

# A good triple of operads

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## Abstract

We show the existence of a triple of operads of the form  $(Zinb, A, Leib)$ , where  $Leib$  is the Leibniz operad,  $Zinb := Leib^1$  denotes its Koszul dual and  $A = Leib^2$  is the operad of Leibniz dialgebras, which is “good” in the sense of Loday [Lod08]. In particular, connected  $(Zinb, Leib^2)$ -bialgebras satisfy generalized versions of PBW and CMM structure theorems analogous to those satisfied by the universal enveloping associative algebra of a Lie algebra seen as a Hopf algebra. In addition, we show that the operadic homology of the enveloping Leibniz dialgebras of a Leibniz algebra is isomorphic to its Leibniz homology of  $\mathfrak{g}$  (over  $\mathbb{Q}$ ), which strengthens the analogy with the Lie algebra case and offers a candidate for the conjectural substitute to Hopf algebras in the Leibniz framework mentioned in [Lod03].

## Notations

$\mathbb{K}$  is a field of characteristic 0, and  $\otimes$  stands for  $\otimes_{\mathbb{K}}$ .  $\Sigma_n$  denotes the symmetric group, that is the group of permutations of the ordered set  $\{1 < 2 < \dots < n\}$ . Given a vector space  $V$ ,  $V^{\otimes n}$  is a representation of  $\Sigma_n$  via

$$\begin{aligned} \phi : \Sigma_n &\rightarrow \text{Aut}_{\mathbb{K}}(V^{\otimes n}) \\ \sigma &\mapsto (v_1 \otimes \dots \otimes v_n \mapsto v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}) \end{aligned}$$

and  $\phi(\sigma)$  will always be replaced by  $\sigma$  in our formulas.  $\bar{T}V$  will stand for the reduced tensor vector space

$$\bar{T}V := \bigoplus_{n \geq 1} V^{\otimes n}$$

Every operad  $A$  will be assumed to verify  $A(0) = 0$  and  $A(1) = \mathbb{K}\text{Id}$ .  $As$ ,  $Com$ ,  $Leib$ , and  $Zinb$  will stand respectively for the operads encoding associative algebras, commutative algebras, right Leibniz algebras, and left Zinbiel algebras.

## 1 The triple $(Zinb, Leib, Vect)$

In [Lod08], J-L Loday explores the notion of generalized bialgebra, and gives several examples of such bialgebras. A type of bialgebra is the datum of

- Two operads  $C$  and  $A$ ,
- A compatibility relation  $\natural$  between each generating operation of  $A$  and each generating cooperation of  $C^c$ , the cooperad associated to  $C$ .

A  $(C^c, A, \natural)$ -bialgebra is then a  $\mathbb{K}$ -module  $H$ , endowed with a  $C$ -coalgebra structure, an  $A$ -algebra structure, and such that the compatibility relation between any  $n$ -ary operation and any  $m$ -ary cooperation holds when applied to any  $n$ -uple of elements of  $H$ .

### Examples 1.0.1.

- *Non-unital associative and coassociative Hopf algebras are bialgebras of type  $(As^c, As, \natural_{nuHopf})$  with  $\natural_{nuHopf}$  given by*

$$\delta \circ \mu = \text{Id} \otimes \text{Id} + (12) + (\text{Id} \otimes \mu) \circ (\delta \otimes \text{Id}) + (\mu \otimes \text{Id}) \circ (23) \circ (\delta \otimes \text{Id}) + (\mu \otimes \text{Id}) \circ (\text{Id} \otimes \delta) + (\text{Id} \otimes \mu) \circ (12) \circ (\text{Id} \otimes \delta) + (\mu \otimes \mu) \circ (23) \circ (\delta \otimes \delta)$$

where  $\delta$  is the generating binary cooperation of  $Com^c$  and  $\mu$  is the generating binary operation of  $As$ . Remark that any unital and counital Hopf algebra  $(H, \mu, \Delta, \eta : \mathbb{K} \rightarrow H, \varepsilon : H \rightarrow \mathbb{K})$  for which the compatibility relation takes the usual form

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (23) \circ (\Delta \otimes \Delta)$$

gives rise to a non-unital one by considering its augmentation ideal  $\text{Ker}(\varepsilon)$  equipped with the reduced coproduct  $\delta$  defined by  $\delta(x) := \Delta(x) - x \otimes 1 - 1 \otimes x$ , for all  $x$  in  $\text{Ker}(\varepsilon)$ .

- Cocommutative associative Hopf algebras, coassociative commutative Hopf algebras, and cocommutative commutative Hopf algebras are respectively bialgebras of type  $(Com^c, As)$ ,  $(As^c, Com)$  and  $(Com^c, Com)$ , when one chooses as compatibility relation the non-unital Hopf relation cited above.
- A  $(Zinb^c, As, \mathfrak{H}_{nusHopf})$ -bialgebra is a  $\mathbb{K}$ -module  $H$  equipped with an associative multiplication  $\mu : H \otimes H \rightarrow H$  and a Zinbiel coproduct  $\delta : H \rightarrow H \otimes H$  such that the following non-unital semi Hopf relation holds  $\mathfrak{H}_{nusHopf}$ :

$$\delta \circ \mu = (12) + (\mu \otimes Id) \circ (\text{Id} \otimes \delta) + (\mu \otimes Id) \circ (23) \circ (\delta^{com} \otimes Id) + (Id \otimes \mu) \circ (12) \circ (Id \otimes \delta) + (\mu \otimes \mu) \circ (23) \circ (\delta^{com} \otimes \delta)$$

where  $\delta^{com}$  is the symmetrized coproduct defined by  $\delta^{com} = \delta + (12) \circ \delta$ .

This example is due to E. Burgunder (see [Bur10]) and, to our knowledge, is the only one involving the operad  $Zinb$  present in the litterature.

**Definition 1.0.2.** [Lod08] Let  $H$  be a  $(C^c, A, \mathfrak{H})$ -bialgebra.

1. Its **primitive part**, denoted by  $\text{Prim}H$  is the submodule of elements  $x$  of  $H$  such that

$$\delta(x) = 0$$

for any cooperation  $\delta$  in  $C^c(n)$ ,  $n \geq 2$ .

