Dequantized Differential Operators between Tensor Densities as Modules over the Lie Algebra of Contact Vector Fields

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Abstract

In recent years, algebras and modules of differential operators have been extensively studied. Equivariant quantization and dequantization establish a tight link between invariant operators connecting modules of differential operators on tensor densities, and module morphisms that connect the corresponding dequantized spaces. In this paper, we investigate dequantized differential operators as modules over a Lie subalgebra of vector fields that preserve an additional structure. More precisely, we take an interest in invariant operators between dequantized spaces, viewed as modules over the Lie subalgebra of infinitesimal contact or projective contact transformations. The principal symbols of these invariant operators are invariant tensor fields. We first provide full description of the algebras of such affine-contact- and contact-invariant tensor fields. These characterizations allow showing that the algebra of projective-contact-invariant operators between dequantized spaces implemented by the same density weight, is generated by the vertical cotangent lift of the contact form and a generalized contact Hamiltonian. As an application, we prove a second key-result, which asserts that the Casimir operator of the Lie algebra of infinitesimal projective contact transformations, is diagonal. Eventually, this upshot is used to depict a family of basic invariant operators between spaces induced by different density weights.

Key-words: Modules of differential operators, tensor densities, contact geometry, invariant operators, representation theory of algebras, equivariant quantization and dequantization

MSC: 13N10, 53D10, 16G30

1 Introduction

Equivariant quantization, in the sense of C. Duval, P. Lecomte, and V. Ovsienko, developed as from 1996, see [LMT96], [LO99], [DLO99], [Lec00], [BM01], [DO01], [BHMP02], [BM06]. This procedure requires equivariance of the quantization map with respect to the action of a finite-dimensional Lie subgroup $G \subset \mathrm{Diff}(\mathbb{R}^n)$ of the symmetry group $\mathrm{Diff}(\mathbb{R}^n)$ of configuration space $\mathbb{R}^n$, or, on the infinitesimal level, with respect to the action of a Lie subalgebra of the Lie algebra of vector fields. Such quantization maps are well-defined globally on manifolds endowed with a flat G-structure and lead to invariant star-products, [LO99],[DLO99]. Equivariant quantization has first been studied on vector spaces, mainly for the projective and conformal groups, then extended in 2001 to arbitrary manifolds, see [Lec01]. In this setting, equivariance with respect to all arguments and for the action of the group of all (local) diffeomorphisms of the manifold (i.e. naturality in the sense of I. Kolář, P. W. Michor, and J. Slovák, [KMS93]) has been ensured via quantization maps that depend on (the projective class of) a connection. Existence of such natural and projectively invariant quantizations has been investigated in several works, [Bor02], [MR05], [Han06].

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From the very beginning, equivariant quantization and symbol calculus, and classification issues in Representation Theory of Algebras appeared as dovetailing topics, see [LMT96], [LO99], [Mat99,1], [BHMP02], [Pon04]. In these works differential operators between sections of vector bundles have been studied and classified as modules over the Lie algebra of vector fields. Except for [Mat99,2], the case of differential operators as representations of a subalgebra of vector fields that preserve some additional structure, was largely uninvestigated. The origin of this paper is the classification problem of differential operators on a contact manifold between tensor densities of possibly different weights (in the frame of equivariant quantization it is natural to consider linear differential operators between densities rather than between functions, as [even mathematical] quantization maps should be valued in a space of operators acting on a Hilbert or preHilbert space), as modules over the Lie subalgebra of contact vector fields.

Let us give a rough description of our approach to the preceding multilayer problem. Further details can be found below. Projectively equivariant quantization establishes a tight connection between the "quantum level"− classification of differential operators as representations of the algebra of contact vector fields, and the "classical level"− quest for intertwining operators between the corresponding modules of symbols over the subalgebra of infinitesimal projective contact transformations. These morphisms (in the category of modules) have (locally) again symbols and these are tensor fields. The principal symbol map intertwines the natural actions on morphisms and tensor fields. Hence, the principal symbol of any "classical" intertwining operator is an invariant tensor field. These invariant fields can be computed. However, it turns out that the obvious technique that should allow lifting invariant tensor fields to "classical" module morphisms is not sufficient for our purpose. The Casimir operator (of the representation of infinitesimal projective contact transformations on symbols) proves to be an efficient additional tool. Calculation of the Casimir itself requires a noncanonical splitting of the module of symbols. This decomposition has been elaborated in a separate paper, see [FMP07].

In the present work, we investigate the "classical level" problem, i.e. we study "dequantized" differential operators between tensor densities as modules over infinitesimal contact transformations.

The paper is self-contained and organized as follows.

In Section 2, we recall essential facts in Contact Geometry, which are relevant to subsequent sections. We place emphasis on global formulæ, as till very recently most of the results were of local nature.

Section 3 provides the whole picture related with infinitesimal projective contact transformations. A good understanding of these upshots is crucial, particularly as regards the calculation, in Section 5, of invariant tensor fields, and in consideration of the computation of the aforementioned Casimir operator, see Section 7.

Coordinate-free approaches to differential operators, their symbols, and all involved actions are detailed in Section 4. This material is of importance with respect to the geometric meaning of several invariant tensor fields constructed later.

In Section 5, we give a full description of the algebra of affine-contact-invariant tensor fields (local investigation), see Theorem 2, and of the algebra of contact-invariant tensor fields (global result), see Theorem 3.

A third main upshot, based on the preceding Section, is the assertion that the algebra of projective-contact-invariant operators between symbol modules "implemented by the same density weight", is generated by two basic operators, the vertical cotangent lift of the contact form and a generalized contact Hamiltonian, both introduced in [FMP07], see Theorem 4, Section 6.

As an application of the aforenoted noncanonical splitting of the module of symbols into submodules, see [FMP07], of Section 3, and of Section 6, we prove in Section 7, that the Casimir operator of the canonical representation of the Lie subalgebra of infinitesimal projective contact transformations on the mentioned symbol space, with respect to the Killing form, is diagonal, see Theorem 5.

Eventually, the computation of this Casimir operator—actually a challenge by itself—allows showing that "basic" projective-contact-invariant operators between symbol modules "implemented by different density weights", are necessarily powers of the generalized Hamiltonian or of the divergence operator, see Section 8.

2 Remarks on Contact Geometry

A contact structure on a manifold M is a co-dimension 1 smooth distribution ξ that is completely nonintegrable. Such a distribution is locally given by the kernel of a nowhere vanishing 1-form α defined up to multiplication by a never vanishing function. Since $(d\alpha)(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$, where notations are self-explaining, *integrability* of ξ, i.e. closeness of sections of ξ under the Lie bracket of vector fields, would require that $d\alpha$ vanish on vectors in ξ. By complete *nonintegrability* we mean that $d\alpha$ is nondegenerate in ξ, for any locally defining 1-form α. It follows that contact manifolds are necessarily odd-dimensional. Eventually we get the following definition.

