EQUIVARIANT QUANTIZATION OF ORBIFOLDS

N. PONCIN*, F. RADOUX**, AND R. WOLAK***

ABSTRACT. Equivariant quantization is a new theory that highlights the role of symmetries in the relationship between classical and quantum dynamical systems. These symmetries are also one of the reasons for the recent interest in quantization of singular spaces, orbifolds, stratified spaces... In this work, we prove existence of an equivariant quantization for orbifolds. Our construction combines an appropriate desingularization of any Riemannian orbifold by a foliated smooth manifold, with the foliated equivariant quantization that we built in [Poncin N, Radoux F, Wolak R, *A first approximation for quantization of singular spaces*, J. Geom. Phys., **59** (4) (2009), pp 503-518]. Further, we suggest definitions of the common geometric objects on orbifolds, which capture the nature of these spaces and guarantee, together with the properties of the mentioned foliated resolution, the needed correspondences between singular objects of the orbifold and the respective foliated objects of its desingularization.

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1. INTRODUCTION

Equivariant quantization, see [15], [16], [6], [14], [2], [7], [1], [3]... is the fruit of a recent research program that aimed at a complete and unambiguous geometric characterization of quantization. The procedure highlights the primary role of symmetries in the relationship between classical and quantum dynamical systems. One of the major achievements of equivariant quantization is the understanding that a fixed G-structure of the configuration space of a mechanical system guarantees existence and uniqueness of a G-equivariant quantization. Roughly and more generally, an equivariant, or better, a natural quantization of a smooth manifold M is a vector space isomorphism

$$Q[\nabla] : \mathrm{Pol}(T^*M) \ni s \to Q[\nabla](s) \in \mathscr{D}(M)$$

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* University of Luxembourg, Campus Limpertsberg, Mathematics Research Unit, 162A, avenue de la Faïencerie, L-1511 Luxembourg City, Grand-Duchy of Luxembourg, E-mail: norbert.poncin@uni.lu.
** University of Liège, Institute of Mathematics, Grande Traverse, 12 - B37, B-4000 Liège, Belgium, E-mail: Fabian.Radoux@ulg.ac.be.
*** Jagiellonian University, ulica Reymonta 4 30-059 Krakow, Poland, E-mail: Robert.Wolak@im.uj.edu.pl.

that maps a smooth function $s \in \mathrm{Pol}(T^*M)$ of the phase space T^*M, which is polynomial along the fibers, to a differential operator $Q[\nabla](s) \in \mathscr{D}(M)$ that acts on functions $f \in C^\infty(M)$ of the configuration space M. The quantization map $Q[\nabla]$ depends on the projective class $[\nabla]$ of an arbitrary torsionless connection ∇ of M, and it is equivariant with respect to the action of local diffeomorphisms ϕ of M, i.e.

$$Q[\phi^*\nabla](\phi^*s)(\phi^*f) = \phi^*(Q[\nabla](s)(f)),$$

$\forall s \in \mathrm{Pol}(T^*M), \forall f \in C^\infty(M)$. Such natural and projectively invariant quantizations, or simply equivariant quantizations, were investigated in several works, see e.g. [4], [17], [9].

On the other hand, *quantization of singular spaces*, see e.g. [5], [11], [12], [13], [10], [18]... is an upcoming topic in Mathematical Physics, in particular in view of the interest of reduction for complex systems with symmetries. More precisely, if a symmetry group acts on the phase space or the configuration space of a general system, the quotient space is usually a singular space, an orbifold or a stratified space... The challenge consists in the quest for a quantization procedure of such singular spaces that in addition commutes with reduction.

It is now quite natural to ask which aspects of the new theory of equivariant quantization – that was recently extended from vector spaces to smooth manifolds – hold true for certain singular spaces. The main result of this work is the proof of existence of *equivariant quantization for orbifolds*.

A first difficulty of the attempt to construct an equivariant quantization on a singular space, is the proper definition of the actors in equivariant quantization – functions, differential operators, symbols, vector fields, differential forms, connections... – for this space. Even in the case of orbifolds no universally accepted definitions can be found in literature. Morevoer, geometric and algebraic definitions do not always coincide as in the classical context. Our method is based on the resolution of orbifolds proposed in [8]. More precisely, we combine this desingularization technique, which allows identifying any Riemannian orbifold V with the leaf space of a foliated smooth manifold $(\tilde{V}, \mathscr{F})$, with the foliated equivariant quantization that we constructed in [19], to build a singular equivariant quantization of orbifolds. To realize this idea, meaningful definitions, which not only capture the nature of orbifolds but ensure simultaneously that singular objects of V are in 1-to-1 correspondence with the respective foliated objects of $(\tilde{V}, \mathscr{F})$, are needed. We show that the chosen foliated resolution of orbifolds has exactly the properties that are necessary for this kind of relationship.

The paper is organized as follows. In the second section, we recall the definitions of foliated objects and of a foliated equivariant quantization. In the third, we detail our geometric definitions of singular objects on orbifolds and study their relevant properties for the singular equivariant quantization problem. We describe and further investigate, in Section 4, the foliated desingularization of a Riemannian orbifold, putting special emphasis on aspects that are of importance for the mentioned appropriate correspondence between foliated and singular objects. The last section deals with existence and the explicit construction of a singular equivariant quantization of Riemannian orbifolds.

2. Foliated quantization

In the sequel, $(M,\mathscr{F})$ denotes an n-dimensional smooth manifold endowed with a regular foliation $\mathscr{F}$ of dimension p and codimension $q=n-p$. Moreover, U is an open set of $(M,\mathscr{F})$.

