STOCHASTIC FLOWS AND AN INTERFACE SDE ON METRIC GRAPHS

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Abstract

We study a stochastic differential equation (SDE) driven by a finite family of independent white noises on a star graph, each of these white noises driving the SDE on a ray of the graph. This equation extends the perturbed Tanaka's equation recently studied by Prokaj [16] and Le Jan-Raimond [11] among others. We prove that there exists a (unique in law) coalescing stochastic flow of mappings solution of this equation. Our proofs involve the study of a Brownian motion in the two dimensional quadrant obliquely reflected at the boundary, with time dependent angle of reflection. Filtering this coalescing flow with respect to the family of white noises yields a Wiener stochastic flow of kernels also solution of this SDE. This Wiener solution is also unique. Moreover, if N denotes the number of rays constituting the star graph, the Wiener solution and the coalescing solution coincide if and only if $N = 2$. When $N \geq 3$, the problem of classifying all solutions is left open. Finally, we define an extension of this equation on more general metric graphs to which we apply some of our previous results [7]. As a consequence, we deduce the existence and uniqueness in law of a flow of mappings and a Wiener flow solutions of this SDE.

1. Introduction and main results

In [16], Prokaj proved that pathwise uniqueness holds for the perturbed Tanaka's equation

$$dX_t = \text{sgn}(X_t)dW_t^1 + \lambda dW_t^2 \tag{1}$$

for all $\lambda \neq 0$ where W^1 and W^2 are two independent Brownian motions. When $\lambda = 1$, after rescaling, setting $W^+ = \frac{W^1+W^2}{\sqrt{2}}$ and $W^- = \frac{W^2-W^1}{\sqrt{2}}$, (1) rewrites

$$dX_t = 1_{\{X_t>0\}}dW_t^+ + 1_{\{X_t\leq 0\}}dW_t^-. \tag{2}$$

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Using different techniques, the same result in the case of (2) has been obtained also by Le Jan and Raimond [11] (see also [4, 14]) and they proved in addition that (2) generates a stochastic coalescing flow. Intuitively, a solution to (2) is a Brownian motion that follows W^+ on its positive excursions and follows W^- on its negative excursions. In this paper, we first consider the analogous SDE on a star graph (by a star graph, we mean a finite number of pieces of $\mathbb{R}_+$ in which all origins are identified). It is clear that a solution has to be a Walsh's Brownian motion on the graph. But it is less clear when it is a strong solution and what are the flows solving this SDE. In this paper, we give a complete answer to the first question and a partial answer to the second one. Then we extend this SDE and these two questions to more general metric graphs : To each edge of the graph is associated a Brownian motion (such that the family of these Brownian motions is independent) and the SDE considered here is such that if X is a solution, then X is a Walsh's Brownian motion which, when moving on some edge, follows the Brownian motion associated to this edge.

1.1. **Notations.**

- If M is a locally compact metric space, $C_b(M)$ and $C_0(M)$ denote respectively the set of bounded continuous functions on M and the set of continuous functions on M vanishing at ∞.
- For $N \geq 2$, a star graph with N rays is a metric graph G with origin denoted by 0 and N edges $(E_i)_{1\leq i\leq N}$ such that $E_i \cap E_j = \{0\}$ if $i \neq j$ and for each i, E_i is isometric to $[0,\infty[$ via a mapping $e_i : [0,\infty[\to E_i$. Define $\sim$ the equivalence relation on G by $x \sim y$ if there exists i such that both x and y belong to E_i, and when it is not the case, we use the notation $x \not\sim y$. Let d be the metric on G such that if $x = e_i(r)$ then $|x| := d(x, 0) = r$, if $x \sim y$ then $d(x, y) = \big||y| - |x|\big|$ and if $x \not\sim y$, $d(x, y) = |x| + |y|$. We equip G with its Borel σ-field $\mathcal{B}(G)$ and use the notations $E_i^* = E_i \setminus \{0\}$ and $G^* = G \setminus \{0\}$.
- Fix $N \geq 2$ and $p_1, \ldots, p_N > 0$ such that $\sum_{i=1}^N p_i = 1$. Let G be a star graph with N rays. We denote by $C_b^2(G^*)$ the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in [1, N]$, $f \circ e_i$ is C^2 on $]0,\infty[$ with bounded first and second derivatives both with finite limits at 0. For $f \in C_b^2(G^*)$ and $x = e_i(r) \in G^*$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$. When $x = 0$ set $f'(0) = \sum_{i=1}^N p_i(f \circ e_i)'(0+)$ and $f''(0) = \sum_{i=1}^N p_i(f \circ e_i)''(0+)$. Set

$$\mathcal{D} = \big\{f \in C_b^2(G^*) : f'(0) = 0\big\}. \tag{3}$$

- The two-dimensional quadrant is the set $\mathcal{Q} := [0,\infty[^2$. Its boundary is denoted by $\partial\mathcal{Q} := \partial_1\mathcal{Q} \cup \partial_2\mathcal{Q}$, where $\partial_1\mathcal{Q} = [0,\infty[\times\{0\}$ and $\partial_2\mathcal{Q} = \{0\}\times[0,\infty[$. We also set $\mathcal{Q}^* = \mathcal{Q}\setminus\{(0,0)\}$.
- For X a continuous semimartingale, we will denote by $L_t(X)$ its symmetric local time process at 0, i.e.

$$L_t(X) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_0^t 1_{\{|X_s|\le\epsilon\}} d\langle X\rangle_s.$$

- For a family of random variables $Z = (Z_{s,t})_{s\le t}$ and a process $X = (X_t)_{t\ge 0}$, we will use the usual notations

$$\mathcal{F}^Z_{s,t} = \sigma(Z_{u,v}, s\le u\le v\le t), \quad \mathcal{F}^X_t = \sigma(X_u, 0\le u\le t).$$

- A filtration generated by a finite or infinite sequence of independent Brownian motions will be called a Brownian filtration.
- The Walsh's Brownian motion on G is the Feller diffusion defined via its Feller semigroup $(P_t, t\ge 0)$ as in [1]: Let $(T^+_t, t\ge 0)$ be the semigroup of reflecting Brownian motion on $\mathbb{R}_+$ and let $(T^0_t, t\ge 0)$ be the semigroup of Brownian motion on $\mathbb{R}_+$ killed at 0, then for $f\in C_0(G)$ and $x\in E_i$, denoting $f_j(r) = f\circ e_j(r)$ for $1\le j\le N$ and $\bar f(r) = \sum_{j=1}^N p_j f_j$,

$$P_t f(x) = T^+_t\bar f(|x|) + T^0_t(f_i - \bar f)(|x|).$$

- For a filtration $(\mathcal{G}_t)_t$, X is a $(\mathcal{G}_t)_t$-Walsh's Brownian motion if it is adapted to $(\mathcal{G}_t)_t$ and if given $\mathcal{G}_t$, $(X_{t+s},\ s\ge 0)$ is a Walsh's Brownian motion started at X_t.

1.2. **The interface SDE on a star graph.** Our main interest in this paper is the following SDE, we call the interface SDE, which is the natural extension of (2) to star graphs.

Definition 1.1. *A solution of the interface SDE (E) on a star graph G is a pair (X,W) of processes defined on a filtered probability space $(\Omega,(\mathcal{F}_t)_t,\mathbb{P})$ such that*

(i) *$W = (W^1,\dots,W^N)$ is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$;*
(ii) *X is a $(\mathcal{F}_t)$-adapted continuous process on G;*
(iii) *For all $f\in\mathcal{D}$,*

$$f(X_t) = f(X_0) + \sum_{i=1}^N\int_0^t f'(X_s)1_{\{X_s\in E_i\}}dW^i_s + \frac{1}{2}\int_0^t f''(X_s)ds. \tag{4}$$

We will say it is a strong solution if X is adapted to the filtration $(\mathcal{F}^W_t)_t$.

Note that it can easily be seen (by choosing for each i a function $f_i \in \mathcal{D}$ such that $f_i(x) = |x|$ if $x \in E_i$) that on E_i, away from 0, X follows the Brownian motion W^i. Our first result is the following

Theorem 1.2. *For all $x \in G$,*

(i) *There exists a solution (X, W) with $X_0 = x$, unique in law, of the SDE (E). Moreover X is a Walsh's Brownian motion.*
(ii) *The solution of the SDE (E) is a strong solution if and only if $N = 2$.*

To prove (ii) when $N = 2$, we will prove that pathwise uniqueness holds for (E). Then, this implies that the solution (X, W) is a strong one. The fact that when $N \geq 3$, (X, W) is not a strong solution is a consequence of a result of Tsirelson [17] (see Theorem 3.6 below) which states that if $N \geq 3$, there does not exist any $(\mathcal{F}_t)_t$-Walsh's Brownian motion on G with $(\mathcal{F}_t)_t$ a Brownian filtration (see also [3]).

When $N = 2$, one can assume $G = \mathbb{R}$, $E_1 =]-\infty, 0]$ and $E_2 = [0, \infty[$. Applying Itô-Tanaka's formula (or Theorem 3.1 below), we see that (E) is equivalent to the skew Brownian motion version of (2):

$$dX_t = 1_{\{X_t>0\}} dW_t^+ + 1_{\{X_t \leq 0\}} dW_t^- + (2p-1) dL_t(X) \tag{5}$$

where $p = p_1$ (note that when $p = 1/2$, (2) and (5) coincide).

In this paper a stochastic flow of mappings as defined by Le Jan and Raimond [12] will be called a SFM. We will be interested in SFM's solving (E) in the following sense.

Definition 1.3. *On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $\mathcal{W} = (W^i, 1 \leq i \leq N)$ be a family of independent real white noises (see* [12, Definition 1.10]*) and φ be a SFM on G. We say that $(\varphi, \mathcal{W})$ solves (E) if for all $s \leq t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$f(\varphi_{s,t}(x)) = f(x) + \sum_{i=1}^{N} \int_s^t (1_{E_i} f')(\varphi_{s,u}(x)) dW_u^i + \frac{1}{2} \int_s^t f''(\varphi_{s,u}(x)) du.$$

We will say it is a Wiener solution if for all $s \leq t$, $\mathcal{F}_{s,t}^{\varphi} \subset \mathcal{F}_{s,t}^{\mathcal{W}}$.

It will be shown that as soon as $(\varphi, \mathcal{W})$ solves (E), we have $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^{\varphi}$ for all $s \leq t$ and thus we may just say φ solves (E). Note that when φ is a Wiener solution, then $\mathcal{F}_{s,t}^{\varphi} = \mathcal{F}_{s,t}^{\mathcal{W}}$ for all $s \leq t$.

We will prove the following

Theorem 1.4. (i) *There exists a SFM φ solution of (E). This solution is unique in law.*

(ii) *The SFM φ is coalescing in the sense that for all $s \in \mathbb{R}$ and $(x,y) \in G^2$, a.s.,*

$$T_s(x,y) = \inf\{t \geq s : \varphi_{s,t}(x) = \varphi_{s,t}(y)\} < \infty$$

and $\varphi_{s,t}(x) = \varphi_{s,t}(y)$ for all $t \geq T_s(x,y)$.

(iii) *The SFM φ is a Wiener solution if and only if $N = 2$.*

Note that (iii) in this theorem is a consequence of (ii) in Theorem 1.2. Let φ be a SFM on G and $\mathcal{W}$ be a family of independent white noises such that $(\varphi, \mathcal{W})$ is a solution to (E). As $\mathcal{F}^{\mathcal{W}}_{s,t} \subset \mathcal{F}^{\varphi}_{s,t}$, Lemma 3.2 in [12] ensures that there exists a stochastic flow of kernels $K^{\mathcal{W}}$ (see [12] for the definition) such that : for all $s \leq t$, $x \in G$, a.s.

$$K^{\mathcal{W}}_{s,t}(x) = E[\delta_{\varphi_{s,t}(x)} | \mathcal{F}^{\mathcal{W}}_{s,t}].$$

A stochastic flow of kernels will be denoted from now on simply by SFK. We will also be interested in SFK's solving (E) in the following sense.

Definition 1.5. *Let K be a SFK on G and $\mathcal{W} = (W^i, 1 \leq i \leq N)$ be a family of independent real white noises. We say that $(K, \mathcal{W})$ solves (E) if for all $s \leq t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^{N} \int_s^t K_{s,u}(1_{E_i} f')(x) dW^i_u + \frac{1}{2} \int_s^t K_{s,u} f''(x) du. \tag{6}$$

We will say it is a Wiener solution if for all $s \leq t$, $\mathcal{F}^K_{s,t} \subset \mathcal{F}^{\mathcal{W}}_{s,t}$.

Since we also have $\mathcal{F}^{\mathcal{W}}_{s,t} \subset \mathcal{F}^K_{s,t}$, we may simply say that K solves (E). Note that when $K = \delta_\varphi$, then K solves (E) if and only if φ also solves (E). In this case, the SFK K will be called a SFM. We have the following

Proposition 1.6. *$K^{\mathcal{W}}$ is the unique (up to modification) Wiener solution of (E).*

We do not give a proof of this proposition here. This can be done following [6, Proposition 8] where this result is proved when all the W^i are equal, or following the proof of [11, Proposition 3.1] where this result is proved in the case of (2).

A consequence of Proposition 1.6 and Theorem 1.4 (iii) is

Corollary 1.7. *$K^{\mathcal{W}}$ is the only SFK solution of (E) if and only if $N = 2$.*

Proof. When $N = 2$, φ is a Wiener solution of (E). Suppose $(K, \mathcal{W})$ is another solution of (E), then $\mathbb{E}[K|\mathcal{W}]$ is a Wiener solution of (E). Since the Wiener solution is unique, for all $s \le t$ and $x \in G$ a.s.

$$\delta_{\varphi_{s,t}(x)} = E[K_{s,t}(x)|\mathcal{F}^{\mathcal{W}}_{s,t}].$$

This yields that $\delta_{\varphi_{s,t}(x)} = K_{s,t}(x)$ a.s. □

For $N \ge 3$, the SDE (E) may have other SFK's solutions different from φ and $K^{\mathcal{W}}$. The problem of a complete classification of the laws of all these flows is left open.

1.3. **Brownian motions with oblique reflections.** To prove Theorems 1.2 and 1.4 we shall study a Brownian motion in the two dimensional quadrant, obliquely reflected at the boundary and with time dependent angles of reflections. We now give an application of our methods to the obliquely reflected Brownian motion defined by Varadhan and Williams in [18].

Fix $\theta_1, \theta_2 \in]0, \frac{\pi}{2}[$ and $x > 0$. Let (B^1, B^2) be a two dimensional Brownian motion and (X, Y) be the reflected Brownian motion in $\mathcal{Q}$ started from $(x, 0)$ with angles of reflections on $\partial_1\mathcal{Q}$ and on $\partial_2\mathcal{Q}$ respectively given by θ_1 and θ_2, and killed at time σ_0, the hitting time of $(0, 0)$. More precisely, for $t < \sigma_0$,

$$\begin{aligned} dX_t &= dB^1_t + dL_t(X) - \tan(\theta_1) dL_t(Y),\ X_0 = x; \\ dY_t &= dB^2_t - \tan(\theta_2) dL_t(X) + dL_t(Y),\ Y_0 = 0. \end{aligned}$$

Denote by $L_t = L_t(X) + L_t(Y)$ the local time accumulated at $\partial\mathcal{Q}$. Then it is known that σ_0 and L_{σ_0} are finite (see [18, 19]). Our next result gives a necessary and sufficient condition for L_{σ_0} to be integrable with an explicit expression of its expectation.