Definition 1. *A contact manifold M is a manifold of odd dimension $2n+1$, together with a smooth distribution ξ of hyperplanes in the tangent bundle of M, such that $\alpha \wedge (d\alpha)^n$ is a nevervanishing top form for any locally defining 1-form α. Distribution ξ is a contact distribution or contact structure on M. A Pfaffian manifold (A. Lichnerowicz) or coorientable contact manifold is a manifold M with odd dimension $2n+1$ endowed with a globally defined differential 1-form α, such that $\alpha \wedge (d\alpha)^n$ is a volume form of M. Form α (which defines of course a contact distribution on M) is called a contact form on M.*

Example 1. Let $(p_1, \ldots, p_n, q^1, \ldots, q^n, t, \tau)$ be canonical coordinates in $\mathbb{R}^{2n+2}$, let $i : \mathbb{R}^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ be the embedding that identifies $\mathbb{R}^{2n+1}$ with the hyperplane $\tau = 1$ of $\mathbb{R}^{2n+2}$, and let σ be the Liouville 1-form of $\mathbb{R}^{2n+2}$ (which induces the canonical symplectic structure of $\mathbb{R}^{2n+2}$). It is easily checked that the pullback

$$\alpha = i^*\sigma = \frac{1}{2}(\sum_{k=1}^{n}(p_k dq^k - q^k dp_k) - dt)$$

of the Liouville form σ by embedding i is a contact form on $\mathbb{R}^{2n+1}$. Any coorientable contact manifold (M, α) can locally be identified with $(\mathbb{R}^{2n+1}, i^*\sigma)$, i.e. Darboux' theorem holds true for contact manifolds.

Remark 1. The preceding extraction of a contact structure from a symplectic structure is the shadow of a tight connection between contact and symplectic manifolds. If (M, α) is a coorientable contact manifold, if $\pi : M \times \mathbb{R} \to M$ is the canonical projection, and s a coordinate function in $\mathbb{R}$, the form $\omega = d(e^s \pi^*\alpha)$ is a symplectic form on $M \times \mathbb{R}$, which is homogeneous with respect to ∂_s, i.e. $L_{\partial_s}\omega = \omega$. This symplectic homogeneous manifold $(M \times \mathbb{R}, \omega, \partial_s)$ is known as the *symplectization* the initial contact manifold (M, α). Actually, there is a 1-to-1 correspondence between coorientable contact structures on M and homogeneous symplectic structures on $M \times \mathbb{R}$ (with vector field ∂_s). This relationship extends from the contact-symplectic to the Jacobi-Poisson setting, see [Lic78], for super-Poissonization, see [GIMPU04]. A coordinate-free description of symplectization is possible. Consider a contact manifold (M, ξ), let $L \subset T^*M$ be the line subbundle of the cotangent bundle, made up by all covectors that vanish on ξ, and denote by L_0 the submanifold of L obtained by removing the 0-section. The restriction to L_0 of the standard symplectic form of T^*M endows L_0 with a symplectic structure, see [Arn89, Ovs05]. Eventually, a contact structure on a manifold M can be viewed as a line subbundle L of the cotangent bundle T^*M such that the restriction to L_0 of the standard symplectic form on T^*M is symplectic. Of course, the contact structure is coorientable if and only if L is trivial.

Example 2. Let $\alpha = i^*\sigma$ be the standard contact form of $\mathbb{R}^{2n+1}$. Set $x = (p, q, t, \tau) = (x', \tau)$. The open half space $\mathbb{R}^{2n+2}_+ = \{(x', \tau) : \tau > 0\}$, endowed with its canonical symplectic structure ω and the Liouville vector field $\Delta = \frac{1}{2}\mathcal{E}$, where $\mathcal{E}$ is the usual Euler field, can be viewed as symplectization of $(\mathbb{R}^{2n+1}, \alpha)$. Indeed, $\mathbb{R}^{2n+2}_+ \simeq \cup_{x' \in \mathbb{R}^{2n+1}} \{x'\} \times \{\tau(x', 1) : \tau > 0\}$ is a line bundle over $\mathbb{R}^{2n+1}$ with fiber coordinate $s = \ln \tau^2$. The projection of this bundle reads $\pi : \mathbb{R}^{2n+2}_+ \ni (x', \tau) \to \tau^{-1}x' \in \mathbb{R}^{2n+1}$. Clearly, ω has degree 1 with respect to Δ and it is easily checked that $\omega = d(e^s \pi^*\alpha)$ and $\Delta = \partial_s$.

In the following, unless otherwise stated, we consider coorientable contact manifolds (or trivial line bundles).

Definition 2. *Let M be a (coorientable) contact manifold. A contact vector field is a vector field X of M that preserves the contact distribution. In other words, for any fixed contact form α, there is a function $f_X \in C^\infty(M)$, such that $L_X\alpha = f_X\alpha$. We denote by* $\mathrm{CVect}(M)$ *the space of contact vector fields of M.*

It is easily seen that space CVect(M) is a Lie subalgebra of the Lie algebra $\mathrm{Vect}(M)$ of all vector fields of M, but not a $C^\infty(M)$-module.

Let us now fix a contact form α on M and view $d\alpha$ as a bundle map $d\alpha : TM \to T^*M$. It follows from the nondegeneracy condition that the kernel $\ker d\alpha$ is a line bundle and that the tangent bundle of M is canonically split: $TM = \ker\alpha \oplus \ker d\alpha$. Moreover,

$$\mathrm{Vect(M)} = \ker\alpha \oplus \ker d\alpha, \tag{1}$$

where α and $d\alpha$ are now viewed as maps between sections. It is clear that there is a unique vector field E, such that $i_E d\alpha = 0$ and $i_E\alpha = 1$ (normalization condition). This field is called the *Reeb vector field*. It is *strongly contact* in the sense that $L_E\alpha = 0$.

Pfaffian structures, just as symplectic structures, can be described by means of contravariant tensor fields. These fields are obtained from α and $d\alpha$ via the musical map $\flat : \mathrm{Vect}(M) \ni X \to (i_X\alpha)\alpha + i_X d\alpha \in \Omega^1(M)$, which is a $C^\infty(M)$-module isomorphism, see e.g. [LLMP99]. The contravariant objects in question are the Reeb vector field $E = \flat^{-1}(\alpha) \in \mathrm{Vect}(M) =: \mathcal{X}^1(M)$ and the bivector field $\Lambda \in \mathcal{X}^2(M)$, defined by

$$\Lambda(\beta,\gamma) = (d\alpha)(\flat^{-1}(\beta), \flat^{-1}(\gamma)),$$

$\beta, \gamma \in \Omega^1(M)$. They verify $[\Lambda,\Lambda]_{\mathrm{SCH}} = 2E\wedge\Lambda$ and $L_E\Lambda = 0$, where $[.,.]_{\mathrm{SCH}}$ is the Schouten-Nijenhuis bracket. Hence, any (coorientable) contact manifold is a *Jacobi manifold*.

Let us recall that Jacobi manifolds are precisely manifolds M endowed with a vector field E and a bivector field Λ that verify the two preceding conditions. The space of functions of a Jacobi manifold (M, Λ, E) carries a Lie algebra structure, defined by

$$\{h, g\} = \Lambda(dh, dg) + hEg - gEh, \tag{2}$$

$h, g \in C^\infty(M)$. The Jacobi identity for this bracket is equivalent with the two conditions $[\Lambda,\Lambda]_{\mathrm{SCH}} = 2E\wedge\Lambda$ and $L_E\Lambda = 0$ for Jacobi manifolds (these conditions can also be expressed in terms of the Nijenhuis-Richardson bracket, see [NR67]). It is well-known that the "Hamiltonian map"

$$X : C^\infty(M) \ni h \to X_h = i_{dh}\Lambda + hE \in \mathrm{Vect}(M) \tag{3}$$

is a Lie algebra homomorphism: $X_{\{h,g\}} = [X_h, X_g]$. If $\dim M = 2n+1$ and $E \wedge \Lambda^n$ is a nowhere vanishing tensor field, manifold M is coorientably contact. Furthermore, if we fix, in the Pfaffian case, a contact form α, we get a Lie algebra isomorphism

$$X : C^\infty(M) \ni h \to X_h \in \mathrm{CVect}(M) \tag{4}$$

between functions and contact vector fields, see [Arn89]. It follows from the above formulæ that $\alpha(X_h) = h$.

