

## ON THE CSÁKI–VINCZE TRANSFORMATION

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### Abstract

Csáki and Vincze have defined in 1961 a discrete transformation  $\mathcal{T}$  which applies to simple random walks and is measure preserving. In this paper, we are interested in ergodic and asymptotic properties of  $\mathcal{T}$ . We prove that  $\mathcal{T}$  is exact:  $\bigcap_{k \geq 1} \sigma(\mathcal{T}^k(S))$  is trivial for each simple random walk  $S$  and give a precise description of the lost information at each step  $k$ . We then show that, in a suitable scaling limit, all iterations of  $\mathcal{T}$  “converge” to the corresponding iterations of the continuous Lévy transform of Brownian motion.

### 1. Introduction and main results

Let  $B$  be a Brownian motion, then  $T(B)_t = \int_0^t \operatorname{sgn}(B_s) dB_s$  is a Brownian motion too. Iterating  $T$  yields a family of Brownian motions  $(B^n)_n$  given by

$$B^0 = B, \quad B^{n+1} = T(B^n).$$

We call  $B^n$  the  $n$ -iterated Lévy transform of  $B$ . At least two transformations of simple random walks have been studied in the literature as discrete analogues to  $T$ . For a simple random walk (SRW)  $S$ , Dubins and Smorodinsky [2] define the Lévy transform  $\Gamma(S)$  of  $S$  as a SRW obtained from the paths

$$n \longmapsto |S_n| - L_n$$

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where  $L$  is a discrete analogue of local time. Their fundamental result says that  $S$  can be recovered from the signs of the excursions of  $S, \Gamma(S), \Gamma^2(S), \dots$  and a fortiori  $\Gamma$  is ergodic. Later, another discrete Lévy transformation  $F$  was given by Fujita [4]:

$$F(S)_{k+1} - F(S)_k = \text{sgn}(S_k)(S_{k+1} - S_k), \quad \text{with the convention } \text{sgn}(0) = -1.$$

However,  $F$  is not ergodic by the main result of [4]. Our main purpose in this paper is to study a transformation already obtained by Csáki and Vincze.

Let  $\mathbb{W} = \mathbb{C}(\mathbb{R}_+, \mathbb{R})$  be the Wiener space equipped with the distance

$$d_U(w, w') = \sum_{n \geq 1} 2^{-n} \left( \sup_{0 \leq t \leq n} |w(t) - w'(t)| \wedge 1 \right).$$

We endow  $\mathbb{E} = \mathbb{W}^{\mathbb{N}}$  with the product metric defined for each  $x = (x_k)_{k \geq 0}$ ,  $y = (y_k)_{k \geq 0}$  by

$$d(x, y) = \sum_{k \geq 0} 2^{-k} (d_U(x_k, y_k) \wedge 1).$$

Thus  $(\mathbb{E}, d)$  is a separable complete metric space.

For each SRW  $S$  and  $h \geq 0$ , we denote by  $\mathcal{T}^h(S)$  the  $h$ -iterated Csáki–Vincze transformation (to be defined in Section 2.1) of  $S$  with the convention  $\mathcal{T}^0(S) = S$ .

Let  $B$  be a Brownian motion defined on  $(\Omega, \mathcal{A}, \mathbb{P})$ . For each  $n \geq 1$ , define  $T_0^n = 0$  and for all  $k \geq 0$ ,

$$T_{k+1}^n = \inf \left\{ t \geq T_k^n : |B_t - B_{T_k^n}| = \frac{1}{\sqrt{n}} \right\}.$$

Then  $S_k^n = \sqrt{n}B_{T_k^n}$ ,  $k \geq 0$ , is a SRW and we have the following

**THEOREM 1.**

- (i) For each SRW  $S$  and  $h \geq 0$ ,  $\mathcal{T}^h(S)$  is independent of  $(S_j, j \leq h)$  and a fortiori

$$\bigcap_{h \geq 0} \sigma(\mathcal{T}^h(S))$$

is trivial.

- (ii) For each  $n \geq 1$ ,  $h \geq 0$  and  $t \geq 0$ , define

$$S^{n,h}(t) = \frac{1}{\sqrt{n}} \mathcal{T}^h(S^n)_{\lfloor nt \rfloor} + \frac{(nt - \lfloor nt \rfloor)}{\sqrt{n}} (\mathcal{T}^h(S^n)_{\lfloor nt \rfloor + 1} - \mathcal{T}^h(S^n)_{\lfloor nt \rfloor}).$$

Then

$$(S^{n,0}, S^{n,1}, S^{n,2}, \dots)$$

converges to

$$(B^0, B^1, B^2, \dots)$$

in probability in  $\mathbb{E}$  as  $n \rightarrow \infty$ .