Let us first recall the definitions of the foliated objects and of the foliated natural and projectively invariant quantization given in [19] :

Definition 1. A *foliated function* f on U is a smooth function $f\in C^\infty(U)$ such that f is constant along the connected components of the traces of the leaves in U. In other words, if $(V,(x,y))$ is a system of adapted coordinates such that $V\cap U\neq\emptyset$, the local form of f on $U\cap V$ depends only on the transverse coordinates y.

We denote by $C^\infty(U,\mathscr{F})$ the algebra of all foliated functions of $(U,\mathscr{F})$.

Definition 2. A *foliated differential operator* D of order $k\in\mathbb{N}$ of U is an endomorphism of the space $C^\infty(U,\mathscr{F})$ of foliated functions, which reads in any system $(V,(x^1,\ldots,x^p,y^1,\ldots,y^q))$ of adapted coordinates in the following way:

$$D|_{U\cap V}=\sum_{|\alpha|\leq k} D_\alpha\,\partial_{y^1}^{\alpha^1}\ldots\partial_{y^q}^{\alpha^q},$$

where the coefficients $D_\alpha\in C^\infty(U\cap V,\mathscr{F})$ are locally defined foliated functions and where k is independent of the considered chart.

We denote by $\mathscr{D}^k(U,\mathscr{F})$ the $C^\infty(U,\mathscr{F})$-module of all k-th order foliated differential operators of $(U,\mathscr{F})$ and set

$$\mathscr{D}(U,\mathscr{F}):=\cup_{k\in\mathbb{N}}\mathscr{D}^k(U,\mathscr{F}).$$

The graded space $\mathscr{S}(U,\mathscr{F})$ associated with the filtered space $\mathscr{D}(U,\mathscr{F})$,

$$\mathscr{S}(U,\mathscr{F}):=\oplus_{k\in\mathbb{N}}\mathscr{S}^k(U,\mathscr{F}):=\oplus_{k\in\mathbb{N}}\mathscr{D}^k(U,\mathscr{F})/\mathscr{D}^{k-1}(U,\mathscr{F}),$$

is the space of *foliated symbols*. The k-th order symbol of a k-th order foliated differential operator D is then simply its class $\sigma_k(D)$ in the k-th term of the symbol space. The principal symbol $[D]$ of D is the symbol $\sigma_k(D)$ with the lowest possible k.

Definition 3. An *adapted vector field* of U is a vector field $X\in\mathrm{Vect}(U)$ such that $[X,Y]\in\Gamma(T\mathscr{F})$, for all $Y\in\Gamma(T\mathscr{F})$.

The space $\mathrm{Vect}_{\mathscr{F}}(U)$ of adapted vector fields is obviously a Lie subalgebra of the Lie algebra $\mathrm{Vect}(U)$ and the space $\Gamma(T\mathscr{F})$ of tangent vector fields is an ideal of $\mathrm{Vect}_{\mathscr{F}}(U)$.

Definition 4. The quotient algebra $\mathrm{Vect}(U,\mathscr{F}):=\mathrm{Vect}_{\mathscr{F}}(U)/\Gamma(T\mathscr{F})$ is the Lie algebra of *foliated vector fields.*

The space $\mathrm{Vect}(U,\mathscr{F})$ is also a $C^\infty(U,\mathscr{F})$-module that acts naturally on $C^\infty(U,\mathscr{F})$.

Proposition 1. *The space* $\mathrm{Vect}(U,\mathscr{F})$ *is isomorphic to the space* $\mathscr{S}^1(U,\mathscr{F})$.

Proof. See [19]. □

Definition 5. A *foliated differential 1-form* of U is a differential 1-form θ of U such that $\mathrm{i}_Y\theta = \mathrm{i}_Y \mathrm{d}\theta = 0$, for all $Y \in \Gamma(T\mathscr{F})$.

We denote by $\Omega^1(U,\mathscr{F})$ the space of all foliated differential 1-forms of U. The interior product of a foliated 1-form with a foliated vector field is a foliated function.

Definition 6. A *foliated torsion-free connection* of U is a bilinear map $\nabla(\mathscr{F}) : \mathrm{Vect}(U,\mathscr{F}) \times \mathrm{Vect}(U,\mathscr{F}) \to \mathrm{Vect}(U,\mathscr{F})$ such that, for all $f \in C^\infty(U,\mathscr{F})$ and all $[X],[Y] \in \mathrm{Vect}(U,\mathscr{F})$, the following conditions are satisfied:

- $\nabla(\mathscr{F})_{f[X]}[Y] = f\nabla(\mathscr{F})_{[X]}[Y]$,
- $\nabla(\mathscr{F})_{[X]}(f[Y]) = ([X].f)[Y] + f\nabla(\mathscr{F})_{[X]}[Y]$,
- $\nabla(\mathscr{F})_{[X]}[Y] = \nabla(\mathscr{F})_{[Y]}[X] + [[X],[Y]]$.

We denote by $\mathscr{C}(U,\mathscr{F})$ the affine space of torsion-free foliated connections of U.

Definition 7. Two foliated connections $\nabla(\mathscr{F})$ and $\nabla'(\mathscr{F})$ of U are *projectively equivalent* if and only if there is a foliated 1-form $\theta \in \Omega^1(U,\mathscr{F})$ such that, for all $[X],[Y] \in \mathrm{Vect}(U,\mathscr{F})$, one has

$$\nabla'(\mathscr{F})_{[X]}[Y] - \nabla(\mathscr{F})_{[X]}[Y] = \theta([X])[Y] + \theta([Y])[X].$$

Definition 8. A *foliated local diffeomorphism* between two foliated manifolds $(M,\mathscr{F})$ and $(M',\mathscr{F}')$ is a smooth mapping $\Phi : M \to M'$ that is locally a diffeomorphism and maps any leaf of $\mathscr{F}$ into a leaf of $\mathscr{F}'$.