The main observation is that Jacobi brackets, see (2), are first order bidifferential operators. This fact is basic in many recent papers, see e.g. [GM03] (inter alia for an elegant approach to graded Jacobi cohomology), or [GIMPU04] (for Poisson-Jacobi reduction).

After the above global formulæ and fundamental facts on Contact Geometry, we continue with other remarks that are of importance for our investigations. The setting is still a $(2n+1)$-dimensional contact manifold M with fixed contact form α. Contraction of the equation $L_{X_h}\alpha = f_{X_h}\alpha$, $h \in C^\infty(M)$, with the Reeb field E leads to $f_{X_h} = E(h)$. If Ω denotes the volume $\Omega = \alpha\wedge(d\alpha)^n$, it is clear that, for any contact vector field X, we have $L_X\Omega = (n+1)f_X\,\Omega$. Hence,

$$\mathrm{div}_\Omega X = (n+1)f_X, \forall X \in \mathrm{CVect}(M), \tag{5}$$

and $\operatorname{div}_\Omega X_h = (n+1)E(h)$, for any $h \in C^\infty(M)$. It follows that for all $h, g \in C^\infty(M)$,

$$\{h, g\} = X_h(g) - gE(h) = X_h(g) - \frac{1}{n+1} g \operatorname{div}_\Omega X_h = L_{X_h}\tilde{g}, \tag{6}$$

where $\tilde{g}$ is function g viewed as tensor density of weight $-1/(n+1)$. Tensor densities will be essential below. For details on densities, we refer the reader to [FMP07]. The afore-depicted Lie algebra isomorphism X between functions and contact vector fields, is also a $\mathrm{CVect}(M)$-module isomorphism, if we substitute the space $\mathcal{F}_{\frac{-1}{n+1}}(M)$ of tensor densities of weight $-1/(n+1)$ for the space of functions (of course, the contact action is $L_{X_{\tilde{h}}}\tilde{g} = \{\tilde{h}, \tilde{g}\}$ on densities, and it is the adjoint action on contact fields). Note that this distinction between functions and densities is necessary only if the module structure is concerned.

We now come back to splitting (1). If we denote by TVect(M) the space of tangent vector fields, i.e. the space $\ker \alpha$ of those vector fields of M that are tangent to the contact distribution, this decomposition also reads

$$\mathrm{Vect}(M) = \mathrm{TVect(M)} \oplus C^\infty(M)E.$$

As abovementioned, our final goal is the solution of the multilayer classification problem of differential operators between tensor densities on a contact manifold, as modules over the Lie algebra of contact vector fields. This question naturally leads to the quest for a splitting of some $\mathrm{CVect}(M)$-modules or $\mathrm{sp}_{(2n+2)}$-modules of symbols, see below, and in particular of the module Vect(M) itself. Space $\mathrm{TVect}(M)$, which is of course not a Lie algebra, is a $C^\infty(M)$-module and a $\mathrm{CVect}(M)$-module. The last upshot follows directly from formula $i_{[X,Y]} = [L_X, i_Y]$, $X, Y \in \mathrm{Vect(M)}$. The second factor $C^\infty(M)E$ however, is visibly not a CVect(M)-module (for instance $[E, X_h] = [X_1, X_h] = X_{E(h)} = i_{d(E(h))}\Lambda + E(h)E$). In [Ovs05], V. Ovsienko proved the noncanonical decomposition

$$\mathrm{Vect}(M) \simeq \mathrm{TVect}(M) \oplus \mathrm{CVect(M)} \tag{7}$$

of $\mathrm{Vect}(M)$ into a direct sum of $\mathrm{CVect}(M)$-modules. An extension of this decomposition, see [FMP07], will be exploited below.

3 Infinitesimal projective contact transformations

Let us first recall that the symplectic algebra $\mathrm{sp}(2n, \mathbb{C})$ is the Lie subalgebra of $\mathrm{gl}(2n, \mathbb{C})$ made up by those matrices S that verify $J_n S + \widetilde{S} J_n = 0$, where J_n is the symplectic unit. This condition exactly means that the symplectic form defined by J_n is invariant under the action of S. Since

$$\mathrm{sp}(2n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathrm{gl}(n, \mathbb{C}), \tilde{B} = B, \tilde{C} = C, D = -\tilde{A} \right\},$$

it is obvious that

$$\epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n}\ (i \le j \le n),\ -\epsilon^{j+n} \otimes e_i - \epsilon^{i+n} \otimes e_j\ (i \le j \le n),\ -\epsilon^j \otimes e_i + \epsilon^{i+n} \otimes e_{j+n}\ (i, j \le n), \tag{8}$$

is a basis of $\mathrm{sp}(2n, \mathbb{C})$. As usual, we denote by $(e_1, \ldots, e_{2n})$ the canonical basis of $\mathbb{C}^{2n}$ and by $(\epsilon^1, \ldots, \epsilon^{2n})$ its dual basis.

Observe now that the Jacobi (or [first] Lagrange) bracket on a contact manifold (M, α) can be built out of contact form α, see Equation (2), or—in view of the aforementioned 1-to-1 correspondence—out of the homogeneous symplectic structure of the symplectization. In the following, we briefly recall the construction via symplectization, see [Mat99,2], of the Lagrange bracket, contact vector fields, and the Lie algebra isomorphism $X : C^\infty(M) \to \mathrm{CVect}(M)$. We then use isomorphism X to depict Lie subalgebras of contact fields, which play a central role in this work. As part of our construction is purely local, we confine ourselves to the Euclidean setting.

Take contact manifold $(\mathbb{R}^{2n+1}, \alpha)$, $\alpha = i^*\sigma$, and its symplectization $(\pi : \mathbb{R}_+^{2n+2} \to \mathbb{R}^{2n+1}, \omega, \Delta)$, see Example 2. Let $\mathcal{H}^\lambda = \mathcal{H}^\lambda(\mathbb{R}_+^{2n+2}) = \{H \in C^\infty(\mathbb{R}_+^{2n+2}) : \Delta H = \lambda H\}$ be the space of homogeneous functions of degree λ. Since ω has degree 1, its contravariant counterpart Π has degree -1, and the corresponding Poisson bracket verifies $\{\mathcal{H}^\lambda, \mathcal{H}^\mu\}_\Pi \subset \mathcal{H}^{\lambda+\mu-1}$. In particular, $\mathcal{H}^1$ is a Lie subalgebra.

If $\mathcal{X}_H = \{H, .\}_\Pi$ is the Hamiltonian vector field of a function $H \in \mathcal{H}^1$, we have $[\Delta, \mathcal{X}_H] = 0$. Hence, $\mathcal{X}_H$ is projectable, i.e. $\pi_*\mathcal{X}_H$ is well-defined. As $\mathcal{X}_H$ is a symplectic vector field, $\pi_*\mathcal{X}_H$ is a contact field. The correspondence $\pi_* \circ \mathcal{X} : \mathcal{H}^1(\mathbb{R}_+^{2n+2}) \to \mathrm{CVect}(\mathbb{R}^{2n+1})$ is obviously a Lie algebra homomorphism. Remark now that a homogeneous function is known on the entire fiber $\pi^{-1}(x') = \tau(x', 1)$ if it is specified on the point $(x', 1)$. Hence, homogeneous functions are in fact functions on the base. The correspondence is (for functions of degree 1) of course

$$\chi : \mathcal{H}^1(\mathbb{R}_+^{2n+2}) \ni H \to h \in C^\infty(\mathbb{R}^{2n+1}), \tag{9}$$

with $h(x') = H(x', 1)$ and $H(x', \tau) = \tau^2 H(\tau^{-1}x', 1) = \tau^2 h(\pi(x', \tau))$ (note that $\Delta = \frac{1}{2}\mathcal{E}$). As every contact vector field is characterized by a unique base function $h \in C^\infty(\mathbb{R}^{2n+1})$, see Equation (4), hence by a unique homogeneous function $H \in \mathcal{H}^1(\mathbb{R}_+^{2n+2})$, morphism $\pi_* \circ \mathcal{X}$ is actually a Lie algebra isomorphism.