Proposition 1.8. *We have that*

$$\mathbb{E}[L_{\sigma_0}] < \infty \text{ if and only if } \tan(\theta_1)\tan(\theta_2) > 1.$$

In this case

$$\mathbb{E}[L_{\sigma_0}] = \frac{x(\tan(\theta_2) + 1)}{\tan(\theta_1)\tan(\theta_2) - 1}.$$

The assumptions on the wedge and the angles considered here are more suitable to our framework but our techniques may be applied to give an expression of $\mathbb{E}[L_{\sigma_0}]$ in other situations.

1.4. **Extension to metric graphs.** Let G be a metric graph (see [7, Section 2.1] for a definition) and denote by V the set of its vertices, and by $\{E_i;\ i \in I\}$ the set of its edges. We assume in this paper that I is finite (in particular V is also finite). To each edge E_i, we associate an isometry $e_i : J_i \to \bar{E}_i$, with $J_i = [0, L_i]$ when $L_i < \infty$ and $J_i = [0, \infty)$ when $L_i = \infty$. When $L_i < \infty$, set $\{g_i, d_i\} = \{e_i(0), e_i(L_i)\}$. When $L_i = \infty$, set $\{g_i, d_i\} = \{e_i(0), \infty\}$. For all $v \in V$, set $I_v^+ = \{i \in I;\ g_i = v\}$, $I_v^- = \{i \in I;\ d_i = v\}$ and $I_v = I_v^+ \cup I_v^-$. Denote by N_v the cardinal of I_v. To each $v \in V$ and $i \in I_v$, we associate a parameter $p_i^v \in [0, 1]$ such that $\sum_{i\in I_v} p_i^v = 1$. Let $G^* = G\setminus V$. We will denote by $C_b^2(G^*)$ the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in I$, $f \circ e_i$ is C^2 on the interior of J_i with bounded first and second derivatives both extendable by continuity to J_i. For $f \in C_b^2(G^*)$ and $x = e_i(r) \in G\setminus V$, set $f'(x) = (f \circ e_i)'(r)$, $f''(x) = (f \circ e_i)''(r)$ and for all $v \in V$, set $f'(v) = \bar{f}'(v)$ and $f''(v) = \bar{f}''(v)$ where for g a real valued continuous function on G^* such that $g \circ e_i$ is extentable by continuity to J_i for all $i \in I$, we set for all $v \in V$,

$$\bar{g}(v) = \sum_{i\in I_v^+} p_i^v (g \circ e_i)(0+) - \sum_{i\in I_v^-} p_i^v (g \circ e_i)(L_i-).$$

Finally set

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : f'(v) = 0 \text{ for all } v \in V \right\}.$$

We can now define the different notions of solutions of an interface SDE on G simply by replacing in Definitions 1.1, 1.3 and 1.5 the set $\{1, \ldots, N\}$ by I and by taking for $\mathcal{D}$ the domain of functions defined above. This SDE will be denoted by $E(G, p)$, where $p := (p_i^v\ :\ v \in V,\ i \in I_v)$, and when there is no ambiguity in the notation, it will simply be denoted by (E).

Note that if (X, W) solves (E), then up to the first hitting time of two different vertices, (X, W) solves an SDE on a star graph. Using this observation and Theorem 1.2, one can prove that

Theorem 1.9. *For all $x \in G$,*

(i) *There exists a solution (X, W) with $X_0 = x$, unique in law, of the SDE (E).*
(ii) *The solution of the SDE (E) is strong if and only if $N_v \leq 2$ for all $v \in V$.*

Our purpose now is to construct and study flows solutions of (E). Our main tools will be Theorems 3.2 and 4.1 in [7].

Let us introduce some more notations. For each $v \in V$, set $G^v = \{v\} \cup \cup_{i\in I_v} E_i$. Then there exists $\hat{G}^v$ a star graph and a mapping i_v :

$G_v \to \hat{G}^v$ such that $i_v : G^v \to i_v(G^v)$ is an isometry with $i_v(E_i) \subset \hat{E}_i^v$ and where $(\hat{E}_i^v)_{i \in I_v}$ is the set of edges of $\hat{G}^v$. Set for all $v \in V$, $p^v := (p_i^v \,:\, i \in I_v)$. For $\mathcal{W} := (W_i)_{i \in I}$ a family of independent white noises, set for all $v \in V$, $\mathcal{W}^v := (W_i^v)_{i \in I_v}$ the family of independent white noises defined by $W_i^v := \epsilon_v^i W_i$, where $\epsilon_v^i = 1$ if $g_i = v$ and $\epsilon_v^i = -1$ if not.

A family of σ-fields $(\mathcal{F}_{s,t};\ s \le t)$ will be said *independent on disjoint time intervals (abbreviated : i.d.i)* as soon as for all $(s_i, t_i)_{1 \le i \le n}$ with $s_i \le t_i \le s_{i+1}$, the σ-fields $(\mathcal{F}_{s_i,t_i})_{1 \le i \le n}$ are independent.

Then, [7, Theorem 3.2] states that to each family of flows $(\hat{K}^v)_{v \in V}$, and to each $\mathcal{W} := (W_i)_{i \in I}$ a family of independent white noises such that

(i) For all $v \in V$, $(\hat{K}^v, \mathcal{W}^v)$ is a solution of $E(\hat{G}^v, p^v)$ on $\hat{G}^v$;
(ii) The family of flows $(\hat{K}^v)_{v \in V}$ is i.d.i. in the sense that the family $\big(\vee_{v \in V} \mathcal{F}_{s,t}^{\hat{K}^v},\ s \le t\big)$ is i.d.i;

one can associate a solution (unique in law) $(K, \mathcal{W})$ of (E).

Conversely [7, Theorem 4.1] states that out of a solution $(K, \mathcal{W})$ of (E), one can construct a family of flows $(\hat{K}^v)_{v \in V}$ for which (i) and (ii) are satisfied and such that the law of $(K, \mathcal{W})$ is uniquely determined by the law of this family. In the following, we will denote by $\hat{\mathbb{P}}^v$ the law of the solution $(\hat{K}^v, \mathcal{W}^v)$. Then $\hat{\mathbb{P}}^v$ is a function of the law of $(K, \mathcal{W})$.

The idea behind these two results is that before passing through two distinct vertices, a "global" flow solution of (E) determines (and is determined by) a "local" flow solution of an interface SDE on a star graph (associated to the vertex that has been visited).

We will prove (see Theorem 5.1) in Section 5 that the i.d.i. condition implies conditional independence with respect to $\mathcal{W}$ of the flows $(\hat{K}^v)_{v \in V}$. This implies the following

Theorem 1.10. *To each family $(\hat{\mathbb{P}}^v)_{v \in V}$, with $\hat{\mathbb{P}}^v$ the law of a solution of $E(\hat{G}^v, p^v)$, is associated one and only one solution of (E).*

Proof. Suppose we are given $(\hat{\mathbb{P}}^v)_{v \in V}$. Then on some probability space it is possible to construct a family of independent white noises $\mathcal{W} = (W^i, i \in I)$ and a family $(\hat{K}^v)_{v \in V}$ of SFK's respectively on $\hat{G}_v$ such that for all $v \in V$, $(\hat{K}^v, \mathcal{W}^v)$ is a solution of $E(\hat{G}^v, p^v)$ distributed as $\hat{\mathbb{P}}^v$ and such that the flows $(\hat{K}^v)_{v \in V}$ are independent given $\mathcal{W}$. In other words,

$$\mathcal{L}\big((\hat{K}^v)_{v \in V} | \mathcal{W}\big) = \prod_{v \in V} \mathcal{L}\big(\hat{K}^v | \mathcal{W}^v\big)$$

where $\mathcal{L}$ stands for the conditional law. This implies in particular that the family $(\hat{K}^v)_{v\in V}$ is i.d.i., and Theorem 3.2 in [7] states that there exists K a SFK on G such that $(K,\mathcal{W})$ solves (E), with K obtained by well concatenating the flows $\hat{K}^v$.

The fact that $(K,\mathcal{W})$ is the only possible (in law) associated solution comes from the fact that the i.d.i. condition implies conditional independence. □

This theorem and the results we obtained on star graphs imply

Theorem 1.11. (i) *There exists a unique (in law) SFM solution of* (E).
(ii) *A SFM solution of* (E) *is a Wiener solution if and only if* $N_v \leq 2$ *for all* $v \in V$.
(iii) *There exists a unique (in law) SFK Wiener solution of* (E).

Proof. Let $(K,\mathcal{W})$ be a solution of (E), and denote by $(\hat{K}^v)_{v\in V}$ the associated family of flows, respective solutions of $E(\hat{G}^v, p^v)$. Note that K is a SFM if and only if the flows $\hat{K}^v$ are SFM's. Now (i) follows from Theorem 1.10 and Theorem 1.4 (i). Note also that K is a Wiener solution if and only if the flows $\hat{K}^v$ are also Wiener solutions. Thus (ii) follows from Theorem 1.4 (iii), and (iii) follows from Theorem 1.10 and Theorem 1.4 (ii). □

Let us now remark that if $(K,\mathcal{W})$ is a solution of (E), then the law of $(K,\mathcal{W})$ depends on the choice of the isometries $(e_i)_{i\in I}$ which define the orientation on G. However the law of K does not depend on this choice, and is thus independent of the orientation of G.

1.5. **Outline of contents.** In Section 2, we study an obliquely reflected Brownian motion in $\mathcal{Q}$, where the angles of reflections depend on time and which is absorbed when it hits the corner. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.4 (i) and (ii), using in particular the results of Section 2. In Section 5, we prove that the i.d.i. condition implies conditional independence thus completing the proof of Theorem 1.10. Finally in Section 6, we discuss some extensions of the present paper.

2. Brownian motion in the quadrant with time dependent angles of reflection

In this section, we study a variation of the obliquely reflected Brownian motion in $\mathcal{Q}$ where the angles of reflections depend on time and which is absorbed when it hits the corner. This process is defined in Section 2.2. We will be interested in the following two questions:

(I) Is the hitting time σ_0 of $(0,0)$ finite a.s.?

(II) Is L_{σ_0}, the local time accumulated at $\partial\mathcal{Q}$ at time σ_0, finite a.s.?

In Sections 2.3 and 2.4, we prove that, under some assumptions on the sequence of the angles of reflections, the answer to these two questions is positive. The tools used are a scaling property and a precise study, done in Section 2.1, of an obliquely reflected Brownian motion on the quadrant started at $(x,0)$, with $x>0$, and stopped when it hits $\{y=0\}$. Finally in Section 2.5, we calculate $\mathbb{E}[L_{\sigma_0}]$.

2.1. **Brownian motion on the half-plane with oblique reflection.** We fix $\theta \in]0,\pi/2[$. Let $Z=(X,Y)$ be the process started from (x,y) in $\mathbb{R}\times\mathbb{R}_+$ obliquely reflected at $\{y=0\}$, with angle of reflection given by θ. More precisely,

$$\begin{aligned} dX_t &= dB^1_t - \tan(\theta)dL_t,\ X_0=x \\ dY_t &= dB^2_t + dL_t,\ Y_0=y \end{aligned}$$

where B^1 and B^2 are two independent Brownian motions and L_t is the local time at 0 of Y. Set $S=\inf\{s:\ X_s=0\}$. Denote by $\mathbb{P}^\theta_x$ the law of $(Z_t;\ t\le S)$ when $y=0$ and $x>0$. Note that for all $t\le S$, $Z_t\in\mathcal{Q}$. Observe that we have the following scaling property:

Proposition 2.1. *For all $x>0$, if the law of $(Z_t;\ t\le S)$ is $\mathbb{P}^\theta_1$, then the law of $(xZ_{x^{-2}t};\ t\le x^2S)$ is $\mathbb{P}^\theta_x$.*

For $z\in\mathbb{C}$, $\arg(z)$, $\mathcal{R}(z)$ and $\mathcal{I}(z)$ will denote respectively the argument, the real part and the imaginary part of z. Following [18], if f is holomorphic on an open set U containing $\mathcal{Q}^*$ such that $f(z)\in\mathbb{R}$ for all $z\in]0,\infty[$, then $\phi(x,y):=\mathcal{R}\big(f(x+iy)e^{-i\theta}\big)$ is harmonic on U. Moreover,

$$v_1(\theta).\nabla\phi(x,0)=0 \text{ for } x>0,\ \text{ where } v_1(\theta)=(-\tan(\theta),1). \tag{7}$$

Indeed, the fact that f is holomorphic with the condition $f(z)\in\mathbb{R}$ for all $z\in]0,\infty[$ implies that $f'(z)\in\mathbb{R}$ for all $z\in]0,\infty[$. Thus

$$\nabla\phi(x,0)=\big(\mathcal{R}\big(f'(x)e^{-i\theta}\big),\mathcal{R}\big(if'(x)e^{-i\theta}\big)\big)=f'(x)(\cos(\theta),\sin(\theta))$$

and (7) follows. These properties imply in particular that $(\phi(Z_{t\wedge S}))_t$ is a local martingale. For $b\in\mathbb{R}$ and $f(z)=z^b$ the function ϕ defined above will be denoted by ϕ_b.

Lemma 2.2. *Let $(Z_t;\ t\le S)$ be a process of law $\mathbb{P}^\theta_x$.*

(i) *If $0<b<1+2\theta/\pi$, then for all $a>x$,*

$$\mathbb{P}\left(\sup_{s\le S}|Z_s|>a\right)\le c_b\left(\frac{x}{a}\right)^b,$$

where $c_b = 1$ *if* $b\pi/2 \leq \theta$ *and* $c_b = \cos(\theta)/\cos(b\pi/2 - \theta)$ *otherwise.*

(ii) *If* $0 < b < 1 - 2\theta/\pi$*, then for all* $a < x$*,*

$$\mathbb{P}\left(\inf_{s\leq S} |Z_s| < a\right) \leq c_b \left(\frac{a}{x}\right)^b,$$

where $c_b = \cos(\theta)/\cos(b\pi/2 + \theta)$*.*

Proof. Using the scaling property we may take $x = 1$. For $a \geq 0$, set $\sigma_a = \inf\{t : |Z_t| = a\}$. Recall that for all $b \in \mathbb{R}$, $(\phi_b(Z_{t\wedge S}))_t$ is a local martingale.

Proof of (i): Fix $a > 1$ and $0 < b < 1+2\theta/\pi$. For $c_b^0 = \inf\{\cos(\theta), \cos(b\pi/2-\theta)\}$ and $t \leq S$, we have

$$c_b^0 |Z_t|^b \leq \phi_b(Z_t) \leq |Z_t|^b.$$

Moreover

$$\mathbb{P}(\sup_{s\leq S} |Z_s| > a) = \mathbb{P}(\sigma_a < S).$$

By the martingale property, for all $t \geq 0$,

$$\cos(\theta) = \phi_b(1) = \mathbb{E}[\phi_b(Z_{t\wedge\sigma_a\wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_b(Z_{t\wedge\sigma_a}) 1_{\{\sigma_a<S\}}].$$

As $t \to \infty$, this last term converges using dominated convergence to

$$\mathbb{E}[\phi_b(Z_{\sigma_a}) 1_{\{\sigma_a<S\}}] \geq c_b^0 a^b \mathbb{P}(\sigma_a < S).$$

This easily implies (i).