2. Let  $F_r H$  be defined by  $F_r H := \{x \in H, \delta(x) = 0, \delta \in C^c(n), n > r\}$ .  $H$  is said to be **connected** if

$$H = \cup_{r \geq 1} F_r H$$

3. The compatibility relation  $\mathfrak{H}$  is **distributive** if for any  $m$ -ary cooperation  $\delta$  in  $C^c(m)$ , and any  $n$ -ary operation  $\mu$  in  $A(n)$ , it takes the form

$$\delta \circ \mu = \sum_{i \in I} (\mu_1^i \otimes \cdots \otimes \mu_m^i) \circ \omega^i \circ (\delta_1^r \otimes \cdots \otimes \delta_n^i)$$

where  $I$  is a finite set of indices, and for any  $i$  in  $I$

$$\begin{cases} \mu_1^i \in A(k_1), \dots, \mu_m^i \in A(k_m) \\ \delta_1^r \in C^c(l_1), \dots, \delta_n^i \in C^c(l_n) \\ k_1 + \dots + k_m = l_1 + \dots + l_n = r_i \\ \omega^i \in \mathbb{K}[\Sigma_{r_i}] \end{cases}$$

It is well known that connected  $(Com^c, Com, \mathfrak{H}_{nuHopf})$ -bialgebras are both free and cofree over their primitive part: this is Hopf-Borel's theorem. This result can be seen as a particular case of the following rigidity theorem:

**Theorem 1.0.3.** [Lod08] Let  $(C^c, A, \mathfrak{H})$  be a type of generalized bialgebra and suppose that the following hypothesis hold

(H0) For any operation  $\mu$  and any cooperation  $\delta$ , there is a distributive compatibility relation,

(H1) For any  $\mathbb{K}$ -module  $V$ , the free  $A$ -algebra  $A(V)$  is naturally equipped with a  $(C^c, A, \mathfrak{H})$ -bialgebra structure,

(H2iso) The natural  $C^c$ -coalgebra map  $\varphi : A(V) \rightarrow C^c(V)$  lifting the projection on the length one summand  $A(V) \rightarrow V = A(1) \otimes V$  is an isomorphism.

Then any connected  $(C^c, A, \mathfrak{H})$ -bialgebra  $H$  is both free and cofree over its primitive part i.e.

$$A(\text{Prim}H) \cong H \cong C^c(\text{Prim}H)$$

where the first isomorphism is an isomorphism of  $A$ -algebras and the second one is an isomorphism of  $C^c$ -coalgebras.

**Examples 1.0.4.**  $(Com^c, Com, \mathfrak{H}_{nuHopf})$  and  $(Zinb^c, As, \mathfrak{H}_{nusHopf})$  both satisfy hypotheses (H0), (H1) and (H2iso).

We are now going to introduce a new type of bialgebras involving the Zinbiel operad.

**Definition 1.0.5.** A  $(Zinb^c, Leib, \mathfrak{H}_{ZL})$ -bialgebra is a  $\mathbb{K}$ -vector space  $H$  endowed with a Zinbiel coproduct  $\delta : H \otimes H \otimes H$  and a right Leibniz bracket  $\nu : H \otimes H \rightarrow H$  such that the following compatibility relation

$$\delta \circ \nu = \text{Id} \otimes \text{Id} + (\text{Id} \otimes \nu) \circ (\delta \otimes \text{Id}) + (\nu \otimes \text{Id}) \circ (23) \circ (\delta \otimes \text{Id}) - \text{Id} \otimes (\nu \circ \delta) \quad \mathfrak{H}_{ZL}$$

holds.

**Proposition 1.0.6.** The free Leibniz algebra  $Leib(V)$  is naturally equipped with a  $(Zinb^c, Leib, \mathfrak{H}_{ZL})$ -bialgebra structure.

*Proof.* Clearly,  $Leib(V) = \bar{TV} = Zinb^c(V)$  as vector spaces which shows that  $Leib(V)$  is endowed with a Zinbiel coalgebra structure  $\delta : Leib(V) \rightarrow Leib(V) \otimes Leib(V)$ . So all we have to check is whether relation  $\emptyset_{ZL}$  is indeed satisfied. Denote by  $[a, b]$  the Leibniz bracket  $\nu(a \otimes b)$  of two elements  $a$  and  $b$  of  $Leib(V)$ . In Sweedler's notation, relation  $\emptyset_{ZL}$  reads

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [b_{(2)}, b_{(1)}] \quad (1)$$

Let's prove that this relation holds by induction on  $|b|$ , the length and  $b$ :

- For  $|b| = 1$ , i.e.  $b \in V$ , (1) holds by definition of the cofree half-shuffle zinbiel coproduct  $\delta$  and because the last term  $a \otimes [b_{(2)}, b_{(1)}]$  vanishes since  $\delta(b) = 0$ .
- For  $|b| > 1$ . Assume that  $\emptyset_{ZL}$  holds for any  $b'$  of length strictly lower than  $|b|$  and for any  $a$ . We can suppose that  $b$  is of the form

$$b = [b', v] \quad , \quad |b'| = |b| - 1, \quad v \in V$$

Thus

$$\begin{aligned} \delta[a, b] &= \delta([a, [b', v]]) \\ &= \delta([[a, b'], v]) - \delta([[a, v], b']) \\ &= [[a, b']_{(1)}, v] \otimes [a, b']_{(2)} + [a, b']_{(1)} \otimes [[a, b']_{(2)}, v] + [a, b'] \otimes v - \delta([[a, v], b']) \end{aligned}$$

But the induction hypothesis allows us to express  $[a, b']_{(1)} \otimes [a, b']_{(2)} = \delta([a, b'])$  and  $\delta([[a, v], b'])$  respectively as

$$[a, b']_{(1)} \otimes [a, b']_{(2)} = a \otimes b' + [a_{(1)}, b'] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b'] - a \otimes [b'_{(2)}, b'_{(1)}]$$

and, using again  $\emptyset_{ZL}$  to rewrite  $\delta[a, v]$ ,

$$\begin{aligned} \delta([[a, v], b']) &= [a, v] \otimes b' + [[a, v]_{(1)}, b'] \otimes [a, v]_{(2)} + [a, v]_{(1)} \otimes [[a, v]_{(2)}, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ &= [a, v] \otimes b' + [[a_{(1)}, v], b'] \otimes a_{(2)} + [a_{(1)}, b'] \otimes [a_{(2)}, v] + [a, b'] \otimes v \\ &\quad + [a_{(1)}, v] \otimes [a_{(2)}, b'] + a_{(1)} \otimes [[a_{(2)}, v], b'] + a \otimes [v, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}] \end{aligned}$$

so that  $\delta[a, b]$  takes the form

$$\begin{aligned} \delta[a, b] &= [a, v] \otimes b' + [[a_{(1)}, b'], v] \otimes a_{(2)} + [a_{(1)}, v] \otimes [a_{(2)}, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ &\quad + a \otimes [b', v] + [a_{(1)}, b'] \otimes [a_{(2)}, v] + a_{(1)} \otimes [[a_{(2)}, b'], v] - a \otimes [[b'_{(2)}, b'_{(1)}], v] + [a, b'] \otimes v \\ &\quad - [a, v] \otimes b' - [[a_{(1)}, v], b'] \otimes a_{(2)} - [a_{(1)}, b'] \otimes [a_{(2)}, v] - [a, b'] \otimes v \\ &\quad - [a_{(1)}, v] \otimes [a_{(2)}, b'] - a_{(1)} \otimes [[a_{(2)}, v], b'] - a \otimes [v, b'] + [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ &= a \otimes [b', v] + \underbrace{[[a_{(1)}, b'], v] \otimes a_{(2)} - [[a_{(1)}, v], b'] \otimes a_{(2)}}_A + \underbrace{a_{(1)} \otimes [[a_{(2)}, b'], v] - a_{(1)} \otimes [[a_{(2)}, v], b']}_B \\ &\quad - a \otimes [[b'_{(2)}, b'_{(1)}], v] - a \otimes [v, b'] \end{aligned}$$

using the right Leibniz relation satisfied by  $[-, -]$  to simplify terms  $A$  and  $B$  in the above expression, and remembering that  $[b', v] = b$  leads to

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [[b'_{(2)}, b'_{(1)}], v] - a \otimes [v, b'] \quad (2)$$

Now remark that, applying once again the induction hypothesis gives

$$b_{(1)} \otimes b_{(2)} = \delta(b) = \delta[b', v] = b' \otimes v + [b'_{(1)}, v] \otimes b'_{(2)} + b'_{(1)} \otimes [b'_{(2)}, v]$$

Thus, using the Leibniz relation to get the second equality,

$$[b_{(2)}, b_{(1)}] = [v, b'] + [b'_{(2)}, [b'_{(1)}, v]] + [[b'_{(2)}, v], b'_{(1)}] = [v, b'] + [[b'_{(2)}, b'_{(1)}], v]$$

Which enables us to rewrite equation (2) as

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [b_{(2)}, b_{(1)}]$$

i.e. (1) is satisfied by  $a$  and  $b$ .