Map χ is a vector space isomorphism that allows to push the Poisson bracket $\{.,.\}_\Pi$ to the base. The resultant bracket $\{h, g\} = \chi\{\chi^{-1}h, \chi^{-1}g\}_\Pi$ is the Lagrange bracket. Eventually, χ is a Lie algebra isomorphism.

$$\begin{tikzcd}
 & \mathcal{H}^1(\mathbb{R}_+^{2n+2}) \arrow[dl, "\chi"'] \arrow[dr, "\pi_* \circ \mathcal{X}"] & \\
C^\infty(\mathbb{R}^{2n+1}) \arrow[rr, "\mathrm{X}"'] & & \mathrm{CVect}(\mathbb{R}^{2n+1})
\end{tikzcd}$$

It is now easily checked that "contact Hamiltonian isomorphism" X is, for any $h \in C^\infty(\mathbb{R}^{2n+1})$, given by

$$X_h = \pi_*\mathcal{X}_{\chi^{-1}h} = \sum_k (\partial_{p_k} h \partial_{q^k} - \partial_{q^k} h \partial_{p_k}) + \mathcal{E}_s h \partial_t - \partial_t h \mathcal{E}_s - 2h\partial_t, \tag{10}$$

where $(p_1, \ldots, p_n, q^1, \ldots, q^n, t) = (p, q, t) = x'$ are canonical coordinates in $\mathbb{R}^{2n+1}$ and where $\mathcal{E}_s = \sum_k (p_k \partial_{p_k} + q^k \partial_{q^k})$ is the spatial Euler field. When comparing this upshot with Equations (3) and (2), we get the explicit local form of the Lagrange bracket.

We now depict the aforementioned Lie subalgebras of contact vector fields as algebras of contact Hamiltonian vector fields of Lie subalgebras of functions. The algebra of Hamiltonian vector fields of the Lie subalgebra $\mathrm{Pol}(\mathbb{R}^{2n+1}) \subset C^\infty(\mathbb{R}^{2n+1})$ of polynomial functions is the Lie subalgebra $\mathrm{CVect}^*(\mathbb{R}^{2n+1})$ of polynomial contact vector fields, i.e. contact vector fields with polynomial coefficients. The space of polynomials admits the decomposition $\mathrm{Pol}(\mathbb{R}^{2n+1}) = \oplus_{r\in\mathbb{N}} \oplus_{k=0}^r \mathcal{P}^{rk}$, where $\mathcal{P}^{rk}$ is the space of polynomials $t^k P_{r-k}(p, q)$ of homogeneous total degree r that have homogeneous degree k in t. The Lie subalgebra

$$\mathrm{Pol}^{\leq 2}(\mathbb{R}^{2n+1}) = \oplus_{r\leq 2} \oplus_{k=0}^r \mathcal{P}^{rk},$$

which corresponds via χ^{-1} to the Lie subalgebra

$$\mathrm{Pol}^2(\mathbb{R}_+^{2n+2}) \subset \mathcal{H}^1(\mathbb{R}_+^{2n+2}),$$

deserves particular attention (note that if we set $\mathfrak{g}_{-2} = \mathcal{P}^{00}$, $\mathfrak{g}_{-1} = \mathcal{P}^{10}$, $\mathfrak{g}_0 = \mathcal{P}^{11} \oplus \mathcal{P}^{20}$, $\mathfrak{g}_1 = \mathcal{P}^{21}$, and $\mathfrak{g}_2 = \mathcal{P}^{22}$, we obtain a grading of $\mathrm{Pol}^{\leq 2}(\mathbb{R}^{2n+1})$ that is compatible with the Lie bracket [when read on the symplectic level, this new grading means that we assign the degree -1 to coordinate τ, degree 1 to t, and degree 0 to any other coordinate]).

Remember now the Lie algebra (anti) isomorphism $J : \mathrm{gl}(m, \mathbb{R}) \ni A \to A^i_j x^j \partial_{x^i} \in \mathrm{Vect}^0(\mathbb{R}^m)$ between the algebras of matrices and of linear vector fields (shifted degree). Its inverse is the Jacobian map $J^{-1} : \mathrm{Vect}^0(\mathbb{R}^m) \ni X \to \partial_x X \in \mathrm{gl}(m, \mathbb{R})$. The Lie subalgebra $\mathrm{Pol}^2(\mathbb{R}^{2n}) = \mathcal{P}^{20}$ is mapped by Lie algebra isomorphism X (remark that on the considered subalgebra $X = \mathcal{X}$ and $\{.,.\} = \{.,.\}_\Pi$) onto a Lie subalgebra of $\mathrm{CVect}(\mathbb{R}^{2n+1})$ and of $\mathrm{Vect}^0(\mathbb{R}^{2n})$, which in turn corresponds through J^{-1} to a Lie subalgebra of $\mathrm{gl}(2n, \mathbb{R})$. A simple computation shows that the natural basis

$$p_i p_j \ (i \leq j \leq n),\ q^i q^j \ (i \leq j \leq n),\ p_j q^i \ (i, j \leq n) \tag{11}$$

of $\mathrm{Pol}^2(\mathbb{R}^{2n})$ is transformed by morphism $J^{-1} \circ \mathcal{X}$ into the above described basis of $\mathrm{sp}(2n, \mathbb{R})$, see Equation (8). It is now clear that $J^{-1} \circ \mathcal{X}$ is a Lie algebra isomorphism between $(\mathrm{Pol}^2(\mathbb{R}^{2n}), \{.,.\}_\Pi)$ and $(\mathrm{sp}(2n, \mathbb{R}), [.,.]_\circ)$, where $[.,.]_\circ$ is the commutator. We denote by sp_{2n} the Lie subalgebra of contact vector fields isomorphic to $\mathrm{Pol}^2(\mathbb{R}^{2n}) \simeq \mathrm{sp}(2n, \mathbb{R})$.

Eventually, we have the following diagram :

$$\begin{tikzcd}
\mathrm{Pol}^2(\mathbb{R}^{2n+2}_+) \arrow[r, "J^{-1} \circ \mathcal{X}"] \arrow[d, "\chi"'] \arrow[dr, "\pi_* \circ \mathcal{X}"] & \mathrm{sp}(2n+2, \mathbb{R}) \arrow[d] \\
\mathrm{Pol}^{\leq 2}(\mathbb{R}^{2n+1}) \arrow[r, "\mathrm{X}"'] & \mathrm{sp}_{2n+2}
\end{tikzcd}$$