Theorem 1(i) says that  $\mathcal{T}$  is exact, but there are more informations in the proof. For instance, the random vectors  $(S_1, S_2, \dots, S_n)$  and  $(S_1, \mathcal{T}(S)_1, \dots, \mathcal{T}^{n-1}(S)_1)$  generate the same  $\sigma$ -field; so the whole path  $(S_n)_{n \geq 1}$  can be encoded in the sequence  $(\mathcal{T}^n(S)_1)_{n \geq 0}$  which is stronger than exactness. From Theorem 1(i), we can deduce the following

**COROLLARY 1.** *Fix  $p \geq 2$  and let  $\alpha^i = (\alpha_n^i)_{n \geq 1}$ ,  $i \in [1, p]$  be  $p$  nonnegative sequences such that*

$$\alpha_n^1 \longrightarrow +\infty, \quad \alpha_n^i - \alpha_n^{i-1} \longrightarrow +\infty \quad \text{as } n \rightarrow \infty \quad \text{for all } i \in [2, p].$$

Let  $S$  be a SRW and  $W_1, \dots, W_{p+1}$  be  $p+1$  independent Brownian motions. For  $n \geq 1, h \geq 0, t \in \mathbb{R}_+$ , define

$$(1) \quad S_n^h(t) = \frac{1}{\sqrt{n}} \mathcal{T}^h(S)_{\lfloor nt \rfloor} + \frac{(nt - \lfloor nt \rfloor)}{\sqrt{n}} (\mathcal{T}^h(S)_{\lfloor nt \rfloor + 1} - \mathcal{T}^h(S)_{\lfloor nt \rfloor}).$$

Then

$$\left( S_n^0, S_n^{\lfloor n\alpha_n^1 \rfloor}, \dots, S_n^{\lfloor n\alpha_n^p \rfloor} \right) \xrightarrow[n \rightarrow +\infty]{law} (W_1, W_2, \dots, W_{p+1}) \quad \text{in } \mathbb{W}^{p+1}.$$

A natural question which is actually motivated by the famous question of ergodicity of the Lévy transformation  $T$  as it will be discussed in Section 3, is to focus on sequences  $(h_n)_n$  tending to  $\infty$  and satisfying

$$(2) \quad \lim_{n \rightarrow \infty} (S_n^{h_n}(t) - B_t^{h_n}) = 0 \quad \text{in probability.}$$

Such sequences exist and when (2) holds, we necessarily have  $\lim_{n \rightarrow \infty} \frac{h_n}{n} = 0$ . This is summarized in the following

**PROPOSITION 1.** *With the same notations of Theorem 1 :*

- (i) *There exists a family  $(\alpha^i)_{i \in \mathbb{N}}$  of nondecreasing sequences  $\alpha^i = (\alpha_n^i)_{n \in \mathbb{N}}$  with values in  $\mathbb{N}$  such that*

$$\alpha_n^0 \longrightarrow +\infty, \quad \alpha_n^i - \alpha_n^{i-1} \longrightarrow +\infty \quad \text{as } n \rightarrow \infty \quad \text{for all } i \geq 1$$

and moreover

$$\lim_{n \rightarrow \infty} (S^{n, \alpha_n^i} - B^{\alpha_n^i}) = 0 \text{ in probability in } \mathbb{W}$$

for all  $i \in \mathbb{N}$ .

- (ii) If  $(h_n)_n$  is any integer-valued sequence such that  $\frac{h_n}{n}$  does not tend to 0, then there exists no  $t > 0$  such that (2) holds.

In the next section, we review the Csáki–Vincze transformation, establish part (i) of Theorem 1 and show that  $(S, \mathcal{T}(S), \dots, \mathcal{T}^h(S), \dots)$  “converges” in law to  $(B, T(B), \dots, T^h(B), \dots)$ . To prove part (ii) of Theorem 1, we use the simple idea: if  $Z_n$  converges in law to a constant  $c$ , then the convergence holds also in probability. The other proofs are based on the crucial property of the transformation  $\mathcal{T}$ :  $\mathcal{T}^h(S)$  is independent of  $\sigma(S_j, j \leq h)$  for each  $h$ . In Section 3, we compare our work with [2] and [4] and discuss the famous question of ergodicity of  $T$ .

## 2. Proofs

### 2.1. The Csáki–Vincze transformation and convergence in law

For the sequel, we recommend to read the pages 113 and 114 in [7] (Theorem 2 below). Some consequences (see Proposition 2 below) have been drawn in [5] (Sections 2.1 and 2.2). We also notice that our stating of this result is slightly different from [7] and leave to the reader to make the obvious analogy.