The proof of (ii) is similar: Fix $a < 1$ and $0 < b < 1 - 2\theta/\pi$. For $c_b^1 = \cos(b\pi/2 + \theta)$ and $t \leq S$,

$$c_b^1 |Z_t|^{-b} \leq \phi_{-b}(Z_t) \leq |Z_t|^{-b}.$$

We also have that

$$\mathbb{P}(\inf_{s\leq S} |Z_s| < a) = \mathbb{P}(\sigma_a < S).$$

By the martingale property, for all $t \geq 0$,

$$\cos(\theta) = \phi_{-b}(1) = \mathbb{E}[\phi_{-b}(Z_{t\wedge\sigma_a\wedge S})]$$

which is larger than

$$\mathbb{E}[\phi_{-b}(Z_{t\wedge\sigma_a}) 1_{\{\sigma_a<S\}}]$$

and this converges as $t \to \infty$ to

$$\mathbb{E}[\phi_{-b}(Z_{\sigma_a}) 1_{\{\sigma_a<S\}}] \geq c_b^1 a^{-b} \mathbb{P}(\sigma_a < S).$$

This easily implies (ii). ∎

Corollary 2.3. *Let $(Z_s;\ s \le S)$ be distributed as $\mathbb{P}_x^\theta$. If $-1+2\theta/\pi < b < 1+2\theta/\pi$, then*

$$\mathbb{E}(\sup_{s\le S}|Z_s|^b) < \infty.$$

Proof. To simplify, assume $x = 1$. For $b \in]0, 1+2\theta/\pi[$, let $b' \in]b, 1+2\theta/\pi[$. Then

$$\begin{aligned}\mathbb{E}(\sup_{s\le S}|Z_s|^b) &= \int_0^\infty \mathbb{P}[\sup_{s\le S}|Z_s| > a^{1/b}]da \\ &\le 1 + c_b\int_1^\infty a^{-b'/b}da < \infty.\end{aligned}$$

For $b \in]-1+2\theta/\pi, 0[$, let $b' \in]-1+2\theta/\pi, b[$. Then

$$\begin{aligned}\mathbb{E}(\sup_{s\le S}|Z_s|^b) &= \int_0^\infty \mathbb{P}[\inf_{s\le S}|Z_s| < a^{1/b}]da \\ &\le 1 + c_b\int_1^\infty a^{-b'/b}da < \infty.\end{aligned}$$

□

Corollary 2.4. *Let $(Z_s;\ s \le S)$ be distributed as $\mathbb{P}_x^\theta$. Let f be an holomorphic function on an open set containing $\mathcal{Q}^*$ such that $f(z) \in \mathbb{R}$ for all $z \in]0,\infty[$. Assume there exists $C > 0$, $b_+ \in]0, 1+2\theta/\pi[$ and $b_- \in]0, 1-2\theta/\pi[$ with*

$$|f(z)| \le C\big(|z|^{-b_-} + |z|^{b_+}\big) \text{ for all } z \in \mathcal{Q}^*.$$

then setting $\phi(x,y) = \mathcal{R}\big(f(x+iy)e^{-i\theta}\big)$, we have

$$E[\phi(iY_S)] = \cos(\theta)f(x).$$

Proof. Recall that $(\phi(Z_{t\wedge S}))_t$ is a local martingale (stopped at time S). Using Corollary 2.3, it is a uniformly integrable martingale. And we conclude using the martingale property. □

Note that the functions $f(z) = z^b$, for $b \in]-1+2\theta/\pi, 1+2\theta/\pi[$, $f(z) = \log(z)^\ell$ for $\ell > 0$ satisfy the assumptions of Corollary 2.4.

Corollary 2.5. *Let $(Z_s;\ s \le S)$ be distributed as $\mathbb{P}_x^\theta$. Then*

- $E[Y_S^b] = x^b\dfrac{\cos(\theta)}{\cos(\theta - b\pi/2)}$ *for* $b \in]-1+2\theta/\pi, 1+2\theta/\pi[$,
- $\mathbb{E}[\log(Y_S)] = \log(x) - \dfrac{\pi}{2}\ \tan(\theta)$,
- $\mathbb{E}[(\log(x^{-1}Y_S))^2] = \dfrac{\pi^2}{4}\big(1 + 2\tan^2(\theta)\big)$.

Proof. The calculation of $\mathbb{E}[Y_S^b]$ is immediate. Using the scaling property one only needs to do the next calculations when $x = 1$. Now, for all $\ell > 0$ and $x = 1$,

$$\mathbb{E}\big[\mathcal{R}\big((\log(Y_S) + i\pi/2)^\ell e^{-i\theta}\big)\big] = 0.$$

Applying this identity for $\ell = 1$, we get the value of $\mathbb{E}[\log(Y_S)]$. For $\ell = 2$, we get

$$\mathbb{E}\big[\big((\log(Y_S))^2 - (\pi/2)^2\big)\cos(\theta) + \pi\log(Y_S)\sin(\theta)\big] = 0.$$

Thus

$$\begin{aligned}\mathbb{E}[(\log(Y_S))^2] &= (\pi/2)^2 - \pi\mathbb{E}[\log(Y_S)]\tan(\theta)\\ &= (\pi/2)^2 + 2(\pi/2)^2(\tan(\theta))^2.\end{aligned}$$

□

2.2. **Brownian motion on the quadrant with time dependent reflections.** In all this section, we fix $z = (x, 0)$ with $x > 0$, and $\theta_{min} \in]0, \frac{\pi}{2}[$. Suppose we are given on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ a sequence of random variables $(\Theta_n)_{n\geq 0}$ and a sequence of processes $(Z^n)_{n\geq 1}$, with $Z^n = \big(Z_t^n = (X_t^n, Y_t^n);\ t \leq S_n\big)$, such that:

(i) With probability 1, for all $n \geq 0$, $\Theta_n \in]\theta_{min}, \frac{\pi}{2}[$.
(ii) Set $U_0 = x$ and for $n \geq 1$, $U_n = Y_{S_n}^n$. Set also for $n \geq 0$,

$$\mathcal{G}_n = \sigma\big((\Theta_k, Z^k);\ 1 \leq k \leq n\big) \vee \sigma(\Theta_0).$$

Then given $\mathcal{G}_n$, Z^{n+1} is distributed as $\mathbb{P}_{U_n}^{\Theta_n}$.

Define for $\theta \in]0, \pi/2[$,

$$v_1(\theta) = (-\tan(\theta), 1) \text{ and } v_2(\theta) = (1, -\tan(\theta)).$$

Our purpose in this section and in Section 2.3 is to construct a process $Z = (X, Y)$, a reflected Brownian motion in $\mathcal{Q}$ stopped at time σ_0, the first hitting time of $(0, 0)$ by Z.

Set $T_0 = 0$ and $T_n = \sum_{k=1}^n S_k$ for $n \geq 1$. For $n \geq 0$, set

$$\begin{aligned}Z_t &= (X_{t-T_{2n}}^{2n+1}, Y_{t-T_{2n}}^{2n+1}) \quad \text{for all } t \in [T_{2n}, T_{2n+1}[,\\ Z_t &= (Y_{t-T_{2n+1}}^{2n+2}, X_{t-T_{2n+1}}^{2n+2}) \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[.\end{aligned}$$

Using this procedure, we have defined a process $(Z_t;\ t < T_\infty)$, where $T_\infty = \lim_{n\to\infty} T_n$. Set for $t \geq T_\infty$, $Z_t = (0, 0)$. Then, by construction, $T_\infty = \sigma_0$. It will be checked in Section 2.3 (see Corollary 2.7) that Z is a continuous process.

Note that there exists B a two-dimensional Brownian motion such that for $n \geq 0$,

$$\begin{aligned} dZ_t &= dB_t + v_1(\Theta_{2n})dL_t^1 \quad \text{for all } t \in [T_{2n}, T_{2n+1}[, \\ dZ_t &= dB_t + v_2(\Theta_{2n+1})dL_t^2 \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[, \end{aligned}$$

with L^1 and L^2 being the local times processes of X and Y. Define $(v_t;\ t < \sigma_0)$ by: for $n \geq 0$

$$\begin{aligned} v_t &= v_1(\Theta_{2n}) \quad \text{for all } t \in [T_{2n}, T_{2n+1}[, \\ v_t &= v_2(\Theta_{2n+1}) \quad \text{for all } t \in [T_{2n+1}, T_{2n+2}[. \end{aligned}$$

Then for all $t < \sigma_0$,

$$Z_t = Z_0 + B_t + \int_0^t v_s dL_s \tag{8}$$

where $Z_0 = (x, 0)$ and $L = L^1 + L^2$ is the accumulated local time at $\partial\mathcal{Q}$ until t.

The purpose of the following sections is to answer the questions (I) and (II) addressed in the beginning of Section 2.

2.3. **The corner is reached.** For $a \geq 0$, set $\sigma_a := \inf\{t;\ |Z_t| = a\}$. Following [18], we will first prove that $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$ for all $K > x$. This is the major difficulty we encountered here although the proof when the angles of reflections remain constant on each boundary is quite easy [18, Lemma 2.1]. The main idea is inspired from [2]. Define for $n \geq 1$, $V_n = \frac{U_n}{U_{n-1}}$. Then using the scaling property (Proposition 2.1) and the strong Markov property, we have that for all $n \geq 0$, given $\mathcal{G}_n$, V_{n+1} is distributed as $\tilde{Y}_{\tilde{S}}$, where $\big((\tilde{X}_t, \tilde{Y}_t);\ t \leq \tilde{S}\big)$ has law $\mathbb{P}_1^{\Theta_n}$.

Lemma 2.6. *With probability* 1, $\sum_{n\geq 0} U_n$ *is finite.*

Proof. For all $n \geq 1$, we have that

$$U_n = x \exp\left(\sum_{k=1}^{n} \log(V_k)\right).$$

We denote by $\mathbb{E}_{\mathcal{G}_k}$ the conditional expectation with respect to $\mathcal{G}_k$. By Corollary 2.5, for all $k \geq 1$, $\mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)] = -\frac{\pi}{2}\ \tan(\Theta_{k-1})$ and $\mathbb{E}_{\mathcal{G}_{k-1}}[(\log(V_k))^2] = \frac{\pi^2}{4}(1 + 2\tan^2(\Theta_{k-1}))$. Note now that

$$\sum_{k=1}^{n} \log(V_k) = M_n + \sum_{k=1}^{n} \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)]$$

where $M_n := \sum_{k=1}^n \big(\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)]\big)$ is a martingale. Denote by $\langle M\rangle_n$ its quadratic variation given by

$$\sum_{k=1}^n \mathbb{E}_{\mathcal{G}_{k-1}}\Big[\big(\log(V_k) - \mathbb{E}_{\mathcal{G}_{k-1}}[\log(V_k)]\big)^2\Big] = \sum_{k=1}^n \frac{\pi^2}{4}\big(1+\tan^2(\Theta_{k-1})\big).$$

Thus $\langle M\rangle_\infty = \infty$ and a.s. $\lim_{n\to\infty} M_n/n = 0$. Since $\inf_{k\geq 0}\Theta_k \geq \theta_{min} > 0$, this easily implies the lemma. □

A first consequence of Lemma 2.6 is

Corollary 2.7. *With probability* 1, $\lim_{t\uparrow\sigma_0} Z_t = (0,0)$.

Proof. For $\epsilon > 0$ and $n \geq 0$, set

$$A_n^\epsilon = \Big\{\sup_{s\in[T_n,T_{n+1}]} |Z_s| > \epsilon\Big\}.$$

By Lemma 2.2 (i), with $b = 1$, for all $n \geq 0$,

$$\mathbb{P}(A_n^\epsilon|\mathcal{G}_n) \leq \sup_{\theta\in]\theta_{min},\frac{\pi}{2}[} \operatorname{cotan}(\theta)\ U_n = \operatorname{cotan}(\theta_{min})\ U_n.$$

Thus by Lemma 2.6, $\sum_n \mathbb{P}(A_n^\epsilon|\mathcal{G}_n) < \infty$ and the corollary follows by applying the conditional Borel-Cantelli lemma. □

Lemma 2.6 will be also used to prove

Lemma 2.8. *For all* $K > x$, $\mathbb{P}(\sigma_0 \wedge \sigma_K < \infty) = 1$.

Proof. For all $n \geq 0$ and $t \in [0, S_{n+1}]$, set

$$W_t^{n+1} \;=\; \cos(\Theta_n)(X_t^{n+1} - U_n) + \sin(\Theta_n)Y_t^{n+1}$$

Recall $\sigma_0 = \lim_{n\to\infty} T_n$. Define the continuous process $(W_t;\ t \leq \sigma_0)$ such that $W_0 = 0$ and for $n \geq 0$ and $t \in]T_n, T_{n+1}]$, $W_t = W_{t-T_n}^{n+1} + W_{T_n}$. Then, it is straightforward to check that $(W_t;\ t \leq \sigma_0)$ is a Brownian motion stopped at σ_0. Since for all $n \geq 0$, $U_n \geq 0$ and $\Theta_n \in]0, \pi/2[$, we get that on the event $\{\sigma_K \geq T_{n+1}\}$,

$$\sup_{t\in[T_n,T_{n+1}]} W_t \leq 2K + W_{T_n}.$$

Thus, on $\{\sigma_K = \infty\}$, $\sup_{t\leq\sigma_0} W_t \leq 2K + \sup_{n\geq 0} W_{T_n}$. Now for all $n \geq 0$, $W_{S_{n+1}}^{n+1} = \sin(\Theta_n)U_{n+1} - \cos(\Theta_n)U_n \leq U_{n+1}$. Note that for all $n \geq 0$,

$$W_{T_{n+1}} - W_{T_n} = W_{S_{n+1}}^{n+1}.$$

This implies that on the event $\{\sigma_K = \infty\}$, $\sup_{t\leq\sigma_0} W_t \leq 2K + \sum_{n\geq 0} U_n$, which is a.s. finite using Lemma 2.6. This shows that a.s. $\{\sigma_K = \infty\} \subset \{\sigma_0 < \infty\}$ and finishes the proof. □

And following [18], we will prove

Theorem 2.9. *With probability* 1*, we have* $\sigma_0 < \infty$*.*

Proof. Set $b = \frac{4\theta_{min}}{\pi}$. Let $\phi(x,y) = \mathcal{R}\big((x+iy)^b e^{-i\theta_{min}}\big)$, then ϕ is harmonic on some open set U containing $\mathcal{Q}^*$. Using $b = \frac{4\theta_{min}}{\pi}$, we have that

$$\begin{aligned}\nabla\phi(x,0) &= bx^{b-1}(cos(\theta_{min}), \sin(\theta_{min})),\\ \nabla\phi(0,y) &= by^{b-1}(sin(\theta_{min}), \cos(\theta_{min})).\end{aligned}$$

Thus for all $t < \sigma_0$ with $Z_t \in \partial\mathcal{Q}$, we have $v_t.\nabla\phi(Z_t) \leq 0$. It follows from (8) and Itô's formula that for all $0 < \epsilon < x < K$ and $t \geq 0$,

$$\mathbb{E}[\phi(Z_{t\wedge\sigma_\epsilon\wedge\sigma_K})] \leq \phi(x,0).$$

Letting $t \to \infty$ and using dominated convergence, we deduce

$$\mathbb{E}[\phi(Z_{\sigma_\epsilon\wedge\sigma_K})] \leq \phi(x,0).$$

Obviously $\phi(z) \geq \cos(\theta_{min})|z|^b$ for all $z \in \mathcal{Q}$. Setting $p_{\epsilon,K} = \mathbb{P}(\sigma_\epsilon < \sigma_K)$, we get

$$\cos(\theta_{min})\big(\epsilon^b p_{\epsilon,K} + K^b(1 - p_{\epsilon,K})\big) \leq x^b.$$

From this, we deduce

$$p_{\epsilon,K} \geq \frac{(K^b - x^b/\cos(\theta_{min}))}{K^b - \epsilon^b}.$$

As in [18], since $\sigma_0 \wedge \sigma_K < \infty$, $\lim_{\epsilon\to 0} p_{\epsilon,K} = \mathbb{P}(\sigma_0 < \sigma_K)$, this yields

$$\mathbb{P}(\sigma_0 < \sigma_K) \geq 1 - \frac{x^b}{K^b\cos(\theta_{min})}. \tag{9}$$

Letting $K \to \infty$, it comes that $\mathbb{P}(\sigma_0 < \infty) = 1$. □

Remark 2.10. *Using the inclusion* $\{\sup_{t<\sigma_0}|Z_t| > \epsilon\} \subset \{\sigma_\epsilon < \sigma_0\}$ *and (9), we deduce that for all* $\epsilon > 0$*,*

$$\lim_{x\to 0+} \mathbb{P}(\sup_{t<\sigma_0}|Z_t| > \epsilon) = 0 \tag{10}$$

This fact will be used in Section 3.