Definition 9. A *foliated natural and projectively invariant quantization* is a map

$$\mathscr{Q}(\mathscr{F}) : \mathscr{C}(M,\mathscr{F}) \times \mathscr{S}(M,\mathscr{F}) \to \mathscr{D}(M,\mathscr{F}),$$

which is defined for any foliated manifold $(M,\mathscr{F})$ and has the following properties:

- $\mathscr{Q}(\mathscr{F})(\nabla(\mathscr{F}))$ is a linear bijection between $\mathscr{S}(M,\mathscr{F})$ and $\mathscr{D}(M,\mathscr{F})$ that verifies $[\mathscr{Q}(\mathscr{F})(\nabla(\mathscr{F}))(S)] = S$, for all $\nabla(\mathscr{F}) \in \mathscr{C}(M,\mathscr{F})$ and all $S \in \mathscr{S}(M,\mathscr{F})$,
- $\mathscr{Q}(\mathscr{F})(\nabla(\mathscr{F})) = \mathscr{Q}(\mathscr{F})(\nabla'(\mathscr{F}))$, if $\nabla(\mathscr{F})$ and $\nabla'(\mathscr{F})$ are projectively equivalent,
- If $\Phi : (M,\mathscr{F}) \to (M',\mathscr{F}')$ is a foliated local diffeomorphism between two foliated manifolds $(M,\mathscr{F})$ and $(M',\mathscr{F}')$, then

  $$\mathscr{Q}(\mathscr{F})(\Phi^*_{\mathscr{C}}\nabla(\mathscr{F}'))(\Phi^*_{\mathscr{S}}S)(\Phi^* f) = \Phi^*(\mathscr{Q}(\mathscr{F}')(\nabla(\mathscr{F}'))(S)(f)),$$

  for all $\nabla(\mathscr{F}') \in \mathscr{C}(M',\mathscr{F}')$, $S \in \mathscr{S}(M',\mathscr{F}')$, $f \in C^\infty(M',\mathscr{F}')$.

Existence of a foliated natural and projectively invariant quantization was proven in [19].

3. Singular objects

Recall first the definition of a Riemannian orbifold.

Definition 10. An n-dimensional ($n \in \mathbb{N}$; smooth or, more precisely, C^∞-smooth) *Riemannian orbifold structure* V on a second countable Hausdorff space $|V|$ is given by the following data:

- An open cover $\{V_i\}_i$ of $|V|$.
- For each $i \in I$, a connected and open subset $U_i \subset \mathbb{R}^n$ with a Riemannian metric h_i; a finite subgroup Γ_i of isometries of the Riemannian manifold (U_i, h_i); an open map $q_i : U_i \to V_i$, called a local uniformization, that induces a homeomorphism from U_i/Γ_i onto V_i.
- For all $x_i \in U_i$ and $x_j \in U_j$ such that $q_i(x_i) = q_j(x_j)$, there exist $W_i \subset U_i$ and $W_j \subset U_j$, open connected neighborhoods of x_i and x_j respectively, and an isometry $\phi_{ji} : W_i \to W_j$, called a change of charts, such that $q_j \phi_{ji} = q_i$ on W_i.

The assumption that the considered smooth orbifold be endowed with a Riemannian metric is not a restriction, since any smooth orbifold admits such a metric. Note further that any open subset U of any n-dimensional Riemannian orbifold, which is defined by an orbifold atlas $\{(U_i, \Gamma_i, q_i)\}_i$, carries an induced n-dimensional Riemannian orbifold structure defined by the atlas $\{(\Omega_i := q_i^{-1}(U \cap V_i), \Gamma_i, q_i|_{\Omega_i})\}_i$.

Definition 11. Let $f : V \to V'$ be a continuous map between two orbifolds V and V'. If for any $x \in V$, there exists a chart (U_i, Γ_i, q_i) around x, i.e. such that $x \in V_i = q_i(U_i)$, a chart (U'_j, Γ'_j, q'_j) around $f(x)$, as well as a function $\tilde{f} \in C^\infty(U_i, U'_j)$, such that $f q_i = q'_j \tilde{f}$, we say that f is a *smooth map*. We denote by $C^\infty(V, V')$ the set of smooth mappings from V to V' and by $\mathrm{Diff}(V, V')$ the set of diffeomorphisms between V and V'.

In particular, a (continuous) function $f : V \to \mathbb{R}$ of an orbifold V is smooth, if for any $x \in V$, there is a chart (U_i, Γ_i, q_i) around x, such that $f q_i \in C^\infty(U_i)$. If U denotes an open subset of V, a (continuous) map $f : U \to \mathbb{R}$ is smooth, if, for any $x \in U$, there exists a chart (U_i, Γ_i, q_i) in the neighborhood of x, such that $f q_i \in C^\infty(q_i^{-1}(U \cap V_i))$. In the following $C^\infty(U)$ denotes the associative commutative algebra of smooth functions on U.

The assumption that $f : V \to \mathbb{R}$ be continuous is redundant here. Indeed, since q_i is surjective, we have $q_i(q_i^{-1} S_i) = S_i$, for any $S_i \subset V_i$. Further, for any open $I \subset \mathbb{R}$, the preimage $q_i^{-1} f|_{V_i}^{-1} I = (f q_i)^{-1} I$ is open and thus $f|_{V_i}^{-1} I = q_i(q_i^{-1} f|_{V_i}^{-1} I)$ is open. Eventually, we get $f^{-1} I = \cup_i f|_{V_i}^{-1} I$ is open in V.