**THEOREM 2** ([7] page 113). *Let  $S = (S_n)_{n \geq 0}$  be a SRW defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  and  $X_i = S_i - S_{i-1}$ ,  $i \geq 1$ . Define  $\tau_0 = 0$  and for  $l \geq 0$ ,*

$$\tau_{l+1} = \min \{i > \tau_l : S_{i-1}S_{i+1} < 0\}.$$

Set

$$\bar{X}_j = \sum_{l \geq 0} (-1)^l X_1 X_{j+1} 1_{\{\tau_{l+1} \leq j \leq \tau_{l+1}\}}.$$

Then  $\bar{S}_0 = 0$ ,  $\bar{S}_n = \bar{X}_1 + \dots + \bar{X}_n$ ,  $n \geq 1$  is a SRW. Moreover if  $Y_n := \bar{S}_n - \min_{k \leq n} \bar{S}_k$ , then for all  $n \in \mathbb{N}$ ,

$$(3) \quad |Y_n - |S_n|| \leq 2.$$

We call  $\bar{S} = \mathcal{T}(S)$ , the Csáki–Vincze transformation of  $S$  (see the figures 1 and 2 below).

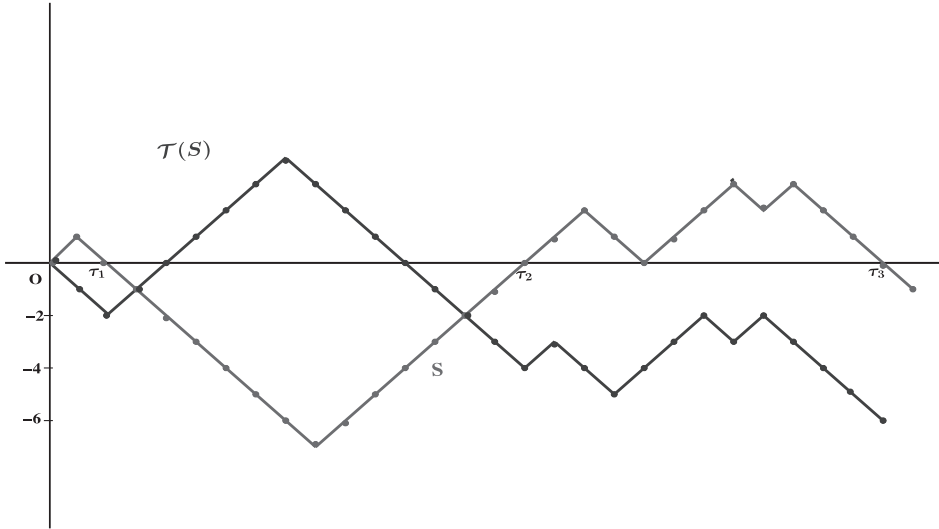


Fig. 1  $S$  and  $\mathcal{T}(S)$

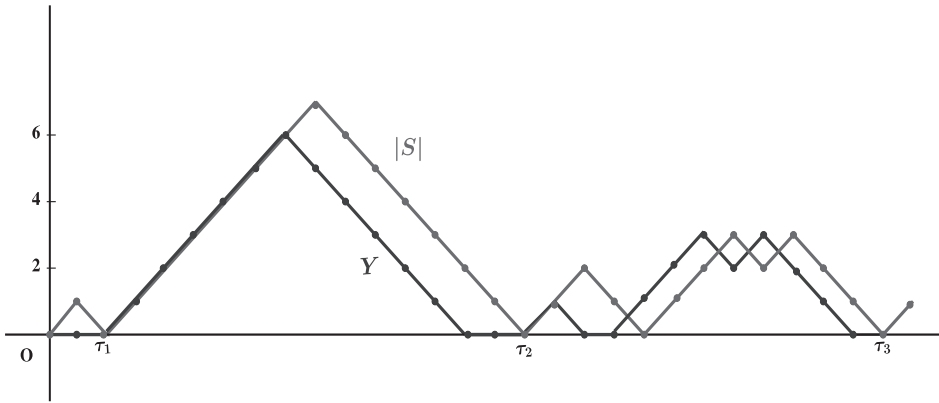


Fig. 2  $|S|$  and  $Y$

Note that  $(-1)^l X_l$  is simply equal to  $\text{sgn}(S)|_{[\tau_l+1, \tau_{l+1}]}$  ( $:= X_{\tau_{l+1}}$ ) which can easily be checked by induction on  $l$ . Thus for all  $j \in [\tau_l + 1, \tau_{l+1}]$ ,

$$\bar{X}_j = \text{sgn}(S)|_{[\tau_l+1, \tau_{l+1}]}(S_{j+1} - S_j)$$

or equivalently

$$(4) \quad \mathcal{T}(S)_j - \mathcal{T}(S)_{j-1} = \text{sgn}(S_{j-\frac{1}{2}})(S_{j+1} - S_j)$$

where  $t \rightarrow S_t$  is the linear interpolation of  $(S_n)_{n \geq 0}$ . Hence, one can expect that  $(S, \mathcal{T}(S))$  will “converge” to  $(B, B^1)$  in a suitable sense. The following proposition has been established in [5]. We give its proof for completeness.

