Geometric structures encoded in the Lie structure of an Atiyah algebroid

Janusz Grabowski,* Alexei Kotov, Norbert Poncin †

Abstract

We investigate Atiyah algebroids, i.e. the infinitesimal objects of principal bundles, from the viewpoint of Lie algebraic approach to space. First we show that if the Lie algebras of smooth sections of two Atiyah algebroids are isomorphic, then the corresponding base manifolds are necessarily diffeomorphic. Further, we give two characterizations of the isomorphisms of the Lie algebras of sections for Atiyah algebroids associated to principal bundles with semisimple structure groups. For instance we prove that in the semisimple case the Lie algebras of sections are isomorphic if and only if the corresponding Lie algebroids are, or, as well, if and only if the integrating principal bundles are locally isomorphic. Finally, we apply these results to describe the isomorphisms of sections in the case of reductive structure groups – surprisingly enough they are no longer determined by vector bundle isomorphisms and involve divergences on the base manifolds.

MSC 2000: 17B40, 55R10, 57R50, 58H05 (primary), 16S60, 17B20, 53D17 (secondary)

Keywords: Algebraic characterization of space, Lie algebroid, Atiyah sequence, principal bundle, Lie algebra of sections, isomorphism, semisimple Lie group, reductive Lie group, divergence.

1 Introduction

The concept of groupoid was introduced in 1926 by the German mathematician Heinrich Brandt. A groupoid is a small category in which any morphism is invertible. Topological and differential groupoids go back to Charles Ehresmann [Ehr59] and are transitive Lie groupoids in the sense of A. Kumpera and A. Weinstein: Lie groupoids are groupoids whose classes Γ_0 of objects and Γ_1 of morphisms are not only sets, but manifolds, the source and target maps s and t are submersions, and all operations are smooth; such a groupoid is called transitive, if any two objects are related by a morphism, i.e. if $(s,t) : \Gamma_1 \to \Gamma_0 \times \Gamma_0$ is surjective. The gauge groupoid of a principal G-bundle P, $\Gamma_0 = P/G$ and $\Gamma_1 = (P \times P)/G$, where the G-action on pairs is componentwise, is the prototype of a transitive Lie groupoid; actually, any transitive Lie groupoid can be viewed as the gauge groupoid of a principal bundle [Lib69].

*The research of J. Grabowski was supported by the Polish Ministry of Science and Higher Education under the grant N N201 416839.

†The research of N. Poncin was supported by the University of Luxembourg, grant SGQnp2008.

The first-order invariants of principal bundles or transitive Lie groupoids $P(M,\pi,G)$ are *Atiyah sequences* of vector bundles

$$0 \to K := P \times_G \mathfrak{g} \to A \stackrel{\pi_*}{\to} TM \to 0,$$

together with the corresponding sequences of modules of sections

$$0 \to \mathscr{K} \to \mathscr{A} \to \mathfrak{X}(M) \to 0\,.$$

They were introduced by M. F. Atiyah [Ati57] in order to investigate the existence of complex analytic connections in fiber bundles. Atiyah sequences were referred to as *Atiyah algebroids* after the introduction of Lie algebroids by J. Pradines [Pra67] in order to grasp the structure of the infinitesimal objects that correspond to Lie groupoids; Lie algebroids unify several well-known passages to the infinitesimal level: from foliation to distribution, from Lie group action to Lie algebra action, from principal bundle to Atiyah sequence, from symplectic Lie groupoid to Poisson manifold, etc. Atiyah sequences of algebroids are in particular transitive Lie algebroids, in the sense that their anchor map is surjective. Transitive Lie algebroids need not be Atiyah algebroids: there are *nonintegrable* transitive Lie algebroids, i.e. that are not associated with any principal bundle [AM85, Kub89]. In general, an Atiyah algebroid is a slightly simpler structure than a principal bundle. For example, there exists a nontrivial principal bundle whose Lie algebroid is trivial [Kub89].

Let us also mention that Atiyah algebroids naturally appear in Theoretical Mechanics. Indeed, subsequently to A. Weinstein's work on the unification of internal and external symmetry [Wei96a] and on the groupoid approach to Mechanics [Wei96b], Lagrangian functions on Lie algebroids were investigated; if we consider a Lagrangian with symmetries on a configuration space that is a principal G-bundle P, i.e. a Lagrangian that is invariant under the action of the structure Lie group G, then the Lagrangian function is defined on $A := TP/G$, i.e. on the Atiyah algebroid associated with this principal bundle [LMM05,] This gives rise to different Lie algebroid generalizations of the Lagrange and Hamilton formalisms [Mar01], [GGU06], [GG08].

In this work, *we investigate Atiyah algebroids from the standpoint of Lie algebraic approach to space*. The general result [GG01, Theorem 8] implies that if the Lie algebras of smooth sections of two Atiyah algebroids $(A_i, [-,-]_i, \pi_{i*})$, $i \in \{1,2\}$, over two differentiable manifolds M_i are isomorphic, then the base manifolds M_i are necessarily diffeomorphic, Section 3, Theorem 1; see also [GP07]. We go further and characterize the isomorphisms of the Lie algebras of smooth sections $(\mathscr{A}_i, [-,-]_i)$, $i \in \{1,2\}$, for Atiyah algebroids associated with principal bundles with semisimple structure groups, Section 4, Theorem 2. *They turn out to be associated with Lie algebroid isomorphisms.* To obtain these results, we identify the maximal finite-codimensional Lie ideals of the kernel $\mathscr{K}$ of the corresponding Atiyah sequence of modules and Lie algebras, Section 4, Theorem 3, and describe the elements of $\mathscr{K}$ and $\mathscr{A}$, which vanish at a given point, in pure Lie algebraic terms, Section 4, Theorems 5 and 6. Next, we prove that Lie algebra isomorphisms between sections come from vector bundle isomorphisms and characterize them. Note that the assumption of semisimplicity is essential to prove that a Lie algebra isomorphism for sections is implemented by an isomorphism of the vector bundles, as shows the case of the Atiyah algebroid $TM \times \mathbb{R}$ of first-order differential operators [GP04]. Combining the semisimple case with the case of first-order differential operators, we describe the isomorphisms for reductive structure groups. They no longer come from vector bundle isomorphisms, as first-order components associated with divergences appear in the picture. Denoting by Z the subbundle $\mathsf{Z} \subset K$ associated with the center $Z\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of the structure group G, our main result can be stated as follows.

Theorem 1. *Let $\Phi : \mathscr{A}_1 \to \mathscr{A}_2$ be an isomorphism of the Lie algebras $(\mathscr{A}_i, [-,-]_i)$ of Atiyah algebroids*

$$0 \to K_i \to A_i \stackrel{\pi_{i*}}{\to} TM_i \to 0$$

with connected reductive structure groups G_i over connected manifolds M_i, $i \in \{1,2\}$. Then the Lie algebroids A_i are isomorphic and Φ is a unique composition $\Phi = \Phi_0 \circ \Phi_1$, where Φ_0 is an isomorphism of the Lie algebras $(\mathscr{A}_i, [-,-]_i)$ associated with a Lie algebroid isomorphism and Φ_1 is an automorphism of the Lie algebra $(\mathscr{A}_1, [-,-]_1)$ of the form

$$\Phi_1(a) = a + \mathrm{div}(\pi_{1*}a) \cdot r\,, \tag{1}$$

where $\mathrm{div} : \mathfrak{X}(M_1) \to C^\infty(M_1)$ *is a divergence operator on M_1 and r is a section of Z_1 represented by the fundamental vector field of an element $r \in Z\mathfrak{g}_1$.*

Let us mention that *algebraic characterization of space* can be traced back to Gel'fand and Kolmogoroff – description of isomorphisms between associative algebras of continuous functions on compact sets – and is concerned with characterization of diverse geometric structures by means of various associative or Lie algebras growing on them. It is well known that isomorphisms of the associative algebras of smooth functions living on second countable manifolds are implemented by diffeomorphisms of the underlying manifolds. This was extended to arbitrary smooth manifolds in [Gra05, Mrc05]. The classical result by Pursell and Shanks [PS54], which states that the Lie algebra structure of the space of compactly supported vector fields characterizes the differential structure of the underlying manifold, is the starting point of a multitude of papers: Koriyama, Maeda, Omori (in these papers the authors investigate whether the manifold structure is encoded in various other complete and transitive Lie algebras of vector fields), see e.g. [Omo76], Amemiya, Masuda, Shiga, Duistermaat, Singer, Grabowski, Poncin (algebras of differential and pseudodifferential operators) [AMS75, DS76, GP04], Abe, Atkin, Grabowski, Fukui, Tomita, Hauser, Müller, Rybicki (algebras in the presence of specific geometric structures), Amemiya, Grabowski (real and analytic cases), see e.g. [Gra78], Skryabin (modular Lie algebras) [Skr87], and Grabowska, Grabowski (Lie algebras associated with Lie algebroids) [GG01]. Our current work fits into line with the preceding papers.

2 Atiyah algebroids, Lie algebra bundles, and Lie algebroids

To ensure independent readability of the present text, we recall some facts about Atiyah and Lie algebroids.

Remark 1. In this article all manifolds are smooth second countable paracompact Hausdorff manifolds of finite and nonzero dimension.

2.1 Atiyah algebroids

Let $P(M,\pi,G)$ be a principal G-bundle $\pi : P \to M$. Set $\mathfrak{g} = \mathrm{Lie}(G)$ and denote by $r_g : P \ni u \to u.g \in P$ the diffeomorphism that is induced by the right action of $g \in G$ on P. The quotient TP/G of the tangent bundle TP by the lifted G-action Tr_g, $g \in G$, is a vector bundle over $P/G \simeq M$, which we denote by A and that is locally diffeomorphic to $A|_U \simeq TU \times \mathfrak{g}$, where U is a chart domain of M over which P is trivial. Sections of A are canonically identified with G-invariant vector fields on P. As G-invariant vector fields are closed with respect to the Lie bracket and are projectable onto M, we have a canonically implemented Lie bracket $[-,-]$ on sections of A and a surjective bundle map $\pi_* : A \to TM$ over the identity, the *anchor* map, which turn the vector bundle A into a Lie algebroid $(A, [-,-], \pi_*)$, called the *Atiyah algebroid* associated with the principal bundle $P(M,\pi,G)$.

Since the anchor map is surjective (the Atiyah algebroid is a *transitive Lie algebroid*) we have the short exact *Atiyah sequence* of vector bundles and bundle maps

$$0 \to K = P \times_G \mathfrak{g} \to A \stackrel{\pi_*}{\to} TM \to 0\,, \tag{2}$$

which is also an exact sequence of morphisms of Lie algebroids.

Let us roughly explain why the kernel $K := \ker \pi_*$ coincides with the associated vector bundle $P \times_G \mathfrak{g}$ over M. At each point $u \in P$ the vertical tangent vectors coincide with the fundamental vectors X^h_u, $h \in \mathfrak{g}$, of the action. Since for each $u \in P$, the map $h \in \mathfrak{g} \to X^h_u \in V_u$ is a vector space isomorphism between the space $\mathfrak{g}$ and the space V_u of vertical vectors at u, and as

$$T_u r_g \, X^h_u = X^{\mathrm{Ad}(g^{-1})h}_{u\cdot g},$$

where Ad is the adjoint representation of G on the vector space $\mathfrak{g}$, the aforementioned equivalence relation identifies $(u,h) \simeq (u.g, \mathrm{Ad}(g^{-1})h)$. Hence, the kernel K is actually the associated vector bundle $P \times_G \mathfrak{g}$.

As soft sheaves over paracompact spaces are acyclic, a short exact sequence of vector bundles over M induces a short exact sequence of the $C^\infty(M)$-modules of sections. Since the module $\mathfrak{X}_G(P)$ of G-invariant vector fields of P coincides with the module $\mathscr{A}$ of sections of A, this new sequence is

$$0 \to \mathscr{K} := \Gamma(K) \simeq C^\infty(P,\mathfrak{g})^G \to \mathscr{A} := \Gamma(A) \simeq \mathfrak{X}_G(P) \stackrel{\pi_*}{\to} \mathscr{V} := \mathfrak{X}(M) \to 0, \tag{3}$$

where $C^\infty(P,\mathfrak{g})^G$ is the module of G-equivariant smooth functions from P to $\mathfrak{g}$, and where $\mathfrak{X}(M)$ is the module of vector fields of M. Sequence (3) is also a short exact sequence of Lie algebras.