This shows that if  $\emptyset_{ZL}$  holds for any  $b'$  such that  $|b'| < |b|$  and any  $a$ , it holds also for  $b$  and any  $a$ . Thus  $\emptyset_{ZL}$  has to hold any elements  $a$  and  $b$  of  $Leib(V)$ .

□

As a direct consequence of this proposition, we have the following

**Theorem 1.0.7.** *The type of bialgebra  $(Zinb^c, Leib, \downarrow_{ZL})$  satisfies hypotheses (H0), (H1) and (H2iso) of the rigidity theorem 1.0.3. Thus any connected  $(Zinb^c, Leib, \downarrow_{ZL})$ -bialgebra is both free and cofree over its primitive part.*

*Proof.* It's clear that  $\downarrow_{ZL}$  is a distributive compatibility relation so (H0) is fulfilled. The fact that (H1) holds is exactly the content of proposition 1.0.6, and hypothesis (H2) is obviously satisfied since the Zinbiel coproduct defined on  $Leib(V) = \overline{TV}$  is by definition the cofree one. □

## 2 Leibniz dialgebras and the triple $(Zinb, Leib^2, Leib)$

**Definition 2.0.8.** *A Leibniz dialgebra (or Leib<sup>2</sup>-algebra) is a  $\mathbb{K}$ -module  $L$  endowed with two linear brackets  $(-, -) : L \otimes L \rightarrow L$  and  $\{-, -\} : L \otimes L \rightarrow L$  such that*

a)  $(-, -)$  is a right Leibniz bracket i.e.

$$((a, b), c) = ((a, c), b) + (a, (b, c))$$

b)  $\{-, -\}$  is a left Leibniz bracket i.e.

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}$$

c)  $\{-, -\}$  is a left derivation of  $(-, -)$  i.e.

$$\{a, (b, c)\} = (\{a, b\}, c) + (b, \{a, c\})$$

d)  $(-, -)$  is a right derivation for  $\{-, -\}$  i.e.

$$(\{a, b\}, c) = \{(a, c), b\} + \{a, (b, c)\}$$

for all  $a, b$  and  $c$  in  $A$ .

**Proposition 2.0.9.** *Let  $(L, (-, -), \{-, -\})$  be a Leibniz dialgebra. Then the linear map  $[-, -] : L \otimes L \rightarrow L$  defined by*

$$[a, b] := (a, b) - \{b, a\}$$

for all  $a$  and  $b$  in  $L$  is a right Leibniz bracket.

*Proof.* Let  $a, b$  and  $c$  be three elements of  $L$ . Then,

$$\begin{aligned} [[a, b], c] &= ([a, b], c) - \{c, [a, b]\} \\ &= ((a, b), c) - (\{b, a\}, c) - \{c, (a, b)\} + \{c, \{b, a\}\} \\ &\stackrel{*}{=} ((a, c), b) + (a, (b, c)) - (\{b, c\}, a) - \{b, (a, c)\} - (\{c, a\}, b) - (a, \{c, b\}) + \{\{c, b\}, a\} + \{b, \{c, a\}\} \\ &= ([a, c], b) + (a, [b, c]) - \{[b, c], a\} - \{b, [a, c]\} \\ &= [[a, c], b] + [a, [b, c]] \end{aligned}$$

where the equality  $*$  is a direct consequence of relations a), b), c) and d) of definition 2.0.8. □

Leibniz dialgebras are algebras over an operad we denote by  $Leib^2$ . This operad is the quotient of  $Free(\vdash, \dashv)$ , the free operad on two binary generators  $\vdash$  and  $\dashv$ , by the operadic ideal  $(R)$  generated the following four relators

$$\begin{aligned} (R1) \quad & \vdash \circ_1 \vdash - \vdash \circ_1 \vdash \quad (23) - \vdash \circ_2 \vdash, \\ (R2) \quad & \dashv \circ_2 \dashv - \dashv \circ_1 \dashv - \dashv \circ_2 \dashv \quad (12), \\ (R3) \quad & \vdash \circ_1 \dashv - \dashv \circ_1 \vdash \quad (23) - \dashv \circ_2 \vdash, \\ (R4) \quad & \dashv \circ_2 \vdash - \vdash \circ_1 \dashv - \vdash \circ_2 \dashv \quad (12). \end{aligned}$$

**Theorem 2.0.10.** *Let  $V$  be a  $\mathbb{K}$ -module and denote by  $Leib^2(V)$  the free Leibniz dialgebra generated by  $V$ . Then*

$$Leib^2(V) \cong \bar{T}\bar{T}V$$

as a  $\mathbb{K}$ -module.

*Proof.* First, notice that the operadic ideal  $(R)$  generated by the four relators  $(R1)$ ,  $(R2)$ ,  $(R3)$  and  $(R4)$  is also generated by the four following ones

$$\begin{aligned} (R1) \quad & \vdash \circ_1 \vdash - \vdash \circ_1 \vdash (23) - \vdash \circ_2 \vdash, \\ (R2) \quad & \dashv \circ_2 \dashv - \dashv \circ_1 \dashv - \dashv \circ_2 \dashv (12), \\ (R3') \quad & \dashv \circ_1 \vdash + \vdash \circ_2 \dashv (123). \\ (R4) \quad & \dashv \circ_2 \vdash - \vdash \circ_1 \dashv - \vdash \circ_2 \dashv (12). \end{aligned}$$

Choosing as leading terms respectively  $\vdash \circ_2 \vdash$ ,  $\dashv \circ_1 \dashv$ ,  $\dashv \circ_1 \vdash$  and  $\dashv \circ_2 \vdash$  leads to the four following rewriting rules:

$$\begin{aligned} (R1) \quad & \vdash \circ_2 \vdash \longmapsto \vdash \circ_1 \vdash - \vdash \circ_1 \vdash (23), \\ (R2) \quad & \dashv \circ_1 \dashv \longmapsto \dashv \circ_2 \dashv - \dashv \circ_2 \dashv (12), \\ (R3') \quad & \dashv \circ_1 \vdash \longmapsto - \vdash \circ_2 \dashv (123). \\ (R4) \quad & \dashv \circ_2 \vdash \longmapsto \vdash \circ_1 \dashv + \vdash \circ_2 \dashv (12). \end{aligned}$$

