mathcal{S}_k(\mathcal{D}(L))$ and a trivialization ς of L over a member U of some appropriate open covering of M. Let us mention that if $P = \sigma_k(D)$, $D \in \mathcal{D}^k(L)$, the restriction $P|_U = (\sigma_k(D))|_U$ is nothing but the well-defined class $\sigma_k(D|_U)$ of the restriction to U of the local operator D. ∎

Theorem 3. *The quantum Poisson algebras $\mathcal{D}(L)$ and $\mathcal{D}(M)$ are isomorphic.*

Proof. Let $L_0 = L\backslash\{0\}$ be the bundle L with removed 0-section and let $\mid L_0 \mid = L_0/\mathbb{Z}_2$ be the quotient of L_0 with respect to the obvious action of the multiplicative group $\mathbb{Z}_2 = \{-1, 1\}$. This quotient $|L_0|$ is an affine bundle of dimension 1 (canonically modelled on the vector bundle $M \times \mathbb{R}$), i.e. a smooth bundle of 1-dimensional affine spaces, such that the passage from one trivialization to another is given by an affine map. More precisely, if ς is a never vanishing section of L over U, the trivialization of $|L_0|$ over U is given by

$$\varphi : U \times \mathbb{R} \ni (x,t) \to e^t\, |\varsigma(x)| \in |L_0|_U\ .$$

Here $|\varsigma(x)|$ denotes the class of $\varsigma(x)$ in the quotient $|L_0|$ and the multiplication by e^t is the canonical multiplication of such a class by a non-zero real number. A change of the non vanishing local section of L is characterized by a non vanishing smooth function $\psi \in C^\infty(U, \mathbb{R}^*)$, hence the corresponding change of trivialization is given by the multiplication by $|\psi|$, which is affine. The fibers of $|L_0|$ are affine lines with a canonical free and transitive action of $\mathbb{R}$ induced by the Liouville vector field of L, $a_x.t = e^t a_x$ for any $t \in \mathbb{R}$ and any a_x in the fiber of $|L_0|$ over $x \in M$. This action turns $|L_0|$ into an $\mathbb{R}$-principal bundle. The compatibility condition $\varphi(x, s+t) = \varphi(x,s).t$ ($x \in U, s,t \in \mathbb{R}$) is obviously verified. Since the fibers are contractible, $|L_0|$ has a global section $|\eta|$. If $(U_\alpha)_{\alpha\in\Lambda}$ is a covering of M by open connected subsets over which L is trivializable, section $|\eta|$ can be viewed as a family $(\{\eta_\alpha, -\eta_\alpha\})_{\alpha\in\Lambda}$ of pairs of non vanishing local sections of L, such that $\{\eta_\alpha, -\eta_\alpha\} = \{\eta_\beta, -\eta_\beta\}$ on $U_\alpha \cap U_\beta$. This follows immediately from the above depicted trivializations. When choosing for each α a representative $\tilde{\eta}_\alpha \in \{\eta_\alpha, -\eta_\alpha\}$, we get a family of nowhere vanishing local sections of L, such that on $U_\alpha \cap U_\beta$, we have $\tilde{\eta}_\alpha = \pm\tilde{\eta}_\beta$ that reflexes the fact that line bundles over M are classified by $H^1(M; \mathbb{Z}_2)$. In view of (11) and (12) we then get a global quantum Poisson algebra isomorphism between $\mathcal{D}(L)$ and $\mathcal{D}(M)$. ∎

The preceding result shows that the Lie algebra $\mathcal{D}(L \to M)$ characterizes the smooth structure of the base manifold M, but does not recognize the topological complications in L.

Corollary 1. *Let $\pi : L \to M$ and $\pi' : L' \to M'$ be two real vector bundles of rank 1 over two smooth manifolds M and M' respectively. The Lie algebras $\mathcal{D}(L \to M)$ and $\mathcal{D}(L' \to M')$ are isomorphic if and only if the base manifolds M and M' are diffeomorphic.*