It is well-known and easily checked that the Lie bracket of any Lie algebroid is local and that a Lie algebroid thus restricts to a Lie algebroid over any open subset of the base of the initial bundle. Let now $U \subset M$ be simultaneously a chart domain of M and a trivialization domain of P: $A|_U \simeq TU \times \mathfrak{g}$ and $\Gamma(A|_U) \simeq \mathfrak{X}(U) \times C^\infty(U,\mathfrak{g})$. As $\pi|_U : P|_U \simeq U \times G \ni (m,g) \to m \in U$, one immediately sees that $\pi_*|_U : TU \times \mathfrak{g} \to TU$ is the projection onto the first factor. Hence, if $(v_1,\gamma_1),(v_2,\gamma_2) \in \mathfrak{X}(U) \times C^\infty(U,\mathfrak{g})$ are sections of A over U, the Lie algebra homomorphism property of the restriction algebroid shows that the first component of their bracket is $[v_1,v_2]_{\mathfrak{X}(U)}$. Further, it follows from the compatibility condition between the module and the Lie structures in a Lie algebroid, that in the kernel $\mathscr{K}$ the Lie bracket is $C^\infty(M)$-bilinear, and thus provides a bracket in each fiber $K_m \simeq \mathfrak{g}$ of $K = P \times_G \mathfrak{g}$. This bracket $[k_m, k'_m] = [k,k']_m$, $k,k' \in \mathscr{K}$, is actually the bracket of $\mathfrak{g}$ [Mac05], so that K becomes a Lie algebra bundle (LAB). Hence, the value at $m \in U$ of the bracket $[(0,\gamma_1),(0,\gamma_2)]$ of the elements $(0,\gamma_i) \in \Gamma(K|_U)$ coincides with the bracket $[\gamma_{1m},\gamma_{2m}]_{\mathfrak{g}}$. Since the module-Lie compatibility condition shows that for any $f \in C^\infty(U)$, we have

$$C^\infty(U,\mathfrak{g}) \ni [(v_1,0),(0,f\gamma_2)] = f[(v_1,0),(0,\gamma_2)] + v_1(f)\gamma_2,$$

we finally realize that

$$[(v_1,\gamma_1),(v_2,\gamma_2)] = ([v_1,v_2]_{\mathfrak{X}(U)}, [\gamma_1,\gamma_2]_{\mathfrak{g}} + v_1(\gamma_2) - v_2(\gamma_1)), \tag{4}$$

where the $\mathfrak{g}$-bracket is computed pointwise, although the exact proof of the last equation is highly nontrivial. This semidirect product and the aforementioned projection onto the first factor define on $TU \times \mathfrak{g}$ a Lie algebroid structure that is called *trivial Lie algebroid* on U with structure algebra $\mathfrak{g}$ (the direct product would not define a Lie algebroid).

2.2 Lie algebra bundles

In the following, K denotes an arbitrary LAB with typical fiber $\mathfrak{g}$ over a manifold M. First remember, see [Mac05, Prop. 3.3.9], that if $\mathfrak{h}$ denotes a Lie subalgebra of $\mathfrak{g}$, there is a sub-LAB H of K with typical fiber $\mathfrak{h}$, whose LAB-atlas is obtained by restriction of the LAB-atlas of K, on the condition that $\mathfrak{h}$ is preserved by any automorphism of $\mathfrak{g}$ – so that the restriction of the transition cocycle of K provides a transition cocycle of H. If $\mathfrak{h}$ is the center $Z\mathfrak{g}$ of $\mathfrak{g}$ (resp. the derived ideal $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$), the corresponding sub-LAB ZK (resp. $[K,K]$) is called the *center sub-LAB* of K (resp. the *derived sub-LAB* of K) and it is denoted shortly by Z (resp. $K^{(1)}$). Moreover, we denote by $\mathscr{K}$ the module $\Gamma(K)$ of smooth global sections of K.

It is known and easily seen that

$$\Gamma(\mathsf{Z}) = Z\Gamma(K) =: \mathscr{Z} \tag{5}$$

(resp. that

$$\Gamma([K,K]) = [\Gamma(K),\Gamma(K)] = [\mathscr{K},\mathscr{K}] \;\;). \tag{6}$$

To understand for instance why $\Gamma([K,K]) \subset [\mathscr{K},\mathscr{K}]$, denote by g_i some global generators of the module $\Gamma(K)$ over $C^\infty(M)$, observe that for any open subset $U \subset M$ any element of $[\Gamma(K|_U),\Gamma(K|_U)]$ reads

$$\sum [k'_U, k''_U] = \sum [k'_U, \sum_i k''^{i}_U g_i|_U] = \sum_i [\sum k''^{i}_U k'_U, g_i|_U] =: \sum_i [k^i_U, g_i|_U], \tag{7}$$

with $k^i_U \in \Gamma(K|_U)$. Take now a partition of unity $(U_\alpha,\varphi_\alpha)$ that is subordinate to a cover by local trivializations of K and let $\delta \in \Gamma([K,K])$. Since $[K,K]|_{U_\alpha}$ is diffeomorphic to $U_\alpha \times [\mathfrak{g},\mathfrak{g}]$, where $\mathfrak{g}$ is finite-dimensional, we have

$$\Gamma([K,K]|_{U_\alpha}) = C^\infty(U_\alpha,[\mathfrak{g},\mathfrak{g}]) = [C^\infty(U_\alpha,\mathfrak{g}), C^\infty(U_\alpha,\mathfrak{g})] = [\Gamma(K|_{U_\alpha}),\Gamma(K|_{U_\alpha})]. \tag{8}$$

When combining the equations (8) and (7), we get

$$\delta = \sum_\alpha \varphi_\alpha \delta|_{U_\alpha} = \sum_i [\sum_\alpha \varphi_\alpha k^i_{U_\alpha}, g_i],$$

with obvious notations. The conclusion $\Gamma([K,K]) \subset [\Gamma(K),\Gamma(K)]$ follows. If $\mathfrak{g}$ is semisimple, Equation (8) shows that locally we have $\Gamma(K|_{U_\alpha}) = [\Gamma(K|_{U_\alpha}),\Gamma(K|_{U_\alpha})]$. When combining this conclusion again with Equation (7), but now for $\delta \in \Gamma(K)$, we see just as before that $\Gamma(K) \subset [\Gamma(K),\Gamma(K)]$. In the semisimple case we thus get

$$\mathscr{K} = \mathscr{K}^{(1)} = [\mathscr{K},\mathscr{K}]. \tag{9}$$

Finally, a LAB K is said to be *reductive*, if all its fibers are, hence, if its typical fiber $\mathfrak{g}$ is. In that case, the structure of the LAB is, exactly as in Lie algebra theory,

$$K = \mathsf{Z} \oplus [K,K]. \tag{10}$$

2.3 Lie algebroids

If $(L,[-,-],\rho)$ and $(L',[-,-]',\rho')$ are two Lie algebroids over two manifolds M and M' respectively, a Lie algebroid morphism F should of course in particular be a vector bundle map $F : L \to L'$ over some smooth map $f : M \to M'$. Since, if f is not a diffeomorphism, such a bundle morphism does not necessarily induce a morphism $F : \mathscr{L} \to \mathscr{L}'$ between the corresponding modules of sections $\mathscr{L} := \Gamma(L)$, $\mathscr{L}' := \Gamma(L')$, the definition of a Lie algebroid morphism is nontrivial in the general situation. However, if $f : M \to M'$ is a diffeomorphism, and especially in the base-preserving case where f is just identity, a *Lie algebroid morphism* can be naturally defined as a vector bundle map $F : L \to L'$ that verifies $\rho' \circ F = f \circ \rho$ and $F[\ell_1,\ell_2] = [F\ell_1,F\ell_2]'$, for any $\ell_1,\ell_2 \in \mathscr{L}$. Of course, such a morphism is called a *Lie algebroid isomorphism*, if $F : L \to L'$ is a vector bundle isomorphism.

It is well known that any short exact sequence of vector spaces and linear maps splits. The same is true for a short exact sequence of vector bundles

$$0 \to E \stackrel{i}{\to} F \stackrel{\rho}{\to} G \to 0$$

over a same manifold and vector bundle maps that cover the identity (it suffices to consider a smooth Riemannian metric in F). In the following, we systematically assume for simplicity reasons that $E \subset F$ is a vector subbundle of F and that i is just the inclusion.

Consider now a transitive Lie algebroid $(L,[-,-],\rho)$ over M and let

$$0 \to K \stackrel{i}{\to} L \stackrel{\rho}{\to} TM \to 0$$

be the corresponding short exact sequence of vector bundles – then K is known to be a LAB. It is customary to refer to a right inverse bundle map or right splitting θ (resp. left inverse bundle map or left splitting j) as a *Lie algebroid connection* of L (resp. *connection form* of L). The point is of course that if the investigated algebroid is the Atiyah algebroid $(A,[-,-],\pi_*)$ of a principal bundle $P(M,\pi,G)$, there is a 1-to-1 correspondence between Lie algebroid connections of A (resp. connection forms of A) and connections of the principal bundle P (resp. connection 1-forms of P) in the traditional sense. The *curvature R_θ of a transitive Lie algebroid connection* θ measures the Lie algebra morphism default of θ, i.e. for any vector fields $X,Y \in \mathfrak{X}(M)$, we set

$$R_\theta(X,Y) := [\theta X,\theta Y] - \theta[X,Y] \in \mathscr{K}.$$

Since R_θ is, as easily seen, $C^\infty(M)$-bilinear, it defines a vector bundle map $R_\theta : TM \times TM \to K$.

An *ideal of a transitive Lie algebroid*

$$0 \to K \xrightarrow{i} L \xrightarrow{\rho} TM \to 0$$

is a sub-LAB H of K, such that, for any $h \in \mathscr{H}$ and $\ell \in \mathscr{L}$, we have $[h,\ell] \in \mathscr{H}$. For instance, the LAB K itself, its center sub-LAB Z and its derived sub-LAB $[K,K]$ are Lie algebroid ideals of L.

Let H be any Lie algebroid ideal of L and consider the short exact sequence of vector bundles

$$0 \to H \xrightarrow{\iota} L \xrightarrow{p} L/H \to 0. \tag{11}$$

Since

$$0 \to \mathscr{H} \xrightarrow{\iota} \mathscr{L} \xrightarrow{p} \Gamma(L/H) \to 0$$

is a short exact sequence of $C^\infty(M)$-modules, the module morphism p induces a canonical isomorphism of modules between $\mathscr{L}/\mathscr{H}$ and $\Gamma(L/H)$. This isomorphism allows transferring the Lie algebra structure $[-,-]^{\tilde{}}$ of $\mathscr{L}/\mathscr{H}$ to $\Gamma(L/H)$. As the vector bundle map $\rho : L \to TM$ factors through the quotient L/H, $\rho = \tilde{\rho} \circ p$, the triplet $(L/H, [-,-]^{\tilde{}}, \tilde{\rho})$ is a transitive Lie algebroid over M – the *quotient Lie algebroid* of L over the Lie algebroid ideal H – and $\ker \tilde{\rho} = K/H$. Moreover, it is clear that the sequence (11) is a short exact sequence of Lie algebroids and Lie algebroid morphisms.

3 Lie algebras of Atiyah algebroids

In the following, we compendiously investigate whether an isomorphism between the Lie algebras of sections of two Atiyah algebroids induces a diffeomorphism between the underlying manifolds.

Consider any smooth Lie algebroid $(L, [-,-], \widehat{-}\,)$ over a smooth manifold M, and set, in order to simplify notations, $\mathscr{N} = C^\infty(M)$ and $\mathscr{L} = \Gamma(L)$. We say that the algebroid L is *strongly nonsingular*, if $\mathscr{N} = \widehat{\mathscr{L}}(\mathscr{N}) := \operatorname{span}\{\widehat{\ell}(f) : \ell \in \mathscr{L}, f \in \mathscr{N}\}$. As, in view of the Serre-Swan theorem, the $\mathscr{N}$-module $\mathscr{L}$ can be characterized as a projective module with a finite number of generators, the module of vector fields $\widehat{\mathscr{L}}$ is finitely generated and the strong nonsingularity can be characterized in geometric terms as the fact that there is a finite number of vector fields from $\widehat{\mathscr{L}}$ which do not all vanish at a single point [Gra93]. In [GG01, Theorem 8] it has been proved that under these conditions isomorphisms between the Lie algebras of Lie algebroids induce diffeomorphisms of the underlying manifolds.