PROPOSITION 2. *With the same notations of Theorem 2, we have*

- (i) *For all  $n \geq 0$ ,  $\sigma(\mathcal{T}(S)_j, j \leq n) \vee \sigma(S_1) = \sigma(S_j, j \leq n+1)$ .*
- (ii)  *$S_1$  is independent of  $\sigma(\mathcal{T}(S))$ .*

PROOF. (i) The inclusion  $\subset$  is clear from (4). Now, for all  $1 \leq j \leq n$ ,  $X_{j+1} = \sum_{l \geq 0} (-1)^l X_1 \bar{X}_j 1_{\{\tau_{l+1} \leq j \leq \tau_{l+1}\}}$ . As a consequence of (iii) and (iv) [7] (page 114), for all  $l \geq 0$ ,

$$\tau_l = \min \{n \geq 0, \mathcal{T}(S)_n = -2l\}.$$

Thus  $\tau_l$  is a stopping time with respect to the natural filtration of  $\mathcal{T}(S)$  and as a result  $\{\tau_l + 1 \leq j \leq \tau_{l+1}\} \in \sigma(\mathcal{T}(S)_h, h \leq j-1)$  which proves the inclusion  $\supset$ .

- (ii) We may write for all  $l \geq 1$ ,

$$\tau_l = \min \{i > \tau_{l-1} : X_1 S_{i-1} X_1 S_{i+1} < 0\}.$$

This shows that  $\mathcal{T}(S)$  is  $\sigma(X_1 X_{j+1}, j \geq 0)$ -measurable and (ii) is proved.  $\square$

Note that

$$\mathcal{T}(S) = \mathcal{T}(-S), \quad \sigma(\mathcal{T}^{h+1}(S)) \subset \sigma(\mathcal{T}^h(S)),$$

which is the analogue of

$$T(B) = T(-B), \quad \sigma(T^{h+1}(B)) \subset \sigma(T^h(B)).$$

The previous proposition yields the following

COROLLARY 2. *For all  $n \geq 0$ ,*

- (i)  *$\sigma(S) = \sigma(\mathcal{T}^n(S)) \vee \sigma(S_k, k \leq n)$ .*
- (ii)  *$\sigma(\mathcal{T}^n(S))$  and  $\sigma(S_k, k \leq n)$  are independent.*
- (iii) *The  $\sigma$ -field*

$$\mathcal{G}^\infty = \bigcap_{n \geq 0} \sigma(\mathcal{T}^n(S))$$

*is  $\mathbb{P}$ -trivial.*

PROOF. Set  $X_i = S_i - S_{i-1}$ ,  $i \geq 1$ .

(i) We apply successively Proposition 2(i) so that for all  $n \geq 1$ ,

$$\begin{aligned} \sigma(S) &= \sigma(\mathcal{T}(S)) \vee \sigma(S_1) \\ &= \sigma(\mathcal{T}^2(S)) \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1) \\ &= \dots \\ &= \sigma(\mathcal{T}^n(S)) \vee \sigma(\mathcal{T}^{n-1}(S)_1) \vee \dots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1). \end{aligned}$$

To deduce (i), it suffices to prove that

$$(5) \quad \sigma(S_k, k \leq n) = \sigma(\mathcal{T}^{n-1}(S)_1) \vee \dots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1).$$

Again Proposition 2(i), yields

$$\begin{aligned} \sigma(S_k, k \leq n) &= \sigma(\mathcal{T}(S)_j, j \leq n-1) \vee \sigma(S_1) \\ &= \sigma(\mathcal{T}^2(S)_j, j \leq n-2) \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1) \\ &= \dots \\ &= \sigma(\mathcal{T}^{n-1}(S)_1) \vee \sigma(\mathcal{T}^{n-2}(S)_1) \dots \vee \sigma(\mathcal{T}(S)_1) \vee \sigma(S_1) \end{aligned}$$

which proves (5) and allows to deduce (i).

(ii) will be proved by induction on  $n$ . For  $n = 0$ , this is clear. Suppose the result holds for  $n$ , then  $S_1, \mathcal{T}^1(S)_1, \dots, \mathcal{T}^{n-1}(S)_1, \mathcal{T}^n(S)$  are independent (recall (5)). Let us prove that  $S_1, \mathcal{T}^1(S)_1, \dots, \mathcal{T}^n(S)_1, \mathcal{T}^{n+1}(S)$  are independent which will imply (ii) by (5). Note that  $\mathcal{T}^n(S)_1$  and  $\mathcal{T}^{n+1}(S)$  are  $\sigma(\mathcal{T}^n(S))$ -measurable. By the induction hypothesis, this shows that  $(S_1, \mathcal{T}^1(S)_1, \dots, \mathcal{T}^{n-1}(S)_1)$  and  $(\mathcal{T}^n(S)_1, \mathcal{T}^{n+1}(S))$  are independent. But  $\mathcal{T}^n(S)_1$  and  $\mathcal{T}^{n+1}(S)$  are also independent by Proposition 2(ii). Hence (ii) holds for  $n+1$  and thus for all  $n$ .