Definition 12. A *differential operator* D of order $k \geq 0$ of an orbifold V is an endomorphism of $C^\infty(V)$, such that we have on all U_i,

$$(Df) q_i = \sum_{|\alpha| \leq k} D_\alpha \, q_i \, \partial_x^\alpha (f q_i),$$

where $D_\alpha \in C^\infty(V_i)$ and where k is independent of the considered chart.

Example. If (U_i, Γ_i, q_i) is an orbifold chart with $q_i(U_i) = V_i$, then the ∂_x^α, $|\alpha| = k \geq 0$, defined for any $f \in C^\infty(V_i)$ by

$$(\partial_x^\alpha f)(q_i(y)) = (\partial_x^\alpha f q_i)(y),$$

$y \in U_i$, is a kth order differential operator of V_i. Indeed, if $x = q_i(gy) \in V_i$, $g \in \Gamma_i$, then g is a diffeomorphism from any neighborhood Ω of y in U_i to the neighborhood $g\Omega$ of gy in U_i, and fq_i has pairwise the same values in Ω and $g\Omega$. It follows that $\partial_x^\alpha f q_i$ associates the same values to y and gy, so that $\partial_x^\alpha f$ is actually a function of V_i. Eventually, a differential operator D of V reads on V_i,

$$Df = \sum_{|\alpha| \leq k} D_\alpha \partial_x^\alpha f.$$

We denote by $\mathscr{D}^k(V)$ the $C^\infty(V)$-module of differential operators of order k of V and by $\mathscr{D}(V) := \bigcup_{i=0}^\infty \mathscr{D}^i(V)$ the Lie algebra of all differential operators of V. As usual, $[\mathscr{D}^i(V), \mathscr{D}^j(V)] \subset \mathscr{D}^{i+j-1}(V)$, so that $\mathscr{D}^1(V)$ is a Lie subalgebra of $\mathscr{D}(V)$ and $C^\infty(V) = \mathscr{D}^0(V) \subset \mathscr{D}^1(V)$ is a Lie ideal of $\mathscr{D}^1(V)$.

Definition 13. The *module of symbols* of degree $k \geq 0$ of V, which we denote by $\mathscr{S}^k(V)$, is equal to $\mathscr{D}^k(V)/\mathscr{D}^{k-1}(V)$. The module $\mathscr{S}(V)$ of all symbols of V is then equal to $\bigoplus_{i=0}^\infty \mathscr{S}^i(V)$.

Definition 14. The *module and Lie algebra of vector fields* of V is given by $\mathrm{Vect}(V) := \mathscr{S}^1(V)$.

Remarks.

- The map $\psi : \mathrm{Vect}(V) \ni [D] \mapsto D - D1 \in \mathscr{D}^1(V)$ is a splitting of the short exact sequence

$$0 \to C^\infty(V) \to \mathscr{D}^1(V) \to \mathrm{Vect}(V) \to 0$$

of $C^\infty(V)$-modules, so that

$$\mathscr{D}^1(V) \simeq C^\infty(V) \oplus \mathrm{Vect}(V).$$

- The local form of a vector field X is

$$Xf = \sum_i X^i \partial_{x^i} f.$$

Definition 15. A *torsion-free connection* ∇ of V is a bilinear map

$$\nabla : \mathrm{Vect}(V) \times \mathrm{Vect}(V) \to \mathrm{Vect}(V),$$

such that

- $\nabla_{fX} Y = f \nabla_X Y$,
- $\nabla_X fY = (Xf)Y + f\nabla_X Y$,
- $\nabla_X Y - \nabla_Y X = [X, Y]$,

for all $X \in \mathrm{Vect}(V)$, $Y \in \mathrm{Vect}(V)$ and $f \in C^\infty(V)$.

We denote by $\mathscr{C}(V)$ the affine subspace of the space of bilinear maps of $\mathrm{Vect}(V)$ that is made up by all torsion-free connections of V.

Definition 16. A *differential one-form* α of V is a linear map from $\mathrm{Vect}(V)$ to $C^\infty(V)$, such that for all $X \in \mathrm{Vect}(V)$, we have on V_i, $\alpha(X) = \sum_j \alpha_j X^j$, where $X = \sum_j X^j \partial_{x^j}$ and $\alpha_j \in C^\infty(V_i)$. We denote by $\Omega^1(V)$ the $C^\infty(V)$-module of differential one-forms of V.

Definition 17. Two torsion-free connections ∇ and ∇' of V are *projectively equivalent* if and only if, for all vector fields $X, Y \in \mathrm{Vect}(V)$,

$$\nabla'_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X,$$

for some one-form α of V.

Definition 18. A *local isometry* between two Riemannian orbifolds V and V' is a smooth map $\varphi \in C^\infty(V, V')$, such that for all $x \in V$, there exists a chart (U_i, Γ_i, q_i) of V, $x \in V_i := q_i(U_i)$, and a chart (U'_j, Γ'_j, q'_j) of V', $V'_j := q'_j(U'_j)$, such that $\varphi \in \mathrm{Diff}(V_i, V'_j)$ admits a lift $\tilde{\varphi} : U_i \to U'_j$, $\varphi q_i = q'_j \tilde{\varphi}$, which is an isometry between the Riemannian manifolds (U_i, h_i) and (U'_j, h'_j), see Definition 10.

In the following definitions φ denotes a local isometry between two Riemannian orbifolds V and V' and notations are those of Definition 18 (possible extensions of these definitions are irrelevant for this paper).

Definition 19. The *pullback of a function* $f \in C^\infty(V'_j)$ is defined by $\varphi^* f := f \circ \varphi \in C^\infty(V_i)$.