To fix notation for further purposes, let us briefly sketch the proof of this fact for Atiyah algebroids. Denote by $I(\mathscr{N})$ (resp. $S(\mathscr{L})$) the set of all maximal finite-codimensional associative ideals of $\mathscr{N}$ (resp. the set of all maximal finite-codimensional Lie subalgebras of $\mathscr{L}$). It was shown in [GG01, Corollary 7] that, in the strongly nonsingular case, we can pass from the associative to the Lie setting via the action of vector fields on functions, i.e., more precisely, that the map

$$I(\mathscr{N}) \ni J \xrightarrow{\simeq} \mathscr{L}_J := \{\ell \in \mathscr{L} : \widehat{\ell}(\mathscr{N}) \subset J\} \in S(\mathscr{L})$$

is a bijection. Since the maximal finite-codimensional ideals of $\mathscr{N}$ "are" exactly the points of M, i.e. as the map

$$M \ni m \xrightarrow{\simeq} J(m) := \{f \in \mathscr{N} : f(m) = 0\} \in I(\mathscr{N}) \tag{12}$$

is bijective [Gra78, Proposition 3.5], we get a 1-to-1 correspondence

$$M \ni m \stackrel{\simeq}{\longrightarrow} \mathscr{L}_m := \mathscr{L}_{J(m)} = \{\ell \in \mathscr{L} : \widehat{\ell}(f)(m) = 0, \forall f \in \mathscr{N}\} = \{\ell \in \mathscr{L} : \widehat{\ell}_m = 0\} \in S(\mathscr{L}).$$

Let $(A, [-,-], \pi_*)$ be an Atiyah algebroid over a manifold M. We use the above notations; further, we set for convenience, $\widehat{-} := \pi_*$. It follows from transitivity that $\widehat{\mathscr{A}} = \mathscr{V} := \mathfrak{X}(M)$, so that A is strongly nonsingular. Since this observation entails that

$$M \ni m \stackrel{\simeq}{\longrightarrow} \mathscr{A}_m = \{a \in \mathscr{A} : \widehat{a}_m = 0\} \in S(\mathscr{A}),$$

the kernel $\mathscr{K}$ of the Atiyah sequence of $C^\infty(M)$-modules and Lie algebras of sections associated with A reads

$$\mathscr{K} = \cap_{m \in M} \{a \in \mathscr{A} : \widehat{a}_m = 0\} = \cap_{T \in S(\mathscr{A})} T.$$

If $\Phi : \mathscr{A}_1 \stackrel{\simeq}{\longrightarrow} \mathscr{A}_2$ is a Lie algebra isomorphism, we have $\Phi(\mathscr{K}_1) = \cap_{T \in S(\mathscr{A}_1)} \Phi(T) = \mathscr{K}_2$, since Φ maps maximal finite-codimensional Lie subalgebras into maximal finite-codimensional Lie subalgebras. Hence, Φ induces an isomorphism $\widetilde{\Phi}$ between the quotient Lie algebras $\mathscr{A}_i / \mathscr{K}_i \simeq \mathscr{V}_i$, $i \in \{1,2\}$. Such an isomorphism however, is implemented by a diffeomorphism $\varphi : M_1 \stackrel{\simeq}{\longrightarrow} M_2$ [Gra78, Corllary 5.8], $(\widetilde{\Phi} v) f = (v(f \circ \varphi)) \circ \varphi^{-1}$, for any $v \in \mathscr{V}_1$ and any $f \in \mathscr{N}_2$. Hence, we can formulate the following Pursell-Shanks type theorem which is a particular case of [GG01, Theorem 8].

Theorem 2. *Let $(A_i, [-,-]_i, \pi_{i*})$ be a smooth Atiyah algebroid over a smooth manifold M_i, $i \in \{1,2\}$. Any isomorphism $\Phi : \mathscr{A}_1 \stackrel{\simeq}{\longrightarrow} \mathscr{A}_2$ between the Lie algebras of smooth sections restricts to an isomorphism $\Phi : \mathscr{K}_1 \stackrel{\simeq}{\longrightarrow} \mathscr{K}_2$ between the kernels of the anchor maps and induces an isomorphism $\widetilde{\Phi} : \mathscr{V}_1 \stackrel{\simeq}{\longrightarrow} \mathscr{V}_2$ between the Lie algebras of vector fields of the base manifolds, which is implemented by a diffeomorphism $\varphi : M_1 \stackrel{\simeq}{\longrightarrow} M_2$, i.e. $\widetilde{\Phi} = \varphi_*$. Furthermore, $\Phi(\mathscr{A}_{1m}) = \mathscr{A}_{2\varphi(m)}$.*

Proof. We only need prove the last claim. It is clear that $\Phi(\mathscr{A}_{1m}) = \mathscr{A}_{2n}$, for some $n \in M_2$. Since the abovementioned isomorphism between $\mathscr{A}_i / \mathscr{K}_i$ and $\mathscr{V}_i$ is $\widetilde{\pi}_{i*} : \mathscr{A}_i / \mathscr{K}_i \ni [a] \stackrel{\simeq}{\longrightarrow} \pi_{i*} a \in \mathscr{V}_i$, we have in fact $\widetilde{\pi}_{2*} \widetilde{\Phi} = \varphi_* \widetilde{\pi}_{1*}$, so that $\widehat{\Phi(a)} = \widetilde{\pi}_{2*} \widetilde{\Phi}[a] = \varphi_* \widehat{a}$. When evaluating both sides of this equation at $\varphi(m)$, we get

$$\widehat{\Phi(a)}_{\varphi(m)} = (\varphi_{*m}) \widehat{a}_m.$$

Hence the result. □

Let us remark that the existence of the Lie algebra isomorphism between $\mathscr{K}_1$ and $\mathscr{K}_2$ does not *a priori* mean that the structure Lie algebras (structure groups) are (locally) isomorphic. We will show later that this isomorphism holds under additional assumptions. Note that the Lie algebras of sections of two commutative Lie algebroids can be isomorphic even when the base manifolds are nondiffeomorphic and the ranks of vector bundles are different.

4 Isomorphisms of Lie algebras of Atiyah algebroids - semisimple structure groups

In this section we take an interest in a possible characterization of isomorphisms $\Phi : \mathscr{A}_1 \stackrel{\simeq}{\longrightarrow} \mathscr{A}_2$ of the Lie algebras of smooth sections $\mathscr{A}_i$ of Atiyah algebroids $(A_i, [-,-]_i, \pi_{i*})$ associated with principal bundles $P_i(M_i,\pi_i,G_i)$, $i\in\{1,2\}$. It seems natural to think that Φ induces a vector bundle isomorphism $\phi : A_1 \xrightarrow{\simeq} A_2$ over a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$ that implements $\widetilde{\Phi} : \mathscr{V}_1 \xrightarrow{\simeq} \mathscr{V}_2$. However, the preceding guess is not true in general, as shows the example of the Atiyah algebroid $TM\times\mathbb{R}$ of first-order differential operators on a manifold M [GP04]. Note that $TM\times\mathbb{R}$ is isomorphic to the Lie algebroid of linear differential operators acting on smooth sections of a real vector bundles of rank 1 over M independently whether the line bundle is trivial or not [GP07]. We can however build the mentioned vector bundle isomorphism under the additional assumption that the structure groups G_i are semisimple.

We are now prepared to state the main result of this section, which yields in particular that the Lie algebra structure of the space of sections of an Atiyah algebroid recognizes not only the smooth structure of the base manifold but also the vector bundle structure of the algebroid.

Theorem 3. *Let A_i, $i\in\{1,2\}$, be smooth Atiyah algebroids associated with principal bundles with semisimple structure groups. Isomorphisms $\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$ of the Lie algebras $\mathscr{A}_i$ of smooth sections of the bundles A_i are in 1-to-1 correspondence with isomorphisms $\phi : A_1 \xrightarrow{\simeq} A_2$ of the corresponding Lie algebroids.*

Remark 2. Finally, the Lie algebra homomorphism property of Φ might be encoded in the similar property of the dual vector bundle isomorphism $\phi^* : A_2^* \xrightarrow{\simeq} A_1^*$ over φ^{-1} (defined in A^*_{2m} by $\phi^*_m := {}^t\phi_{\varphi^{-1}(m)}$, where notations are self-explaining) with respect to the linear Poisson structure of A_i^* that is associated with the Lie algebroid structure of A_i.

Let us recall the basic results pertaining to the mentioned Poisson structure of the dual vector bundle $\mathfrak{p} : L^* \to M$ of any smooth Lie algebroid $(L,[-,-],\widehat{-})$ over a smooth manifold M. We set again $\mathscr{L} = \Gamma(L)$. The map $-^\bullet : \mathscr{L} \ni \ell \to \ell^\bullet \in C^\infty(L^*)$, which is defined, for any $\lambda \in L^*_m$, $m\in M$, by $\ell^\bullet(\lambda) = \lambda(\ell_m) \in \mathbb{R}$ and associates to any smooth section of L a smooth function of L^* that is linear in the fibers, is visibly an injective and nonsurjective linear mapping. It is well-known, see e.g. [Mar08], that there is a unique Poisson structure $\{-,-\}$ on L^*, such that $-^\bullet$ is a Lie algebra homomorphism, i.e. $\{\ell^\bullet,\ell'^\bullet\} = [\ell,\ell']^\bullet$, for any $\ell,\ell'\in\mathscr{L}$. Of course, this implies that $\{-,-\}$ is a linear Poisson bracket. Moreover, it follows from the module-Lie compatibility condition in the Lie algebroid and the Leibniz property of the Poisson bracket that necessarily $\{\ell^\bullet, g\circ\mathfrak{p}\} = \widehat{\ell}(g)\circ\mathfrak{p}$ and $\{f\circ\mathfrak{p}, g\circ\mathfrak{p}\} = 0$, for any $f,g\in C^\infty(M)$ and any $\ell\in\mathscr{L}$.

Roughly speaking, since a fiber A_{im}, $m\in M_i$, of A_i can be viewed as the quotient space $\mathscr{A}_i/\mathscr{A}_i(m)$, where $\mathscr{A}_i(m) = \{a\in\mathscr{A}_i : a_m = 0\}$, any Lie algebra isomorphism $\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$ induces isomorphisms $\phi_m : A_{1m} \xrightarrow{\simeq} A_{2\varphi(m)}$, where $\varphi : M_1 \xrightarrow{\simeq} M_2$ is the diffeomorphism generated by Theorem 2, if $\mathscr{A}_1(m)$ and $\mathscr{A}_2(\varphi(m))$ can be characterized in Lie algebraic terms.

To simplify notations, we drop in the following index i. Equation (4) implies that the value at $m\in M$ of the bracket $[a,a']$ of two sections $a,a'\in\mathscr{A}$ is given by

$$[a,a']_m = [a|_U, a'|_U]_m = ([v_1,v_2]_{\mathfrak{X}(U)}, [\gamma_1,\gamma_2]_{\mathfrak{g}} + v_1(\gamma_2) - v_2(\gamma_1))_m, \tag{13}$$

where U denotes a chart domain of M and a trivialization domain of the principal bundle $P(M,\pi,G)$, where $\mathfrak{g} = \mathrm{Lie}(G)$, and where $a|_U = (v_1,\gamma_1)$, $a'|_U = (v_2,\gamma_2) \in \mathfrak{X}(U)\times C^\infty(U,\mathfrak{g})$. As $a\in\mathscr{A}(m)$ if and only if $v_{1m} = \gamma_{1m} = 0$ and as $a'\in\mathscr{K}$ entails $0 = (\pi_* a')|_U = \pi_*|_U(v_2,\gamma_2) = v_2$, it follows from

Equation (13) that the $\mathscr{A}(m)$ could be the (maximal) Lie subalgebras of $\mathscr{A}$, such that $[\mathscr{A}(m),\mathscr{K}] \subset \mathscr{K}(m) = \{k \in \mathscr{K} : k_m = 0\}$. Hence, the $\mathscr{A}(m)$ are characterized in Lie algebraic terms, if the $\mathscr{K}(m)$ are. Equation (13) allows seeing that the $\mathscr{K}(m)$ are finite-codimensional Lie ideals of $\mathscr{K}$. The fact that, for any point $m \in M$ and any (maximal) ideal $\mathfrak{g}_0 \subset \mathfrak{g}$, the space $\mathscr{K}(m,\mathfrak{g}_0) := \{k \in \mathscr{K} : k_m \in \mathfrak{g}_0\}$ is a (maximal) finite-codimensional Lie ideal of $\mathscr{K}$, suggests that $\mathscr{K}(m)$ is the intersection of all maximal finite-codimensional Lie ideals $\mathscr{K}(m,\mathfrak{g}_0)$, as, for a semisimple Lie group G, the intersection of all maximal ideals $\mathfrak{g}_0$ of $\mathfrak{g}$ vanishes.

Remark 3. Let us stress that in the remainder of this section the structure group of the principal bundle, which gives rise to the considered Atiyah algebroid, is assumed to be semisimple. If G is a semisimple Lie group, its Lie algebra $\mathfrak{g}$ has no nonzero solvable ideals, its radical $\mathfrak{r}$ vanishes and $\mathfrak{g}^{(1)} = \mathfrak{g}$. The latter clearly entails $\mathscr{K}^{(1)} = \mathscr{K}$, see Equation (9).