One has to check that any critical monomial is confluent (see [Hof10] or Chap. 8 of [LV12]).  $\square$

**Corollary 2.0.11.**  *$Leib^2$  is a Koszul operad.*

*Proof.* See [Hof10] or Chap. 8 of [LV12].  $\square$

**Definition 2.0.12.** *A  $(Zinb^c, Leib^2, \mathcal{Q}_{ZL^2})$ -bialgebra is a vector space  $H$  endowed with a Zinbiel coproduct  $\delta : H \rightarrow H \otimes H$  and a  $Leib^2$ -algebra structure given by brackets  $(-, -) : L \otimes L \rightarrow L$  and  $\{-, -\} : L \otimes L \rightarrow L$  such that*

- $(H, \delta, (-, -))$  is a  $(Zinb^c, Leib, \mathcal{Q}_{ZL^2})$ -bialgebra i.e.  $\delta$  and  $(-, -)$  satisfy relation  $\mathcal{Q}_{ZL}$  of definition 1.0.5,
- $(H, \delta, \{-, -\} \circ (12))$  is also a  $(Zinb^c, Leib, \mathcal{Q}_{ZL^2})$ -bialgebra.

**Proposition 2.0.13.** *The free Leibniz dialgebra  $Leib^2(V)$  generated by a vector space  $V$  is naturally equipped with a Zinbiel coproduct  $\delta : Leib^2(V) \rightarrow Leib^2(V)^{\otimes 2}$  which turns it into a  $(Zinb^c, Leib^2, \mathcal{Q}_{ZL^2})$ -bialgebra.*

*Proof.* Set  $\delta(v) = 0$  for any  $v$  in  $V$  and note that  $\delta$  is then fully determined by induction on the length of monomials since if it exists, it has to satisfy  $\mathcal{Q}_{ZL^2}$ . Moreover,  $\mathcal{Q}_{ZL^2}$  indeed well-defines a linear map  $\delta$  on  $Leib^2(V)$ , because proposition 2.0.10 implies that we have a PBW basis of  $Leib^2(V)$  consisting of monomials of the form

$$(\{v_1^1, \dots, v_{i_1}^1\}, \{v_1^2, \dots, v_{i_2}^2\}, \dots, \{v_1^n, \dots, v_{i_n}^n\})$$

where  $n$  ranges over all positive numbers and the  $v_j^k$  run over some fixed basis of  $V$ , where  $(a_1, \dots, a_n)$  stands for the right-iterated bracket

$$(((\dots(a_1, a_2), a_3), \dots), a_n)$$

and  $\{v_1, \dots, v_k\}$  for the iterated left-bracket

$$\{v_1, \{v_2, \{\dots, \{v_{k-1}, v_k\} \dots\}\}\}$$

Obviously, the naturality of  $\delta$  comes from the fact that its definition doesn't depend on the choice of basis of  $V$  we are using.

Let us prove that  $\delta$  is a left-Zinbiel coproduct, i.e. that it satisfies

$$(\delta \otimes \text{Id})\delta = (\text{Id} \otimes \delta^{com}) \circ \delta \tag{3}$$

where, as usual,  $\delta^{com} := \delta + (12) \circ \delta$ .

This we do one more time by induction on length, using the natural grading of  $Leib^2(V)$  given by the PBW basis for which  $(\{v_1^1, \dots, v_{i_1}^1\}, \{v_1^2, \dots, v_{i_2}^2\}, \dots, \{v_1^n, \dots, v_{i_n}^n\})$  has length  $i_1 + \dots + i_n$ .

- Clearly, (3) holds on elements of  $V$ .

- Assume that (3) holds on any monomial of length strictly lower than  $n$ . Any PBW-monomial  $a$  of length  $n$  is of the form

$$a = (a', w) \quad , \quad |a'| = n - p < n - 1, \quad w = \{v_1, \dots, v_p\}, \quad v_1, \dots, v_p \in V$$

or of the form

$$b = \{v, b'\} \quad , \quad b' = \{v_1, \dots, v_{n-1}\}, \quad v, v_1, \dots, v_{n-1} \in V$$

But, since  $\delta$  is defined so that it satisfies  $\mathcal{Q}_{ZL^2}$  on  $(a', w)$  and on  $\{v, b'\}$ ,

$$\delta(a', w) = a' \otimes w + (a'_{(1)}, w) \otimes a'_{(2)} + a'_{(1)} \otimes (a'_{(2)}, w) - a' \otimes (w_{(2)}, w_{(1)})$$

and

$$\delta\{v, b'\} = b' \otimes v + \{v, b'_{(1)}\} \otimes b'_{(2)} + b'_{(1)} \otimes \{v, b'_{(2)}\}$$

Applying  $\delta \otimes \text{Id}$  to both equations and using relation  $\mathcal{Q}_{ZL^2}$  again gives respectively

$$\begin{aligned} (\delta \otimes \text{Id}) \circ \delta(a', w) &= a'_{(1)} \otimes a'_{(2)} \otimes w + (a'_{(1)}, w)_{(1)} \otimes (a'_{(1)}, w)_{(2)} \otimes a'_{(2)} \\ &\quad + a'_{(1)(1)} \otimes a'_{(1)(2)} \otimes (a'_{(2)}, w) - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)}) \\ &= a'_{(1)} \otimes a'_{(2)} \otimes w + a'_{(1)} \otimes w \otimes a'_{(2)} + (a'_{(1)(1)}, w) \otimes a'_{(1)(2)} \otimes a'_{(2)} + a'_{(1)(1)} \otimes (a'_{(1)(2)}, w) \otimes a'_{(2)} \\ &\quad - a'_{(1)} \otimes (w_{(2)}, w_{(1)}) \otimes a'_{(2)} + a'_{(1)(1)} \otimes a'_{(1)(2)} \otimes (a'_{(2)}, w) - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)}) \end{aligned}$$

and

$$\begin{aligned} (\delta \otimes \text{Id}) \circ \delta\{v, b'\} &= b'_{(1)} \otimes b'_{(2)} \otimes v + \{v, b'_{(1)}\}_{(1)} \otimes \{v, b'_{(1)}\}_{(2)} \otimes b'_{(2)} + b'_{(1)(1)} \otimes b'_{(1)(2)} \otimes \{v, b'_{(2)}\} \\ &= b'_{(1)} \otimes b'_{(2)} \otimes v + b'_{(1)} \otimes v \otimes b'_{(2)} + \{v, b'_{(1)(1)}\} \otimes b'_{(1)(2)} \otimes b'_{(2)} + b'_{(1)(1)} \otimes \{v, b'_{(1)(2)}\} \otimes b'_{(2)} \\ &\quad + b'_{(1)(1)} \otimes b'_{(1)(2)} \otimes \{v, b'_{(2)}\} \end{aligned}$$