(iii) Let  $A \in \mathcal{G}^\infty$  and fix  $n \geq 1$ . Then  $A \in \sigma(\mathcal{T}^n(S))$  and we deduce from (ii) that  $A$  is independent of  $\sigma(S_k, k \leq n)$ . Since this holds for all  $n$ ,  $A$  is independent of  $\sigma(S)$ . As  $\mathcal{G}^\infty \subset \sigma(S)$ ,  $A$  is therefore independent of itself.  $\square$

Let  $S$  be a SRW defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  and recall the definition of  $S_n^h(t)$  from (1). On  $\mathbb{E}$ , define

$$Z^n(t_0, t_1, \dots, t_h, \dots) = \left( S_n^0(t_0), S_n^1(t_1), \dots, S_n^h(t_h), \dots \right)$$

and let  $\mathbb{P}_n$  be the law of  $Z^n$ .

LEMMA 1. *The family  $\{\mathbb{P}_n, n \geq 1\}$  is tight on  $\mathbb{E}$ .*

PROOF. By Donsker theorem for each  $h$ ,  $S_n^h$  converges in law to standard Brownian motion as  $n \rightarrow \infty$ . Thus the law of each coordinate of  $Z^n$  is tight on  $\mathbb{W}$  which is sufficient to get the result (see [3] page 107).  $\square$

**The limit process.** Fix a sequence  $(m_n, n \in \mathbb{N})$  such that  $Z^{m_n} \xrightarrow[n \rightarrow +\infty]{\text{law}} Z$  in  $\mathbb{E}$  where

$$Z = (B^{(0)}, B^{(1)}, \dots, B^{(h)}, \dots)$$

is the limit process. Note that  $B^{(0)}$  is a Brownian motion. From (3), we have  $\forall n \geq 1, t \geq 0$

$$\left| |S_n^0(t)| - (S_n^1(t) - \min_{0 \leq u \leq t} S_n^1(u)) \right| \leq \frac{2012}{\sqrt{n}}.$$

Letting  $n \rightarrow \infty$ , we get

$$|B_t^{(0)}| = B_t^{(1)} - \min_{0 \leq u \leq t} B_u^{(1)}.$$

Tanaka's formula for local time gives

$$|B_t^{(0)}| = \int_0^t \text{sgn}(B_u^{(0)}) dB_u^{(0)} + L_t(B^{(0)}) = B_t^{(1)} - \min_{0 \leq u \leq t} B_u^{(1)},$$

where  $L_t(B^{(0)})$  is the local time at 0 of  $B^{(0)}$  and so

$$B_t^{(1)} = \int_0^t \text{sgn}(B_s^{(0)}) dB_s^{(0)}.$$

The same reasoning shows that for all  $h \geq 1$ ,

$$B_t^{(h+1)} = \int_0^t \text{sgn}(B_s^{(h)}) dB_s^{(h)}.$$

Thus the law of  $Z$  is independent of the sequence  $(m_n, n \in \mathbb{N})$  and therefore

$$(6) \quad (S_n^0, S_n^1, \dots, S_n^h, \dots) \xrightarrow[n \rightarrow +\infty]{\text{law}} (B, B^1, \dots, B^h, \dots) \text{ in } \mathbb{E}$$

where  $B$  is a Brownian motion.



## 2.2. Convergence in probability

Let  $B$  be a Brownian motion and recall the notations in Theorem 1. For each  $n \geq 1$ , define

$$U^n = (B, S^{n,0}, B^1, S^{n,1}, \dots, B^h, S^{n,h}, \dots).$$

and let  $\mathbb{Q}_n$  be the law of  $U^n$ . Since  $\mathcal{T}^h(S^n)$  is a simple random walk for each  $(h, n)$ , a similar argument as in the proof of Lemma 1 shows that  $\{\mathbb{Q}_n, n \geq 1\}$  is tight on  $\mathbb{E}$ . Fix a sequence  $(m_n, n \in \mathbb{N})$  such that  $U^{m_n} \xrightarrow[n \rightarrow +\infty]{\text{law}} U$  in  $\mathbb{E}$ . Using (6), we see that there exist two Brownian motions  $C$  and  $D$  such that

$$U = (C, D, C^1, D^1, \dots, C^h, D^h, \dots).$$

It is easy to check that if  $\varphi : \mathbb{W} \rightarrow \mathbb{R}$  is bounded and uniformly continuous, then  $\Psi(f, g) = \varphi(f - g)$  defined for all  $(f, g) \in \mathbb{W}^2$  is also bounded uniformly continuous which comes from

$$d_U(f - f', g - g') = d_U(f - g, f' - g') \quad \text{for all } f, f', g, g' \in \mathbb{W}.$$