Definition 20. The *pullback of a kth order differential operator* $D \in \mathscr{D}^k(V'_j)$, $\varphi^*_{\mathscr{D}} D \in \mathscr{D}^k(V_i)$, is defined by

$$(\varphi^*_{\mathscr{D}} D) f := \varphi^*(D(\varphi^{-1*} f)),$$

for all $f \in C^\infty(V_i)$.

Indeed, we have $\varphi^*_{\mathscr{D}} D \in \mathrm{End}(C^\infty(V_i))$ and, since $(U_i, \Gamma_i, \varphi q_i)$ is a compatible orbifold chart of V'_j, we get on U_i

$$((\varphi^*_{\mathscr{D}} D) f) q_i = (D(f \varphi^{-1})) \varphi q_i = \sum_{|\alpha| \le k} D_\alpha \varphi q_i \, \partial_x^\alpha (f q_i),$$

with $D_\alpha \varphi \in C^\infty(V_i)$.

It is easily checked that $\varphi^*_{\mathscr{D}}$ is a Lie algebra isomorphism between $\mathscr{D}(V'_j)$ and $\mathscr{D}(V_i)$.

Thanks to the fact that $\varphi^*_{\mathscr{D}}$ preserves the order of the differential operators, one can give the following definition:

Definition 21. If $S \in \mathscr{S}^k(V'_j)$ and if $S = [D]$ with $D \in \mathscr{D}^k(V'_j)$, we define the *symbol pullback* of S by

$$\varphi^*_{\mathscr{S}} S := [\varphi^*_{\mathscr{D}} D] \in \mathscr{S}^k(V_i).$$

Definition 22. The *pullback map of vector fields* is

$$\varphi^*_{\mathrm{Vect}} := \varphi^*_{\mathscr{S}}|_{\mathrm{Vect}(V'_j)}.$$

Note first that if we identify the $C^\infty(V)$-module Vect(V) with the submodule $\mathtt{Vect}(V) := \psi(\mathrm{Vect}(V))$ of $\mathscr{D}^1(V)$, see above, we have

$$\varphi^*_{\mathrm{Vect}} = \varphi^*_{\mathscr{D}}|_{\mathtt{Vect}(V'_j)}. \tag{1}$$

It follows immediately from the preceding definitions and the Lie algebra isomorphism property of $\varphi^*_{\mathscr{D}}$ that $\varphi^*_{\mathrm{Vect}}$ is a Lie algebra isomorphism between Vect(V'_j) and Vect(V_i). Further, for any $f \in C^\infty(V'_j)$ and any $X \in \mathrm{Vect}(V'_j)$, we have

$$\varphi^*_{\mathrm{Vect}}(fX) = (\varphi^* f)(\varphi^*_{\mathrm{Vect}} X),$$

and, in view of Equation (1), we also get

$$(\varphi^*_{\mathrm{Vect}} X)(\varphi^* f) = \varphi^*(Xf).$$

Definition 23. The *pullback map of torsion-free connections* $\varphi^*_{\mathscr{C}} : \mathscr{C}(V'_j) \to \mathscr{C}(V_i)$ is defined in this way:

$$(\varphi^*_{\mathscr{C}} \nabla)_X Y := \varphi^*_{\mathrm{Vect}}(\nabla_{\varphi^{-1*}_{\mathrm{Vect}} X} \varphi^{-1*}_{\mathrm{Vect}} Y),$$

for all $\nabla \in \mathscr{C}(V'_j)$, $X, Y \in \mathrm{Vect}(V_i)$.

Remark that the just defined pullback of a torsion-free connection is again a torsion-free connection, due to the preceding properties of the pullback map for vector fields.

Definition 24. A *natural and projectively invariant quantization Q of orbifolds* associates to any Riemannian orbifold V a map

$$Q_V : \mathscr{C}(V) \times \mathscr{S}(V) \to \mathscr{D}(V),$$

such that

- $Q_V(\nabla)$ is a linear bijection between $\mathscr{S}(V)$ and $\mathscr{D}(V)$, such that
$$[Q_V(\nabla)(S)] = S,$$
for all $\nabla \in \mathscr{C}(V)$ and all $S \in \mathscr{S}^k(V)$,
- $Q_V(\nabla) = Q_V(\nabla')$, if ∇ and ∇' are projectively equivalent,
- if $\varphi : V \to V'$ is a local isometry between two Riemannian orbifolds V and V', then
$$Q_{V_i}(\varphi^*_{\mathscr{C}} \nabla)(\varphi^*_{\mathscr{S}} S)(\varphi^* f) = \varphi^* \left(Q_{V'_j}(\nabla)(S)(f) \right),$$
for all $\nabla \in \mathscr{C}(V'_j)$, $S \in \mathscr{S}(V'_j)$, $f \in C^\infty(V'_j)$.

4. Resolution of a Riemannian orbifold

For any n-dimensional Riemannian orbifold V, it is possible to build a foliated manifold $\tilde{V}$, whose leaf space can be identified with V. This construction is explained in details e.g. in [8]. Let us briefly recall it here.

For any local uniformization $q_i : U_i \to V_i$, we denote by $\tilde{U}_i(U_i, \pi_i, O(n))$, where $O(n)$ is the orthogonal group of degree n, the principal bundle of orthonormal frames of the Riemannian manifold (U_i, h_i). The Γ_i-action on U_i lifts in an obvious way to $\tilde{U}_i$: if $\tilde{u}_i = (\tilde{u}_{i,1}, \ldots, \tilde{u}_{i,n}) \in \tilde{U}_i$ is an orthonormal frame over $x_i \in U_i$ and if

$g_i \in \Gamma_i$ is an isometry of (U_i, h_i), then $g_i\tilde{u}_i := (g_{i*}\tilde{u}_{i,1}, \ldots, g_{i*}\tilde{u}_{i,n})$ is an orthonormal frame over $g_i x_i \in U_i$. This lifted action is free, since an isometry is characterized by its derivative at one point (more precisely, the map that associates to any $g_i \in \Gamma_i$ an element $g_i \in \mathrm{Aut}(\tilde{U}_i)$ of the automorphism group of the fiber bundle $\tilde{U}_i$ is a group monomorphism). The quotient $\tilde{V}_i := \tilde{U}_i/\Gamma_i$ is an ordinary smooth manifold. Indeed, as Γ_i is a finite group, its action on $\tilde{U}_i$ is also properly discontinuous.