Before passing to main theorems, let us start with a series of lemmata.

Lemma 1. *The kernel $\mathscr{K} = \Gamma(P \times_G \mathfrak{g})$ of the Atiyah sequence of modules and Lie algebras, which is implemented by a principal bundle P with semisimple structure group G, is an infinite-dimensional Lie algebra, such that $\mathscr{K}^{(1)} = \mathscr{K}$.*

In the following, a "maximal object with a given property" is an object that is not strictly contained in any proper object with the same property.

Lemma 2. *Let $\mathscr{K}$ be as detailed in Lemma 1. Any maximal finite-codimensional Lie ideal $\mathscr{K}_0$ in $\mathscr{K}$ is modular, i.e. $\mathscr{N}\mathscr{K}_0 = \mathscr{K}_0$.*

Proof. Since
$$[\mathscr{N}\mathscr{K}_0,\mathscr{K}] = [\mathscr{K}_0,\mathscr{N}\mathscr{K}] \subset \mathscr{K}_0 \subset \mathscr{N}\mathscr{K}_0, \tag{14}$$
the space $\mathscr{N}\mathscr{K}_0$ is a finite-codimensional Lie ideal in $\mathscr{K}$. Hence, either $\mathscr{N}\mathscr{K}_0 = \mathscr{K}_0$ or $\mathscr{N}\mathscr{K}_0 = \mathscr{K}$. In the second case, Equation (14) shows that $\mathscr{K}^{(1)} \subset \mathscr{K}_0$ – a contradiction in view of the semisimplicity assumption for G and the maximality assumption for $\mathscr{K}_0$. □

Lemma 3. *Let $\mathscr{N} = C^\infty(M)$ be the associative commutative unital algebra of smooth functions of a manifold M and let $I \subset \mathscr{N}$. Denote by* $\operatorname{Spec}(\mathscr{N},I)$ *the set of all maximal finite-codimensional ideals of $\mathscr{N}$ that contain I and set $\bar{I} = \cap_{J\in \operatorname{Spec}(\mathscr{N},I)} J$. Finally, if I is an ideal in $\mathscr{N}$, let $\sqrt{I} = \{f \in \mathscr{N} : f^n \in I$, for some $n > 0\}$ be the radical ideal of I. If I is s-codimensional in $\mathscr{N}$, $s > 0$, there are points $m_1,\ldots,m_\ell \in M$, $1 \le \ell \le s$, such that*
$$\sqrt{I} = \bar{I} = \cap_{i=1}^{\ell} J(m_i) = \cap_{i=1}^{\ell}\{f \in \mathscr{N} : f(m_i) = 0\}. \tag{15}$$
Moreover, we have $\sqrt{I}^{\,s} \subset I$.

Proof. Since the points $m \in M$ are in 1-to-1 correspondence with the maximal finite-codimensional ideals $J(m) \subset \mathscr{N}$, see Equation (12), we have $\bar{I} = \cap_{m\in M: J(m)\supset I} J(m)$. Since each finite-codimensional ideal is included in at least one maximal finite-codimensional ideal, the preceding intersection is not the intersection of the empty family. Further, as $\mathscr{N}/\cap_{i=1}^{\ell} J(m_i) \simeq \mathbb{R}^\ell$ and as $s = \dim \mathscr{N}/I \ge \dim \mathscr{N}/\bar{I}$, it is clear that I cannot be contained in more than s maximal finite-codimensional ideals, so that $\bar{I} = \cap_{i=1}^{\ell} J(m_i)$, $\ell \in \{1,\ldots,s\}$ and $\operatorname{codim} \bar{I} = \ell \ge 1$.

If the descending series of finite-codimensional ideals

$$\bar{I} = I + \bar{I} \supset I + \bar{I}^2 \supset \ldots \supset I + \bar{I}^s \supset I + \bar{I}^{s+1} \supset I$$

were strictly decreasing (except maybe for the last inclusion), we could not have $\operatorname{codim} \bar{I} \geq 1$. Hence,

$$I + \bar{I}^n = I + \bar{I}^{n+1}, \tag{16}$$

for some $n \in \{1, \ldots, s\}$. Remember now Nakayama's lemma that holds true for any commutative ring R with identity 1, any ideal $\mathfrak{I}$ in R, and any finitely-generated module M over R, and which states that if $\mathfrak{I}M = M$, there is $r \in R$, $r \sim 1$ modulo $\mathfrak{I}$, such that $rM = 0$. If $\mathfrak{I}$ is included in the Jacobson radical of R, i.e. in the intersection of all maximal ideals of R, then r is invertible and $M = 0$. When applying this result to $R = \mathscr{N}/I$, $\mathfrak{I} = \bar{I}/I$, and $M = (I + \bar{I}^n)/I$, where M is actually finite-dimensional and where $\mathfrak{I}M = M$ in view of Equation (16), we get $\bar{I}^n \subset I$, so that $\bar{I} \subset \sqrt{I}$. The fact that for all $J \in \operatorname{Spec}(\mathscr{N}, I)$ the inclusion $\sqrt{I} \subset \sqrt{J} = \sqrt{J(m)} = J(m) = J$ holds true, entails that $\sqrt{I} \subset \bar{I}$. Finally, $\sqrt{I} = \bar{I}$ and $\sqrt{I}^s \subset \sqrt{I}^n = \bar{I}^n \subset I$. □

Theorem 4. *Let $\mathscr{K}$ be the kernel of the Atiyah sequence of modules and Lie algebras associated with a principal bundle with semisimple structure group G over a manifold M. Then, the maximal finite-codimensional Lie ideals of $\mathscr{K}$ are exactly the ideals of the form $\mathscr{K}(m, \mathfrak{g}_0) = \{k \in \mathscr{K} : k_m \in \mathfrak{g}_0\}$, where $m \in M$ and where $\mathfrak{g}_0$ is a maximal Lie ideal of $\mathfrak{g} = \operatorname{Lie}(G)$.*

Proof. Take a maximal finite-codimensional Lie ideal $\mathscr{K}_0$ in $\mathscr{K}$. The $\mathscr{N}$-module structure of $\mathscr{K}_0$, see Lemma 2, allows switching from the Lie algebraic to the associative context and then applying Lemma 3. Indeed, in view of this modularity, the space $I_k := \{f \in \mathscr{N} : fk \in \mathscr{K}_0\}$ is, for any $k \in \mathscr{K}$, an associative ideal in $\mathscr{N}$, which is finite-codimensional as it is the kernel of the linear map $\ell : \mathscr{N} \ni f \to \widetilde{fk} \in \mathscr{K}/\mathscr{K}_0$ with values in a finite-dimensional space. If $\widetilde{k}_1, \ldots, \widetilde{k}_q$ is a basis of the space $\mathscr{K}/\mathscr{K}_0$, the intersection $I = \cap_{i=1}^q I_{k_i}$ is an associative ideal in $\mathscr{N}$, verifies $I\mathscr{K} \subset \mathscr{K}_0$, and is nonzero- and finite-codimensional ($\operatorname{codim} I \leq \sum_{i=1}^q \operatorname{codim} I_{k_i}$). Hence, Lemma 3 is valid for this ideal I; in the sequel, we use the notations of this lemma.

It is obvious that the space $\sqrt{I}\mathscr{K}$, which is made up by finite sums of products fk, $f \in \sqrt{I}$, $k \in \mathscr{K}$, is a Lie ideal in $\mathscr{K}$ and consists of the sections in $\mathscr{K}$ which vanish at the points $m_1, \ldots, m_\ell$ that provide the radical $\sqrt{I}$. Indeed, if we take a partition of unity $(U_\mathfrak{a}, \gamma_\mathfrak{a})_{\mathfrak{a} \in \mathfrak{A}}$ that is subordinate to a cover by domains of local coordinates, such a section is locally, in each $U_\mathfrak{a}$, a combination of local sections and local functions that vanish at the $m_i \in U_\mathfrak{a}$. It suffices then to use the partition $\gamma_\mathfrak{a}$ to show that the considered section is actually an element of $\sqrt{I}\mathscr{K}$.

This characterization of the elements of the Lie ideal $\sqrt{I}\mathscr{K}$ in $\mathscr{K}$ allows proving that there exists a class in the Lie algebra $\mathscr{K}/\sqrt{I}\mathscr{K}$ that does not contain any element of $\mathscr{K}_0$.

Indeed, if any class contained an element of $\mathscr{K}_0$, then, for any $k \in \mathscr{K}$, we would have a series of equations

$$\begin{aligned} k &= k_0 + \textstyle\sum_{i_1} r_{i_1} k_{i_1}, \\ k_{i_1} &= k_{0 i_1} + \textstyle\sum_{i_2} r_{i_1 i_2} k_{i_1 i_2}, \\ &\ldots \\ k_{i_1 \ldots i_{s-1}} &= k_{0 i_1 \ldots i_{s-1}} + \textstyle\sum_{i_s} r_{i_1 \ldots i_s} k_{i_1 \ldots i_s}, \end{aligned}$$

with $k_{0i_1\dots i_{u-1}} \in \mathscr{K}_0$, $r_{i_1\dots i_u} \in \sqrt{I}$, and $k_{i_1\dots i_u} \in \mathscr{K}$. Thus,

$$k = \left(k_0 + \sum_{i_1} r_{i_1} k_{0i_1} + \ldots + \sum_{i_1\dots i_{s-1}} r_{i_1} \ldots r_{i_1\dots i_{s-1}} k_{0i_1\dots i_{s-1}} \right) + \sum_{i_1\dots i_s} r_{i_1} \ldots r_{i_1\dots i_s} k_{i_1\dots i_s},$$

and, since $\mathscr{K}_0$ is modular in view of Lemma 2, the parenthesis in the RHS of this equation is an element of $\mathscr{K}_0$, whereas the last term is in $\mathscr{K}_0$ as well, since $\sqrt{I}^s \subset I$, due to Lemma 3, and $I\mathscr{K} \subset \mathscr{K}_0$. It follows that $\mathscr{K} \subset \mathscr{K}_0$ - a contradiction, since $\mathscr{K}_0$ is maximal by assumption.

We just proved that there exists at least one element $k \in \mathscr{K}$, such that for any $k_0 \in \mathscr{K}_0$, we have $k - k_0 \notin \sqrt{I}\mathscr{K}$. But then, there is at least one point $m \in M$, such that the space $\mathfrak{g}_0 := \{k_{0m} : k_0 \in \mathscr{K}_0\}$ is a proper subspace of $\mathfrak{g}$.

In fact, otherwise $\mathfrak{g}_0$ would vanish for any $m \in M$ or would coincide with $\mathfrak{g}$ for any $m \in M$. As the first alternative is impossible, it follows that for any $m \in M$ and any $\kappa \in \mathscr{K}$, there exists $\kappa_0 \in \mathscr{K}_0$, such that $\kappa_m = \kappa_{0m}$. This assertion holds true in particular for the points $m_1, \ldots, m_\ell \in M$ and the above section $k \in \mathscr{K}$; there are $k_{0i} \in \mathscr{K}_0$, such that $k_{m_i} = k_{0i,m_i}$. Take now open subsets $U_i \subset M$ that contain m_i and have pairwise empty intersections, as well as bump functions $\alpha_i \in \mathscr{N}$ with value 1 in a neighborhood of m_i and compact support in U_i, and set finally $k_0 = \sum_i \alpha_i k_{0i} \in \mathscr{K}_0$. It is clear that $k_{0,m_j} = k_{m_j}$, so that $k - k_0 \in \mathscr{K}$ vanishes at $m_1, \ldots, m_\ell$ and is thus an element of $\sqrt{I}\mathscr{K}$. Since this is impossible, there actually exists at least one point $m \in M$, such that $\mathfrak{g}_0 := \{k_{0m} : k_0 \in \mathscr{K}_0\}$ is proper in $\mathfrak{g}$.

It is readily checked that the subspace $\mathfrak{g}_0$ is a Lie ideal in $\mathfrak{g}$. Note further that for any point $p \in M$ and any Lie ideal $\mathfrak{h}_0 \subset \mathfrak{g}$, the subspace $\mathscr{K}(p, \mathfrak{h}_0) := \{k \in \mathscr{K} : k_p \in \mathfrak{h}_0\}$ is a Lie ideal in $\mathscr{K}$, and that this ideal is finite-codimensional since the space $\mathscr{K}/\mathscr{K}(p, \mathfrak{h}_0)$ is isomorphic to $\mathfrak{g}/\mathfrak{h}_0$. As obviously $\mathscr{K}_0 \subset \mathscr{K}(m, \mathfrak{g}_0)$, the preceding observation entails that the maximal finite-codimensional ideal $\mathscr{K}_0$ is included in the finite-codimensional ideal $\mathscr{K}(m, \mathfrak{g}_0)$, which is in turn strictly included in $\mathscr{K}$, since $\mathfrak{g}_0$ is proper in $\mathfrak{g}$. Hence, $\mathscr{K}_0 = \mathscr{K}(m, \mathfrak{g}_0)$. Finally, the ideal $\mathfrak{g}_0$ is maximal in $\mathfrak{g}$; otherwise, it would be strictly included in a proper ideal $\mathfrak{h}_0$, so that $\mathscr{K}_0 = \mathscr{K}(m, \mathfrak{g}_0)$ would on his part be strictly included in the proper finite-codimensional ideal $\mathscr{K}(m, \mathfrak{h}_0)$. This concludes the proof of the first part of Theorem 4.