Proof. Immediate consequence of Theorem 3 and of [GP03, Theo. 6]. ∎

Note that the essential fact is that $\mathcal{D}(L \to M)$ and $\mathcal{D}(L' \to M)$ are always isomorphic (even as qPa), independently whether L and L' are isomorphic as vector bundles or not. However these isomorphisms are not canonical ones, depending on the choice of the section $|\eta|$ of $|L_0|$. This observation, together with the description of automorphisms and derivations of $\mathcal{D}(M)$ and $\mathcal{D}^1(M)$ [GP03, GP05b], gives automatically the obvious description of automorphisms and derivations of $\mathcal{D}(L \to M)$ and $\mathcal{D}^1(L \to M)$. The only difference is that diffeomorphims ϕ of M (resp. vector fields on M) do not define automorphisms (resp. derivations) of $\mathcal{D}(L \to M)$ and $\mathcal{D}^1(L \to M)$ canonically, but in the way depending on the choice of the section $|\eta|$ of $|L_0|$. For example, the derivation $C_X^{|\eta|}$ associated with a vector field X on M and a choice of $|\eta|$ is defined locally by

$$C_X^{|\eta|}(D)(f\eta_\alpha) = [X(D_\alpha(f)) - D_\alpha(X(f))]\eta_\alpha,$$

where $D_\alpha(f)\eta_\alpha = D(f\eta_\alpha)$. This definition is correct, since nothing changes when we choose $-\eta_\alpha$ instead of η_α.

References

[Abe82] Abe K, *Pursell-Shanks type theorem for orbit spaces and G-manifolds*, Publ. Res. Inst. Math. Sci. , **18** (1982), pp. 265-282

[AMR88] Abraham R, Marsden J E, Ratiu T S, *Manifolds, Tensor Analysis, and Applications*, second edition, Applied Mathematical Sciences nr. 75, Springer Verlag, New York (1988)

[AAL02] Agrebaoui B, Ammar F, Lecomte P, *On the cohomology of the spaces of differential operators acting on skew-symmetric tensor fields or on forms, as modules of the Lie algebra of vector fields*, electronic version at http://arXiv.org/abs/math.DG/0208251v1

[Ame75] Amemiya I, *Lie algebra of vector fields and complex structure*, J. Math. Soc. Japan, **27** (1975), pp. 545-549

[AG90] Atkin C J, Grabowski J, *Homomorphisms of the Lie algebras associated with a symplectic manifold*, Compos. Math., **76** (1990), pp. 315-348

[ADML74] Avez A, Diaz-Miranda A, Lichnerowicz A, *Sur l'algèbre des automorphismes infinitèsimaux d'une variété symplectique*, J. Diff. Geom. **9** (1974), 1-40

[BHMP02] Boniver F, Hansoul S, Mathonet P, Poncin N, *Equivariant symbol calculus for differential operators acting on forms*, Lett. Math. Phys., **62** (2002), pp. 219-232

[CMZ97] Cohen P, Manin Yu, Zagier D, *Automorphic pseudodifferential operators, Algebraic Aspects of Integrable Systems*, Progr. Nonlinear Differential Equations Appl., **26** (1997), pp. 17-47, Birkhâuser Verlag

[DWL81] De Wilde M, Lecomte P, *Some Characterizations of Differential Operators on Vector Bundles*, In: E.B. Christoffel, Eds: Butzer P, Feher F, Brikhâuser Verlag, Basel (1981), pp. 543-549

[DWL83] De Wilde M, Lecomte P, *Cohomology of the Lie algebra of smooth vector fields of a manifold, associated to the Lie derivative of smooth forms*, J. Pure Appl. Math., **9, 62, no 2** (1983), pp 197-214

[Dir58] Dirac P A M, *The Principles of Quantum Mechanics*, Chap. IV, Oxford University Press, Oxford (1958)

[DO97] Duval C, Ovsienko V, *Space of second order linear differential operators as a module over the Lie algebra of vector fields*, Adv. in Math., **132**(2) (1997), pp. 316-333

[FL80] Flato M, Lichnerowicz A, *Cohomologie des représentations définies par la dérivation de Lie et à valeurs dans les formes, de l'algèbre de Lie des champs de vecteurs d'une variété différentiable. Premiers espaces de cohomologie. Applications.*, C.R.Acad.Sci., Sér. A **291** (1980), pp. 331-335

[Fuc87] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, consutants Bureau, New York (1987)

[GO96] Gargoubi H, Ovsienko V, *Space of linear differential operators on the real line as a module over the Lie algebra of vector fields*, Internat. Math. Res. Notices, **5** (1996), pp. 235-251

[GG01] Gotay M, Grabowski J, *On quantizing nilpotent and solvable basic algebras*, Canadian Math. Bull. **44** (2001), pp. 140-149

[Gra78] Grabowski J, *Isomorphisms and ideals of the Lie algebras of vector fields*, Invent. math., **50** (1978), pp. 13-33

[Gra81] Grabowski J, *Derivations of the Lie algebras of analytic vector fields*, Compositio Math. **43** (1981), pp. 239-252

[Gra86] Grabowski J, *Derivations of the Lie algebras associated with a symplectic structure*, In: "Differential Geometry and Its Applications", (Eds.), Proceedings of the Conference, Brno (1986), pp. 117-125