2.4. **The local time process.** Following Williams [19], we prove in this section that

Theorem 2.11. *With probability* 1*,* $L_{\sigma_0} := \lim_{t\uparrow\sigma_0} L_t$ *is finite.*

Proof. In what follows, we refer to the proof of Theorem 1 in [19] for more details. Let $\tilde{\theta} \in]0, \theta_{min} \wedge \pi/4[$ and set $\tilde{b} = \frac{4\tilde{\theta}}{\pi}$. Le $\tilde{\phi}$ be defined as the function ϕ in the proof of Theorem 2.9, with the parameters (b, θ_{min}) replaced by $(\tilde{b}, \tilde{\theta})$. Then there exists $c > 0$ such that for all t for which $Z_t \in \partial\mathcal{Q}$, we have $v_t.\nabla\phi(Z_t) \leq -c|Z_t|^{\tilde{b}-1}$. For each $\gamma > 0$, define $f_\gamma = e^{-\gamma\phi}$. Then f_γ is twice continuously differentiable in $\mathcal{Q}^*$ and

$$\Delta f_\gamma(z) = \gamma^2 f_\gamma(z)(\tilde{b}|z|^{\tilde{b}-1})^2 \text{ for } z \in \mathcal{Q}^*.$$

Moreover for all t such that $Z_t \in \partial\mathcal{Q}$,

$$v_t.\nabla f_\gamma(Z_t) \;=\; -\gamma f_\gamma(Z_t)\big(v_t.\nabla\phi(Z_t)\big).$$

For $t < \sigma_0$, set

$$A_t = -\gamma \int_0^t (v_s.\nabla\phi(Z_s))dL_s + \frac{\gamma^2}{2}\int_0^t (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds.$$

and $A_{\sigma_0} = \lim_{t\uparrow\sigma_0} A_t$. Then

$$A_{\sigma_0} \;\geq\; c\gamma \int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s + \frac{\gamma^2}{2}\int_0^{\sigma_0} (\tilde{b}|Z_s|^{\tilde{b}-1})^2 ds.$$

Itô's formula implies that for $t < \sigma_0$,

$$f_\gamma(Z_t)e^{-A_t} = f_\gamma(Z_0) + \int_0^t e^{-A_s}(\nabla f_\gamma(Z_s).dB_s).$$

Taking the expectation, we get

$$\mathbb{E}\Big[\exp\Big(-c\gamma\int_0^{\sigma_0} |Z_s|^{\tilde{b}-1} dL_s\Big)\Big] \geq f_\gamma(Z_0).$$

This easily implies that for all $r > 0$,

$$\mathbb{E}\Big[\exp\Big(-\gamma c r^{\tilde{b}-1}\int_0^{\sigma_0} 1_{\{|Z_s|\leq r\}} dL_s\Big)\Big] \geq f_\gamma(Z_0).$$

Letting $\gamma \downarrow 0$, we get a.s.

$$\int_0^{\sigma_0} 1_{\{|Z_s|\leq r\}} dL_s < \infty. \tag{11}$$

Let $S_r = \sup\{t \geq 0 : |Z_t| > r\}$, then by the continuity of Z, $S_r < \sigma_0$ and thus $L_{S_r} < \infty$. By combining this with (11), we get $L_{\sigma_0} < \infty$. $\square$

2.5. **On the integrability of L_{σ_0}.** In this section, Proposition 1.8 is proved. We use the notation of Section 2.2 in which the process Z is constructed. Note that $L_{\sigma_0} = \sum_{n=1}^\infty L^n_{S_n}$, where L^n is the local time at 0 of Y^n and where $Z^n = (X^n, Y^n)$. Recall that for $n \geq 0$, given $\mathcal{G}_n$, the law of Z^{n+1} is $\mathbb{P}^{\Theta_n}_{U_n}$, where $U_0 = x$ and $U_n = Y^n_{S_n}$ for $n \geq 1$.

Let $Z^0 = (X^0_t, Y^0_t)_{t\leq S^0}$ be a process of law $\mathbb{P}^\theta_x$. Then, if $L^0_t = L_t(Y^0)$, for all $t \geq 0$, $Y^0_{t\wedge S^0} = B^2_{t\wedge S^0} + L^0_{t\wedge S^0}$ where $(B^2_{t\wedge S^0})_t$ is a Brownian motion stopped at time S^0. Thus $\mathbb{E}[Y^0_{t\wedge S^0}] = \mathbb{E}[L^0_{t\wedge S^0}]$. Taking the limit as $t \to \infty$ and using Corollary 2.3 leads to $\mathbb{E}[L^0_{S^0}] = \mathbb{E}[Y^0_{S^0}]$. But $\mathbb{E}[Y^0_{S^0}] = x \text{ cotan}(\theta)$ by Corollary 2.5 and this implies that

$$\mathbb{E}[L^{n+1}_{S_{n+1}} | \mathcal{G}_n] = U_n \text{ cotan}(\Theta_n).$$

Consequently

$$\mathbb{E}[L_{\sigma_0}] = \sum_{n\geq 0} \mathbb{E}[U_n \text{ cotan}(\Theta_n)].$$

Assume that for all n, U_n and Θ_n are independent, then

$$\mathbb{E}[U_n \text{ cotan}(\Theta_n)] = \mathbb{E}[\text{cotan}(\Theta_n)]\mathbb{E}[U_n] = \cdots = x \prod_{k=0}^{n} \mathbb{E}[\text{cotan}(\Theta_k)].$$

Therefore

$$\mathbb{E}[L_{\sigma_0}] = x \sum_{n\geq 0} \prod_{k=0}^{n} \mathbb{E}[\text{cotan}(\Theta_k)].$$

This gives a necessary and sufficient condition to have $\mathbb{E}[L_{\sigma_0}] < \infty$.

Assume that $\Theta_n = \theta \in]0, \pi/2[$ for all n, we get

$$\mathbb{E}[L_{\sigma_0}] = x \sum_{n\geq 0} \big(\text{cotan}(\theta)\big)^{n+1}$$

which is finite if and only if $\theta \in]\pi/4, \pi/2[$. In this case

$$\mathbb{E}[L_{\sigma_0}] = \frac{x \text{ cotan}(\theta)}{1 - \text{cotan}(\theta)} = \frac{x}{\tan(\theta) - 1}.$$

Assume that $\Theta_{2n} = \theta_1$ and $\Theta_{2n+1} = \theta_2$. Set $c_1 = \text{cotan}(\theta_1)$ and $c_2 = \text{cotan}(\theta_2)$. Then

$$\begin{aligned}
\mathbb{E}[L_{\sigma_0}] &= x(c_1 + c_1c_2 + c_1^2c_2 + c_1^2c_2^2 + \cdots) \\
&= c_1\big(1 + c_2 + c_1c_2 + c_1c_2^2 + \cdots\big) \\
&= c_1\big((1 + c_2) + (1 + c_2)c_1c_2 + \cdots\big)
\end{aligned}$$

which is finite if and only if $c_1c_2 < 1$. In this case, we have

$$\mathbb{E}[L_{\sigma_0}] = \frac{xc_1(1 + c_2)}{1 - c_1c_2}$$

and Proposition 1.8 is proved.

3. Proof of Theorem 1.2

Theorem 1.2 (i) is proved in Section 3.1. For the construction of a solution, we will use Freidlin-Sheu formula for Walsh's Brownian motion (see Theorem 3.1 below). The uniqueness in law of the solution of (E) will follow from the fact that Walsh's Brownian motion is the unique solution of a martingale problem.

Theorem 1.2 (ii) is proved in Section 3.2. To prove pathwise uniqueness for (E) when $N = 2$, we proceed as in [4] using the local times techniques introduced in [10, 15]. The fact that the solution of (E) is not strong when $N \geq 3$ is a consequence of a theorem by Tsirelson (see Theorem 3.6 below).

We prove Theorem 1.2 only for $x = 0$, the case $x \neq 0$ following easily.

3.1. **Proof of Theorem 1.2 (i).**

Let us recall Freidlin-Sheu formula (see [5] and also [6, Theorem 3]).

Theorem 3.1. [5] *Let $(X_t)_{t\geq 0}$ be a Walsh's Brownian motion on G started from X_0 and $B^X_t = |X|_t - |X_0| - L_t(|X|)$. Then B^X is a Brownian motion and for all $f \in C^2_b(G^*)$, we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dB^X_s + \frac{1}{2}\int_0^t f''(X_s)ds + f'(0)L_t(|X|).$$

We call B^X the Brownian motion associated to X.

Remark that in this formula the local martingale part of $f(X_t)$ is always a stochastic integral with respect to B^X. This is an expected fact since B^X has the martingale representation property for $(\mathcal{F}^X_t)_t$ ([1, Theorem 4.1]). This martingale representation property will be used to prove the uniqueness in law of the solutions to (E).

3.1.1. *Construction of a solution to* (E).

Let X be a Walsh's Brownian motion with $X_0 = 0$ and let B^X be the Brownian motion associated to X. Take a N-dimensional Brownian motion $V = (V^1, \cdots, V^N)$ independent of X. Let $(\mathcal{F}_t)$ denote the filtration generated by X and V. For $i \in [1, N]$, define

$$W^i_t = \int_0^t 1_{\{X_s \in E_i\}} dB^X_s + \int_0^t 1_{\{X_s \notin E_i\}} dV^i_s.$$

Then $W := (W^1, \cdots, W^N)$ is a N-dimensional $(\mathcal{F}_t)$-Brownian motion by Lévy's theorem and

$$B^X_t = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW^i_s.$$

Then, using Theorem 3.1, (X,W) solves (E). Denote by μ the law of (X,W).

3.1.2. *Uniqueness in law.* To prove the uniqueness in law, we will apply two lemmas. The first lemma states that the Walsh's Brownian motion is the unique solution of a martingale problem. The second lemma gives conditions that ensure that a Walsh's Brownian motion is independent of a given family of Brownian motions.

Lemma 3.2. *Let $(\mathcal{F}_t)$ be a filtration and let X be a G-valued $(\mathcal{F}_t)$-adapted and continuous process such that for all $f \in \mathcal{D}$,*

$$M_t^f := f(X_t) - f(x) - \frac{1}{2}\int_0^t f''(X_s)ds. \tag{12}$$

is a martingale with respect to $(\mathcal{F}_t)$, then X is a $(\mathcal{F}_t)$-Walsh's Brownian motion.

Proof. We exactly follow the proof of [1, Theorem 3.2] and only check that with our conventions $f'(0) = f''(0) = 0$ for $f \in \mathcal{D}$, we avoid all trivial solutions to the previous martingale problem (with the hypothesis of Theorem 3.2 of [1], the trivial process $X_t = 0$ is a possible solution of the martingale problem (3.3) in [1]). For $i \in [1,N]$, set $q_i = 1 - p_i$ and let f_i and g_i be defined by

$$\begin{aligned} f_i(x) &= q_i|x|1_{\{x\in E_i\}} - p_i|x|1_{\{x\notin E_i\}}, \\ g_i(x) &= \big(f_i(x)\big)^2 = q_i^2|x|^2 1_{\{x\in E_i\}} + p_i^2|x|^2 1_{\{x\notin E_i\}}. \end{aligned}$$

Then f_i and g_i are C^2 on G^*. We have $f_i'(x) = q_i$ for $x \in E_i^*$, $f_i'(x) = -p_i$ for $x \notin E_i$ and $f_i'(0) = 0$. Moreover, for all $x \in G$, $f_i''(x) = 0$. We also have $g_i'(x) = 2q_i^2|x|$ for $x \in E_i^*$, $g_i'(x) = 2p_i^2|x|$ for $x \notin E_i$ and $g_i'(0) = 0$. Moreover, $g_i''(x) = 2q_i^2$ for $x \in E_i^*$, $g_i''(x) = 2p_i^2$ for $x \notin E_i$ and $g_i''(0) = 2p_iq_i$. Set $Y_t^i := f_i(Z_t)$. Although f_i is not bounded, by a localization argument, we have that Y_t^i is a local martingale. Using the function g_i, we also have that $(Y_t^i)^2 - \frac{1}{2}\int_0^t g_i''(Z_s)ds$ is a local martingale. Thus

$$\langle Y^i\rangle_t = \int_0^t \big(q_i^2 1_{\{Z_s\in E_i^*\}} + p_i^2 1_{\{Z_s\notin E_i\}} + p_iq_i 1_{\{Z_s=0\}}\big)ds.$$

Set

$$U_t^i = \int_0^t \big(q_i^{-1}1_{\{Y_s^i>0\}} + p_i^{-1}1_{\{Y_s^i<0\}} + \big(p_iq_i\big)^{-1/2}1_{\{Y_s^i=0\}}\big)dY_s^i.$$

Then U_t^i is a local martingale with $\langle U^i\rangle_t = t$; that is U_t^i is a Brownian motion. Let $\phi(y) = q_i 1_{\{y>0\}} + p_i 1_{\{y<0\}} + \sqrt{p_iq_i}1_{\{y=0\}}$. Then Y^i is a solution of the stochastic differential equation

$$Y_t^i = Y_0^i + \int_0^t \phi(Y_s^i) dU_s^i.$$

As in [1], the solution of this SDE is pathwise unique and following the end of the proof of [1, Theorem 3.2], we arrive at

$$\mathbb{E}[f(Z_t)|\mathcal{F}_s] = P_{t-s} f(Z_s)$$

for all $s \leq t$ and $f : G \to \mathbb{R}$ a bounded measurable where P_t is the semigroup of the Walsh's Brownian motion. □

Lemma 3.3. *Let $(\mathcal{G}_t)$ be a filtration. Let X be a $(\mathcal{G}_t)$-Walsh's Brownian motion, B^X its associated Brownian motion and $B = (B^1, \cdots, B^d)$ be a $(\mathcal{G}_t)$-Brownian motion in $\mathbb{R}^d$, with $d \geq 1$. If B^X and B are independent, then X and B are independent.*

Proof. Let U be a bounded $\sigma(B)$-measurable random variable. Then

$$U = \mathbb{E}[U] + \sum_{i=1}^d \int_0^\infty H_s^i dB_s^i$$

with H^i predictable for the filtration $\mathcal{F}_\cdot^B$ and $E[\int_0^\infty (H_s^i)^2 ds] < \infty$. Let U' be a bounded $\sigma(X)$-measurable random variable. Since B^X has the martingale representation property for $\mathcal{F}_\cdot^X$ [1, Theorem 4.1], we deduce that

$$U' = \mathbb{E}[U'] + \int_0^\infty H_s dB_s^X$$

with H predictable for $\mathcal{F}_\cdot^X$ and $\mathbb{E}[\int_0^\infty (H_s)^2 ds] < \infty$. Then H and $(H^i)_{1 \leq i \leq d}$ are also predictable for $(\mathcal{G}_t)$. It is also easy to check that B^X is a $(\mathcal{G}_t)$-Brownian motion. Now

$$\begin{aligned}
\mathbb{E}[UU'] &= \mathbb{E}[U]\mathbb{E}[U'] + \mathbb{E}\left[\sum_{i=1}^d \int_0^\infty H_s^i dB_s^i \int_0^\infty H_s dB_s^X\right] \\
&= \mathbb{E}[U]\mathbb{E}[U'] + \sum_{i=1}^d \mathbb{E}\left[\int_0^\infty H_s^i H_s d\langle B^i, B^X\rangle_s\right] \\
&= \mathbb{E}[U]\mathbb{E}[U'].
\end{aligned}$$

□

Let (X, W) be a solution of (E), defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$, and such that $X_0 = 0$. Without loss of generality, we can assume that $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$. For all $f \in \mathcal{D}$, $\sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i$ is a martingale, and therefore X is a solution to the martingale problem of Lemma 3.2. Thus X is a Walsh's Brownian motion. Let B be a

Brownian motion independent of (X, W), denote by B^X the Brownian motion associated to X and set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^B$. Note that B^X is a $(\mathcal{G}_t)$-Brownian motion. For $i \in [1, N]$, define

$$V_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s + \int_0^t 1_{\{X_s \notin E_i\}} dW_s^i.$$

Then $V := (V^1, \cdots, V^N)$ is a N-dimensional $(\mathcal{G}_t)$-Brownian motion independent of B^X. By the previous lemma V is also independent of X. It is easy to check that for all $i \in [1, N]$,

$$W_t^i = \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \int_0^t 1_{\{X_s \notin E_i\}} dV_s^i.$$

This proves that the law of (X, W) is μ.