Thus if  $(F_n, G_n)$  converges in law to  $(F, G)$  in  $\mathbb{W}^2$ , then  $F_n - G_n$  converges in law to  $F - G$  in  $\mathbb{W}$ . Applying this, we see that  $B - S^{n,0}$  converges in law to  $C - D$ . On the other hand,  $B - S^{n,0}$  converges to 0 (in  $\mathbb{W}$ ) in probability (see [6] page 39). Consequently  $C = D$  and

$$U^n \xrightarrow[n \rightarrow +\infty]{\text{law}} (B, B, B^1, B^1, \dots, B^h, B^h, \dots) \quad \text{in } \mathbb{E}.$$

In particular for each  $h$ ,  $S^{n,h} - B^h$  converges in law to 0 as  $n \rightarrow \infty$ , that is  $S^{n,h}$  converges to  $B^h$  in probability as  $n \rightarrow \infty$ . Now the following equivalences are easy

- (i)  $\lim_{n \rightarrow \infty} (S^{n,0}, S^{n,1}, S^{n,2}, \dots) = (B^0, B^1, B^2, \dots)$  in probability in  $\mathbb{E}$ .
- (ii) For each  $h$ ,  $\lim_{n \rightarrow \infty} S^{n,h} = B^h$  in probability in  $\mathbb{W}$ .

Since we have proved (ii), Theorem 1 holds.

## 2.3. Proof of Corollary 1

(i) Let  $S$  be a SRW and  $W_1, W_2, \dots, W_{p+1}$  be  $p + 1$  independent Brownian motions (not necessarily defined on the same probability space as  $S$ ). Fix

$$0 \leq t_1^0 \leq \dots \leq t_{i_0}^0, \quad 0 \leq t_1^1 \leq \dots \leq t_{i_1}^1, \dots, \quad 0 \leq t_1^p \leq \dots \leq t_{i_p}^p.$$

By Corollary 2(ii), for  $n$  large enough (such that  $\lfloor nt_{i_0}^0 \rfloor + 1 \leq \lfloor n\alpha_n^1 \rfloor$ ),  $(S_n^0(t_1^0), \dots, S_n^0(t_{i_0}^0))$  which is  $\sigma(S_j, j \leq \lfloor nt_{i_0}^0 \rfloor + 1)$ -measurable, is independent of  $\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)$ . Thus  $(S_n^0(t_1^0), \dots, S_n^0(t_{i_0}^0))$  is independent of  $(S_n^{\lfloor n\alpha_n^1 \rfloor}, \dots, S_n^{\lfloor n\alpha_n^p \rfloor})$  and similarly  $\mathcal{T}^{\lfloor n\alpha_n^2 \rfloor}(S)$  is independent of  $\sigma(\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)_j, j \leq \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor)$ . Again, for  $n$  large (such that  $\lfloor nt_{i_1}^1 \rfloor + 1 \leq \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor$ ),  $(S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \dots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1))$  is  $\sigma(\mathcal{T}^{\lfloor n\alpha_n^1 \rfloor}(S)_j, j \leq \lfloor n\alpha_n^2 \rfloor - \lfloor n\alpha_n^1 \rfloor)$ -measurable and therefore is independent of  $(S_n^{\lfloor n\alpha_n^2 \rfloor}, \dots, S_n^{\lfloor n\alpha_n^p \rfloor})$ . By induction on  $p$ , for  $n$  large enough,

$$\begin{aligned} & (S_n^0(t_1^0), \dots, S_n^0(t_{i_0}^0)), \quad \left( S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \dots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1) \right), \dots, \\ & \left( S_n^{\lfloor n\alpha_n^p \rfloor}(t_1^p), \dots, S_n^{\lfloor n\alpha_n^p \rfloor}(t_{i_p}^p) \right) \end{aligned}$$

are independent and this yields the convergence in law of

$$\begin{aligned} & \left( S_n^0(t_1^0), \dots, S_n^0(t_{i_0}^0), S_n^{\lfloor n\alpha_n^1 \rfloor}(t_1^1), \dots, S_n^{\lfloor n\alpha_n^1 \rfloor}(t_{i_1}^1), \dots, \right. \\ & \left. S_n^{\lfloor n\alpha_n^p \rfloor}(t_1^p), \dots, S_n^{\lfloor n\alpha_n^p \rfloor}(t_{i_p}^p) \right) \end{aligned}$$

to

$$(W_1(t_1^0), \dots, W_1(t_{i_0}^0), W_2(t_1^1), \dots, W_2(t_{i_1}^1), \dots, W_{p+1}(t_1^p), \dots, W_{p+1}(t_{i_p}^p)).$$

Thus the convergence of the finite dimensional marginals holds and the proof is completed.