It is obvious that the right bottom algebra is a Lie subalgebra of contact vector fields that is isomorphic with $\mathrm{sp}(2n+2, \mathbb{R})$: hence the notation. The right vertical arrow refers to the embedding of $\mathrm{sp}(2n+2, \mathbb{R})$ into $\mathrm{CVect}(\mathbb{R}^{2n+1})$ that can be realized just as the projective embedding of $\mathrm{sl}(m+1, \mathbb{R})$ into $\mathrm{Vect}(\mathbb{R}^m)$. More precisely, the linear symplectic group $\mathrm{SP}(2n+2, \mathbb{R})$ naturally acts on $\mathbb{R}^{2n+2}$ by linear symplectomorphisms. The projection $\rho(S)(x') := \pi(S.(x', 1))$, $S \in \mathrm{SP}(2n+2, \mathbb{R})$, $x' \in \mathbb{R}^{2n+1}$, of this action "." induces a "local" action on $\mathbb{R}^{2n+1}$. The tangent action to projection ρ is a Lie algebra homomorphism that maps the symplectic algebra $\mathrm{sp}(2n+2, \mathbb{R})$ into contact vector fields $\mathrm{CVect}(\mathbb{R}^{2n+1})$. We refer to the Lie subalgebra generated by the fundamental vector fields associated with this infinitesimal action as the algebra of *infinitesimal projective contact transformations*. This algebra sp_{2n+2} is a maximal proper Lie subalgebra of $\mathrm{CVect}^*(\mathbb{R}^{2n+1})$ (just as the projective embedding $\mathrm{sl}_{\mathrm{m}+1}$ of $\mathrm{sl}(m+1, \mathbb{R})$ is a maximal proper Lie subalgebra of $\mathrm{Vect}^*(\mathbb{R}^m)$). Over a Darboux chart, any $(2n+1)$-dimensional contact manifold can be identified with $(\mathbb{R}^{2n+1}, i^*\sigma)$. It is therefore natural to consider sp_{2n+2} as a subalgebra of vector fields over the chart.

Eventually, a basis of sp_{2n+2} can be deduced via isomorphism X from the canonical basis of

$$\mathrm{Pol}^{\leq 2}(\mathbb{R}^{2n+1}) = \oplus_{r \leq 2} \oplus_{k=0}^r \mathcal{P}^{rk}.$$

Using Equation (10), we immediately verify that the contact Hamiltonian vector fields of $1 \in \mathcal{P}^{00}$ and $p_i, q^i \in \mathcal{P}^{10}$ are

$$X_1 = -2\partial_t = E, X_{p_i} = \partial_{q^i} - p_i \partial_t, X_{q^i} = -\partial_{p_i} - q^i \partial_t. \tag{12}$$

These fields generate a Lie algebra $\tilde{\mathfrak{h}}_{n,1}$ that is isomorphic to the Heisenberg algebra $\mathfrak{h}_n$. Let us recall that the Heisenberg algebra $\mathfrak{h}_n$ is a nilpotent Lie algebra with basis vectors $(a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ that verify the commutation relations

$$[a_i, b_j] = \delta_{ij} c, [a_i, a_j] = [b_i, b_j] = [a_i, c] = [b_i, c] = 0.$$

Similarly the Hamiltonian vector field of $t \in \mathcal{P}^{11}$ is the modified Euler field

$$X_t = -\mathcal{E}_s - 2t\partial_t, \tag{13}$$

and the Hamiltonian vector fields of $p_ip_j, q^iq^j, p_jq^i \in \mathcal{P}^{20}$ are

$$X_{p_ip_j} = p_j\partial_{q^i} + p_i\partial_{q^j}\ (i \leq j \leq n),\ X_{q^iq^j} = -q^j\partial_{p_i} - q^i\partial_{p_j}\ (i \leq j \leq n),\ X_{p_jq^i} = q^i\partial_{q^j} - p_j\partial_{p_i}\ (i, j \leq n). \tag{14}$$

These fields form the basis of sp_{2n} that corresponds via J to the basis of $\mathrm{sp}(2n, \mathbb{R})$ specified in Equation (8). Finally, $p_it, q^it \in \mathcal{P}^{21}$, and $t^2 \in \mathcal{P}^{22}$ induce the fields

$$X_{p_it} = t(\partial_{q^i} - p_i\partial_t) - p_i\mathcal{E}_s = t\partial_{q^i} - p_i\mathcal{E},\ X_{q^it} = t(-\partial_{p_i} - q^i\partial_t) - q^i\mathcal{E}_s = -t\partial_{p_i} - q^i\mathcal{E},\ X_{t^2} = -2t\mathcal{E}. \tag{15}$$

Again these fields generate a Lie algebra $\tilde{\mathfrak{h}}_{n,2}$ that is a model of the Heisenberg algebra $\mathfrak{h}_n$.

Observe also that the contact Hamiltonian vector field of a member of $\mathcal{P}^{rk}$ $(r \geq 3)$ is a polynomial contact field of degree r. Finally, the algebra $\tilde{\mathfrak{h}}_{n,1} \oplus \mathbb{R}X_t \oplus \mathrm{sp}_{2n}$ is the algebra $\mathrm{AVect}(\mathbb{R}^{2n+1}) \cap \mathrm{CVect}(\mathbb{R}^{2n+1})$ of affine contact vector fields.

4 Differential operators, symbols, actions, tensor densities

Let $\pi : E \to M$ and $\tau : F \to M$ be two (finite rank) vector bundles over a (smooth m-dimensional) manifold M.

We denote by $\mathcal{D}_k(E, F)$, $k \in \mathbb{N}$, the space of *kth order linear differential operators* between the spaces $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$ of smooth global sections of E and F (in the following we simply write $\Gamma(E)$ or $\Gamma(F)$), i.e. the space of the linear maps $D \in \mathrm{Hom}_{\mathbb{R}}(\Gamma(E), \Gamma(F))$ that factor through the kth jet bundle J^kE (i.e. for which there is a bundle map $\tilde{D} : J^kE \to F$, such that $D = \tilde{D} \circ i^0$, where $i^0 : E \to J^kE$ is the canonical injection and where the RHS is viewed as a map between sections). It is obvious that $0-$ order differential operators are just the sections $\Gamma(\mathrm{Hom(E, F)}) \simeq \Gamma(E^* \otimes F)$ and that the space $\mathcal{D}(E, F) = \cup_k \mathcal{D}_k(E, F)$ of all linear differential operators between E and F (or better between $\Gamma(E)$ and $\Gamma(F)$) is filtered by the order of differentiation.

The *kth order principal symbol* $\sigma_k(D)$ of an operator $D \in \mathcal{D}_k(E, F)$ is the map $\tilde{D} \circ i^k$, where i^k denotes the canonical injection $i^k : \mathcal{S}^kT^*M \otimes E \to J^kE$. Actually, this compound map is a bundle morphism $\sigma_k(D) : \mathcal{S}^kT^*M \otimes E \to F$, or, equivalently, a section $\sigma_k(D) \in \Gamma(\mathcal{S}^kTM \otimes E^* \otimes F)$. In the following, we call symbol space (associated with $\mathcal{D}(E, F)$), and denote by $\mathcal{S}(E, F)$, the graded space $\mathcal{S}(E, F) = \oplus_k \mathcal{S}^k(E, F)$, where $\mathcal{S}^k(E, F) := \Gamma(\mathcal{S}^kTM \otimes E^* \otimes F)$. Since $\sigma_k : \mathcal{D}_k(E, F) \to \mathcal{S}^k(E, F)$ is a linear surjection, it induces a vector space isomorphism between the graded space associated with the filtered space $\mathcal{D}(E, F)$ and the graded space $\mathcal{S}(E, F)$.