Similarly, any change of charts $\phi_{ji} : W_i \subset U_i \to W_j \subset U_j$ lifts to a fiber bundle isomorphism

$$\tilde{\phi}_{ji} : \tilde{W}_i \subset \tilde{U}_i \to \tilde{W}_j \subset \tilde{U}_j,\ \tilde{w}_i \mapsto (\phi_{ji*}\tilde{w}_{i,1}, \ldots, \phi_{ji*}\tilde{w}_{i,n}).$$

Define now a projection

$$p_i : \tilde{V}_i \to V_i : [\tilde{u}_i] \mapsto q_i\, \pi_i\, \tilde{u}_i,$$

where $[.]$ denotes of course a class of the quotient $\tilde{V}_i$. It is obviously well-defined. Our goal is to glue the $\tilde{V}_i$ by means of gluing diffeomorphisms

$$\tilde{f}_{ji} : p_i^{-1}(V_{ji}) \subset \tilde{V}_i \to p_j^{-1}(V_{ji}) \subset \tilde{V}_j,$$

where $V_{ji} = V_j \cap V_i$, which verify the usual cocycle condition. Let $[\tilde{u}_i] \in p_i^{-1}(V_{ji})$. Choose a representative $\tilde{u}_i$ (resp. $g_i\tilde{u}_i$), as well as a change of charts $\phi_{ji} : W_i \subset U_i \to W_j \subset U_j$ such that $\pi_i\, \tilde{u}_i \in W_i$ (resp. $\phi'_{ji} : g_i W_i \subset U_i \to W'_j \subset U_j$), and set

$$\tilde{f}_{ji}[\tilde{u}_i] = [\tilde{\phi}_{ji}\, \tilde{u}_i] \in \tilde{V}_j \text{ (resp. } \tilde{f}_{ji}[\tilde{u}_i] = [\tilde{\phi}'_{ji}\, g_i\, \tilde{u}_i] \in \tilde{V}_j).$$

Observe that

$$p_j[\tilde{\phi}_{ji}\tilde{u}_i] = q_j\, \pi_j\, \tilde{\phi}_{ji}\, \tilde{u}_i = q_j\, \phi_{ji}\, \pi_i\, \tilde{u}_i = q_i\, \pi_i\, \tilde{u}_i = p_i[\tilde{u}_i] \in V_{ji}, \tag{2}$$

and that the map $\tilde{f}_{ji}$ is well-defined, since the two chart changes ϕ_{ji} and $\phi'_{ji}\, g_i$ defined on W_i coincide up to $g_j \in \Gamma_j$. Eventually, it is well-known that the chart changes ϕ_{ji} verify the cocycle equation $g_{ijk}\phi_{ki} = \phi_{kj}\phi_{ji}$, $g_{ijk} \in \Gamma_k$; this entails that the same equation holds true for the lifts $\tilde{\phi}_{ji}$ and thus that we have $\tilde{f}_{ki} = \tilde{f}_{kj}\tilde{f}_{ji}$. Hence, if we glue the $\tilde{V}_i$ according to the $\tilde{f}_{ji}$, we get a smooth manifold $\tilde{V}$ of dimension $n(n+1)/2$.

Let now $\tilde{V}_i \ni [\tilde{u}_i] \simeq [\tilde{\phi}_{ji}\tilde{u}_i] \in \tilde{V}_j$ be an element of $\tilde{V}$. It follows from Equation (2) that the local projections $p_i : \tilde{V}_i \to V_i$ define a global projection $p : \tilde{V} \to V$. Moreover, the manifold $\tilde{V}$ admits a right $O(n)$-action. Indeed, for any i, the canonical "matrix product" right action of $M \in O(n)$ on an orthonormal frame $\tilde{u}_i \in \tilde{U}_i$ is an orthonormal frame over the same point. Since clearly $(g_i\tilde{u}_i)M = g_i(\tilde{u}_i M)$, this $O(n)$-action on $\tilde{U}_i$ induces an action on $\tilde{V}_i$, given by $[\tilde{u}_i]M := [\tilde{u}_i M]$. Thanks to the fact that we also have $(\tilde{\phi}_{ji}\tilde{u}_i)M = \tilde{\phi}_{ji}(\tilde{u}_i M)$, we get a global $O(n)$-action on $\tilde{V}$. The orbits of this action, which coincide with the fibers of the projection $p : \tilde{V} \to V$, are known to be the leaves of a regular foliation $\mathscr{F}$ on $\tilde{V}$.