Let now $m \in M$ and let $\mathfrak{g}_0$ be a maximal Lie ideal in $\mathfrak{g}$. We already mentioned that $\mathscr{K}(m, \mathfrak{g}_0)$ is then a finite-codimensional Lie ideal in $\mathscr{K}$. If this ideal is not maximal it is strictly included in a proper finite-codimensional ideal $\mathscr{K}_1$, which can of course be assumed to be maximal. But in this case, the first part of the theorem implies that $\mathscr{K}_1 = \mathscr{K}(p, \mathfrak{h}_0)$, for some $p \in M$ and some maximal ideal $\mathfrak{h}_0$ in $\mathfrak{g}$. Hence, $k_m \in \mathfrak{g}_0$ entails $k_p \in \mathfrak{h}_0$, for any $k \in \mathscr{K}$. From this it first follows that $m = p$. Indeed, otherwise we take $k \in \mathscr{K}$, such that $k_p \notin \mathfrak{h}_0$, as well as a bump function $\alpha \in C^\infty(M)$ that has value 0 in a neighborhood of m and value 1 in some neighborhood of p. The fact that $(\alpha k)_m \in \mathfrak{g}_0$ and $(\alpha k)_p \notin \mathfrak{h}_0$ then constitutes a contradiction. We now see that the maximal ideal $\mathfrak{g}_0$ is included in the (maximal and thus) proper ideal $\mathfrak{h}_0$. This shows that $\mathfrak{g}_0 = \mathfrak{h}_0$ and, since $m = p$, that $\mathscr{K}(m, \mathfrak{g}_0) = \mathscr{K}(p, \mathfrak{h}_0) = \mathscr{K}_1$, so that $\mathscr{K}(m, \mathfrak{g}_0)$ is actually maximal. □

The space $\mathscr{K}(m, \mathfrak{g}_0)$ is a maximal finite-codimensional Lie ideal in the Lie ideal $\mathscr{K}$ in $\mathscr{A}$. The next proposition provides the normalizer $N(\mathscr{A}, \mathscr{K}(m, \mathfrak{g}_0))$ of $\mathscr{K}(m, \mathfrak{g}_0)$ in $\mathscr{A}$, i.e. the biggest Lie subalgebra of $\mathscr{A}$ that admits $\mathscr{K}(m, \mathfrak{g}_0)$ as a Lie ideal.

Proposition 1. *The normalizer in $\mathscr{A}$ of any maximal finite-codimensional Lie ideal $\mathscr{K}(m,\mathfrak{g}_0)$ in $\mathscr{K}$ coincides with the maximal finite-codimensional Lie subalgebra $\mathscr{A}_m = \{a \in \mathscr{A} : \widehat{a}_m = 0\}$.*

Proof. Let us look for all the $a \in \mathscr{A}$, such that $[a,k_0] \in \mathscr{K}(m,\mathfrak{g}_0)$, for any $k_0 \in \mathscr{K}(m,\mathfrak{g}_0)$. Since this condition only involves the value of the local bracket $[-,-]$ at the point m, we consider a chart domain U of M around m that is simultaneously a trivialization domain of principal bundle that gives rise to the Atiyah sequence; we choose the local coordinates $x = (x_1,\ldots,x_n)$ in U in such a way that $x(m) = 0$. If we set $a|_U = (v,\gamma)$ and $k_0|_U = (0,\eta)$, the condition reads

$$[\gamma_m,\eta_m]_{\mathfrak{g}} + v_m(\eta)|_m \in \mathfrak{g}_0,$$

for any $\eta \in C^\infty(U,\mathfrak{g})$ such that $\eta_m \in \mathfrak{g}_0$. Since the bracket in the first term is always in $\mathfrak{g}_0$, the condition means that $v_m = 0$, as we easily see when taking $\eta = x_k e$, where $e \in \mathfrak{g} \setminus \mathfrak{g}_0$ is a nonzero vector. Finally, the normalizer is exactly $\mathscr{A}_m = \{a \in \mathscr{A} : \widehat{a}_m = 0\}$. □

Theorem 5. *Let $0 \to \mathscr{K} \to \mathscr{A} \to \mathscr{V} \to 0$ be the Atiyah sequence of modules an Lie algebras associated to a principal bundle with semisimple structure group over a manifold M. The intersections of all maximal finite-codimensional Lie ideals in $\mathscr{K}$ which are normalized by a given maximal finite-codimensional Lie subalgebra in $\mathscr{A}$ are exactly of the form $\mathscr{K}(m) = \{k \in \mathscr{K} : k_m = 0\}$, $m \in M$, where m is the point that characterizes the chosen subalgebra of $\mathscr{A}$.*

Proof. There is a unique $m \in M$ such that $\mathscr{A}_m$ coincides with the chosen maximal finite-codimensional Lie subalgebra. In view of the preceding proposition, the corresponding normalized maximal finite-codimensional Lie ideals of $\mathscr{K}$ are exactly the $\mathscr{K}(m,\mathfrak{g}_0)$, where $\mathfrak{g}_0$ runs through the maximal ideals of $\mathfrak{g}$. Hence, the wanted intersection is $\mathscr{K}(m) := \{k \in \mathscr{K} : k_m = 0\}$, as the intersection of all maximal ideals vanishes in a semisimple Lie algebra. □

Theorem 6. *Consider an Atiyah sequence as in Theorem 5. The Lie subalgebras $\mathscr{A}(m) := \{a \in \mathscr{A} : a_m = 0\}$, $m \in M$, of $\mathscr{A}$ can be characterized as the maximal Lie subalgebras $\mathscr{A}_0$ in $\mathscr{A}$ such that $[\mathscr{A}_0,\mathscr{K}]$ is included in a certain $\mathscr{K}(m)$, $m \in M$. In other words, the Lie subalgebras $\mathscr{A}(m)$, $m \in M$, are exactly the maximal Lie subalgebras $\mathscr{A}_0$ in $\mathscr{A}$ such that $[\mathscr{A}_0,\mathscr{K}]$ is included in the intersection of all maximal finite-codimensional Lie ideals in $\mathscr{K}$, which are normalized by a given maximal finite-codimensional Lie subalgebra in $\mathscr{A}$. The maximal finite-codimensional Lie subalgebra that corresponds to $\mathscr{A}(m)$ is precisely $\mathscr{A}_m$.*

Remark 4. Let us emphasize that in this statement "maximal" means "maximal in the class of all Lie subalgebras $\mathscr{A}_0$ of $\mathscr{A}$ that have the property $[\mathscr{A}_0,\mathscr{K}] \subset \mathscr{K}(m)$, for some $m \in M$".

Proof. Let $a \in \mathscr{A}$, $k \in \mathscr{K}$, and $m \in M$. Consider again a chart and trivialization domain U around m and set $a|_U = (v,\gamma)$ and $k|_U = (0,\eta)$. Then,

$$[a,k]_m = [\gamma_m,\eta_m]_{\mathfrak{g}} + v_m(\eta)|_m. \tag{17}$$

Equation (17) entails that any Lie subalgebra $\mathscr{A}(m)$ verifies $[\mathscr{A}(m),\mathscr{K}] \subset \mathscr{K}(m)$.

Conversely, let $\mathscr{A}_0$ be any maximal Lie subalgebra of $\mathscr{A}$ such that $[\mathscr{A}_0,\mathscr{K}] \subset \mathscr{K}(m)$, for a certain $m \in M$. It follows from Equation (17), written for $\eta \in \mathfrak{g} \subset C^\infty(U,\mathfrak{g})$, that any $a \in \mathscr{A}_0$ has a vanishing second component γ_m, since the center of a semisimple Lie algebra vanishes. When writing now

Equation (17) for an arbitrary function $\eta \in C^\infty(U,\mathfrak{g})$, we get $v_m(\eta)|_m = 0$. Thus $v_m = 0$ and $\mathscr{A}_0 \subset \mathscr{A}(m)$. If $\mathscr{A}_0 \subsetneq \mathscr{A}(m)$, we have $\mathscr{A}_0 \subsetneq \mathscr{A}(m) \subset \mathscr{A}_m \subsetneq \mathscr{A}$, since $\mathscr{A}_m$ is maximal. As $\mathscr{A}_0$ is maximal, it follows that $\mathscr{A}_0 = \mathscr{A}(m)$.

It is now easily seen that $\mathscr{A}(m)$ is maximal among the Lie subalgebras $\mathscr{A}_0$ of $\mathscr{A}$ such that $[\mathscr{A}_0, \mathscr{K}] \subset \mathscr{K}(p)$, for some $p \in M$. Indeed, since $\mathscr{A}/\mathscr{A}(m) \simeq A_m$, where A_m denotes the fiber at m of the Atiyah algebroid A, the space $\mathscr{A}(m)$ is of course finite-codimensional. So if $\mathscr{A}(m)$ is strictly included in a proper Lie subalgebra $\mathscr{A}_0$ such that $[\mathscr{A}_0, \mathscr{K}] \subset \mathscr{K}(p)$, $p \in M$, we can assume that $\mathscr{A}_0$ is maximal in the considered class. But then $\mathscr{A}_0 = \mathscr{A}(p)$, $p \in M$, and $\mathscr{A}(m) \subsetneq \mathscr{A}(p)$. The usual argument based upon a smooth function that has value 0 at m and value 1 at p then shows that $m = p$ and that $\mathscr{A}(m)$ is maximal. □

We are now able to provide the proof of Theorem 3.

Proof of Theorem 3. Let $\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$ be a Lie algebra isomorphism.

Since Theorem 6 characterizes the subalgebras $\mathscr{A}(m)$, $m \in M$, in pure Lie algebraic terms, isomorphism Φ transforms an $\mathscr{A}_1(m)$ into an $\mathscr{A}_2(p)$. Therefore, $\mathscr{A}_2(p) = \Phi(\mathscr{A}_1(m)) \subset \Phi(\mathscr{A}_{1\,m}) = \mathscr{A}_{2\,\varphi(m)}$, where φ is the diffeomorphism that is implemented by Φ, see Theorem 2. But then $p = \varphi(m)$. Indeed, let $v \in \mathscr{V}_2$ be a vector field of M_2 such that $v_{\varphi(m)} \neq 0$ and take $a \in \mathscr{A}_2$ with anchor $\widehat{a} = v$. If $p \neq \varphi(m)$, we can choose a function α with value 0 around p and value 1 around $\varphi(m)$. Thus $\alpha a \in \mathscr{A}_2(p) \subset \mathscr{A}_{2\,\varphi(m)}$, so that $\widehat{a}_{\varphi(m)} = v_{\varphi(m)} = 0$. It follows that

$$\Phi(\mathscr{A}_1(m)) = \mathscr{A}_2(\varphi(m)). \tag{18}$$

As the fiber $A_{i\,m}$ is isomorphic to the vector space $\mathscr{A}_i/\mathscr{A}_i(m)$, the preceding equation entails that Φ induces linear maps $\phi_m : A_{1\,m} \to A_{2\,\varphi(m)}$, as well as a smooth vector bundle map $\phi : A_1 \to A_2$ over the diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$. Smoothness of ϕ is a consequence of the fact that the map Φ, which ϕ induces between sections, transforms smooth sections into smooth sections. The bundle map ϕ is actually a vector bundle isomorphism over φ, since ϕ_m is bijective, due to bijectivity of Φ and Equation (18). □

To conclude we combine the preceding results with those of [Kub89] and get the following characterization of isomorphisms of Lie algebras of semisimple Atiyah algebroids.