Roughly spoken, an *equivariant or natural quantization* is a vector space isomorphism $Q : \mathcal{S}(E, F) \to \mathcal{D}(E, F)$ that verifies some normalization condition and intertwines the actions on $\mathcal{S}(E, F)$ and $\mathcal{D}(E, F)$ of some symmetry group G of base manifold M. However, in order to define such actions, the action ϕ^M of G on M should lift to E (and F) as an action ϕ^E (resp. ϕ^F) of G by vector bundle maps $\phi^E_g : E \to E$, $g \in G$, over the corresponding maps $\phi^M_g : M \to M$. Actually, the action $\phi^{\Gamma(E)}$ of G on $\Gamma(E)$ can then be defined by

$$(\phi^{\Gamma(E)}_g s)_x := \phi^E_{g^{-1}} s_{\phi^M_g(x)},$$

$g \in G$, $s \in \Gamma(E)$, $x \in M$, and the action $\phi^{\mathcal{D}}$ of G on $\mathcal{D}(E, F)$ is

$$\phi^{\mathcal{D}}_g D := \phi^{\Gamma(F)}_g \circ D \circ \phi^{\Gamma(E)}_{g^{-1}},$$

for any $g \in G, D \in \mathcal{D}(E, F)$. Eventually, there is also a canonical action $\phi^{\mathcal{S}}$ on symbols. Indeed, for any $g \in G$, $P \in \mathcal{S}^k(E, F) = \Gamma(\mathcal{S}^kTM \otimes E^* \otimes F)$, $x \in M$, $e \in E_x$, it suffices to set

$$(\phi^{\mathcal{S}}_g P)_x(e) := (\mathcal{S}^kT\phi^M_{g^{-1}} \otimes \phi^F_{g^{-1}})P_{\phi^M_g(x)}(\phi^E_g e).$$

The appropriate setting for such investigations is the framework of natural functors (for all questions related with natural functors and natural operations, we refer the reader to [KMS93], for a functorial approach to natural quantization, see [Bor02]). Indeed, let $\mathbb{F}$ and $\mathbb{F}'$ be two natural vector bundle functors and consider differential operators and symbols between the vector bundles $E = \mathbb{F}M$ and $F = \mathbb{F}'M$ over an m-dimensional smooth manifold M. If now $\phi_g^M : M \to M$ is a local diffeomorphism, then $\phi_g^E = \mathbb{F}\phi_g^M : \mathbb{F}M \to \mathbb{F}M$ (resp. $\phi_g^F = \mathbb{F}'\phi_g^M$) is a vector bundle map over ϕ_g^M. Hence, actions on the base lift canonically and actions of the group $\mathrm{Diff}(M)$ of local diffeomorphisms of M (and of the algebra $\mathrm{Vect}(M)$ of vector fields of M) can be defined (as detailed above) on sections $\Gamma(\mathbb{F}M)$ and $\Gamma(\mathbb{F}'M)$, as well as on differential operators $\mathcal{D}(\mathbb{F}M, \mathbb{F}'M)$ and symbols $\mathcal{S}(\mathbb{F}M, \mathbb{F}'M)$ between these spaces of sections.

Remember now that there is a 1-to-1 correspondence between representations of the jet group G_m^r on vector spaces V and natural vector bundle functors $\mathbb{F}$ of order r on the category of m-dimensional smooth manifolds M, see [KMS93, Proposition 14.8]. The objects of such a functor are the vector bundles $\mathbb{F}M = P^rM \times_{G_m^r} V$ associated with the rth order frame bundles P^rM. So the canonical representation of $G_m^1 = \mathrm{GL}(m, \mathbb{R})$ on the (rank 1) vector spaces $\Delta^\lambda \mathbb{R}^m$ ($\lambda \in \mathbb{R}$) of λ-densities on $\mathbb{R}^m$ induces a 1-parameter family of natural 1st order vector bundle functors $\mathbb{F}_\lambda$. Hence, we get $\mathrm{Diff}(M)$- and $\mathrm{Vect}(M)$-actions on sections of the (trivial) line bundles

$$\mathbb{F}_\lambda M = P^1M \times_{\mathrm{GL}(m,\mathbb{R})} \Delta^\lambda \mathbb{R}^m = \Delta^\lambda TM$$

of λ-densities of M, i.e. on tensor densities $\mathcal{F}_\lambda(M) := \Gamma(\mathbb{F}_\lambda M)$ of order λ of M. As aforementioned these actions generate actions on differential operators $\mathcal{D}_{\lambda\mu}(M) := \mathcal{D}(\mathbb{F}_\lambda M, \mathbb{F}_\mu M)$ between tensor densities of weights λ and μ, and on the corresponding symbols $\mathcal{S}_\delta(M) := \mathcal{S}(\mathbb{F}_\lambda M, \mathbb{F}_\mu M) = \Gamma(\mathcal{S}TM \otimes \mathbb{F}_\lambda^* M \otimes \mathbb{F}_\mu M) = \Gamma(\mathcal{S}TM \otimes \mathbb{F}_\delta M)$, where $\delta = \mu - \lambda$.

Primarily the local forms of these actions are well-known. Below, we focus on the algebra actions rather than on the group actions. Let us recall that triviality of the line bundles $\mathbb{F}_\lambda M$ has been proven via construction of a nowhere vanishing section ρ_0 of $\mathbb{F}_1 M = \Delta^1 TM$ that has at each point only strictly positive values. If the considered manifold M is orientable, we can set $\rho_0 = |\Omega|$, where Ω is a volume of M. Let us choose such a trivialization ρ_0. The correspondences $\tau_0^\lambda : C^\infty(M) \ni f \to f\rho_0^\lambda \in \mathcal{F}_\lambda(M)$ are then vector space isomorphisms and the actions L^λ of vector fields on the spaces $\mathcal{F}_\lambda(M)$ are, for any $X \in \mathrm{Vect}(M)$ and any $f \in C^\infty(M)$, given by

$$L_X^\lambda(f\rho_0^\lambda) = (X(f) + \lambda f \operatorname{div}_{\rho_0} X)\rho_0^\lambda.$$

Furthermore, for any $X \in \mathrm{Vect}(M)$, $D \in \mathcal{D}_{\lambda\mu}(M)$, and $P \in \mathcal{S}_\delta(M)$, we have

$$\mathcal{L}_X D = L_X^\mu \circ D - D \circ L_X^\lambda,$$

and

$$L_X P = X^\sharp(P) + \delta P \operatorname{div}_{\rho_0} X, \tag{16}$$

where $X^\sharp$ denotes the cotangent lift of X and where we have omitted in the LHS the dependance of the actions on λ, μ, and on δ, respectively.

5 Invariant tensor fields

Consider a $(2n+1)$-dimensional smooth Hausdorff second countable (coorientable) contact manifold (M, α). Let us recall that this work is originated from the classification problem of the spaces $(\mathcal{D}_{\lambda\mu}(M), \mathcal{L})$ as modules over the Lie algebra $\mathrm{CVect}(M)$ of contact vector fields. A first approximation is the computation of the intertwining operators T between the corresponding $\mathrm{CVect}(M)$-modules $(\mathcal{S}_\delta(M), L)$. Note that locally these symbol spaces are also modules over the Lie subalgebra $\mathrm{sp}_{2n+2} \subset \mathrm{CVect}(M)$. If such a module morphism

$$T : \mathcal{S}_\delta^\ell(M) = \Gamma(\mathcal{S}^\ell TM \otimes \mathbb{F}_\delta M) \to \mathcal{S}_\varepsilon^m(M) = \Gamma(\mathcal{S}^m TM \otimes \mathbb{F}_\varepsilon M)$$

is a kth order differential operator, its principal symbol $\sigma_k(T) \in \Gamma(\mathcal{S}^k TM \otimes \mathcal{S}^m TM \otimes \mathcal{S}^\ell T^*M \otimes \mathbb{F}_\nu M)$, $\nu = \varepsilon - \delta$, is (roughly spoken) again invariant, see below. Hence, the quest for tensor fields in the preceding symbol space

$$\mathcal{S}_{\ell;\nu}^{km}(M) := \Gamma(\mathcal{S}^k TM \otimes \mathcal{S}^m TM \otimes \mathcal{S}^\ell T^*M \otimes \mathbb{F}_\nu M), \tag{17}$$

$k, m, \ell \in \mathbb{N}$, $\nu \in \mathbb{R}$, which are $\mathrm{CVect}(M)$- and, locally, sp_{2n+2}-invariant for the canonical action.