3.2. **Proof of Theorem 1.2 (ii).**

3.2.1. *The case $N = 2$.* To prove that the solution is a strong one, it suffices to prove that pathwise uniqueness holds for (E). Fix $p \in]0, 1[$, and set $\beta = \frac{1-p}{p}$.

Lemma 3.4. *Let B^+ and B^- be two independent Brownian motions. Let also X and Y be two continuous processes, with $Y_t = \beta X_t 1_{\{X \geq 0\}} + X_t 1_{\{X_t < 0\}}$. Then (X, B^+, B^-) is a solution to (E) or equivalently of*

$$dX_t = 1_{\{X_t > 0\}} dB_t^+ + 1_{\{X_t \leq 0\}} dB_t^- + (2p - 1) dL_t(X) \tag{13}$$

if and only if (Y, B^+, B^-) is a solution of the following SDE

$$dY_t = \beta 1_{\{Y_t > 0\}} dB_t^+ + 1_{\{Y_t \leq 0\}} dB_t^-. \tag{14}$$

Proof. Suppose (X, B^+, B^-) solves (13). Set $B_t = \int_0^t 1_{\{X_s > 0\}} dB_s^+ + 1_{\{X_s \leq 0\}} dB_s^-$. Then B_t is a Brownian motion, and (X, B) is a solution of the SDE $X_t = B_t + (2p - 1) L_t(X)$. It is well known (see for example Section 5.2 in the survey [13]) that (Y, B) solves

$$dY_t = \beta 1_{\{Y_t > 0\}} dB_t + 1_{\{Y_t \leq 0\}} dB_t$$

and thus that (Y, B^+, B^-) solves (14). The converse can be proved in the same way. □

Proposition 3.5. *Pathwise uniqueness holds for (E).*

Proof. Lemma 3.4 implies that the proposition holds if pathwise uniqueness holds for (14). Let (Y, B^+, B^-) and (Y', B^+, B^-) be two solutions of (14) with $Y_0 = Y'_0 = 0$. Set $\mathrm{sgn}(y) = \mathbf{1}_{\{y>0\}} - \mathbf{1}_{\{y<0\}}$. We shall use the same techniques as in [4] (see also [10, 15]) and first prove that a.s.

$$\int_{]0,+\infty]} L_t^a(Y - Y') \frac{da}{a} < \infty. \tag{15}$$

By the occupation times formula

$$\int_{]0,+\infty]} L_t^a(Y-Y')\frac{da}{a} = \int_0^t 1_{\{Y_s-Y'_s>0\}}\frac{d\langle Y-Y'\rangle_s}{Y_s-Y'_s}.$$

It is easily verified that

$$d\langle Y-Y'\rangle_s \leq C\big|\mathrm{sgn}(Y_s)-\mathrm{sgn}(Y'_s)\big|ds$$

where $C=(1+\beta^2)/2$. Let $(f_n)_n \subset C^1(\mathbb{R})$ such that $f_n \to \mathrm{sgn}$ pointwise and $(f_n)_n$ is uniformly bounded in total variation. By Fatou's lemma, we get

$$\begin{aligned}\int_{]0,+\infty]} L_t^a(Y-Y')\frac{da}{a} &\leq C\liminf_n \int_0^t 1_{\{Y_s-Y'_s>0\}}\frac{|f_n(Y_s)-f_n(Y'_s)|}{Y_s-Y'_s}ds\\ &\leq C\liminf_n \int_0^t 1_{\{Y_s-Y'_s>0\}}\left|\int_0^1 f'_n(Z_s^u)du\right|ds\end{aligned}$$

where

$$Z_s^u = (1-u)Y_s + uY'_s.$$

It is easy to check the existence of a constant $A>0$ such that for all $s\geq 0$ and $u\in[0,1]$, $\frac{d}{du}\langle Z^u\rangle_s \geq A^{-1}$. Hence, setting $C'=A\times C$, we have

$$\begin{aligned}\int_{]0,+\infty]} L_t^a(Y-Y')\frac{da}{a} &\leq C'\liminf_n \int_0^1\int_0^t \big|f'_n(Z_s^u)\big|d\langle Z^u\rangle_s du\\ &\leq C'\liminf_n \int_0^1\int_{\mathbb{R}} \big|f'_n(a)\big|L_t^a(Z^u)dadu.\end{aligned}$$

Now taking the expectation and using Fatou's lemma, we get

$$\mathbb{E}\left[\int_{]0,+\infty]} L_t^a(Y-Y')\frac{da}{a}\right] \leq C'\liminf_n \int_{\mathbb{R}}\big|f'_n(a)\big|da \sup_{a\in\mathbb{R},u\in[0,1]} \mathbb{E}\big[L_t^a(Z^u)\big].$$

It remains to prove that $\sup_{a\in\mathbb{R},u\in[0,1]}\mathbb{E}\big[L_t^a(Z^u)\big]<\infty$. By Tanaka's formula, we have

$$\begin{aligned}\mathbb{E}\big[L_t^a(Z^u)\big] &= \mathbb{E}\big[\big|Z_t^u-a\big|\big]-\mathbb{E}\big[\big|Z_0^u-a\big|\big]-\mathbb{E}\left[\int_0^t \mathrm{sgn}(Z_s^u-a)dZ_s^u\right]\\ &\leq E\big[\big|Z_t^u-Z_0^u\big|\big].\end{aligned}$$

It is easy to check that the right-hand side is uniformly bounded with respect to (a,u) which permits to deduce (15). Consequently, since $\lim_{a\downarrow 0} L^a(Y-Y') = L^0(Y-Y')$, (15) implies that $L_t^0(Y-Y')=0$ and thus by Tanaka's formula, $|Y-Y'|$ is a local martingale which is also a nonnegative supermartingale, with $|Y_0-Y'_0|=0$ and finally Y and Y' are indistinguishable. □

3.2.2. *The case $N \geq 3$.* Let (X, W) be a solution to (E). Then X is a $(\mathcal{F}_t)$-Walsh's Brownian motion, where $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$. If (X, W) is a strong solution we thus have that X is a $(\mathcal{F}_t^W)$-Walsh's Brownian motion, which is impossible when $N \geq 3$ because of the following Tsirelson's theorem :

Theorem 3.6. [17] *There does not exist any $(\mathcal{G}_t)_t$-Walsh's Brownian motion on a star graph with three or more rays with $(\mathcal{G}_t)_t$ a Brownian filtration.*

4. Proof of Theorem 1.4

In this section, we prove the assertions (i) and (ii) of Theorem 1.4. We first construct a coalescing SFM solution of (E). To construct this SFM, we will use the following

Theorem 4.1. [12] *Let $(P^{(n)}, n \geq 1)$ be a consistent family of Feller semigroups acting respectively on $C_0(M^n)$ where M is a locally compact metric space such that*

$$P_t^{(2)} f^{\otimes 2}(x, x) = P_t^{(1)} f^2(x) \text{ for all } f \in C_0(M),\ x \in M,\ t \geq 0. \tag{16}$$

Then there exists a (unique in law) SFM $\varphi = (\varphi_{s,t})_{s \leq t}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that

$$P_t^{(n)} f(x) = \mathbb{E}[f(\varphi_{0,t}(x_1), \cdots, \varphi_{0,t}(x_n))]$$

for all $n \geq 1$, $t \geq 0$, $f \in C_0(M^n)$ and $x \in M^n$.

To apply this theorem, we construct a consistent family of n-point motions (i.e. the Markov process associated to $P^{(n)}$) up to their first coalescing times in Section 4.1. After associating to the two-point motion an obliquely reflected Brownian motion in $\mathcal{Q}$ in Section 4.2, we prove the coalescing property in Section 4.3 and the Feller property in Section 4.4. It is then possible to apply Theorem 4.1 and as a result we get a flow φ. In Section 4.4, we also show that φ solves (E). Finally, we prove in Section 4.5 that φ is the unique SFM solving (E).

Note finaly that in the case of Le Jan and Raimond [11], all the angles of reflection of the obliquely reflected Brownian motion associated to the two-point motion are equal to $\pi/4$. This simplifies greatly the study of Section 2.

4.1. Construction of the n-point motion up to the first coalescing time.

Fix $x_1, \cdots, x_n \in G$ such that $|x_1| < \cdots < |x_n|$ and let (X, W) be a solution of the SDE (E), with $X_0 = x_1$.

Set, for $t \geq 0$, $X_t^{1,0} = X_t$ and for all $j \in [2, n]$, if $x_j \in E_i$, define

$$X_t^{j,0} = e_i(|x_j| + W_t^i).$$

Set

$$\tau_1 = \inf\{t \geq 0 : \exists j \neq 1 : X_t^{j,0} = 0\}.$$

For $t \leq \tau_1$, set $X_t^{(n)} = (X_t^{1,0}, \cdots, X_t^{n,0})$.

Assume now that $(\tau_k)_{k \leq \ell}$ and $(X_t^{(n)})_{t \leq \tau_l}$ have been defined such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$ is an increasing sequence of stopping times with respect to the filtration associated to $(X_t^{(n)})_{t \leq \tau_l}$;
- for all k, there exists an integer j_k such that $X_{\tau_k}^{j_k} = 0$.

Now introduce an independent solution (X, W) of the SDE (E), with $X_0 = 0$. Define $(X_t^{(n)})_{t \in [\tau_\ell, \tau_{\ell+1}]}$ by analogy with the construction of $(X_t^{(n)})_{t \in [0, \tau_1]}$ by replacing $(x_1, \cdots, x_n)$ with $(X_{\tau_l}^{j_1^\ell}, \cdots, X_{\tau_l}^{j_n^\ell})$, where $(j_1^\ell, \ldots, j_n^\ell)$ are such that

$$0 = |X_{\tau_l}^{j_1^\ell}| < \cdots < |X_{\tau_l}^{j_n^\ell}|.$$

Thus, we have defined $X_t^{(n)}$ for all $t < \tau_\infty$, where $\tau_\infty := \lim_{l \to \infty} \tau_l$.

We denote by $\mathbb{P}_x^{(n),0}$ the law of $(X_t^{(n)})_{t < \tau_\infty}$. Notice that if we set $X^{(n)} = (X^1, \ldots, X^n)$, then for all i and all ℓ, $(X_{t \wedge \tau_\ell}^i)$ is a Walsh's Brownian motion stopped at time τ_ℓ. Thus a.s. on the event $\{\tau_\infty < \infty\}$, $X_{\tau_\infty}^{(n)} := \lim_{t \uparrow \tau_\infty} X_t^{(n)}$ exists. Note also that a.s. on the event $\{\tau_\infty < \infty\}$, there exist $i \neq j$ such that $X_{\tau_\ell}^i = X_{\tau_{\ell+1}}^j = 0$ for infinitely many ℓ's. This implies that a.s. on the event $\{\tau_\infty < \infty\}$, there exist $i \neq j$ such that $\lim_{t \uparrow \tau_\infty} X_t^i = \lim_{t \uparrow \tau_\infty} X_t^j = 0$, and thus that $X_{\tau_\infty}^{(n)} \in \Delta_n$, with $\Delta_n := \{(x_1, \cdots, x_n) \in G^n : \exists i \neq j, x_i = x_j\}$. Now, by construction, τ_∞ coincides with

$$T_{\Delta_n} = \inf\{t \geq 0 : X_t^{(n)} \in \Delta_n\}. \tag{17}$$

Note that in the particular case $n = 2$, on the event $\{\tau_\infty < \infty\}$, a.s. $X_{\tau_\infty}^{(2)} = (0, 0)$. We will prove in Section 4.3 that $\tau_\infty < \infty$ a.s.

4.2. **An obliquely reflected Brownian motion associated to the 2-point motion.** Fix $x \in G$, and let i such that $x \in E_i$. Recall the construction of (X, Y) of law $\mathbb{P}_{(x,0)}^{(2)}$. We have $\tau_0 = 0$ and for $k \geq 0$,

$$\begin{aligned} \tau_{2k+1} &= \inf\{t \geq \tau_{2k} : X_t = 0\}, \\ \tau_{2k+2} &= \inf\{t \geq \tau_{2k+1} : Y_t = 0\}. \end{aligned}$$

For $n \geq 0$, let i_{2n} and i_{2n+1} be in $\{1, \dots, N\}$ such that $X_{\tau_{2n}} \in E_{i_{2n}}$ and $Y_{\tau_{2n+1}} \in E_{i_{2n+1}}$. Then for $n \geq 0$,

$$\begin{aligned} X_t &= e_{i_{2n}}\left(|X_{\tau_{2n}}| + W_t^{i_{2n}} - W_{\tau_{2n}}^{i_{2n}}\right) & \text{for } t \in [\tau_{2n}, \tau_{2n+1}[, \\ Y_t &= e_{i_{2n+1}}\left(|Y_{\tau_{2n+1}}| + W_t^{i_{2n+1}} - W_{\tau_{2n+1}}^{i_{2n+1}}\right) & \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}[. \end{aligned}$$

Define, for $i \in [1, N]$, $f^i : G \to \mathbb{R}$ by

$$f^i(x) = -|x| \text{ if } x \in E_i \quad \text{and} \quad f^i(x) = |x| \text{ if not.}$$

Define now $(U_t, V_t)_{t<\tau_\infty}$ such that for $n \geq 0$

$$(U_t, V_t) = \begin{cases} (|X_t|, f^{i_{2n}}(Y_t)) & \text{for } t \in [\tau_{2n}, \tau_{2n+1}[; \\ (f^{i_{2n+1}}(X_t), |Y_t|) & \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}[. \end{cases}$$

Remark that $(U_t, V_t)_{t<\tau_\infty}$ is a continuous process with values in $\{(u, v) \in \mathbb{R}^2 : \ u + v > 0\}$ and such that for all $n \geq 0$, $U_{\tau_{2n}} > 0$, $V_{\tau_{2n}} = 0$, $U_{\tau_{2n+1}} = 0$ and $V_{\tau_{2n+1}} > 0$. Note that the excursions of this process outside of $\mathcal{Q}$ occur on straight lines parallel to $\{y = -x\}$.