We can find an atlas of $\tilde{V}$ made up by charts that are adapted to $\mathscr{F}$. It suffices to build such an atlas for $\tilde{V}_i = \tilde{U}_i/\Gamma_i$ by means of the general technique for quotients of manifolds by free and properly discontinuous group actions. Let $[\tilde{u}_i] \in \tilde{V}_i$ and let $\tilde{U}$ be a neighborhood of $\tilde{u}_i$ in $\tilde{U}_i$ such that $g_i\tilde{U} \cap \tilde{U} = \emptyset$, for all $g_i \in \Gamma_i$ different from the identity. Such a neighborhood exists since the action of Γ_i is properly discontinuous. We may assume that $\tilde{U}$ is contained in an open of trivialization. For any $[\tilde{u}] \in [\tilde{U}]$, there is a unique representative, say $\tilde{u}$, in $\tilde{U}$. The coordinates of $[\tilde{u}]$ are then $(M_{\tilde{u}}, \pi_i \tilde{u})$, where $M_{\tilde{u}} \in O(n)$ is the orthogonal matrix associated to $\tilde{u}$ via the trivialization. It is a matter of common knowledge that the coordinate systems

$$\psi : [\tilde{U}] \ni [\tilde{u}] \mapsto (M_{\tilde{u}}, \pi_i \tilde{u}) \in O(n) \times \pi_i \tilde{U}$$

form an atlas of $\tilde{V}_i$. Further, they are obviously adapted to $\mathscr{F}$, the transverse coordinates of $[\tilde{u}]$ being the components of $\pi_i \tilde{u}$.

Observe that $p[\tilde{U}] = q_i \pi_i \tilde{U}$ is an open subset of the orbifold V_i defined by the chart (U_i, Γ_i, q_i), so that it is itself an orbifold for the chart $(\Omega_i := q_i^{-1}(q_i \pi_i \tilde{U}), \Gamma_i, q_i|_{\Omega_i})$.

5. Singular quantization

In the following V denotes a Riemannian orbifold and $(\tilde{V}, \mathscr{F})$ is its foliated resolution.

Proposition 2. *The map*

$$p^* : C^\infty(V) \ni f \mapsto f\,p \in C^\infty(\tilde{V}, \mathscr{F})$$

is a linear isomorphism.

Proof. If $f \in C^\infty(V)$, it is clear that $f\,p$ is a foliated function. Since around any point of $\tilde{V}$ there is a chart $([\tilde{U}], \psi)$, such that

$$f\,p\,\psi^{-1} = f\,q_i\,\pi_i\,\psi^{-1} = f\,q_i\,\mathrm{pr}_2,$$

where pr_2 is the projection from $O(n) \times U$ onto U, the function $f\,p$ is also smooth. Conversely, a foliated function gives rise to a function of the leaf space, i.e. to a function of V. □

Proposition 3. *The map*

$$p^*_{\mathscr{D}} : \mathscr{D}^k(V) \ni D \mapsto p^*\,D\,p^{*-1} \in \mathscr{D}^k(\tilde{V}, \mathscr{F})$$

is a linear isomorphism and even a Lie algebra isomorphism between $\mathscr{D}(V)$ and $\mathscr{D}(\tilde{V}, \mathscr{F})$.

Proof. The unique point that requires an explanation is the fact that the conjugate operator has the appropriate local form. This question is actually just a matter of notations. Observe that if the variable $[\tilde{u}]$ runs through an adapted chart domain $[\tilde{U}] \subset \tilde{V}_i$, then $p[\tilde{u}] = q_i \pi_i \tilde{u} =: q_i y$, where the transverse coordinates $y = (y^1, \ldots, y^n)$ run through the corresponding open subset $U \subset U_i$. Further, as aforementioned, a foliated function $g[\tilde{u}]$ factors in the form $\tilde{g}p[\tilde{u}] = \tilde{g}\,q_i y$, where $\tilde{g}$ is a singular function. Hence, if $D \in \mathscr{D}^k(V)$, the value at g of the endomorphism $p^*_{\mathscr{D}} D = p^*\,D\,p^{*-1}$ locally reads

$$(p^*_{\mathscr{D}} D)(g)[\tilde{u}] = D(\tilde{g})\,p[\tilde{u}] = D(\tilde{g})\,q_i y = \textstyle\sum_{|\alpha| \le k} \tilde{D}_\alpha q_i y\ \partial_y^\alpha(\tilde{g}\,q_i y)$$

$$= \textstyle\sum_{|\alpha| \le k} \tilde{D}_\alpha p[\tilde{u}]\,\partial_y^\alpha(\tilde{g}p[\tilde{u}]) = \sum_{|\alpha| \le k} \tilde{D}_\alpha p(M, y)\ \partial_y^\alpha(g(M, y)),$$

where we identified the point $[\tilde{u}]$ with its coordinates (M, y). □

Proposition 4. *The map*

$$p^*_{\mathscr{S}} : \mathscr{S}^k(V) \ni [D] \mapsto [p^*_{\mathscr{D}} D] \in \mathscr{S}^k(\tilde{V}, \mathscr{F})$$

is a linear isomorphism.

Proof. Obvious. □

The restriction of the mapping $p^*_{\mathscr{S}}$ to $\mathscr{S}^1(V)$ is of course a Lie algebra isomorphism p^*_{Vect} between $\mathrm{Vect}(V)$ and $\mathrm{Vect}(\tilde{V}, \mathscr{F})$. Furthermore, just as for the pullback by a local isometry, we have $p^*_{\mathrm{Vect}}(fX) = (p^* f)(p^*_{\mathrm{Vect}} X)$ and $(p^*_{\mathrm{Vect}} X)(p^* f) = p^*(Xf)$, for all $f \in C^\infty(V)$ and all $X \in \mathrm{Vect}(V)$.

Remark : One can easily show that the previous results can be extended to the case where $\tilde{V}$ is replaced by an open set $\tilde{\Omega}$ of $\tilde{V}$ and where V is replaced by $p(\tilde{\Omega})$.

Lemma 5. *There exists a pullback p^*_Ω that maps singular 1-forms of V to foliated 1-forms of $(\tilde{V}, \mathscr{F})$ and verifies*

$$(p^*_\Omega \alpha)(X) = p^*(\alpha(p^{*-1}_{\mathrm{Vect}}(X)),$$

for all $\alpha \in \Omega^1(V)$ and all $X \in \mathrm{Vect}(\tilde{V}, \mathscr{F})$.