Theorem 7. *Let A_i, $i \in \{1,2\}$, be smooth Atiyah algebroids associated with principal bundles $P_i(M_i, \pi_i, G_i)$ with semisimple structure groups. The Lie algebras $\mathscr{A}_i$ of smooth sections of the bundles A_i are isomorphic if and only if the Lie algebroids A_i are isomorphic, or, as well, if and only if the principal bundles P_i are locally isomorphic. In particular, the structural groups are locally isomorphic.*

5 Isomorphisms of Lie algebras of Atiyah algebroids - reductive structure groups

Let

$$0 \to K \overset{i}{\to} A \overset{\pi_*}{\to} TM \to 0 \tag{19}$$

be an Atiyah algebroid associated to a principal bundle $P(M,\pi,G)$ with connected *reductive structure group* G. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ then canonically splits into its Abelian center $Z\mathfrak{g}$ and its semisimple derived ideal $[\mathfrak{g},\mathfrak{g}]$. It follows from Equation (10) that the kernel K is similarly split,

$$K = \mathsf{Z} \oplus [K,K],$$

where we wrote simply Z for ZK, and from Equations (5) and (6) that

$$\mathscr{K} = \Gamma(K) = \Gamma(\mathsf{Z}) \oplus \Gamma([K,K]) = \mathscr{Z} \oplus [\mathscr{K},\mathscr{K}].$$

Proposition 2. *The center sub-LAB $Z = P \times_G Z\mathfrak{g}$ of the kernel LAB $K = P \times_G \mathfrak{g}$ of the Atiyah sequence of a principal bundle $P(M,\pi,G)$ with connected structure Lie group G is the trivial bundle $Z = M \times Z\mathfrak{g}$ and it admits a global frame made up by constant functions $\mathfrak{c}_1,\ldots,\mathfrak{c}_k \in C^\infty(M,Z\mathfrak{g}) \simeq C^\infty(M,\mathbb{R}^k)$.*

Proof. Let us first recall that the adjoint action of a connected finite-dimensional Lie group G on the center $Z\mathfrak{g}$ of its Lie algebra is trivial.

Triviality of the adjoint action on the kernel entails that the sections of $\mathsf{Z} = P \times_G Z\mathfrak{g}$, i.e. the G-invariant functions from P to $Z\mathfrak{g}$, are exactly the functions from M to $Z\mathfrak{g}$. Hence, any basis $\mathfrak{z}_1,\ldots,\mathfrak{z}_k$ of the vector space $Z\mathfrak{g}$ corresponds to global sections $\mathfrak{c}_1,\ldots,\mathfrak{c}_k$ of Z that obviously form a global frame. □

Since Z is a Lie algebroid ideal of A, the sequence

$$0 \to \mathsf{Z} \xrightarrow{i} A \xrightarrow{p} A/\mathsf{Z} \to 0, \tag{20}$$

is a short exact sequence of Lie algebroids and the transitive quotient Lie algebroid $\widetilde{A} := A/\mathsf{Z}$ is associated with the short exact sequence

$$0 \to K/\mathsf{Z} \simeq [K,K] \xrightarrow{\tilde{i}} \widetilde{A} \xrightarrow{\tilde{\pi}_*} TM \to 0, \tag{21}$$

see Section 2.3. Moreover, the isotropy algebra of $\widetilde{A}$, the quotient algebra $\widetilde{\mathfrak{g}} := \mathfrak{g}/Z\mathfrak{g}$, is semisimple. We will write $\widetilde{\mathscr{A}} := \Gamma(A/\mathsf{Z}) = \mathscr{A}/\mathscr{Z}$ for the Lie algebra of sections of this Lie algebroid and $\hat{a}$ for the anchor $\tilde{\pi}_*(\tilde{a})$ of a section $\tilde{a} \in \widetilde{\mathscr{A}}$.

Remark 5. It is known [Kub89] that any transitive Lie algebroid with semisimple LAB is the Atiyah algebroid of some principal bundle. Hence, the quotient Lie algebroid $\widetilde{A} = A/\mathsf{Z}$ is an Atiyah algebroid, namely that of the principal bundle $P/ZG(M,\tilde{\pi},G/ZG)$, where notations are self-explaining and where G/ZG is semisimple.

In the sequel, we use the global frame $\mathfrak{c}_1,\ldots,\mathfrak{c}_k \in C^\infty(M,Z\mathfrak{g})$ of $\mathscr{Z}$ made up by constant functions.

Lemma 4. *For any sections $a \in \mathscr{A}$ and $c = \sum_i f^i \mathfrak{c}_i \in \mathscr{Z}$, $f^i \in C^\infty(M)$, the adjoint action on c by a and the canonical action by $\pi_* a \in \mathfrak{X}(M)$ coincide:*

$$[a,c] = (\pi_* a)(c) = \sum_i (\pi_* a)(f^i)\mathfrak{c}_i.$$

Proof. Since $\mathfrak{c}_i$ is a constant $Z\mathfrak{g}$-valued function, (it follows for instance from the local form of the Atiyah algebroid bracket that) we have $[a, \mathfrak{c}_i] = 0$. The lemma is then an immediate consequence of the Leibniz property of the Lie bracket $[-,-]$. □

Proposition 3. *The set $\mathscr{Z}(m)$, $m \in M$, of all the sections in $\mathscr{Z}$ that vanish at m is given in Lie algebraic terms by*

$$\mathscr{Z}(m) = [\mathscr{A}_m, \mathscr{Z}],$$

where $\mathscr{A}_m = \{a \in \mathscr{A} : (\pi_* a)_m = 0\}$.

Proof. Lemma 4 entails that the inclusion $\supset$ holds true. As for the converse inclusion, remember first that any function of M can be written as a sum of Lie derivatives [Gra78]; in particular, there are vector fields $X_j \in \mathfrak{X}(M)$ and functions $g^j \in C^\infty(M)$ such that $1 = \sum_j L_{X_j} g^j$. Let now $c = \sum_i f^i \mathfrak{c}_i \in \mathscr{Z}(m)$, so that $f^i(m) = 0$, and set $Y^i_j = f^i X_j \in \mathfrak{X}(M)$, so that $f^i = \sum_j L_{Y^i_j} g^j$ and $Y^i_{j,m} = 0$. If $a^i_j \in \mathscr{A}$ denotes any preimage of Y^i_j by π_*, we have

$$c = \sum_{ij} (\pi_* a^i_j)(g^j)\mathfrak{c}_i = \sum_{ij} [a^i_j, g^j \mathfrak{c}_i] \in [\mathscr{A}_m, \mathscr{Z}].$$

□

In the following, we investigate isomorphisms

$$\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$$

of Lie algebras $\mathscr{A}_i$ of Atiyah algebroids A_i associated to principal bundles $P_i(M_i, \pi_i, G_i)$ with connected *reductive structure groups* G_i. Let $0 \to K_i \to A_i \to TM_i \to 0$ be the corresponding Atiyah sequences. In view of Theorem 2, we have $\Phi(\mathscr{K}_1) = \mathscr{K}_2$, so that moreover $\Phi(\mathscr{Z}_1) = \mathscr{Z}_2$. Hence, the isomorphism $\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$ induces an isomorphism

$$\Phi^0 : \mathscr{Z}_1 \xrightarrow{\simeq} \mathscr{Z}_2$$

of the centers $\mathscr{Z}_i$ and an isomorphism

$$\Phi^s : \widetilde{\mathscr{A}_1} \xrightarrow{\simeq} \widetilde{\mathscr{A}_2}$$

of the Lie algebras $\widetilde{\mathscr{A}_i} = \Gamma(\widetilde{A}_i)$ of the Atiyah algebroids $\widetilde{A}_i = A_i/\mathsf{Z}_i$ implemented by principal bundles with the semisimple structure groups G_i/ZG_i. Note that, in view of Theorem 7, the Lie algebra isomorphism Φ^s is implemented by a Lie algebroid isomorphism $\phi^s : \widetilde{A}_1 \xrightarrow{\simeq} \widetilde{A}_2$ covering a diffeomorphism $\varphi : M_1 \xrightarrow{\simeq} M_2$.

Proposition 4. *The Abelian Lie algebra isomorphism $\Phi^0 : \mathscr{Z}_1 \xrightarrow{\simeq} \mathscr{Z}_2$ is implemented by a vector bundle isomorphism $\phi^0 : Z_1 \xrightarrow{\simeq} Z_2$ that covers φ. Moreover, for any $c_1 = \sum_i f^i \mathfrak{c}_{1i} \in \mathscr{Z}_1$, the map Φ^0 is given by*

$$\Phi^0\left(\sum_i f^i \mathfrak{c}_{1i}\right) = \sum_{ij} I^i_j \left(f^j \circ \varphi^{-1}\right) \mathfrak{c}_{2i}, \tag{22}$$

where $I = (I^i_j) \in \mathrm{GL}(k, \mathbb{R})$.

Proof. Since, due to Proposition 3, the isomorphism Φ^0 generates a bijection between the sets $\{\mathscr{Z}_1(m_1) : m_1 \in M_1\}$ and $\{\mathscr{Z}_2(m_2) : m_2 \in M_2\}$, it is implemented by a vector bundle isomorphism $\phi^0 : \mathsf{Z}_1 \xrightarrow{\simeq} \mathsf{Z}_2$ between the trivial center bundles, which covers a diffeomorphism $\varphi^0 : M_1 \xrightarrow{\simeq} M_2$:

$$\phi^0 : \mathsf{Z}_1 \ni (m_1, z_1) \xrightarrow{\simeq} (\varphi^0(m_1), \mathscr{I}(m_1)z_1) \in \mathsf{Z}_2,$$

where $\mathscr{I}(m_1)$ is a vector space isomorphism from $Z\mathfrak{g}_1$ onto $Z\mathfrak{g}_2$. Therefore,

$$\Phi^0\left(\sum_i f^i \mathfrak{c}_{1i}\right) = \phi^0\left(\sum_i f^i (\varphi^0)^{-1} \mathfrak{c}_{1i}\right) = \sum_{ij} I_i^j (\varphi^0)^{-1} \cdot f^i (\varphi^0)^{-1}\, \mathfrak{c}_{2j}, \tag{23}$$

where compositions are understood and where $I \in C^\infty(M_1, \mathrm{GL}(k,\mathbb{R}))$ denotes the matrix of $\mathscr{I}$ in the bases $(\mathfrak{c}_{1i})$ and $(\mathfrak{c}_{2i})$.

Let $a_1 \in \mathscr{A}_1$, $c_1 \in \mathscr{Z}_1$ and set $\pi_{1*}a_1 =: X_1$. The Lie algebra morphism property of Φ then reads

$$\Phi^0(L_{X_1} c_1) = \Phi((\pi_{1*}a_1)(c_1)) = \Phi[a_1, c_1] = [\Phi a_1, \Phi^0 c_1] = (\pi_{2*}\Phi a_1)(\Phi^0 c_1).$$

As Theorem 2 implies that Φ^s induces a Lie algebra isomorphism $\widetilde{\Phi}^s = \varphi_*$ between the algebras of vector fields, we get

$$\pi_{2*}\Phi a_1 = \tilde{\pi}_{2*} p_2 \Phi a_1 = \tilde{\pi}_{2*}\Phi^s p_1 a_1 = \widetilde{\Phi}^s(\tilde{\pi}_{1*} p_1 a_1) = \varphi_* \pi_{1*} a_1 = \varphi_* X_1.$$

The combination of the last two identities finally gives

$$\Phi^0(L_{X_1} c_1) = L_{\varphi_* X_1}(\Phi^0 c_1). \tag{24}$$

To simplify notations, denote by f the $\mathbb{R}^k$-valued function with components f^i. The combination of the equations (24) and (23) then leads to

$$L_{\varphi_* X_1}\left(I(\varphi^0)^{-1} \cdot f(\varphi^0)^{-1}\right) = I(\varphi^0)^{-1} \cdot (L_{X_1} f)(\varphi^0)^{-1} = I(\varphi^0)^{-1} \cdot \varphi^0_* L_{X_1} f. \tag{25}$$

For any vector field $X_1 \in \mathfrak{X}(M_1)$ that vanishes at an arbitrarily chosen point $m_1 \in M_1$, if we write Equation (25) at φm_1, we get $(L_{X_1} f)((\varphi^0)^{-1}\varphi m_1) = 0$, for any $f \in C^\infty(M_1, \mathbb{R}^k)$. It follows that X_1 vanishes at $(\varphi^0)^{-1}\varphi m_1$, if, as assumed, X_1 vanishes at m_1. Hence, $\varphi = \varphi^0$.

Equation (25) gives now

$$L_{\varphi_* X_1}\left(I\varphi^{-1} \cdot \varphi_* f\right) = I\varphi^{-1} \cdot L_{\varphi_* X_1} \varphi_* f,$$

so that the matrix I is actually constant. □

We now aim at writing $\Phi : \mathscr{A}_1 \xrightarrow{\simeq} \mathscr{A}_2$ by means of $\Phi^0 : \mathscr{Z}_1 \xrightarrow{\simeq} \mathscr{Z}_2$ and $\Phi^s : \widetilde{\mathscr{A}_1} \xrightarrow{\simeq} \widetilde{\mathscr{A}_2}$. This requires the use of a connection.