Let, for $n \geq 0$,

$$\Theta_n = \arctan\left(\frac{p_{i_n}}{1 - p_{i_n}}\right).$$

Define for $t < \tau_\infty$,

$$A(t) = \int_0^t 1_{\{(U_s, V_s) \in \mathcal{Q}\}} ds = \int_0^t 1_{\{X_s \not\sim Y_s\}} ds.$$

Set $\gamma(t) = \inf\{s \geq 0 : A(s) > t\}$. Set for $n \geq 0$, $T_n = A(\tau_n)$ and $S_{n+1} = T_{n+1} - T_n$. Define for $t < T_\infty := \lim_{n\to\infty} T_n$,

$$(U_t^r, V_t^r) = (U_{\gamma(t)}, V_{\gamma(t)})$$

and for $t \geq T_\infty$, $(U_t^r, V_t^r) = (0, 0)$. Note that $T_{2n+1} = \inf\{t \geq T_{2n} : V_t^r = 0\}$ and $T_{2n+2} = \inf\{t \geq T_{2n+1} : U_t^r = 0\}$ and that $\gamma(T_n) = \tau_n$.

Lemma 4.2. *Given Θ_0, the law of $(U_t^r, V_t^r)_{t \leq S_1}$ is $\mathbb{P}_{|x|}^{\Theta_0}$.*

The proof of this lemma is given at the end of this section.

Notice that since a.s. $|Y_{\tau_1}| = V_{T_1}^r \neq 0$, then the sequence $(\tau_k)_k$ defined above is a.s. strictly increasing. It is also a sequence of stopping times with respect to the filtration $\mathcal{F}_t = \sigma\big((X_s, Y_s);\ s \leq t\big), t \geq 0$.

Define the sequence of processes $(Z^n)_{n\geq 1}$ such that for $n \geq 0$,

$$\begin{aligned} Z^{2n+1} &= (U_{t+T_{2n}}^r, V_{t+T_{2n}}^r)_{t \leq S_{2n+1}}, \\ Z^{2n+2} &= (V_{t+T_{2n+1}}^r, U_{t+T_{2n+1}}^r)_{t \leq S_{2n+2}}. \end{aligned}$$

Set also for $n \geq 0$, $U_{2n} = U_{T_{2n}}^r$ and $U_{2n+1} = V_{T_{2n+1}}^r$.

Applying Lemma 4.2 and using the strong Markov property at the stopping times τ_n, with the fact that if (X,Y) is distributed as $\mathbb{P}^{(2),0}_{(x,y)}$, then (Y,X) is distributed as $\mathbb{P}^{(2),0}_{(y,x)}$, one has the following

Lemma 4.3. *For all $n\geq 0$, given $\mathcal{F}_{\tau_n}$, the law of Z^{n+1} is $\mathbb{P}^{\Theta_n}_{U_n}$.*

This lemma shows that the sequences $(\Theta_n)_{n\geq 0}$ and $(Z^n)_{n\geq 1}$ satisfy (i) and (ii) in the beginning of Section 2.2 since for all $n\geq 0$,

$$\mathcal{G}_n=\sigma\big((\Theta_k,Z^k);\ 1\leq k\leq n\big)\vee\sigma(\Theta_0)\subset\mathcal{F}_{\tau_n}.$$

Thus $(U^r_t,V^r_t)_{t<T_\infty}$ is a Brownian motion in $\mathcal{Q}^*$ started from $(|x|,0)$, with time dependent angle of reflections at the boundaries given by $(\Theta_n)_{n\geq 0}$ and stopped when it hits $(0,0)$, as defined in Section 2. In particular, (U^r,V^r) is a continuous process and $\lim_{t\uparrow T_\infty}(U^r_t,V^r_t)=(0,0)$. We will now denote the process (U^r,V^r) by Z .

Remark 4.4. *Note that $(i_n)_{n\geq 0}$ is an homogeneous Markov chain started from $i_0=1$ with transition matrix $(P_{i,j})$ given by : for $(i,j)\in[1,N]^2$, $P_{i,j}=\frac{p_j}{\sum_{k\neq i}p_k}$. Remark also that given $\mathcal{G}_n$, Z^{n+1} and i_{n+1} are independent and a fortiori Z^{n+1} and Θ_{n+1} are also independent.*

Proof of Lemma 4.2. Let i be such that $x\in E_i$. Let (Y,W) be a solution of (E) with $Y_0=0$ and define $X_t=e_i(|x|+W^i_t)$ for $0\leq t\leq\tau_1$ where $\tau_1=\inf\{s\geq 0:|x|+W^i_s=0\}$. Set for $t\geq 0$, $(U_t,V_t):=(|x|+W^i_t,f^i(Y_t))$ where $f^i(y)=|y|\mathbf{1}_{y\notin E_i}-|y|\mathbf{1}_{y\in E_i}$. Note that for $t\leq\tau_1$, $U_t=|X_t|$.

Since Y is a Walsh's Brownian motion started at 0, it is well known that V is a skew Brownian motion with parameter $1-p_i$. This can be seen using Freidlin-Sheu formula, which shows that

$$V_t=\int_0^t\big(\mathbf{1}_{\{V_s>0\}}-\mathbf{1}_{\{V_s\leq 0\}}\big)dB^Y_s+(1-2p_i)L_t(V). \tag{18}$$

Define $A(t)=\int_0^t\mathbf{1}_{\{V_s\geq 0\}}ds=\int_0^t\mathbf{1}_{\{Y_s\notin E_i\}}ds$ and $\gamma(t)=\inf\{s\geq 0: A(s)>t\}$. It is also well known that $V^r_t:=V_{\gamma(t)}$ is a reflecting Brownian motion on $\mathbb{R}_+$. Set $M_t=\int_0^t 1_{\{V_s>0\}}dV_s=\int_0^t 1_{\{Y_s\notin E_i\}}dB^Y_s$. Then $B^2_t:=M_{\gamma(t)}$ is a Brownian motion. We also have that $V_t\vee 0=M_t+(1-p_i)L_t(V)$, which implies that $V^r_t=B^2_t+(1-p_i)L_{\gamma(t)}(V)$ and therefore that $L_t(V^r)=(1-p_i)L_{\gamma(t)}(V)$. Note finally that $L(V)=L(|Y|)$.

Set for $t\geq 0$, $B^1_t=\int_0^{\gamma(t)}1_{\{V_s>0\}}dW^i_s$. By Lévy's theorem B^1 and B^2 are two independent Brownian motions. Finally, set $U^r_t=U_{\gamma(t)}$. Then $(U^r_t,V^r_t)_{t\leq\gamma(\tau_1)}$ is equal in law to the process $(U^r_t,V^r_t)_{t\leq S_1}$ given in the statement of Lemma 4.2.

Lemma 4.2 is a direct consequence of the following.

Lemma 4.5. *For all $t \geq 0$,*

$$\begin{aligned} U^r_t &= |x| + B^1_t - \frac{p_i}{1-p_i} L_t(V^r) \\ V^r_t &= B^2_t + L_t(V^r). \end{aligned}$$

Proof. We closely follow the proof of [11, Lemma 4.3]. Let $\epsilon > 0$ and define the sequences of stopping times σ^ϵ_k and τ^ϵ_k such that $\tau^\epsilon_0 = 0$ and for $k \geq 0$,

$$\begin{aligned} \sigma^\epsilon_k &= \inf\{t \geq \tau^\epsilon_k;\ V_t = -\epsilon\}, \\ \tau^\epsilon_{k+1} &= \inf\{t \geq \sigma^\epsilon_k;\ V_t = 0\}. \end{aligned}$$

Note first that (18) implies that

$$\sum_{k\geq 0} \left(V_{\sigma^\epsilon_k \wedge \gamma(t)} - V_{\tau^\epsilon_k \wedge \gamma(t)}\right)$$

converges in probability as $\epsilon \to 0$ towards $B^2_t + (1-2p_i)L_{\gamma(t)}(V)$. Since $V^r_t = B^2_t + (1-p_i)L_{\gamma(t)}(V)$, if we set

$$L^{\epsilon,r}_t = \sum_{k\geq 0} \left(V_{\tau^\epsilon_{k+1} \wedge \gamma(t)} - V_{\sigma^\epsilon_k \wedge \gamma(t)}\right),$$

as $\epsilon \to 0$, $L^{\epsilon,r}_t$ converges towards $p_i L_{\gamma(t)}(V)$ in probability. Now for $t > 0$,

$$U^r_t = |x| + \sum_{k\geq 0} \left(U_{\tau^\epsilon_{k+1} \wedge \gamma(t)} - U_{\tau^\epsilon_k \wedge \gamma(t)}\right).$$

Set for $t \geq 0$,

$$B^{\epsilon,1}_t = \sum_{k\geq 0} \left(W^i_{\sigma^\epsilon_k \wedge \gamma(t)} - W^i_{\tau^\epsilon_k \wedge \gamma(t)}\right).$$

Note that $d(U_s + V_s) = \sum_{j\neq i} \mathbf{1}_{\{Y_s \in E_j\}} dW^j_s$ and thus when $Y_s \in E^*_i$ (i.e. when V_s is negative), $U_s + V_s$ remains constant, and we have

$$\begin{aligned} U^r_t &= |x| + \sum_{k\geq 0} \left(U_{\tau^\epsilon_{k+1} \wedge \gamma(t)} - U_{\sigma^\epsilon_k \wedge \gamma(t)}\right) + \sum_{k\geq 0} \left(U_{\sigma^\epsilon_k \wedge \gamma(t)} - U_{\tau^\epsilon_k \wedge \gamma(t)}\right) \\ &= |x| - L^{\epsilon,r}_t + B^{\epsilon,1}_t. \end{aligned}$$

Since $B^{\epsilon,1}_t$ converges in probability towards B^1_t, we get

$$U^r_t = |x| + B^1_t - p_i L_{\gamma(t)}(V).$$

And we conclude using that $L_t(V^r) = (1-p_i)L_{\gamma(t)}(V)$. □

4.3. **Coalescing property.** Our purpose in this section is to prove that τ_∞ defined above is finite a.s. By symmetry and the strong Markov property, it suffices to prove this for $n = 2$ and $(X_0, Y_0) = (x, 0)$ for some $x \in G^*$. We use the notations of Section 4.2.

Proposition 4.6. *With probability* 1, $\tau_\infty < \infty$.

Proof. Since $(|X_t|, t \leq \tau_\infty)$ is a reflected Brownian motion stopped at time τ_∞, it will suffice to prove that a.s. $L_{\tau_\infty}(|X|) < \infty$. Denote by L^1_t and L^2_t the local times accumulated by Z respectively on $\{u = 0\}$ and $\{v = 0\}$ up to t and $L_t = L^1_t + L^2_t$. First, note that for $t \leq S_1$, $L_t(V^r) = (1-p_i)L_{\gamma(t)}(V) = (1-p_i)L_{\gamma(t)}(|Y|)$. Thus $L_{\tau_1}(|Y|) = \frac{L_{S_1}(V^r)}{1-p_i}$. Note also that $L_{\tau_1}(|X|) = 0$. Thus

$$L_{\tau_1}(|X|) + L_{\tau_1}(|Y|) = \frac{L_{S_1}}{1 - p_i}.$$

By induction, we get that

$$L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) \;=\; \sum_{n\geq 0} \frac{L_{T_{n+1}} - L_{T_n}}{1 - p_{i_n}} \;\leq\; C\, L_{T_\infty}$$

with $C = \sup_{\{1\leq i\leq N\}}(1 - p_i)^{-1}$. By Theorem 2.11 a.s. $L_{T_\infty} < \infty$, and so $L_{\tau_\infty}(|X|) + L_{\tau_\infty}(|Y|) < \infty$. □

The fact that when $n \geq 3$, $\tau_\infty < \infty$ a.s., with τ_∞ defined in Section 4.1, is an immediate consequence of Proposition 4.6.

4.4. **Construction of φ.** Let $(P^{(n)}, n \geq 1)$ be the unique consistent family of Markovian semigroups such that

(i) $P^{(1)}$ is the semigroup of the Walsh's Brownian motion on G.
(ii) The n-point motion of $P^{(n)}$ started from $x \in G^n$ up to its entrance time in Δ_n is distributed as $\mathbb{P}^{(n),0}_x$.
(iii) The n-point motion $(X^1, \ldots, X^n)$ of $P^{(n)}$ is such that if $X^i_s = X^j_s$ then $X^i_t = X^j_t$ for all $t \geq s$.

We will prove that all $P^{(n)}$ are Feller and that (16) holds. By [12, Lemma 1.11], this amounts to check the following condition.

Lemma 4.7. *Let* (X, Y) *be the two point motion associated to* $P^{(2)}$, *then for all positive* $\epsilon > 0$

$$\lim_{d(x,y)\to 0} \mathbb{P}^{(2),0}_{(x,y)}[d(X_t, Y_t) > \epsilon] = 0.$$

Proof. As in the proof of Proposition 4.6, we take $y = 0$. Then using the same notations, for all positive ϵ, $\{d(X_t, Y_t) > \epsilon\} \subset \{\sup_{t<\sigma_0} |Z_t| > \epsilon\}$. Now the result of the lemma follows from Remark 2.10. □

By Theorem 4.1, a SFM φ can be associated to $(P^{(n)})_n$.

Proposition 4.8. *Let φ be a SFM associated to $(P^{(n)})_n$. Then there exists a family of independent white noises $\mathcal{W} = (W^i, 1 \le i \le N)$ such that*

(i) $\mathcal{F}^{\mathcal{W}}_{s,t} \subset \mathcal{F}^{\varphi}_{s,t}$ *for all $s \le t$ and*
(ii) $(\varphi, \mathcal{W})$ *solves (E).*

Proof. Let $V_{s,\cdot}(x)$ be the Brownian motion associated to $\varphi_{s,\cdot}(x)$. For all $i \in [1, N]$ and $s \le t$, set

$$W^i_{s,t} = \lim_{|x|\to\infty, x\in E_i, |x|\in\mathbb{Q}} V_{s,t}(x).$$

For all $i \in [1, N]$ and $s \le t$, with probability 1, this limit exists. Indeed if $x, y \in E_i$ are such that $|x| \le |y|$, then a.s. $V_{s,t}(x) = V_{s,t}(y)$ for all $s \le t \le \tau^x_s = \inf\{u \ge s;\ \varphi_{s,u}(x) = 0\}$. Moreover $W^i = (W^i_{s,t}, s < t)$ is a real white noise. Indeed, W^i is centered and Gaussian, and by the flow property of φ and using $\varphi_{s,u}(x) = e_i(|x| + W^i_{s,u})$ if $s \le u \le \tau^x_s$ and $x \in E_i$, we have $W^i_{s,u} = W^i_{s,t} + W^i_{t,u}$. It is also clear that W^i has independent increments with respect to (s, t). Thus, W^i is a real white noise. The fact that $\mathcal{W} = (W^i, 1 \le i \le N)$ is a family of independent real white noises easily holds.