[Gra88] Grabowski J, *Free subgroups of diffeomorphism groups*, Fundamenta Math. **131** (1988), pp. 103-121

[Gra93] Grabowski J, *Lie algebras of vector fields and generalized foliations*, Publ. Matem., **37** (1993), pp. 359-367

[GP03] Grabowski J, Poncin N, *Automorphisms of quantum and classical Poisson algebras*, Comp. Math., **140** (2004), pp. 511-527

[GP05a] Grabowski J, Poncin N, *Lie - algebraic characterizations of manifolds*, Central Europ. J. of Math., **2(5)** (2005), pp. 811-825

[GP05b] Grabowski J, Poncin N, *Derivations of the Lie algebras of differential operators*, Indag. Math., N.S., **16(2)** (2005), pp. 181-200

[GMM03] Grabowski J, Marmo G, Michor P W, *Homology and modular classes of Lie algebroids*, Ann. Inst. Fourier. (to appear), (electronic version at http://www.arxiv.org/abs/math.DG/0310072v2)

[HM93] Hauser H, Müller G, *Affine varieties and Lie algebras of vector fields*, Manusc. Math., **80** (1993), pp. 309-337

[Jos70] Joseph A, *Derivations of Lie Brackets and Canonical Quantization*, Commun. Math. Phys., **17** (1970), pp. 210-232

[Kan75] Kanie Y, *Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations. Case of classical type*, Publ. RIMS, Kyoto Univ. **11** (1975), pp. 213-245

[Kan78] Kanie Y, *Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations. Foliated case*, Publ. RIMS, Kyoto Univ. **14** (1978), pp. 487-501

[Kan79] Kanie Y, *Some Lie algebras of vector fields on foliated manifolds and their derivation algebras*, Proc. Japan Acad. **55**, Ser. A (1979), pp. 409-411

[Kan81] Kanie Y, *Some Lie algebras of vector fields on foliated manifolds and their derivations: Case of partially classical type*, Nagoya Math. J. **82** (1981), pp. 175-207

[Kop70] Kopell N, *Commuting diffeomorphisms*, Proc. Symp. Pure Math. , Amer. Math. Soc., **14** (1970), pp. 165-184

[KM97a] Kriegl A, Michor P W, *Regular infinite dimensional Lie groups*, J. Lie Theory **7** (1997), pp. 61-99

[KM97b] Kriegl A, Michor P W, *The Convenient Setting of Global Analysis*, Math. Surv. and Monogr. **53**, Amer. Math. Soc., Providence (1997)

[LMT96] Lecomte P, Mathonet P, Tousset E, *Comparison of some modules of the Lie algebra of vector fields*, Indag. Math., **7(4)** (1996), pp. 461-471

[LO99] Lecomte P, Ovsienko V, *Projectively equivariant symbol calculus*, Lett. Math. Phys., **49** (1999), pp. 173-196

[Lec02] Lecomte P, *Sur les champs de densités sur une variété*, preprint, University of Liège (2003) (electornic version at http://www.ulg.ac.be/geothalg/QPE/index.html)

[Los70] Losik M V, *On the cohomologies of infinite dimensional Lie algebras of vector fields*, Func. An. Appl., **4, 2** (1970), pp 127-135

[Omo76] Omori H, *Infinite dimensional Lie transformation groups*, Lect. Notes in Math., **427** (1976), Springer Verlag

[Pal73] Palis J, *Vector fields generate few diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1973), pp. 503-505

[Pee60] Peetre J, *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand., **7** (1959), pp. 211-218, **8** (1960), pp. 116-120

[Pon99] Poncin N, *Cohomologie de l'algèbre de Lie des opérateurs différentiels sur une variété, à coefficients dans les fonctions*, C.R.A.S. Paris, **328** Série I (1999), pp. 789-794

[Pon02] Poncin N, *Equivariant Operators between some Modules of the Lie Algebra of Vector Fields*, Comm. Alg., **32, 7** (2004), pp. 2559-2572

[PS54] Shanks M E, Pursell L E, *The Lie algebra of a smooth manifold*, Proc. Amer. Math. Soc., **5** (1954), pp. 468-472

[Skr87] Skryabin S M, *The regular Lie rings of derivations of commutative rings*, preprint WINITI 4403-W87 (1987)

[Tak73] Takens F, *Derivation of vector fields*, Compositio Math. **26** (1973), pp. 151-158

[Whi36] Whitney H, *Differentiable manifolds*, Ann. Math., **37** (1936), pp. 645-680

[Wol69] Wollenberg L S, *Derivations of the Lie algebra of polynomials under Poisson bracket*, Proc. Amer. Math. Soc. **20** (1969), pp. 315-320

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