Proof. Let $\alpha \in \Omega^1(V)$ and $X \in \mathrm{Vect}(\tilde{V})$. Note that for the moment we do not assume that X is foliated. For any chart $([\tilde{U}], (M, y))$ of $\tilde{V}$ adapted to $\mathscr{F}$, we can apply the preceding pullback results to the orbifold $p[\tilde{U}]$. If X reads $X = \sum_\iota X^\iota \partial_{M^\iota} + \sum_{\mathfrak{i}} X^{\mathfrak{i}} \partial_{y^{\mathfrak{i}}}$ in $[\tilde{U}]$, we thus can set

$$(p^*_\Omega \alpha)(X)|_{[\tilde{U}]} := \sum_{\mathfrak{i}} X^{\mathfrak{i}} p^*(\alpha(p^{*-1}_{\mathrm{Vect}}[\partial_{y^{\mathfrak{i}}}])) \in C^\infty([\tilde{U}]),$$

where the second factors of the RHS are foliated locally defined functions. One can quite easily prove that the functions $(p^*_\Omega \alpha)(X)|_{[\tilde{U}]}$ can be glued and yield a global function $(p^*_\Omega \alpha)(X)$ of $\tilde{V}$, since, if (N, z) are other adapted coordinates, we have $z = z(y)$. It follows that $p^*_\Omega \alpha$ is a differential 1-form of $\tilde{V}$, which is clearly foliated in view of the preceding definition. Observe eventually that for foliated vector fields X, the RHS of the defining equation reads

$$p^*(\alpha(p^{*-1}_{\mathrm{Vect}}[\sum_{\mathfrak{i}} X^{\mathfrak{i}} \partial_{y^{\mathfrak{i}}}])).$$

□

Proposition 6. *The map*

$$p^*_{\mathscr{C}} : \mathscr{C}(V) \ni \nabla \mapsto p^*_{\mathscr{C}} \nabla \in \mathscr{C}(\tilde{V}, \mathscr{F}),$$

where $p^*_{\mathscr{C}} \nabla$ is defined by

$$(p^*_{\mathscr{C}} \nabla)_X Y = p^*_{\mathrm{Vect}}(\nabla_{p^{*-1}_{\mathrm{Vect}} X} p^{*-1}_{\mathrm{Vect}} Y),$$

transforms projective classes of singular torsion-free connections in projective classes of foliated torsion-free connections.

Proof. The result is a consequence of the preceding propositions. □

Theorem 7. *There exists a natural and projectively invariant quantization of orbifolds. If Q denotes this quantization and if V is a Riemannian orbifold, the map Q_V is, for any singular connection $\nabla \in \mathscr{C}(V)$ and any singular symbol $S \in \mathscr{S}(V)$, defined by:*

$$Q_V(\nabla)(S) := p_{\mathscr{D}}^{*-1}\left(\mathscr{Q}(\mathscr{F})(p_{\mathscr{C}}^{*}\nabla)(p_{\mathscr{S}}^{*}S)\right),$$

where $\mathscr{Q}(\mathscr{F})$ is the map associated by the foliated natural and projectively invariant quantization $\mathscr{Q}$ to the foliated manifold $(\tilde{V},\mathscr{F})$.

Proof. The unique required property of Q, which is not obvious in view of the above propositions and of the properties of $\mathscr{Q}$, is its naturality.

Let $\varphi : V \to V'$ be a local isometry between two Riemannian orbifolds V, V' and let $\tilde{\varphi} : U_i \to U'_j$ be the isometry that lifts the diffeomorphism $\varphi : V_i \to V'_j$. Then $\tilde{\varphi}_* : \tilde{U}_i \to \tilde{U}'_j$ is a bundle isomorphism over $\tilde{\varphi}$, which, in view of standard arguments, induces a diffeomorphism $\Phi : \tilde{V}_i \to \tilde{V}'_j$, $\Phi[\tilde{u}_i] = [\tilde{\varphi}_*\tilde{u}_i]$. It follows that

$$p'\Phi[\tilde{u}_i] = q'_j\pi'_j\tilde{\varphi}_*\tilde{u}_i = q'_j\tilde{\varphi}\pi_i\tilde{u}_i = \varphi q_i\pi_i\tilde{u}_i = \varphi p[\tilde{u}_i],$$

so that

$$p'\Phi = \varphi p, \tag{3}$$

where notations are self-explaining. Further, $\Phi : \tilde{V}_i \to \tilde{V}'_j$ is a foliated local diffeomorphism between $(\tilde{V}_i,\mathscr{F})$ and $(\tilde{V}'_j,\mathscr{F}')$. Indeed, it maps any leaf $p^{-1}v_i$, $v_i \in V_i$ of $\mathscr{F}$ into a leaf of $\mathscr{F}'$, since $p'\Phi p^{-1}v_i = \varphi p p^{-1}v_i = \{\varphi v_i\}$.

It is straightforwardly checked that equation (3) entails

$$p^*\varphi^* = \Phi^* p'^*,\ p^*_{\mathscr{S}}\varphi^*_{\mathscr{S}} = \Phi^*_{\mathscr{S}} p'^*_{\mathscr{S}},\ p^*_{\mathscr{C}}\varphi^*_{\mathscr{C}} = \Phi^*_{\mathscr{C}} p'^*_{\mathscr{C}}. \tag{4}$$

The definition of the singular quantization (which implies a similar equation for $Q_V(\nabla)(S)(f)$), the commutation relations (4), and the naturality of the foliated quantization finally show that the singular quantization is natural as well.

□

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