Lemma 5. *Any right splitting ∇ of the vector bundle sequence (19), i.e. any connection of an Atiyah algebroid A with connected reductive structure group, naturally induces a right splitting $\tilde{\nabla}$ of the sequence (21) and a right splitting $\bar{\nabla}$ of the sequence (20). Moreover, for any $k^{(1)} \in [K,K]$, we have $\bar{\nabla} p(k^{(1)}) = k^{(1)}$. The preceding splitting allows identifying $\widetilde{A}$ with a vector subbundle $\bar{\nabla}(\widetilde{A})$ of A that verifies $A = Z \oplus \bar{\nabla}(\widetilde{A})$ (as vector bundles) and $[K,K] \subset \bar{\nabla}(\widetilde{A})$.*

Proof. The first claim is clear, it suffices to set $\tilde{\nabla} = p \circ \nabla$. As for $\bar{\nabla}$, observe that, if $'$ (resp. $''$) is the projection of $K = \mathsf{Z} \oplus [K,K]$ onto Z (resp. $[K,K]$) and if $p(a) \in \widetilde{A}$, the difference $a - \nabla\pi_* a$ belongs to K and the sum $\nabla\pi_* a + (a - \nabla\pi_* a)''$ is well-defined in $\widetilde{A}$. The bundle map

$$\bar{\nabla} : \widetilde{A} \ni p(a) \to \nabla\pi_* a + (a - \nabla\pi_* a)'' \in A$$

is then the searched splitting $\bar{\nabla}$, since

$$p(\bar{\nabla} p(a)) = p(\nabla\pi_* a + (a - \nabla\pi_* a)'') = p(\nabla\pi_* a + (a - \nabla\pi_* a)' + (a - \nabla\pi_* a)'') = p(a).$$

The assertion $\bar{\nabla} \circ p = \mathrm{id}$ on [K,K] immediately follows from the definition of $\bar{\nabla}$, whereas the last part of the lemma is obvious. □

Lemma 6. *Let ∇ be any connection of an Atiyah algebroid $A \to M$ with connected reductive structure group. If we transfer the Lie algebroid bracket on $\mathscr{A} = \mathscr{Z} \oplus \bar{\nabla}(\widetilde{\mathscr{A}})$ from $\mathscr{A}$ to $\mathscr{Z} \oplus \widetilde{\mathscr{A}}$, it reads, for any $c, c' \in \mathscr{Z}$ and $\widetilde{a}, \widetilde{a}' \in \widetilde{\mathscr{A}}$,*

$$[\![c + \widetilde{a}, c' + \widetilde{a}']\!]_\omega = \hat{a}(c') - \hat{a}'(c) + \omega(\hat{a}, \hat{a}') + [\widetilde{a}, \widetilde{a}']^{\tilde{}}, \tag{26}$$

where, for any $X, Y \in \mathfrak{X}(M)$,

$$\omega(X,Y) = \sum_j \omega^j(X,Y)\mathfrak{c}_j, \tag{27}$$

for some closed 2-forms ω^j on M. In other words, any Atiyah algebroid $(A, [-,-], \pi_*)$ over M with reductive structure group is isomorphic to a model reductive Atiyah algebroid $(Z \oplus_M \widetilde{A}, [\![-,-]\!]_\omega, \widetilde{\pi}_*^0)$, *where Z is a trivial bundle and $\widetilde{A}$ is an Atiyah algebroid over M with semisimple structure group, with the Lie bracket of sections (26) associated with a closed 2-form ω on M with values in $\mathsf{Z}\mathfrak{g}$, and with the anchor map $\widetilde{\pi}_*^0(c + \tilde{a}) = \widetilde{\pi}_*(\tilde{a}) = \hat{a}$.*

Proof. It is well-known that the curvature $R_{\bar{\nabla}}$ of $\bar{\nabla}$ is a closed Lie algebroid 2-form of $\widetilde{A}$ valued in Z, so that

$$\Omega_\nabla := R_{\bar{\nabla}} \in \Gamma(\wedge^2 \widetilde{A}^* \otimes \mathsf{Z}) = \Gamma(\wedge^2 \widetilde{A}^* \otimes \mathsf{Z}\mathfrak{g}),$$

see e.g. Section 2.3. To get the transferred bracket, note that the first term $[c, c']$, $c, c' \in \mathscr{Z}$, of $[c + \bar{\nabla} p(a), c' + \bar{\nabla} p(a')]$ vanishes, that the second and third terms are of the type

$$[\bar{\nabla} p(a), c'] = (\pi_* \bar{\nabla} p(a))(c') = (\tilde{\pi}_* p(a))(c') = (\pi_* a)(c') \in \mathscr{Z},$$

due to Lemma 4, whereas the fourth term reads

$$[\bar{\nabla} p(a), \bar{\nabla} p(a')] = \Omega_\nabla(p(a), p(a')) + \bar{\nabla}[p(a), p(a')]^{\tilde{}} \in \mathscr{Z} \oplus \bar{\nabla}(\widetilde{\mathscr{A}}).$$

Hence the announced result up to the third term of the RHS of Equation (26).

We can conclude that the curvature Ω_∇ is defined on $\mathfrak{X}(M) \simeq \widetilde{\mathscr{A}} / \widetilde{\mathscr{K}}$, where $\widetilde{\mathscr{K}} := \mathscr{K} / \mathscr{Z}$, if we prove that it vanishes once one of the arguments is in $[\mathscr{K}, \mathscr{K}] = \mathscr{K}^{(1)} \simeq p(\mathscr{K}^{(1)}) = \widetilde{\mathscr{K}}$. However, since $\bar{\nabla} \circ p = \mathrm{id}$ on $[K,K]$ and since $[K,K]$ is a Lie algebroid ideal in A, we have

$$\Omega_\nabla(p(a), p(k^{(1)})) = [\bar{\nabla} p(a), k^{(1)}] - \bar{\nabla} p[a, k^{(1)}] = [\bar{\nabla} p(a) - a, k^{(1)}] = 0, \tag{28}$$

where the last member vanishes since by definition $\bar{\nabla}p(a)-a=(\nabla\pi_*a-a)'\in\mathscr{Z}$. The resulting Z-valued (i.e. $Z\mathfrak{g}$-valued or $\mathbb{R}^k$-valued) 2-form $\omega=\omega_\nabla$ of M is still closed. Indeed, this form can be computed, for any $X,Y\in\mathfrak{X}(M)$, by $\omega_\nabla(X,Y)=\Omega_\nabla(\tilde{\nabla}X,\tilde{\nabla}Y)$, since $\tilde{\pi}_*\tilde{\nabla}X=X$. The de Rham differential $(d\omega_\nabla)(X,Y,Z)$ is thus made up by two types of terms,

$$X.\omega_\nabla(Y,Z)=(\tilde{\pi}_*\tilde{\nabla}X).\Omega_\nabla(\tilde{\nabla}Y,\tilde{\nabla}Z)$$

and

$$\omega_\nabla([X,Y],Z)=\Omega_\nabla(\tilde{\nabla}[X,Y],\tilde{\nabla}Z)+\Omega_\nabla(R_{\tilde{\nabla}}(X,Y),\tilde{\nabla}Z)=\Omega_\nabla([\tilde{\nabla}X,\tilde{\nabla}Y]^{\tilde{}},\tilde{\nabla}Z),$$

where the first equality is due to Equation (28) and to the fact $R_{\tilde{\nabla}}(X,Y)\in p(\mathscr{K}^{(1)})$. Finally, the considered de Rham differential of ω_∇ coincides with the Lie algebroid differential $(d\Omega_\nabla)(\tilde{\nabla}X,\tilde{\nabla}Y,\tilde{\nabla}Z)=0$ of the closed form Ω_∇ of the Lie algebroid $(\widetilde{A},[-,-]^{\tilde{}},\tilde{\pi}_*)$. □

Just as the Lie bracket, the Lie isomorphism $\Phi:\mathscr{A}_1\xrightarrow{\simeq}\mathscr{A}_2$ catches a twist when read through the isomorphism

$$\mathscr{A}_i=\mathscr{Z}_i\oplus\bar{\nabla}_i(\widetilde{\mathscr{A}_i})\simeq\mathscr{Z}_i\oplus\widetilde{\mathscr{A}_i}. \tag{29}$$

Before formulating our main result, let us recall that on every manifold M there exists a *divergence*, i.e. a linear operator $\mathrm{div}:\mathfrak{X}(M)\to C^\infty(M)$, which is a cocycle, i.e.

$$\mathrm{div}[X,Y]=X\left(\mathrm{div}(Y)\right)-Y\left(\mathrm{div}(X)\right),$$

and that verifies

$$\mathrm{div}(fX)=f\,\mathrm{div}(X)+X(f)$$

for any $X,Y\in\mathfrak{X}(M)$ and $f\in C^\infty(M)$. For details pertaining to divergence operators on an arbitrary manifold, we refer the reader to [GP04].

Theorem 8. *Let $\Phi:\mathscr{A}_1\xrightarrow{\simeq}\mathscr{A}_2$ be an isomorphism of the Lie algebras $\mathscr{A}_i$ of model reductive Atiyah algebroids $(A_i=Z_i\oplus\widetilde{A}_i,[\![-,-]\!]_{\omega_i},\widetilde{\pi}^0_{i*})$ with connected reductive structure groups G_i over connected manifolds M_i, $i\in\{1,2\}$, and let* div *be a fixed divergence on M_1. Then, there are*

- *a Lie algebroid isomorphism $\phi^s:\widetilde{A}_1\xrightarrow{\simeq}\widetilde{A}_2$ covering a diffeomorphism $\varphi:M_1\xrightarrow{\simeq}M_2$ and inducing a Lie algebra isomorphism $\Phi^s:\widetilde{\mathscr{A}_1}\xrightarrow{\simeq}\widetilde{\mathscr{A}_2}$,*
- *a vector bundle isomorphism $\phi^0:Z_1\xrightarrow{\simeq}Z_2$ covering the same diffeomorphism φ and inducing a linear isomorphism $\Phi^0:\mathscr{Z}_1\xrightarrow{\simeq}\mathscr{Z}_2$,*
- *a one-form η on M_1 with values in $Z\mathfrak{g}_1$ satisfying*

$$d\eta=\omega_1-\phi^{0*}\omega_2\,,$$

- *an element $r\in Z\mathfrak{g}_1$ representing a section of Z_1,*

such that

$$\Phi(c+\widetilde{a})=\Phi^0\left(c+\eta\left(\hat{a}\right)+\mathrm{div}(\hat{a})\cdot r\right)+\Phi^s(\widetilde{a})\,. \tag{30}$$

Conversely, every mapping of the form (30) with Φ^s, Φ^0, η, and r satisfying the above conditions is a Lie algebra isomorphism.

Proof. We first show that
$$\Phi(c+\widetilde{a}) = \Phi^0(c+F(\widetilde{\pi}_{1*}\widetilde{a}))+\Phi^s(\widetilde{a}), \tag{31}$$
where $\Phi^0 : \mathscr{Z}_1 \xrightarrow{\simeq} \mathscr{Z}_2$ and $\Phi^s : \widetilde{\mathscr{A}_1} \xrightarrow{\simeq} \widetilde{\mathscr{A}_2}$ are the canonically Φ-induced isomorphisms between the centers $\mathscr{Z}_i$ and the Lie algebras $\widetilde{\mathscr{A}_i}$ of semisimple Atiyah algebroids, respectively, see Proposition 4, Theorem 3 and Theorem 7, and where $F : \mathfrak{X}(M_1) \to \mathscr{Z}_1$ is a linear map. Indeed, let $\mathrm{pr}_{\mathscr{Z}_2} : \mathscr{Z}_2 \oplus \widetilde{\mathscr{A}_2} \to \mathscr{Z}_2$ be the canonical projection. Since
$$\Phi(c+\tilde{a}) = \left(\Phi^0(c)+\mathrm{pr}_{\mathscr{Z}_2}(\Phi(\tilde{a}))\right)+\Phi^s(\tilde{a})\,,$$
it suffices to set $F := (\Phi^0)^{-1}\mathrm{pr}_{\mathscr{Z}_2}\Phi : \widetilde{\mathscr{A}_1} \to \mathscr{Z}_1$ and to prove that this linear map factors through the quotient $\widetilde{\mathscr{A}_1}/\widetilde{\mathscr{K}_1} \simeq \mathfrak{X}(M_1)$, i.e. to show that $\mathrm{pr}_{\mathscr{Z}_2}\Phi$ vanishes on $\widetilde{\mathscr{K}_1} \simeq \widetilde{\mathscr{K}_1}^{(1)}$. Indeed, when using Equation (26), as well as the fact that the $\mathsf{Z}_i \oplus \tilde{K}_i$ are the kernel LABs of the Atiyah algebroids $\mathsf{Z}_i \oplus \tilde{A}_i$, we get
$$\Phi\left(\widetilde{\mathscr{K}_1}^{(1)}\right) = \Phi\left((\mathscr{Z}_1 \oplus \widetilde{\mathscr{K}_1})^{(1)}\right) = (\mathscr{Z}_2 \oplus \widetilde{\mathscr{K}_2})^{(1)} = \widetilde{\mathscr{K}_2}^{(1)}\,,$$
so that $\Phi(\tilde{k}) \in \widetilde{\mathscr{A}_2}$, for each $\tilde{k} \in \widetilde{\mathscr{K}_1}$. It suffices now to put $F(\widetilde{\pi}_{1*}\widetilde{a}) = (\Phi^0)^{-1}\left(\mathrm{pr}_{\mathscr{Z}_2}(\Phi(\tilde{a}))\right)$. Note that the map Φ defined by (31) is always a linear isomorphism.