Now notice that the coZinbiel relation reads

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} + c_{(1)} \otimes c_{(2)(2)} \otimes c_{(2)(1)}$$

in Sweedler's notation. Applying it taking  $c = a'$  and  $c = b'$  in the two preceding equations, and using  $\mathcal{Q}_{ZL^2}$  to simplify the terms leads to

$$\begin{aligned} (\delta \otimes \text{Id}) \circ \delta(a', w) &= a'_{(1)} \otimes a'_{(2)} \otimes w + a'_{(1)} \otimes w \otimes a'_{(2)} - a'_{(1)} \otimes (w_{(2)}, w_{(1)}) \otimes a'_{(2)} - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)}) \\ &\quad + (a'_{(1)}, w) \otimes a'_{(2)(1)} \otimes a'_{(2)(2)} + (a'_{(1)}, w) \otimes a'_{(2)(2)} \otimes a'_{(2)(1)} \\ &\quad + a'_{(1)} \otimes (a'_{(2)(1)}, w) \otimes a'_{(2)(2)} + a'_{(1)} \otimes (a'_{(2)(2)}, w) \otimes a'_{(2)(1)} \\ &\quad + a'_{(1)} \otimes a'_{(2)(1)} \otimes (a'_{(2)(2)}, w) + a'_{(1)} \otimes a'_{(2)(2)} \otimes (a'_{(2)(1)}, w) \\ &= (a'_{(1)}, w) \otimes \delta^{com} a'_{(2)} + a'_{(1)} \otimes \delta^{com}(a'_{(2)}, w) \\ &= (\text{Id} \otimes \delta^{com})((a'_{(1)}, w) \otimes a'_{(2)} + a'_{(1)} \otimes (a'_{(2)}, w)) \\ &= (\text{Id} \otimes \delta^{com}) \circ \delta(a', w) = (\text{Id} \otimes \delta^{com}) \circ \delta(a) \end{aligned}$$

and similarly, since  $\delta(v) = 0$ , to

$$\begin{aligned} (\delta \otimes \text{Id}) \circ \delta\{v, b'\} &= b'_{(1)} \otimes b'_{(2)} \otimes v + b'_{(1)} \otimes v \otimes b'_{(2)} \\ &\quad + \{v, b'_{(1)}\} \otimes b'_{(2)(1)} \otimes b'_{(2)(2)} + \{v, b'_{(1)}\} \otimes b'_{(2)(2)} \otimes b'_{(2)(1)} \\ &\quad + b'_{(1)} \otimes \{v, b'_{(2)(1)}\} \otimes b'_{(2)(2)} + b'_{(1)} \otimes \{v, b'_{(2)(2)}\} \otimes b'_{(2)(1)} \\ &\quad + b'_{(1)} \otimes b'_{(2)(1)} \otimes \{v, b'_{(2)(2)}\} + b'_{(1)} \otimes b'_{(2)(2)} \otimes \{v, b'_{(2)(1)}\} \\ &= (\text{Id} \otimes \delta^{com})(\{v, b'_{(1)}\} \otimes b'_{(2)} + b'_{(1)} \otimes \{v, b'_{(2)}\}) \\ &= (\text{Id} \otimes \delta^{com}) \circ \delta(\{v, b'\}) = (\text{Id} \otimes \delta^{com}) \circ \delta(b) \end{aligned}$$

which proves that if (3) holds on words of length strictly lower than  $n$ , it holds for words of length  $n$ . By induction on  $n$ , this shows that  $\delta$  is a coZinbiel coproduct.

Now notice that we have defined  $\delta$  so that it satisfies  $\mathcal{Q}_{ZL^2}$  a priori only on PBW monomials, so it remains to prove that the compatibility relation holds on arbitrary brackets, that is on brackets of PBW monomials. The fact that  $(-, -)$  and  $\delta$  satisfy  $\mathcal{Q}_{ZL^2}$  can be proved performing exactly the same computation that the one we gave in the proof of proposition 1.0.6, so we only check here that  $\{-, -\}$  also satisfies  $\mathcal{Q}_{ZL^2}$ .

Let  $a$  and  $b$  be PBW monomials.

- If  $b$  is of the form  $\{v_1, \dots, v_n\}$ ,  $\emptyset_{ZL^2}$  holds because  $(a, b)$  is also a PBW monomial.
- If not,  $b$  can be written as  $b = (b', w)$ , with  $b'$  a PBW-monomial, i.e. an iterated bracket  $(-, -, \dots, -)$ , with necessarily one less entry than  $b$  and on which we can assume that  $\emptyset_{ZL^2}$  holds, and where  $w$  is of the form  $w = \{v_1, \dots, v_n\}$ .

$$\begin{aligned}
\delta(a, b) &= \delta(a, (b', w)) \\
&= \delta((a, b'), w) - \delta((a, w), b') \\
&= (a, b') \otimes w + ((a, b')_{(1)}, w) \otimes (a, b')_{(2)} + (a, b')_{(1)} \otimes ((a, b')_{(2)}, w) - (a, b') \otimes (w_{(2)}, w_{(1)}) \\
&\quad - (a, w) \otimes b' - ((a, w)_{(1)}, b') \otimes (a, w)_{(2)} - (a, w)_{(1)} \otimes ((a, w)_{(2)}, b') + (a, w) \otimes (b'_{(2)}, b'_{(1)})
\end{aligned}$$

□

We now determine the primitive operad  $\text{Prim}_{\text{Zinb}^c}(\text{Leib}^2)$ . The method we use is the one employed by Loday in [Petit Livre Bleu] to show that the primitive operad of the type  $(As^c, Dup)$  is  $Mag$ .

**Proposition 2.0.14.** *Let  $\mathfrak{g}$  be a right Leibniz algebra with bracket  $[-, -]_{\mathfrak{g}}$ , and define three linear maps  $(-, -) : \bar{T}\mathfrak{g} \otimes \bar{T}\mathfrak{g} \rightarrow \bar{T}\mathfrak{g}$ ,  $\{-, -\} : \bar{T}\mathfrak{g} \otimes \bar{T}\mathfrak{g} \rightarrow \bar{T}\mathfrak{g}$  and  $\delta : \bar{T}\mathfrak{g} \rightarrow \bar{T}\mathfrak{g} \otimes \bar{T}\mathfrak{g}$  by setting:*

- $\delta$  is the cofree conZinbiel coproduct obtained by identifying  $\bar{T}\mathfrak{g}$  with  $\text{Zinb}^c(\mathfrak{g})$ , the cofree coZinbiel coalgebra cogenerated by the vector space  $\mathfrak{g}$ ,
- $(-, -)$  is the free right Leibniz bracket obtained by identifying  $\bar{T}\mathfrak{g}$  with  $\text{Leib}(\mathfrak{g})$ , the free Leibniz algebra generated  $\mathfrak{g}$ .
- $\{a, b\} := (b, a) - [b, a]$  for all  $a$  and  $b$  in  $\bar{T}\mathfrak{g}$ ,

where  $[-, -]$  is the unique bracket defined inductively by

- $[g, h] = [g, h]_{\mathfrak{g}}$  if  $g$  and  $h$  are in  $\mathfrak{g}$ ,
- $[(a, g), b] = ([a, b], g) + (a, [g, b])$  for all  $a, b$  in  $\bar{T}\mathfrak{g}$  and  $g$  in  $\mathfrak{g}$ ,
- $[g, (b, h)] = (a, (b, h)) + (a, (h, b)) - (a, [h, b])$  for all  $b$  in  $\bar{T}\mathfrak{g}$  and  $g, h$  in  $\mathfrak{g}$ .