## 2.4. Proof of Proposition 1

To prove part (i), we need the following lemma which may be found in [1] page 32 in more generality:

**LEMMA 2.** *If  $(u_{k,n})_{k,n \in \mathbb{N}}$  is a nonnegative and bounded doubly indexed sequence such that for all  $k$ ,  $\lim_{n \rightarrow \infty} u_{k,n} = 0$ , then there exists a nondecreasing sequence  $(k_n)_n$  such that  $\lim_{n \rightarrow \infty} k_n = +\infty$  and  $\lim_{n \rightarrow \infty} u_{k_n,n} = 0$ .*

**PROOF.** By induction on  $p$ , we construct an increasing sequence  $(n_p)_{p \in \mathbb{N}}$  such that  $u_{p,n} < 2^{-p}$  for all  $n \geq n_p$ . Now define

$$k_n = \begin{cases} n & \text{if } 0 \leq n \leq n_0 \\ p & \text{if } n_p \leq n < n_{p+1} \text{ for some } p \in \mathbb{N}. \end{cases}$$

Clearly  $n \mapsto k_n$  is nondecreasing and  $\lim_{n \rightarrow \infty} k_n = +\infty$ . Moreover for all  $p$  and  $n \geq n_p$ , we have  $u_{k_n, n} < 2^{-p}$ . Thus for all  $p$ ,  $0 \leq \limsup_{n \rightarrow \infty} u_{k_n, n} \leq 2^{-p}$  and since  $p$  is arbitrary, the lemma is proved.  $\square$

The previous lemma applied to

$$u_{k, n} = E[d_U(S^{n, k}, B^k) \wedge 1],$$

guarantees the existence of a nondecreasing sequence  $(\alpha_n^0)_n$  with values in  $\mathbb{N}$  such that  $\lim_{n \rightarrow \infty} \alpha_n^0 = +\infty$  and

$$(7) \quad \lim_{n \rightarrow \infty} (S^{n, \alpha_n^0} - B^{\alpha_n^0}) = 0 \text{ in probability in } \mathbb{W}.$$

Now set

$$V^n = (B^{\alpha_n^0}, S^{n, \alpha_n^0}, B^{\alpha_n^0+1}, S^{n, \alpha_n^0+1}, \dots, B^{\alpha_n^0+h}, S^{n, \alpha_n^0+h}, \dots).$$

Using the same idea as in Section 2.2 and the relation (7), we prove that for all  $j \in \mathbb{N}$ ,

$$(8) \quad \lim_{n \rightarrow \infty} (S^{n, \alpha_n^0+j} - B^{\alpha_n^0+j}) = 0 \text{ in probability in } \mathbb{W}.$$

Equivalently: for all  $j \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} u_{j, n}^0 = 0 \text{ where } u_{j, n}^0 = E[d_U(S^{n, \alpha_n^0+j}, B^{\alpha_n^0+j}) \wedge 1].$$

By Lemma 2 again, there exists a nondecreasing sequence  $(\beta_n^0)_n$  with values in  $\mathbb{N}$  such that  $\lim_{n \rightarrow \infty} \beta_n^0 = +\infty$  and

$$(9) \quad \lim_{n \rightarrow \infty} (S^{n, \alpha_n^0+\beta_n^0} - B^{\alpha_n^0+\beta_n^0}) = 0 \text{ in probability in } \mathbb{W}.$$

Define  $\alpha_n^1 = \alpha_n^0 + \beta_n^0$ . Now using (9) and the same preceding idea, we construct  $\alpha^2$  and all the  $(\alpha^i)_i$  by the same way. Thus part (i) of Proposition 1 is proved.

To prove (ii), write

$$\mathcal{T}(S^n)_{j+1} - \mathcal{T}(S^n)_j = \text{sgn}(S_{j+\frac{1}{2}}^n)(S_{j+2}^n - S_{j+1}^n).$$

Thus for each  $k \geq 1$  and  $i \geq 1$ ,

$$\mathcal{T}^k(S^n)_i = \sum_{j=0}^{i-1} P^{n,k,j}(S_{j+k+1}^n - S_{j+k}^n), \quad \text{with}$$

$$P^{n,k,j} = \prod_{l=1}^k \operatorname{sgn} \left( \mathcal{T}^{k-l}(S^n)_{j+l-\frac{1}{2}} \right).$$