Now, we will show that it is a Lie algebra isomorphism if and only if, for any vector field X, $F(X) = \eta(X)+\mathrm{div}(X)\cdot r$, with η and r as described in the theorem.

When applying Equations (26) and (31), as well as the Lie algebra morphism property of Φ^s, we find that the Lie algebra morphism property
$$\Phi\left([\![c+\widetilde{a},c'+\widetilde{a}']\!]_{\omega_1}\right) = [\![\Phi(c+\widetilde{a}),\Phi(c'+\widetilde{a}')]\!]_{\omega_2}$$
reads
$$\begin{gathered}\Phi^0\left(\hat{a}(c')-\hat{a}'(c)+\omega_1(\hat{a},\hat{a}')+F(\widetilde{\pi}_{1*}[\widetilde{a},\widetilde{a}']\tilde{\ })\right) = \\ (\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}))\left(\Phi^0(c'+F(\hat{a}'))\right)-(\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}'))\left(\Phi^0(c+F(\hat{a}))\right)+\omega_2(\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}),\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}')).\end{gathered} \tag{32}$$
Note now that, since Φ^s induces a Lie algebra isomorphism $\tilde{\Phi}^s = \varphi_*$ implemented by a diffeomorphism φ, we have
$$\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}) = \varphi_*\widetilde{\pi}_{1*}\widetilde{a} = \varphi_*\hat{a}\,,$$
and finally combine the last identity with Equation (24). This leads to
$$(\widetilde{\pi}_{2*}\Phi^s(\widetilde{a}))(\Phi^0(c'+F(\hat{a}'))) = (\varphi_*\hat{a})(\Phi^0(c'+F(\hat{a}'))) = \Phi^0(\hat{a}(c'+F(\hat{a}'))),$$
so that the morphism condition (32) is equivalent to
$$\omega_2(\varphi_*X,\varphi_*X')-\Phi^0\left(\omega_1(X,X')-dF(X,X')\right) = 0\,, \tag{33}$$
where
$$dF(X,X') = L_X(F(X'))-L_{X'}(F(X))-F([X,X'])\,,$$
for all vector fields X,X' of M_1. When decomposing $\omega_1 = \sum_\ell \omega_1^\ell \mathfrak{c}_{1\ell}$, $\omega_2 = \sum_\ell \omega_2^\ell \mathfrak{c}_{2\ell}$, $F = \sum_\ell F^\ell \mathfrak{c}_{1\ell}$ in the corresponding global frames, and when observing that
$$(\varphi^*\omega_2)(X,X') = \omega_2(\varphi_*X,\varphi_*X')\circ\varphi \quad \text{and} \quad \Phi^0\left(\sum_i f^i\mathfrak{c}_{1i}\right)\circ\varphi = \sum_{ij} I^i_j f^j \mathfrak{c}_{2i},$$

we can rewrite (33) in the form

$$\mathrm{d}F_i^m = \omega_1^m - \sum_\ell (I^{-1})^m_\ell \varphi^* \omega_2^\ell, \quad m \in \{1, \ldots, k\}. \tag{34}$$

Equation (34) implies that each $F_i^m : \mathfrak{X}(M_1) \to C^\infty(M_1)$ is a local, thus locally, a differential operator. Indeed, if a vector field Y vanishes in a neighborhood of a point $m_1 \in M_1$, it can be written in the form $Y = \sum_k [X_k, X_k']$ with vector fields X_k, X_k' that vanish in a neighborhood of m_1. It follows from (34) that $F_i^m(Y)$ equals to

$$\sum_k F_i^m([X_k, X_k']) = \sum_k \left(X_k(F_i^m(X_k')) - X_k'(F_i^m(X_k)) - \left(\omega_1^m - \sum_\ell (I^{-1})^m_\ell \varphi^* \omega_2^\ell \right) (X_k, X_k') \right),$$

so that $F_i^m(Y)$ vanishes in a neighborhood of m_1.

Let $U_\mathfrak{a}$, $\mathfrak{a} \in \mathfrak{A}$, be an open covering of M_1 by contractible charts. Since the 2-form $\beta^m = \omega_1^m - \sum_\ell (I^{-1})^m_\ell \varphi^* \omega_2^\ell$ of M_1 is closed, there is, for every $\mathfrak{a} \in \mathfrak{A}$, a one-form $\alpha_\mathfrak{a}^m$ on $U_\mathfrak{a}$ such that $\beta^m|U_\mathfrak{a} = \mathrm{d}\alpha_\mathfrak{a}^m$. The linear map $F^m|U_\mathfrak{a} - \alpha_\mathfrak{a}^m : \mathfrak{X}(U_\mathfrak{a}) \to C^\infty(U_\mathfrak{a})$ is therefore a 1-cocycle of the Chevalley-Eilenberg cohomology of vector fields represented upon functions, i.e., see [GP04], [Pon98], $F^m|U_\mathfrak{a} - \alpha_\mathfrak{a}^m = r_\mathfrak{a}^m \operatorname{div} + \gamma_\mathfrak{a}^m$, for some $r_\mathfrak{a}^m \in \mathbb{R}$ and some closed 1-form $\gamma_\mathfrak{a}^m \in \Omega^1(U_\mathfrak{a})$. In other words, $F^m|U_\mathfrak{a} = r_\mathfrak{a}^m \operatorname{div} + \eta_\mathfrak{a}^m$, where the one-form $\eta_\mathfrak{a}^m = \gamma_\mathfrak{a}^m + \alpha_\mathfrak{a}^m$. Since on an intersection $U_\mathfrak{a} \cap U_{\mathfrak{a}'}$ the constants $r_\mathfrak{a}^m$ and $r_{\mathfrak{a}'}^m$ must coincide with a single r^m (as M_1 is connected), the one-forms $\eta_\mathfrak{a}^m$ and $\eta_{\mathfrak{a}'}^m$ coincide as well. Hence $F^m = r^m \operatorname{div} + \eta^m$. Since div is a cocycle, we have $d\eta^m = \beta^m$. □

It is clear that if we choose $r = 0$ we get an isomorphism of Lie algebroids $\Phi_0 : \mathscr{A}_1 \ni c + \tilde{a} \xrightarrow{\simeq} \Phi^0(c + \eta(\hat{a})) + \Phi^s(\tilde{a}) \in \mathscr{A}_2$. It follows that $\Phi = \Phi_0 \Phi_1$, where $\Phi_1 : \mathscr{A}_1 \ni c + \tilde{a} \xrightarrow{\simeq} c + \tilde{a} + \operatorname{div}(\hat{a}) \cdot r \in \mathscr{A}_1$ is an automorphism of $\mathscr{A}_1$. When identifying the Atiyah algebroids with the model algebroids, we get Theorem 1.

References

[AM85] Almeida R., Molino P., *Suites d'Atiyah et feuilletages transversalement complets*, C. R. Acad. Sci. Paris Ser. I, Math. **300** (1985), 13–15

[AMS75] Amemiya I., Masuda K., Shiga K., *Lie algebras of differential operators*, Osaka J. Math. **12** (1975), 139–172

[Ati57] Atiyah M. F., *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207

[DS76] Duistermaat J. J., Singer I. M., *Order-preserving isomorphisms between algebras of pseudo-differential operators*, Comm. Pure Appl. Math. **29(1)** (1976), 39–47

[Ehr59] Ehresmann C., *Catégories topologiques et catégories différentiables*, Colloq. Géom. Diff. Globale, Bruxelles (1959), 137–150

[Gra78] Grabowski J., *Isomorphisms and Ideals of the Lie Algebras of Vector Fields*, Inventiones math. **50** (1978), 13–33

[Gra93] Grabowski J., *Lie algebras of vector fields and generalized foliations*, Publ. Math. **37** (1993), 359–367

[Gra05] Grabowski J., *Isomorphisms of algebras of smooth functions revisited*, Arch. Math. (Basel) **85** (2005), 190–196

[GG01] Grabowska K., Grabowski J., *Lie algebras of Lie algebroids*, in: Lie Algebroids and Related Topics in Differential Geometry, J. Kubarski et al. (eds.), Banach Center Publ. **54** (2001), 43–50

[GG08] Grabowska K., Grabowski J., *Variational calculus with constraints on general algebroids*, J. Phys A: Math. Theor. **41** (2008), 175204, 25 pp

[GGU06] Grabowska K., Grabowski J., Urbański P., *Geometrical Mechanics on Algebroids*, Int. J. Geom. Meth. Mod. Phys. **3** (2006), 559–575

[GP04] Grabowski J., Poncin N., *Automorphisms of quantum and classical Poisson algebras*, Compositio Math. **140** (2004), 511–527

[GP05] Grabowski J., Poncin N., *Derivations of the Lie algebras of differential operators*, Indag. Mathem. **16(2)** (2005), 181–200

[GP07] Grabowski J., Poncin N., *On Quantum and Classical Poisson Algebras*, Banach Center Publ. **76** (2007), 313–324

[Kub89] Kubarski J., *Lie algebroid of a principal bundle*, Publ. Dép. Math. Univ. C. Bernard **A, 89-1** (1989), 1–66

[LMM05] de Léon M., Marrero J. C., Martinez E., *Lagrangian submanifolds and dynamics on Lie algebroids*, J. Phys. A: Math Gen. **38(24)** (2005), 241–308

[Lib69] Libermann P., *Sur les prolongements des fibrés principaux et groupoïdes différentiables*, Séminaire Analyse Globale Montréal (1969), 7–108

[Mac05] Mackenzie K., *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press **213** (2005)

[Mar08] Marle C.-M., *Calculus on Lie algebroids, Lie groupoids and Poisson manifolds*, Dissertationes Math. **457** (2008), 57 pp

[Mar01] Martínez E., *Lagrangian Mechanics on Lie Algebroids*, Acta Appl. Math. **67** (2001), 295–320

[Mes05] Mestdag T., *A Lie algebroid approach to Lagrangian systems with symmetry*, Diff. Geo. appl. (2005), 523–535

[Mrc05] Mrcun J., *On isomorphisms of algebras of smooth functions*, Proc. Amer. Math. Soc. **133** (2005), 3109–3113 (electronic)

[Omo76] Omori H., *Infinite dimensional Lie transformation groups*, Lect. Notes in Math. **427** (1976), Springer Verlag

[Pon98] Poncin N., *Premier et deuxième espaces de cohomologie de l'algèbre de Lie des opérateurs différentiels sur une variété, à coefficients dans les fonctions*, Bull. Soc. Roy. Sci. Liège **67** **(6)** (1998), 291–337

[Pra67] Pradines J., *Théorie de Lie pour les groupoïdes différentiables, Calcul différentiel dans la catégorie des groupoïdes infinitésimaux*, C. R. A. S. Paris **264, Sér. A** (1967), 246–248

[PS54] Pursell L. E., Shanks M. E., *The Lie algebra of a smooth manifold*, Proc. Amer. Math. Soc. **5** (1954), 468–472

[Skr87] Skryabin S. M., *The regular Lie rings of derivations of commutative rings*, preprint WINITI 4403-W87 (1987)

[Wei96a] Weinstein A., *Groupoids: unifying internal and external symmetry*, AMS Notices **43** (1996), 744–752

[Wei96b] Weinstein A., *Lagrangian mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207–231

Janusz GRABOWSKI
Polish Academy of Sciences, Institute of Mathematics
Śniadeckich 8, P.O. Box 21, 00-956 Warsaw, Poland
Email: jagrab@impan.pl

Alexei KOTOV
University of Luxembourg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
Email: alexei.kotov@uni.lu

Norbert PONCIN
University of Luxembourg, Mathematics Research Unit
6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg City, Grand-Duchy of Luxembourg
E-mail: norbert.poncin@uni.lu