Then  $(\bar{T}\mathfrak{g}, (-, -), \{-, -\}, \delta)$  is a  $(\text{Zinb}^c, \text{Leib}^2, \emptyset_{ZL^2})$ -bialgebra.

Applying the preceding proposition to the case  $\mathfrak{g} = \text{Leib}(V)$ , we get the following:

**Theorem 2.0.15.** *The unique map of Leibniz dialgebras  $\psi : \text{Leib}^2(V) \rightarrow \bar{T}\text{Leib}(V)$  lifting the canonical inclusion  $V \rightarrow \bar{T}\text{Leib}(V)$  is an isomorphism of coZinbiel coalgebras.*

**Corollary 2.0.16.**

$$\text{Prim}_{\text{Zinb}^c}(\text{Leib}^2) = \text{Leib}$$

**Definition 2.0.17.** • An **ideal** of a Leibniz dialgebra  $(L, (-, -), \{-, -\})$  is a linear subspace  $I$  such that  $(I, L)$ ,  $(L, I)$ ,  $\{I, L\}$  and  $\{L, I\}$  are all contained in  $I$ .

- Let  $(\mathfrak{g}, [-, -]_{\mathfrak{g}})$  be a Leibniz algebra. The **universal enveloping Leibniz dialgebra** of  $\mathfrak{g}$ , denoted  $U_{LD}(\mathfrak{g})$ , is the quotient of the free Leibniz dialgebra generated by  $\mathfrak{g}$  by the ideal  $I$  generated by elements of the form  $(g, h) - \{h, g\} - [g, h]_{\mathfrak{g}}$ , i.e.

$$U_{LD}(\mathfrak{g}) := \text{Leib}^2(\mathfrak{g}) / \langle (g, h) - \{h, g\} - [g, h]_{\mathfrak{g}}, g, h \in \mathfrak{g} \rangle$$

**Theorem 2.0.18.** *The triple  $(\text{Zinb}, \text{Leib}^2, \text{Leib})$  is a good triple of operads.*

### 3 Homology of Leibniz dialgebras

**Definition 3.0.19.** A **Zinbiel dialgebra** (or  $\text{Zinb}^2$ -algebra) is a vector space  $Z$  equipped with two products  $\succ: Z \otimes Z \rightarrow Z$  and  $\prec: Z \otimes Z \rightarrow Z$  satisfying the four following relations

1.  $(a \prec b) \prec c = a \prec (b \prec c + c \prec b)$ ,
2.  $a \succ (b \succ c) = (a \succ b + b \succ a) \succ c$ ,
3.  $(a \prec b) \succ c = a \prec (b \succ c) + a \succ (c \prec b)$ ,
4.  $a \succ (b \prec c) = (a \succ b) \prec c + (b \prec a) \succ c$ ,

for all  $a, b$  and  $c$  in  $Z$ .

Zinbiel dialgebras are encoded by an algebraic operad we denote  $\text{Zinb}^2$ .

**Proposition 3.0.20.**  $Zinb^2$  is the Koszul dual of  $Leib^2$ , i.e.

$$(Leib^2)^! = Zinb^2$$

The knowledge of the Koszul dual of  $Leib^2$  enables us to determine the operadic homology theory of Leibniz dialgebras:

**Definition 3.0.21.** Let  $(L, (-, -), \{-, -\})$  be a Leibniz dialgebra. The **Leibniz dialgebra chain complex** of  $L$ , denoted by  $CLD_*(L)$ , is the graded vector space defined in degree  $n$  by

$$CLD_n(L) := \mathbb{K}[\{0, 1\}^{n-1}] \otimes L^{\otimes n}$$

equipped with the differential  $d^{LD} : CLD_*(L) \rightarrow CLD_{*-1}(L)$  defined in degree  $n$  by

$$d^{LD}((\varepsilon_2, \dots, \varepsilon_n) \langle x_1 | \dots | x_n \rangle) := \sum_{1 \leq i < j \leq n} (-1)^j (\varepsilon_2, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1 | \dots | x_{i-1} | (x_i, x_j)_{\varepsilon_j} | \dots | \widehat{x}_j | \dots | x_n \rangle$$

for all  $\varepsilon_2, \dots, \varepsilon_n$  in  $\{0, 1\}$ , and  $x_1, \dots, x_n$  in  $L$ , where the bracket  $(x, y)_\varepsilon$  is defined by

$$(x, y)_\varepsilon := \begin{cases} (x, y) & \text{if } \varepsilon = 1, \\ \{y, x\} & \text{if } \varepsilon = 0. \end{cases}$$

and where the notation  $(\varepsilon_2, \dots, \varepsilon_n) \langle x_1 | \dots | x_n \rangle$  stands for the length  $n$  elementary tensor  $(\varepsilon_2, \dots, \varepsilon_n) \otimes x_1 \otimes x_2 \otimes \dots \otimes x_n$  and the symbol  $\widehat{x}_j$  means as usual that  $x_j$  has been omitted.

**Proposition 3.0.22.** The graded vector space  $CLD_*(L)$  equipped with the degree  $-1$  map  $d^{LD}$  defined above is indeed a chain complex, i.e.

$$d^{LD} \circ d^{LD} = 0$$

Moreover, its homology, that we denote  $HLD_*(L) := H_*(CLD_*(L), d^{LD})$ , is the operadic homology of the  $L$  prescribed by the theory of operads, meaning that it can be obtained as the homology of the (graded) cofree  $Zinb^2$ -coalgebra cogenerated by  $L[1]$  endowed with the unique coderivation extending the canonical twisting cochain  $\kappa : (Zinb^2)^c(L[1]) \rightarrow L[1]$ .

The canonical morphism of operads  $Leib \rightarrow Leib^2$  induces a morphism of operads  $(Leib^2)^! \rightarrow Leib^!$  and thus, for any Leibniz dialgebra  $(L, (-, -), \{-, -\})$ , a morphism of chain complexes

$$CL_*(L_{Leib}) \rightarrow CLD_*(L)$$

where  $L_{Leib}$  denotes the underlying Leibniz algebra obtained from  $L$  by only remembering the Leibniz bracket  $[-, -] := (-, -) - \{-, -\} \circ (12)$ .

Taking  $L = U_{LD}(\mathfrak{g})$  for some Leibniz algebra  $\mathfrak{g}$  and precomposing this morphism with the one induced at the level of chain complexes by the inclusion of Leibniz algebras  $\mathfrak{g} \hookrightarrow (U_{LD}(\mathfrak{g}))_{Leib}$ , we get a morphism of chain complexes

$$\phi : CL_*(\mathfrak{g}) \rightarrow CLD_*(U_{LD}(\mathfrak{g})) \quad (4)$$

**Proposition 3.0.23.** The morphism of chain complexes  $\phi : CL_*(\mathfrak{g}) \rightarrow CLD_*(U_{LD}(\mathfrak{g}))$  defined above is given in degree  $n$  by the following explicit formula:

$$\phi(\langle g_1 | \dots | g_n \rangle) = \sum_{\varepsilon \in \{0, 1\}^{n-1}} (-1)^{c(\varepsilon)} \varepsilon \langle g_1 | \dots | g_n \rangle$$

where the integer  $c(\varepsilon)$  is the number of 0's in the multi-index  $\varepsilon$ .