For $x \in G$ and $t \ge 0$,

$$\langle W^i_{s,\cdot}, V_{s,\cdot}(x)\rangle_t = \lim_{|y|\to\infty, y\in E_i, |y|\in\mathbb{Q}} \langle V_{s,\cdot}(y), V_{s,\cdot}(x)\rangle_t = \int_s^t 1_{\{\varphi_{s,u}(x)\in E_i\}} du.$$

This yields

$$V_{s,t}(x) = \sum_{i=1}^N \int_s^t 1_{\{\varphi_{s,u}(x)\in E_i\}} dW^i_u.$$

By Theorem 3.1, we deduce that $(\varphi, \mathcal{W})$ solves (E). □

Denote by $\mathbb{P}_E$ the law of $(\varphi, \mathcal{W})$.

4.5. **Uniqueness in law of a SFM solution of (E).** In this section, we show that the SFM φ constructed in Section 4.4 is the only SFM solution of (E). More precisely, we show

Proposition 4.9. *Let $(\varphi, \mathcal{W})$ be a solution of (E), with φ a SFM. Then the law of $(\varphi, \mathcal{W})$ is $\mathbb{P}_E$.*

Proof. We start by showing

Lemma 4.10. *For all $x = e_i(r) \in G$, we have $\varphi_{s,t}(x) = e_i(r + W^i_{s,t})$ for all $s \le t \le \tau^x_s = \inf\{t \ge s : \varphi_{s,t}(x) = 0\}$. In particular for all $1 \le i \le N$, $s \le t$, we have $\mathcal{F}^{W^i}_{s,t} \subset \mathcal{F}^{\varphi}_{s,t}$.*

Proof. Let $f \in \mathcal{D}$ such that $f(x) = |x|$ for all $x \in E_i$. By applying f in (E), we deduce the first claim. The second claim is then an immediate consequence by taking a sequence $(x_k)_k \subset E_i$ converging to ∞. □

With this lemma and Theorem 1.2 we prove the following

Lemma 4.11. *Let $x = (x_1, \cdots, x_n) \in G^n$. Let $S = \inf\{t \geq 0 : (\varphi_{0,t}(x_1), \cdots, \varphi_{0,t}(x_n)) \in \Delta_n\}$. Then $(\varphi_{0,t}(x_1), \cdots, \varphi_{0,t}(x_n))_{t \leq S}$ is distributed like $\mathbb{P}_x^{(n),0}$.*

Proof. Suppose $|x_1| < \cdots < |x_n|$. For $k \in [1, n]$, set $Y_t^k = \varphi_{0,t}(x_k)$ and $Y_t^{(n)} = (Y_t^1, \ldots, Y_t^n)$. Set for $i \in [1, N]$, $W_t^i = W_{0,t}^i$ and $W_t = (W_t^1, \ldots, W_t^n)$. Note that for all $k \in [1, n]$, (Y^k, W) is a solution of (E). Set

$$\sigma_1 = \inf\{t \geq 0 : \exists k \neq 1 : Y_t^k = 0\}$$

and for $\ell \geq 1$, set

$$\sigma_{\ell+1} = \inf\{t \geq \sigma_\ell : \exists k \in [1, n] : Y_t^k = 0, Y_{\sigma_\ell}^k \neq 0\}.$$

Let $S^n = \lim_{\ell \to \infty} \sigma_\ell$, then $S^n = S = \inf\{t : Y_t^{(n)} \in \Delta_n\}$. From Theorem 1.2, the law of (Y^1, W) is uniquely determined. Now, for $k \in [2, n]$ with $x_k \in E_i$, we have that for $t \leq \sigma_1$, $Y_t^k = e_i(|x_k| + W_t^i)$. This shows that $(Y_t^{(n)})_{t \leq \sigma_1}$ is distributed as $(X_t^{(n)})_{t \leq \tau_1}$, constructed in Subsection 4.1. Adapting the previous argument on the time interval $[\sigma_\ell, \sigma_{\ell+1}]$, we show that for all $\ell \geq 1$, $(Y_t^{(n)})_{t \leq \sigma_\ell}$ is distributed as $(X_t^{(n)})_{t \leq \tau_\ell}$. This thus shows the lemma. □

Lemma 4.11 permits to conclude the proof of Proposition 4.9. Indeed, the law of a SFM is uniquely determined by its family of n-point motions $X^{(n)}$. Using the fact that Δ_n is an absorbing set for $X^{(n)}$, the strong Markov property at time $T^n = \inf\{t;\ X_t^{(n)} \in \Delta_n\}$ and the consistency of the family of n-point motions, we see that the law of a SFM is uniquely determined by its family of n-point motions stopped at its first entrance time in Δ_n. □

5. Extension to metric graphs

In this section, we consider a family of star graphs $(G_k)_{1 \leq k \leq K}$ with $G_k = \cup_{i \in I_k} E_k^i$ and for each $1 \leq k \leq K$, a family of parameters $p_k := (p_k^i)_{i \in I_k}$ associated to G_k such that $0 \leq p_k^i \leq 1$ and $\sum_{i \in I_k} p_k^i = 1$. Set $I = \{(i, k) : i \in I_k, 1 \leq k \leq K\}$. For $i \in \cup_k I_k$, set $k(i) = \inf\{k,\ i \in I_k\}$. Let $\mathcal{W} = \{W_k^i,\ (i, k) \in I\}$ be a family of white noises such that

- $\{W_{k(i)}^i,\ i \in \cup_k I_k\}$ is a family of independent white noises.
- For all $1 \leq k \neq \ell \leq K$, and $i \in I_k \cap I_\ell$, $W_k^i + W_\ell^i = 0$.

In particular, for all $1 \leq k \leq K$, $\mathcal{W}_k := \{W_k^i,\ i \in I_k\}$ is a family independent white noises.

Theorem 5.1. *Let $(K_k)_{1\leq k\leq K}$ be a family of SFK's respectively defined on G_k. Assume that*

- *For all k, (K_k, W_k) solves $E(G_k, p_k)$,*
- *$(\mathcal{F}_{s,t} := \vee_k \mathcal{F}_{s,t}^{K_k})_{s\leq t}$ is i.d.i.*

Then, the flows $(K_k)_{1\leq k\leq K}$ are independent given $\mathcal{W}$.

The rest of this section will consist in proving Theorem 5.1, and the assumptions of the theorem are supposed to be satisfied. For a SFK K, $K(t)$ and $K(s,t)$ respectively denote $K_{0,t}$ and $K_{s,t}$, for a white noise W, $W(t)$ and $W(s,t)$ denote $W_{0,t}$ and $W_{s,t}$, and for a semigroup P, $P(t)$ denotes P_t.

5.1. **Feller semigroups.** Let $n := (n_k)_{1\leq k\leq K}$ be a family of nonnegative integers and set $G^{(n)} := \prod_k G_k^{n_k}$. For $t \geq 0$, $x := (x_k)_{1\leq k\leq K} \in G^{(n)}$ and $w \in \mathbb{R}^{|I|}$, set for $f \in C_b(G^{(n)})$ and $g \in C_b(\mathbb{R}^{|I|})$,

$$\mathbb{Q}_t^{(n)}(f \otimes g)(x, w) = \mathbb{E}\big[\big(\otimes_k (K_k(t))^{\otimes n_k}\big) f(x) g(w + W(t))\big].$$

Note that the i.d.i property implies that $\mathbb{Q}^{(n)}$ defines a Feller semigroup on $G \times \mathbb{R}^{|I|}$. Denote by $\mathbb{Q}_{(x,w)}^{(n)}$ the law of the diffusion started at (x, w) associated to this semigroup.

Define also for all k, $\mathbb{Q}_t^{(k,n_k)}$ the Feller semigroup on $G_k^{n_k} \times \mathbb{R}^{|I|}$ such that for $f_k \in C_b(G_k^{n_k})$ and $g \in C_b(\mathbb{R}^{|I|})$,

$$\mathbb{Q}_t^{(k,n_k)}(f_k \otimes g)(x_k, w) = \mathbb{E}\big[\big(K_k(t))^{\otimes n_k} f_k\big)(x_k) g(w + W(t))\big].$$

Denote as above by $\mathbb{Q}_{(x_k,w)}^{(k,n_k)}$ the law of the diffusion started at (x_k, w) associated to this semigroup.

Let (X, W) be a diffusion of law $\mathbb{Q}_{(x,0)}^{(n)}$, then for all (i, k), (X_k^i, W) is a diffusion of law $\mathbb{Q}_{(x_k^i,0)}^{(k,1)}$ and (X_k^i, W_k) is a solution of $E(G_k, p_k)$ with $X_k^i(0) = x_k^i$. This fact can easily be seen as a consequence of

Lemma 5.2. *For all $k \in \{1, \ldots, K\}$ and all $x \in G_k$, if (X, W) is a diffusion of law $\mathbb{Q}_{(x,0)}^{(k,1)}$, then (X, W) is a solution of $E(G_k, p_k)$.*

Proof. In the following, set $G = G_k$, $p = p_k$, $\mathbb{Q}_t = \mathbb{Q}_t^{(k,1)}$ and $N = |I_k|$. It is obvious that W is a N-dimensional Brownian motion. It is also clear that X is a Walsh's Brownian motion. Denote by B^X the Brownian motion associated to X. Then by Freidlin-Sheu formula, (X, W) solves $E(G, p)$ as soon as $B_t^X = \sum_i \int_0^t 1_{\{X_s \in E_i\}} dW_s^i$. It is enough to prove $\langle B^X, W^i \rangle_t = \int_0^t 1_{\{X_s \in E_i\}} ds$ for all i.

Recall the definition of $\mathcal{D}$ from (3) and set $\mathcal{D}_1 = \{f \in \mathcal{D} :\ f, f', f'' \in C_0(G)\}$. Denote by A the generator of $\mathbb{Q}_t$ and $\mathcal{D}(A)$ its domain, then $\mathcal{D}_1 \otimes C_0^2(\mathbb{R}^N) \subset \mathcal{D}(A)$ and for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$,

$$A(f\otimes g)(x,w) = \frac{1}{2}f(x)\Delta g(w) + \frac{1}{2}f''(x)g(w) + \sum_{i=1}^N (f'1_{E_i})(x)\frac{\partial g}{\partial w^i}(w).$$

Thus for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$,

$$f(X_t)g(W_t) - \int_0^t A(f\otimes g)(X_s,W_s)ds \text{ is a martingale.} \tag{19}$$

On the other hand, (19), Freidlin-Sheu and Itô's formulas imply that

$$\sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s)\frac{\partial g}{\partial w^i}(W_s)ds = \sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s)\frac{\partial g}{\partial w^i}(W_s)d\langle B^X, W_i\rangle_s.$$

Since this holds for all $f \in \mathcal{D}_1$ and all $g \in C_0^2(\mathbb{R}^N)$, we get that $\langle B^X, W^i\rangle_t = \int_0^t 1_{\{X_s\in E_i\}}ds$ for all i. □

5.2. **A sufficient condition for conditional independence.** For $x = (x_k)_k \in G^{(n)}$ and $w \in \mathbb{R}^{|I|}$, define $\mathbb{P}^{(n)}_{(x,w)}$ the law of $(X_1,\dots,X_K,W)$ such that $X_1,\dots,X_K$ are independent given W and for all k, (X_k,W) is distributed as $\mathbb{Q}^{(k,n_k)}_{(x_k,w)}$. Denote by $\mathbb{E}^{(n)}_{(x,w)}$ the expectation with respect to $\mathbb{P}^{(n)}_{(x,w)}$. Denote also by $\mathbb{E}^{(k,n_k)}_{(x_k,w)}$ the expectation with respect to $\mathbb{P}^{(k,n_k)}_{(x_k,w)}$. When Z is a $\sigma(W)$-measurable random variable, with W a white noise, we simply denote $\mathbb{E}^{(n)}_{(x,0)}[Z]$ and $\mathbb{E}^{(k,n_k)}_{(x_k,w)}[Z]$ by $\mathbb{E}[Z]$.

Proposition 5.3. *If for all $n := (n_k)_k$ and all $x := (x_k)_k$,*

$$\mathbb{Q}^{(n)}_{(x,0)} = \mathbb{P}^{(n)}_{(x,0)}, \tag{20}$$

then the flows $(K_k)_k$ are independent given W.

Proof. For $n := (n_k)_k$, $x := (x_k)_k$, $f = \otimes_k f_k$, with $f_k \in C_0(G_k^{n_k})$,

$$\begin{aligned}\mathbb{E}\big[\prod_k (K_k(t))^{\otimes n_k} f_k(x_k)\big] &= \mathbb{Q}_t^{(n)}(f\otimes 1)(x,0)\\ &= \mathbb{E}^{(n)}_{(x,0)}\big[\prod_k f_k(X_k(t))\big]\\ &= \mathbb{E}\big[\prod_k \mathbb{E}^{(k,n_k)}_{(x_k,0)}[f_k(X_k(t))|W]\big].\end{aligned}$$

Then the proposition follows from the fact that

$$\mathbb{E}^{(k,n_k)}_{(x_k,0)}[f_k(X_k(t))|W] = \mathbb{E}[(K_k(t))^{\otimes n_k} f_k(x_k)|W]. \tag{21}$$

Let $g_0, \dots g_J$ be in $C_0(\mathbb{R}^{|I|})$ and let $0 = t_0 < \cdots < t_J = t$. Then setting, for all $g \in C_0(\mathbb{R}^{|I|})$, $h \in C_0(G_k^{n_k} \times \mathbb{R}^{|I|})$ and all $t \geq 0$, $\mathbb{Q}_t^g h(x,w) = g(w)\mathbb{Q}_t^{(k,n_k)} h(x,w)$, one has (to lighten the notation below, f_k, x_k, X_k and $(K_k(t))^{\otimes n_k}$ are denoted by f, x, X and K_t)

$$\begin{aligned}\mathbb{E}\big[K_t f(x) \prod_{0\leq j\leq J} g_j(W(t_j))\big] &= \mathbb{Q}_{t_1}^{g_0} \cdots \mathbb{Q}_{t_J - t_{J-1}}^{g_{J-1}} (f \otimes g_J)(x,0) \\ &= \mathbb{E}_{(x,0)}^{(k,n_k)}\big[f_k(X(t)) \prod_{0\leq j\leq J} g_j(W(t_j))\big] \\ &= \mathbb{E}\big[\mathbb{E}_{(x,0)}^{(k,n_k)}[f(X(t))|W] \prod_{0\leq j\leq J} g_j(W(t_j))\big],\end{aligned}$$

which suffices to deduce (21). □

Note that the Feller property implies that (20) is satisfied for all n and all x if it is satisfied for all n and all x in a dense subset of $G^{(n)}$.