5.1 CVect(M)- and $\mathbf{sp_{2n+2}}$-invariants

In the following, we need a result on Taylor expansions. If $f \in C^\infty(\mathbb{R}^m)$ and $x_0 \in \mathbb{R}^m$, we denote by $t^k_{x_0}(f)$ the kth order Taylor expansion of f at x_0. We use the same notation for Taylor expansions of vector fields.

Proposition 1. *For every $f \in C^\infty(\mathbb{R}^{2n+1})$ and $x_0 \in \mathbb{R}^{2n+1}$, we have*

$$t^1_{x_0}(X_f) = t^1_{x_0}(X_{t^2_{x_0}(f)}).$$

Proof. First note that, in any coordinate system $(x^1, \ldots, x^{2n+1})$, if $\mathcal{E} = x^i\partial_{x^i}$ is the Euler field, one has, for all $k \geq 1$,

$$\begin{cases} \partial_{x^i} \circ t^k_{x_0} & = & t^{k-1}_{x_0} \circ \partial_{x^i}, \\ (\mathcal{E} - k\,\mathrm{id}) \circ t^k_{x_0} & = & t^{k-1}_{x_0} \circ (\mathcal{E} - k\,\mathrm{id}). \end{cases}$$

This allows to show that

$$t^1_{x_0}(X_f) = X_{t^2_{x_0}(f)} + [t^1_{x_0}(\partial_{x^{2n+1}} f) - (\partial_{x^{2n+1}} f)(x_0)][\mathcal{E} - \mathcal{E}_{x_0}].$$

The result follows, since the last term of the RHS and its partial derivatives vanish at x_0. ∎

It is clear that the spaces $\Gamma(\otimes^p_q TM \otimes \mathbb{F}_\nu M)$ are again representations of the Lie algebra $\mathrm{CVect}(M)$.

Theorem 1. *Let M be a (coorientable) contact manifold of dimension $2n+1$. A tensor field $u \in \Gamma(\otimes^p_q TM \otimes \mathbb{F}_\nu M)$ is $\mathrm{CVect}(M)$-invariant if and only if, over any Darboux chart, u is sp_{2n+2}-invariant.*

Proof. The Lie derivative

$$L : \mathrm{Vect}(M) \times \Gamma(\otimes^p_q TM \otimes \mathbb{F}_\nu M) \to \Gamma(\otimes^p_q TM \otimes \mathbb{F}_\nu M) : (X, u) \to L_X u$$

is a differential operator that has order 1 in the first argument. In other words, the value $(L_X u)_{x_0}$, $x_0 \in M$, only depends on the first jet $j^1_{x_0}(X)$ of X at x_0. Hence, the result follows from Proposition 1. ∎

5.2 Particular invariants

We continue to work on a $(2n+1)$-dimensional coorientable contact manifold (M, α) endowed with a fixed contact form, and describe basic contact invariant or locally affine contact invariant tensor fields in $\mathcal{S}(M) := \oplus_{km\ell\nu}\mathcal{S}^{km}_{\ell;\nu}(M)$, see Equation (17). Observe that locally we can view the elements of $\mathcal{S}^{km}_{\ell;\nu}(M)$ as polynomials of homogeneous degrees k, m, and ℓ in fiber variables ξ, η, and Y, with coefficients in $\mathcal{F}_\nu(M)$.

1. The *identity endomorphism* $u \in \Gamma(TM \otimes T^*M)$ of TM can be viewed as a $\mathrm{Vect}(M)$-invariant element $u_1 \in \mathcal{S}^{10}_{1;0}(M)$ and as a $\mathrm{Vect}(M)$-invariant element $u_2 \in \mathcal{S}^{01}_{1;0}(M)$. Locally, these two invariant tensor fields read $u_1 : (\xi, \eta, Y) \to \langle Y, \xi\rangle$ and $u_2 : (\xi, \eta, Y) \to \langle Y, \eta\rangle$, where $\langle ., .\rangle$ denotes the contraction.

2. *Contact form* α induces a $\mathrm{CVect}(M)$-invariant tensor field

   $$u_3 = \alpha \otimes |\Omega|^{-I} \in \mathcal{S}^{00}_{1;-\mathrm{I}}(M),$$

   $\Omega = \alpha \wedge (d\alpha)^n$, $\mathrm{I} = \frac{1}{n+1}$. Invariance of u_3 with respect to the action of contact fields is a direct consequence of Equation (5). The local form of u_3 is $u_3 : (\xi, \eta, Y) \to \alpha(Y)|\Omega|^{-\mathrm{I}}$.

3. The next invariant tensor field is implemented by the *Lagrange bracket*. Let us first mention that our construction of the Lagrange bracket on $(\mathbb{R}^{2n+1}, i^*\sigma)$, as pullback of the Poisson bracket of the symplectization of this contact structure, can be generalized to an arbitrary contact manifold M, see [OR92, Section 10.2, Corollary 2]: the Poisson bracket on the symplectization of M defines a bracket $\{., .\}$, called Lagrange bracket, on the space $\mathcal{F}_{-\mathrm{I}}(M) = \Gamma(\mathbb{F}_{-\mathrm{I}} M)$ of $(-\mathrm{I})$-tensor

densities of M. This bracket is a first order bidifferential operator between $\mathcal{F}_{-\mathrm{I}}(M) \times \mathcal{F}_{-\mathrm{I}}(M)$ and $\mathcal{F}_{-\mathrm{I}}(M)$. Its principal symbol is defined just along the same lines than the principal symbol of a differential operator, see Section 4. Hence, this symbol $\sigma_{11}(\{.,.\})$ is a tensor field

$$L_1 := \sigma_{11}(\{.,.\}) \in \Gamma(TM \otimes TM \otimes \mathbb{F}_{\mathrm{I}}M) = \mathcal{S}^{11}_{0;\mathrm{I}}(M).$$

It is a basic fact in equivariant quantization that the principal symbol of a (multi)differential operator between tensor densities intertwines the actions L and $\mathcal{L}$ of vector fields on symbols and operators (a short computation in local coordinates also allows to assure oneself of this fact). $\mathrm{CVect}(M)$-invariance of L_1 then follows from contact invariance of $\{.,.\}$ that in turn is nothing but a reformulation of the Jacobi identity, see Equation (6). The local polynomial form of L_1 follows from Equation (10): $L_1 : (\xi, \eta, Y) \to (\sum_k(\xi_{p_k}\eta_{q^k} - \xi_{q^k}\eta_{p_k}) + \eta_t\langle\mathcal{E}_s, \xi_s\rangle - \xi_t\langle\mathcal{E}_s, \eta_s\rangle)|\Omega|^{\mathrm{I}}$, where $\beta_s = \beta - \beta_t dt$, for any one form β.

4. Eventually, the *Reeb vector field* E also induces an invariant tensor field $u = E \otimes |\Omega|^{\mathrm{I}} \in \Gamma(TM \otimes \mathbb{F}_{\mathrm{I}}M)$, which can be viewed as an element $u_4 \in \mathcal{S}^{10}_{0;\mathrm{I}}(M)$ and as an element $u_5 \in \mathcal{S}^{01}_{0;\mathrm{I}}(M)$. Since $[X_f, E] = -i_{d(E(f))}\Lambda - E(f)E$ (confer end Section 2), it is easily seen that fields u_4 and u_5 are not contact invariant, but only (locally) affine contact invariant (confer end Section 3). They (locally) read $u_4 : (\xi, \eta, Y) \to -2\xi_t|\Omega|^{\mathrm{I}}$ and $u_5 : (\xi, \eta, Y) \to -2\eta_t|\Omega|^{\mathrm{I}}$.