Notice that  $(Leib(V), [-, -])$ , the free Leibniz algebra generated by some vector space  $V$ , can be seen as a Leibniz dialgebra by setting  $(-, -) := [-, -]$  and  $\{-, -\} := [-, -] \circ (12)$ .

**Proposition 3.0.24.** The universal enveloping dialgebra of a vector space  $V$  seen as abelian Leibniz algebra is  $Leib(V)$  endowed with the dialgebra structure defined above.

**Proposition 3.0.25.** For any vector space  $V$  seen as an abelian Leibniz algebra, the map  $\phi$  of (4) induces an isomorphism in homology i.e.

$$H_n(\phi) : HL_n(V) = V^{\otimes n} \xrightarrow{\cong} HLD_n(U_{LD}(V)) = HLD_n(Leib(V))$$

is an isomorphism for all  $n \geq 0$

*Proof.* Let  $n$  be a positive integer and recall that, as a vector space,  $Leib(V) = \bar{T}V$  and  $V \subset Leib(V)$ , and that the  $Leib^2$ -brackets are given by  $(-, -)_1 = (-, -)_0 = [-, -]$ , the free Leibniz bracket on  $Leib(V)$ .

The fact that  $HLD_n(Leib(V))$  is "smaller" than  $\mathbb{K}[\{0, 1\}^{n-1}] \otimes V^{\otimes n}$  is a consequence of the following

**Lemma 3.0.26.** *Any  $n$ -cycle in  $CLD_n(Leib(V))$  is homologeous to a  $n$ -cycle in  $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes n} \subset CLD_n(Leib(V))$*

Before proving the lemma, let us show how to use it to establish the proposition. Fix an integer  $n$ .

First notice that  $H_n(\phi)$  is clearly injective because  $\phi$  takes its values in  $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes n}$  which is not hit by  $d^{LD}$  since any tensor in the image of an element under  $d^{LD}$  has to contain a factor of length greater than 2.

Let us show that  $H_n(\phi)$  is surjective, i.e. that any cycle  $\omega$  in  $CLD_n(Leib(V))$  is, modulo some boundary, in the image of  $\phi$ . Thanks to lemma 3.0.26, we can restrict to the case when  $\omega$  is a cycle of the form

$$\omega = \sum_{\varepsilon} \varepsilon \langle v_1^{\varepsilon} | \cdots | v_n^{\varepsilon} \rangle$$

where the  $v_i^{\varepsilon}$ 's are in  $V$ .

*Proof of lemma 3.0.26.* Let  $\omega = \sum_{\varepsilon} \varepsilon \langle x_1^{\varepsilon} | \cdots | x_n^{\varepsilon} \rangle$  be an arbitrary element in  $CLD_n(Leib(V))$ , where the  $\varepsilon$ 's are multi-indices in  $\{0,1\}^{n-1}$  (possibly redundant) and the  $x_i$ 's are elements of  $Leib(V) = \widehat{TV}$ . Since for any  $\varepsilon = (\varepsilon_2 \cdots, \varepsilon_n)$ , any  $y$  in  $Leib(V)$  and  $v \in V$

$$\begin{aligned} d^{LD}((\varepsilon_2, \dots, \varepsilon_n, 1) \langle x_1 | \cdots | x_{n-1} | y | v \rangle) &= (-1)^n \varepsilon \langle x_1 | \cdots | x_{n-1} | [y, v] \rangle + \sum_{\varepsilon'} \varepsilon' \langle z_1^{\varepsilon'} | \cdots | z_{n-1}^{\varepsilon'} | y \rangle \\ &\quad + \sum_{\varepsilon''} \varepsilon'' \langle z_1^{\varepsilon''} | \cdots | z_{n-1}^{\varepsilon''} | v \rangle \end{aligned}$$

where the  $z_i^{\varepsilon'}$ 's and the  $z_i^{\varepsilon''}$ 's are elements of  $Leib(V)$ , we can assume that  $x_n^{\varepsilon}$  is in  $V$  for all  $\varepsilon$  (because applying the preceding equality recursively to lower the degree of the last factor shows that  $\omega$  is at least homologeous to a sum of tensors having this property).

Now suppose that  $\omega$  is homologeous to a cycle of the form

$$\omega' := \sum_{\varepsilon} \varepsilon \langle x_1^{\varepsilon} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | v_n^{\varepsilon} \rangle$$

where  $k < n$  and the  $v_i^{\varepsilon}$ 's have length 1. Then

$$\begin{aligned} 0 = d^{LD} \omega = d^{LD} \omega' &= \sum_{\varepsilon} \sum_{i < j \leq k} (-1)^j \varepsilon^j \langle x_1^{\varepsilon} | \cdots | [x_i^{\varepsilon}, x_j^{\varepsilon}] | \cdots | \widehat{x_j^{\varepsilon}} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | v_n^{\varepsilon} \rangle \\ &\quad + \sum_{\varepsilon} \sum_{i \leq k < j} (-1)^j \varepsilon^j \langle x_1^{\varepsilon} | \cdots | [x_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} \rangle \\ &\quad + \sum_{\varepsilon} \sum_{k < i < j} (-1)^j \varepsilon^j \langle x_1^{\varepsilon} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | [v_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} \rangle \end{aligned}$$

where  $\varepsilon^j := (\varepsilon_2, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_n)$  if  $\varepsilon = (\varepsilon_2, \dots, \varepsilon_n)$ . But looking at the elements located at the  $k$ -th place and at the  $i$ -th place of each tensor appearing in the right-hand side of the preceding equation, we can see that for length reasons, it implies

$$(S) \begin{cases} \sum_{\substack{\varepsilon \\ |x_k^{\varepsilon}| > 1}} \sum_{k < j} (-1)^j \varepsilon^j \langle x_1^{\varepsilon} | \cdots | [x_k^{\varepsilon}, v_j^{\varepsilon}] | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} \rangle = 0 \\ \sum_{\substack{\varepsilon \\ |x_k^{\varepsilon}| > 1}} \sum_{k < i < j} (-1)^j \varepsilon^j \langle x_1^{\varepsilon} | \cdots | x_k^{\varepsilon} | \cdots | [v_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} \rangle = 0 \end{cases}$$

Now write any  $x_k^{\varepsilon}$  of length greater than 1 as a bracket of the form  $x_k^{\varepsilon} = [y_k^{\varepsilon}, v_k^{\varepsilon}]$ , with  $v_k^{\varepsilon}$  in  $V$  and  $y_k^{\varepsilon}$  in  $Leib(V)$ . As the Leibniz bracket  $[-, -]$  is the free one, we can replace every  $x_k^{\varepsilon}$  by  $y_k^{\varepsilon} | v_k^{\varepsilon}$  and any  $\varepsilon^j$  by  $(\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_n)$  in the second equation of (S) so that

$$\sum_{\substack{\varepsilon \\ |x_k^{\varepsilon}| > 1}} \sum_{k < i < j} (-1)^j (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon_j}, \dots, \varepsilon_n) \langle x_1^{\varepsilon} | \cdots | y_k^{\varepsilon} | v_k^{\varepsilon} | \cdots | [v_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} \rangle = 0 \quad (5)$$