Denote by  $(\mathcal{F}_t)_{t \geq 0}$  the natural filtration of  $B$ , then  $P^{n,k,j}$  is the product of  $k$  random signs which are  $\mathcal{F}_{T_{j+k}^n}$ -measurable. This yields

$$\mathbb{E}[P^{n,k,j}(S_{j+k+1}^n - S_{j+k}^n) | \mathcal{F}_{T_{j+k}^n}] = P^{n,k,j} \mathbb{E}[\sqrt{n}(B_{T_{j+k+1}^n} - B_{T_{j+k}^n}) | \mathcal{F}_{T_{j+k}^n}] = 0.$$

Consequently  $\mathbb{E}[\mathcal{T}^k(S^n)_i | \mathcal{F}_{T_{j+k}^n}] = 0$  and a fortiori

$$\mathbb{E}[S^{n,k}(t) | \mathcal{F}_{T_k^n}] = 0 \quad \text{for all } n, k \text{ and } t.$$

Suppose there exists  $t > 0$  (which is fixed from now) such that

$$\lim_{n \rightarrow \infty} (S^{n,h_n}(t) - B_t^{h_n}) = 0$$

in probability; we will show that  $\frac{h_n}{n}$  must tend to 0. By Burkholder's inequality, we have

$$\mathbb{E}[|S_j|^p] \leq C_p \mathbb{E} \left[ (S_1^2 + (S_2 - S_1)^2 + \cdots + (S_j - S_{j-1})^2)^{\frac{p}{2}} \right] = C_p j^{\frac{p}{2}}.$$

Hence the  $L^p$ -norm of  $S^{n,h_n}(t)$  is bounded uniformly in  $n$  and the same is true for the  $L^p$ -norm of  $B_t^{h_n}$  because  $B^{h_n}$  is a Brownian motion. As a consequence,  $S^{n,h_n}(t) - B_t^{h_n}$  tends to 0 also in  $L^p$ -spaces.

From  $\mathbb{E}[S^{n,h_n}(t) | \mathcal{F}_{T_{h_n}^n}] = 0$  and using the  $L^2$ -continuity of conditional expectations, we get

$$\mathbb{E}[B_t^{h_n} | \mathcal{F}_{T_{h_n}^n}] \rightarrow 0 \quad \text{in } L^2.$$

Since  $M_s = B_{t \wedge s}^{h_n}$  is a square-integrable  $\mathcal{F}$ -martingale; we have  $\mathbb{E}[B_t^{h_n} | \mathcal{F}_{T_{h_n}^n}] = B_{t \wedge T_{h_n}^n}^{h_n}$  and  $B_{t \wedge T_{h_n}^n}^{h_n}$  must therefore tend to 0 in  $L^2$ . So

$$0 = \lim_{n \rightarrow \infty} \mathbb{E} \left[ (B_{t \wedge T_{h_n}^n}^{h_n})^2 \right] = \lim_{n \rightarrow \infty} \mathbb{E}[t \wedge T_{h_n}^n].$$

This means that  $T_{h_n}^n \rightarrow 0$  in probability. Now recall the following

LEMMA 3 (see [6] page 39). *The sequence of continuous-time processes  $(\Delta^n)_n$  defined by*

$$\Delta^n(t) = \frac{k}{n} \text{ for } t \in [T_k^n, T_{k+1}^n[$$

*converges uniformly on compacts in probability to the identity process  $t$ .*

This Lemma implies that  $\Delta^n(T_{h_n}^n) \rightarrow 0$  in probability. But  $\Delta^n(T_{h_n}^n) = \frac{h_n}{n}$ , so that  $\frac{h_n}{n} \rightarrow 0$ .

### 3. Concluding remarks

We first notice that with a little more work, Theorem 1(ii) remains true when the Csáki–Vincze transform is replaced with the Dubins–Smorodinsky, Fujita transform or any other “reasonable” discrete version. In Proposition 1, it is clear that there is no contradiction between the two statements. In fact, by the proof of Lemma 2, the sequences  $(\alpha_i)_{i \in \mathbb{N}}$  are constructed such that  $0 \leq \alpha_n^0 \leq n$  and  $0 \leq \alpha_n^i - \alpha_n^{i-1} \leq n$  for all  $i$  and  $n$ . Let us now explain our interest in relation (2). Suppose that (2) were true for a sequence  $(h_n)_n$  with  $\frac{h_n}{n} \rightarrow \infty$ , then using the convergence of  $S^{n,0}$  to  $B$  and Corollary 2(ii) applied to  $S^n$ , this would imply that  $(B, B^{h_n})$  converge in law to a 2-dimensional Brownian motion which is equivalent to the ergodicity of the continuous Lévy transformation on path space (see Proposition 17 in [8]). Corollary 1 asserts that this convergence holds in discrete time. However as proved before, such a sequence  $(h_n)_n$  does not exist and so the possible ergodicity of  $T$  cannot be established by arguments involving asymptotics of  $\mathcal{T}^n$ . Thus the impression that a thorough study of good discrete versions could lead to a better understanding of the conjectured ergodicity of  $T$  may be misleading.

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