5.3. **Uniqueness up to the first meeting time at** 0**.** Take $n = (n_k)_k$ and choose $x = (x_k)_k$ such that for all $k \neq \ell$, $0 \notin \{x_k^i, 1 \leq i \leq n_k\} \cap \{x_l^j, 1 \leq j \leq n_l\}$. In the following, (X, W) will be distributed as $\mathbb{P}_{(x,0)}^{(n)}$ or as $\mathbb{Q}_{(x,0)}^{(n)}$ with $X = (X_1, \cdots, X_K)$, and $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by (X, W). For $t \geq 0$, let $R_k(t) := \{X_k^i(t) \mid 1 \leq i \leq n_k\}$. Define the sequence of stopping times $(\sigma_j)_{j\geq 0}$ such that $\sigma_0 = 0$ and for all $j \geq 0$,

$$\sigma_{j+1} = \inf\{t \geq \sigma_j \mid \exists k,\ 0 \in R_k(t) \text{ and } 0 \notin R_k(\sigma_j)\}. \tag{22}$$

Using the strong Markov property, it is easy to see that for all $j \geq 1$, there is only one k such that $0 \in R_k(\sigma_j)$ and that the sequence $(\sigma_j)_{j\geq 1}$ is strictly increasing. Denote by $\sigma_\infty = \lim_{j\to\infty} \sigma_j$.

Proposition 5.4. *The law of* $(X(t), W(t))_{t<\sigma_\infty}$ *is the same under* $\mathbb{Q}_{(x,0)}^{(n)}$ *and under* $\mathbb{P}_{(x,0)}^{(n)}$.

Proof. Let (X, W) be distributed as $\mathbb{Q}_{(x,0)}^{(n)}$. Without loss of generality, assume there exists ℓ such that $0 \in R_\ell(0)$ and $0 \notin \cup_{k\neq\ell} R_k(0)$. Then $\{(X_k(t), t \leq \sigma_1),\ k \neq \ell\}$ is $\sigma(W)$-measurable and therefore $\{(X_k(t), t \leq \sigma_1),\ 1 \leq k \leq K\}$ is a family of independent random variables given W. So the conditional law of this family given W is the same as the conditional law of $\{(X_k(t), t \leq \sigma_1),\ 1 \leq k \leq K\}$ given W when (X, W) is distributed as $\mathbb{P}_{(x,0)}^{(n)}$. Denote this law by $\mu(x, W)$. Using the strong Markov property at time σ_n, we get that given $\mathcal{F}_{\sigma_n}$ and W, the law of $\{(X_k(t + \sigma_n), t \leq \sigma_{n+1} - \sigma_n),\ 1 \leq k \leq K\}$ is $\mu(X(\sigma_n), W(\cdot +$

$\sigma_n)-W(\sigma_n))$. Since this characterizes the law of $(X(t),W(t))_{t<\sigma_\infty}$, the proposition is proved. □

This proposition implies in particular that if $\mathbb{P}^{(n)}_{(x,0)}(\sigma_\infty=\infty)=1$, then $\mathbb{Q}^{(n)}_{(x,0)}=\mathbb{P}^{(n)}_{(x,0)}$.

5.4. **The meeting time at** 0 **is infinite.** Our purpose here is to prove the following

Proposition 5.5. *For all* $n=(n_k)_k$ *and all* $x=(x_k)$ *such that for all* $k\neq\ell$, $0\notin\{x_k^i,1\le i\le n_k\}\cap\{x_l^j,1\le j\le n_l\}$, *we have*

$$\mathbb{P}^{(n)}_{(x,0)}(\sigma_\infty=\infty)=1.$$

Proof. Assume $K=2$, $n_1=n_2=1$ and take $x=(x_1,x_2)$ such that $x_1\neq 0$ and $x_2=0$. It is easy to see that if the proposition holds in this particular case, then it also holds in the general case. We use in the following the notations of Subsection 5.3. Note that

$$\sigma_\infty=\inf\{t\ge 0: X_1(t)=X_2(t)=0\}$$

and that for $k\in\{1,2\}$, (X_k,W_k) is a solution of $E(G_k,p_k)$. Set $I^c=I_1\cap I_2$ and set for $k\in\{1,2\}$, $I_k^0:=I_k\backslash I^c$. Recall that for $i\in I^c$, $W_1^i+W_2^i=0$. For $k\in\{1,2\}$ and $i\in I_k$, set $\theta_k^i=\arctan\left(\frac{1-p_k^i}{p_k^i}\right)$. For $k\in\{1,2\}$ and $i\notin I_k$, set $\theta_k^i=0$.

Say $x_1\sim x_2$ if there exists $i\in I^c$ such that $x_1\in E_1^i$ and $x_2\in E_2^i$ and say $x_1\not\sim x_2$ otherwise. Recall the definition of $(\sigma_n)_n$: $\sigma_0=0$ and for all $\ell\ge 0$,

$$\begin{aligned}\sigma_{2\ell+1} &= \inf\{t\ge\sigma_{2\ell}: X_1(t)=0\},\\ \sigma_{2\ell+2} &= \inf\{t\ge\sigma_{2k+1}: X_2(t)=0\}.\end{aligned}$$

For all $\ell\ge 0$, set $U^0_{2\ell}=|X_1(\sigma_{2\ell})|$ and $U^0_{2\ell+1}=|X_2(\sigma_{2\ell+1})|$. Define also $i_{2\ell}$ and $i_{2\ell+1}$ such that $X_1(\sigma_{2\ell})\in E_1^{i_{2\ell}}$ and $X_2(\sigma_{2\ell+1})\in E_2^{i_{2\ell+1}}$.

Set for all $t\ge 0$

$$A_t=\int_0^t 1_{\{X_1(s)\not\sim X_2(s)\}}ds$$

and

$$\gamma_t=\inf\{s\ge 0: A_s>t\},\quad \mathcal{G}_t=\mathcal{F}_{\gamma_t}.$$

Set for $n\ge 0$, $S_n=A_{\sigma_n}$ and $S_\infty=\lim_{n\to\infty}S_n=A_{\sigma_\infty}$.

Our purpose now is to define a càdlàg process $(U_t,V_t)_{t<S_\infty}$, continuous on $[S_{n-1},S_n)$ for all $n\ge 1$, taking its values in the quadrant $\mathcal{Q}$ and such that $(U_{S_n},V_{S_n})=(U_n^0,0)$ for all $n\ge 0$. Its left limit will be denoted by $(U_{t-},V_{t-})=\lim_{s\uparrow t}(U_s,V_s), t<S_\infty$.

Let us explain the procedure for $n = 0$. We have $U_0 = |x_1|$, $X_1(t) \in E_1^{i_0}$ and $|X_1(t)| = |x_1| + W_1^{i_0}(t)$ for $t \le \sigma_1$.

Now we discuss the following two cases:
First case: $i_0 \in I_1^0$, then for all $t < \sigma_1$, $X_2(t) \not\sim X_1(t)$, $A(t) = t$ and $S_1 = \sigma_1$. Set for $t \le S_1$, $U_t = |X_1(t)|$ and $V_t = |X_2(t)|$. Then $(U_t, V_t)_{t \le S_1}$ is a reflected Brownian motion in the quadrant $\mathcal{Q}$ started at $(|x_1|, 0)$ and stopped when hitting $\{x = 0\}$. Here the reflection is normal and the angle of reflection is $\theta_2^i = 0$.
Second case: $i_0 \in I^c$. For $t \le \sigma_1$, set $X_t = |X_1(t)|$ and

$$Y_t = |X_2(t)|\big(1_{\{X_2(t) \not\sim X_1(t)\}} - 1_{\{X_2(t) \sim X_1(t)\}}\big).$$

For $t \le \sigma_1$, $A_t = \int_0^t 1_{\{Y_s > 0\}} ds$ and $1_{\{Y_t < 0\}} d(X_t + Y_t) = 0$. The process $(X_t, Y_t)_{t \le \sigma_1}$ behaves as a two dimensional Brownian motion in the interior of $\mathcal{Q}$ and outside $\mathcal{Q}$, it evolves on straight lines parallel to $\{y = x\}$. Finally set $\big(U_t, V_t\big) = \big(X_{\gamma_t}, Y_{\gamma_t}\big)$, for $t \le S_1$.

Lemma 5.6. *Assume $i_0 \in I^c$. There exist two independent $(\mathcal{G}_t)_t$-Brownian motions B^1 and B^2 such that for $t < S_1$,*

$$\begin{aligned} dU_t &= dB_t^1 + \tan\big(\theta_2^{i_0}\big) dL_t(V), \\ dV_t &= dB_t^2 + dL_t(V). \end{aligned}$$

Moreover $U_t > 0$ for all $t < S_1$, $V_{S_1-} = 0$ if $X_2(\sigma_1) \in E_2^{i_0}$ and $U_{S_1-} = 0$ if not.

The proof of this lemma follows exactly Lemma 4.5 and is omitted. Note that this lemma also holds if $i_0 \notin I^c$.

In the same way, we construct $(U_t, V_t)_{t \in [S_{n-1}, S_n)}$ such that conditionally on $\mathcal{F}_{\sigma_{n-1}}$, $(U_t, V_t)_{t \in [S_{n-1}, S_n)}$ is an obliquely reflected Brownian motion in the quadrant $\mathcal{Q}$ started at $(U_{n-1}^0, 0)$ at time S_{n-1}, stopped at time S_n such that $U_t > 0$ for all $t < S_\infty$, and

- If n is odd, $V_{S_n-} = 0$ if $X_2(\sigma_n) \in E_2^{i_{n-1}}$ and $U_{S_n-} = 0$ if not.
- If n is even, $V_{S_n-} = 0$ if $X_1(\sigma_n) \in E_1^{i_{n-1}}$ and $U_{S_n-} = 0$ if not.

Moreover the reflection occurs only at the boundary $\{y = 0\}$, and the angle of reflection is $-\theta_1^{i_n}$ if n is odd and is $-\theta_2^{i_n}$ if n is even.

When $t \in [S_{n-1}, S_n)$, we denote by Θ_t this angle of reflection. Note that $U_n^0 = U_{S_n}$ when $U_{S_n-} > 0$ and $U_n^0 = V_{S_n}$ when $U_{S_n-} = 0$.

Define the sequence $(T_j)_{j \ge 0}$ by $T_0 = 0$ and for all $j \ge 0$

$$T_{j+1} = \inf\{S_n : \ S_n > T_j \text{ and } U_{S_n-} = 0\}.$$

Set $T_\infty = \lim_{j \to \infty} T_j = S_\infty = A_{\sigma_\infty}$. Then $(U_t, V_t)_{t < S_\infty}$ is continuous expect at the times T_j, $j \ge 1$. Moreover, we have the following

Lemma 5.7. *The process* (U_t, V_t) *is* $(\mathcal{G}_t)_t$*-adapted. There exist two independent* $(\mathcal{G}_t)_t$*-Brownian motions* B^1 *and* B^2*, such that for all* $t \in [T_j, T_{j+1})$*,*

$$\begin{aligned} U_t &= U_{T_j} + \int_{T_j}^t \big(dB_s^1 - \tan\big(\Theta_s\big) dL_s(V)\big), \\ V_t &= \int_{T_j}^t \big(dB_s^2 + dL_s(V)\big). \end{aligned}$$

Moreover $U_{T_j} = V_{T_j -}$, $U_{T_{j+1}-} = 0$ *and* $U_t > 0$ *for all* $t < S_\infty$*.*

The process (U, V) is not continuous, but out of it, one can construct an obliquely reflected Brownian motion on $\mathcal{Q}$ $(Z_t^r = (X_t^r, Y_t^r), t < T_\infty)$, and satisfying the following lemma.

Lemma 5.8. *For all* $n \geq 0$, $Y_{T_{2n}}^r = U_{T_{2n}}$, $X_{T_{2n+1}}^r = U_{T_{2n+1}}$

$$\begin{aligned} T_{2n+1} &= \inf\{t \geq T_{2n} : Y_t^r = 0\}, \\ T_{2n+2} &= \inf\{t \geq T_{2n+1} : X_t^r = 0\}, \\ T_\infty &= \inf\{t \geq 0 : Z_t^r = (0,0)\}, \end{aligned}$$

and for all $t < T_\infty$*,*

$$\begin{aligned} dX_t^r &= dB_t^1 + dL_t(X^r) - \tan\big(\Theta_t\big) dL_t(Y^r), \\ dY_t^r &= dB_t^2 - \tan\big(\Theta_t\big) dL_t(X^r) + dL_t(Y^r). \end{aligned}$$

Moreover $T_\infty = \infty$ *implies* $\sigma_\infty = \infty$*.*

To conclude the proof of Proposition 5.5, it remains to prove that a.s. $T_\infty = \infty$. We exactly follow [9, Page 161]. For all $a \geq 0$, define

$$\tau_a = \inf\{t \geq 0 : |Z_t^r| = a\}.$$

Take $\epsilon < |x_1| < A$ and set $\tau_{\epsilon,A} = \tau_\epsilon \wedge \tau_A$, then by Itô's formula, setting $R_t = |Z_t^r|$, we have that, for all $t \geq 0$,

$$\log(R_{t \wedge \tau_{\epsilon,A}}) = \log(|x_1|) + M_t + C_t$$

where M is a martingale started from 0 and C is a nonnegative nondecreasing process (using that $\Theta_t \leq 0$). Thus by letting $t \to \infty$, we get $E[\log(R_{\tau_{\epsilon,A}})] \geq \log(|x_1|)$. So

$$\log(\epsilon)\mathbb{P}(\tau_\epsilon < \tau_A) + \log(A)(1 - \mathbb{P}(\tau_\epsilon < \tau_A)) \geq \log(|x_1|)$$

and consequently

$$\mathbb{P}(\tau_\epsilon < \tau_A) \leq \frac{\log(A) - \log(|x_1|)}{\log(A) - \log(\epsilon)}.$$

Replacing ϵ with $\epsilon(A) = (\frac{1}{A})^A$, yields

$$\mathbb{P}(T_\infty < \infty) = \lim_{A\to\infty} \mathbb{P}(T_\infty < \tau_A) \leq \lim_{A\to\infty} \mathbb{P}(\tau_{\epsilon(A)} < \tau_A) = 0.$$

□

6. Final remarks

It is also possible to extend the framework of the present paper to the case of a metric star graph with an infinite family of rays $G = \cup_{n\in\mathbb{N}} E_n$. We only give a formal discussion of the problem. Suppose we are given a family $p = (p_n)_{n\in\mathbb{N}} \subset]0,1[$ such that $\sum_n p_n = 1$. Then the law of the Walsh's Brownian motion associated to p is still defined via its semigroup (see the introduction). It satisfies also a Freidlin-Sheu formula similar to the finite case (see [8]):

$$df(Z_t) = f'(Z_t)dB_t^Z + \frac{1}{2}f''(Z_t)dt$$

where B^Z is again the martingale part of $|Z|$ and f runs over an appropriate domain of test functions $\mathcal{D}$. Now suppose given a family $(W^n)_{n\in\mathbb{N}}$ of independent Brownian motions, then the natural extension of (E) associated to p is the following

$$df(Z_t) = \sum_n (f'1_{E_n})(Z_t)dW_t^n + \frac{1}{2}f''(Z_t)dt, \ f \in \mathcal{D}$$

which we denote again by (E). The Brownian motion B^Z has also the martingale representation property for $(\mathcal{F}^Z)_t$ [3, Proposition 19 (ii)]. Thus following our arguments, under some conditions on Z, the law of any solution $(Z, W^n, n \in \mathbb{N})$ to (E) is unique. One could also investigate stochastic flows solutions of (E). However, in contrast to the discrete case, here we have

$$\inf \Big\{ \arctan\Big(\frac{p_n}{1-p_n}\Big) : n \in \mathbb{N} \Big\} = 0.$$

This is the new difficulty with respect to the present paper. We leave the question of existence of a SFM in this case open.

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