Similarly, the definition of the free bracket enables us to replace each  $[x_k^\varepsilon, v_j^\varepsilon] = [[y_k^\varepsilon, v_k^\varepsilon], v_j^\varepsilon]$  by  $y_k^\varepsilon |v_k^\varepsilon| v_j^\varepsilon$  and any  $\varepsilon^j$  by  $(\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n)$  in the first equation of (S) to get

$$\sum_{|x_k^\varepsilon| > 1} \sum_{k < j} (-1)^j (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | y_k^\varepsilon | v_k^\varepsilon | v_j^\varepsilon | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle = 0 \quad (6)$$

which implies both

$$\sum_{|x_k^\varepsilon| > 1} \sum_{k < j} (-1)^j (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | y_k^\varepsilon | [v_k^\varepsilon, v_j^\varepsilon] | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle = 0 \quad (7)$$

and

$$\sum_{|x_k^\varepsilon| > 1} \sum_{k < j} (-1)^j (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | [y_k^\varepsilon, v_j^\varepsilon] | v_k^\varepsilon | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle = 0 \quad (8)$$

by applying respectively  $\text{Id}^{\otimes k} \otimes [-, -] \otimes \text{Id}^{\otimes n-k-1}$  and  $\text{Id}^{\otimes k-1} \otimes [-, -] \otimes \text{Id}^{\otimes n-k} \circ (k \ k + 1)$  to (6).

Now define the  $n + 1$ -chain  $\alpha$  by

$$\alpha := \sum_{|x_k^\varepsilon| > 1} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | y_k^\varepsilon | v_k^\varepsilon | \dots | v_n^\varepsilon \rangle$$

So that

$$\begin{aligned} d^{LD}(\alpha) &= \sum_{|x_k^\varepsilon| > 1} \sum_{k < i < j} (-1)^{j+1} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | y_k^\varepsilon | v_k^\varepsilon | \dots | [v_i^\varepsilon, v_j^\varepsilon] | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ \sum_{|x_k^\varepsilon| > 1} \sum_{k < j} (-1)^{j+1} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | y_k^\varepsilon | [v_k^\varepsilon, v_j^\varepsilon] | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ \sum_{|x_k^\varepsilon| > 1} \sum_{k < j} (-1)^{j+1} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | [y_k^\varepsilon, v_j^\varepsilon] | v_k^\varepsilon | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ \sum_{|x_k^\varepsilon| > 1} \sum_{i < k < j} (-1)^{j+1} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | [x_i^\varepsilon, v_j^\varepsilon] | \dots | y_k^\varepsilon | v_k^\varepsilon | \dots | \widehat{v}_j^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ (-1)^{k+1} \sum_{|x_k^\varepsilon| > 1} \varepsilon \langle x_1^\varepsilon | \dots | x_k^\varepsilon | v_{k+1}^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ (-1)^{k+1} \sum_{|x_k^\varepsilon| > 1} \sum_{i < k} \varepsilon \langle x_1^\varepsilon | \dots | [x_i^\varepsilon, v_k^\varepsilon] | \dots | y_k^\varepsilon | v_{k+1}^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ (-1)^k \sum_{|x_k^\varepsilon| > 1} \sum_{i < k} (\varepsilon_2, \dots, \varepsilon_{k-1}, 1, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | [x_i^\varepsilon, y_k^\varepsilon] | \dots | x_{k-1}^\varepsilon | v_k^\varepsilon | \dots | v_n^\varepsilon \rangle \\ &+ \sum_{|x_k^\varepsilon| > 1} \sum_{i < j < k} (-1)^j (\varepsilon_2, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_k, 1, \dots, \varepsilon_n) \langle x_1^\varepsilon | \dots | [x_i^\varepsilon, x_j^\varepsilon] | \dots | \widehat{x}_j^\varepsilon | \dots | y_k^\varepsilon | v_k^\varepsilon | \dots | v_n^\varepsilon \rangle \end{aligned}$$

Equations (5), (7) and (8) imply that the first three sums in this expression have to vanish, which leads to

$$\begin{aligned}
\sum_{|x_k^\varepsilon|>1} \varepsilon < x_1^\varepsilon | \cdots | x_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon > &= (-1)^k d^{LD}(\alpha) \\
&+ \sum_{|x_k^\varepsilon|>1} \sum_{i<k<j} (-1)^{j+k} (\varepsilon_2, \dots, \varepsilon_k, 1, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_j^\varepsilon] \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | \widehat{v}_j^\varepsilon | \cdots | v_n^\varepsilon > \\
&+ \sum_{|x_k^\varepsilon|>1} \sum_{i<k} \varepsilon < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_k^\varepsilon] \cdots | y_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon > \\
&- \sum_{|x_k^\varepsilon|>1} \sum_{i<k} (\varepsilon_2, \dots, \varepsilon_{k-1}, 1, \dots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, y_k^\varepsilon] \cdots | x_{k-1}^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon > \\
&- \sum_{|x_k^\varepsilon|>1} \sum_{i<j<k} (-1)^{j+k} (\varepsilon_2, \dots, \widehat{\varepsilon}_j, \dots, \varepsilon_k, 1, \dots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, x_j^\varepsilon] \cdots | \widehat{x}_j^\varepsilon | \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon >
\end{aligned}$$

This proves that  $\sum_{|x_k^\varepsilon|>1} \varepsilon < x_1^\varepsilon | \cdots | x_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon >$  is homologous to a chain of the form

$$\sum_{\varepsilon'} \varepsilon' < x_1^{\varepsilon'} | \cdots | x_k^{\varepsilon'} | v_{k+1}^{\varepsilon'} | \cdots | v_n^{\varepsilon'} >$$

such that

$$\max_{\varepsilon'} |x_k^{\varepsilon'}| < \max_{\varepsilon} |x_k^\varepsilon| =: N_k(\omega')$$

and thus so is  $\omega'$ . By decreasing induction on  $N_k(\omega')$ , this proves that  $\omega'$  is homologous to a cycle of the form

$$\sum_{\varepsilon''} \varepsilon'' < x_1^{\varepsilon''} | \cdots | x_{k-1}^{\varepsilon''} | v_k^{\varepsilon''} | \cdots | v_n^{\varepsilon''} > \quad , v_i^{\varepsilon''} \in V$$

and thus so is  $\omega$ .

By decreasing induction on the integer  $k$ , this shows that  $\omega$  is indeed homologous to a cycle in elements of  $V$ , i.e. in  $\mathbb{K}\{0, 1\}^{n-1} \otimes V^\otimes \subset CLD_n(Leib(V))$ , which concludes the proof of the lemma. □

□

**Theorem 3.0.27.** *For any Leibniz algebra  $\mathfrak{g}$ ,*

$$HLD_*(U_{LD}(\mathfrak{g})) \cong HL_*(\mathfrak{g})$$

*Proof.* The general case can be reduced to the abelian one by the following standard spectral sequence argument: □

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