5.3 Classifications

In this subsection we classify affine contact invariant and contact invariant tensor fields.

Theorem 2. *The polynomials u_i, $i \in \{1, \ldots, 5\}$, and L_1 generate the algebra of* $\mathrm{AVect}(\mathbb{R}^{2n+1}) \cap \mathrm{CVect}(\mathbb{R}^{2n+1})$*-invariant polynomials in* $\mathcal{S}(\mathbb{R}^{2n+1})$.

Proof. In the following we refer to the algebra of invariant polynomials generated by $u_1, \ldots, u_5$, and L_1 as the space of *classical invariant polynomials*. In order to show that there are no other invariants, we prove that the dimensions of the subspace $\mathcal{S}_1$ of classical invariant polynomials in $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ and of the subspace $\mathcal{S}_2$ of all invariant polynomials inside $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ coincide, for any fixed (k, m, ℓ, ν).

Since the polynomials

$$u_1^a\, u_2^b\, u_3^c\, u_4^d\, u_5^e\, L_1^f \in \mathcal{S}^{a+d+f,b+e+f}_{a+b+c;\mathrm{I}(d+e+f-c)}(\mathbb{R}^{2n+1})$$

are independent and belong to $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ if and only if $(a, b, c, d, e, f) \in \mathbb{N}^6$ is a solution of

$$(S_1) : \begin{cases} a + d + f & = & k \\ b + e + f & = & m \\ a + b + c & = & \ell \\ d + e + f - c & = & (n+1)\nu \end{cases},$$

the dimension of $\mathcal{S}_1$ is exactly the number (which is clearly finite) of solutions in $\mathbb{N}^6$ of system (S_1). $(\star)$

Let now $Q \in \mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ be an arbitrary invariant polynomial and set

$$Q(\xi, \eta, Y) = \sum_{i=0}^{k}\sum_{j=0}^{m}\sum_{r=0}^{\ell} \xi_t^i \eta_t^j (Y^t)^r Q_{i,j,r}(\xi_s, \eta_s, Y_s),$$

where polynomial $Q_{i,j,r}$ is homogeneous of degree $(k-i, m-j, \ell-r)$. The degree defined by

$$\mathrm{Deg}(\xi_t^i \eta_t^j (Y^t)^r Q_{i,j,r}(\xi_s, \eta_s, Y_s)) := i + j - r$$

will be basic in our investigation. Let us recall that the obvious extension of action (16) to $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ reads, for all vector fields $X \in \mathrm{Vect}(\mathbb{R}^{2n+1})$,

$$L_X Q = X(Q) - \partial_j X^i \eta_i \partial_{\eta_j} Q - \partial_j X^i \xi_i \partial_{\xi_j} Q + \partial_j X^i Y^j \partial_{Y^i} Q + \nu \partial_i X^i Q, \qquad (18)$$

where ∂_k denotes the derivative with respect to the kth coordinate of $\mathbb{R}^{2n+1}$. It is easily checked that the invariance conditions with respect to the contact Hamiltonian vector fields X_1, $X_{p_\mathfrak{a}}$, $X_{q^\mathfrak{a}}$, and X_t ($\mathfrak{a} \in \{1, \ldots, n\}$), see Section 3, read

$$\left\{\begin{array}{lcl} \partial_t Q_{i,j,r} & = & 0 \\ \partial_{p_\mathfrak{a}} Q_{i,j,r} - \partial_{\xi_{q^\mathfrak{a}}} Q_{i-1,j,r} - \partial_{\eta_{q^\mathfrak{a}}} Q_{i,j-1,r} + (r+1) Y^{q^\mathfrak{a}} Q_{i,j,r+1} & = & 0 \\ \partial_{q^\mathfrak{a}} Q_{i,j,r} + \partial_{\xi_{p_\mathfrak{a}}} Q_{i-1,j,r} + \partial_{\eta_{p_\mathfrak{a}}} Q_{i,j-1,r} - (r+1) Y^{p_\mathfrak{a}} Q_{i,j,r+1} & = & 0 \\ \mathcal{E}_s(Q_{i,j,r}) - \left[(k+m-\ell) + (i+j-r) - 2(n+1)\nu\right] Q_{i,j,r} & = & 0 \end{array}\right. .$$

Let d_0 be the lowest degree Deg in Q. If $Q_{i,j,r}$ is part of a term of degree d_0, the first three equations of the above system imply that $Q_{i,j,r}$ has constant coefficients. The fourth equation entails that $d_0 = 2(n+1)\nu - (k+m-\ell)$. An easy induction shows that all polynomials $Q_{i,j,r}$ have polynomial coefficients and that they are completely determined by the lowest degree terms. So, the dimension of the space $\mathcal{S}_2$ of invariant polynomials in $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ is at most the dimension of the space of lowest degree terms. $(\star\star)$

We now take a closer look at these lowest degree terms

$$\sum_{i+j-r=d_0} \xi_t^i \eta_t^j (Y^t)^r Q_{i,j,r}(\xi_s, \eta_s, Y_s), \tag{19}$$

where the polynomials $Q_{i,j,r}$ have constant coefficients, and use the invariance conditions with respect to the algebra sp_{2n}. Observe first that the Lie derivatives in the direction of the fields of this algebra preserve the degree Deg. Indeed, for any field X of the basis of sp_{2n}, see Section 3, the derivatives $\partial_j X^i$ vanish for X^t and for ∂_t. Hence, every polynomial $Q_{i,j,r}$ in (19) must be sp_{2n}-invariant. As these polynomials have constant coefficients and the considered vector fields have vanishing divergence, this means that any $Q_{i,j,r}$ in (19) is invariant for the canonical $\mathrm{sp}(2n, \mathbb{R})$-action. When applying a classical result of Weyl, [Wey46], we conclude that each polynomial $Q_{i,j,r}$ in (19) is a polynomial in the variables $\langle Y_s, \xi_s \rangle$, $\langle Y_s, \eta_s \rangle$, and $\Pi(\xi_s, \eta_s)$, where $\Pi = \sum_\mathfrak{a} \partial_{p_\mathfrak{a}} \wedge \partial_{q^\mathfrak{a}}$. Eventually, the lowest degree terms of Q read

$$\sum_{i,j,r,\alpha,\beta,\gamma \in \mathbb{N}} c_{ijr\alpha\beta\gamma} \;\; \xi_t^i \eta_t^j (Y^t)^r \langle Y_s, \xi_s \rangle^\alpha \langle Y_s, \eta_s \rangle^\beta \Pi(\xi_s, \eta_s)^\gamma,$$

where $c_{ijr\alpha\beta\gamma} \in \mathbb{R}$ and where $(i, j, r, \alpha, \beta, \gamma) \in \mathbb{N}^6$ is a solution of the system

$$(S_2) : \left\{\begin{array}{lcl} \alpha + \gamma & = & k - i \\ \beta + \gamma & = & m - j \\ \alpha + \beta & = & \ell - r \\ i + j - r & = & 2(n+1)\nu - (k+m-\ell) \end{array}\right. .$$

This system implies in particular that $(n+1)\nu = k + m - \ell - \gamma$ is an integer. It is easily checked that system (S_2) is equivalent to system (S_1). When taking into account upshots $(\star)$ and $(\star\star)$, we finally see that the dimension of the space $\mathcal{S}_2$ of all invariant polynomials in $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$ is at most the dimension of the space $\mathcal{S}_1$ of classical invariant polynomials in $\mathcal{S}^{km}_{\ell;\nu}(\mathbb{R}^{2n+1})$. ∎

As a corollary, we get the following

Theorem 3. *For every (coorientable) contact manifold M, the fields u_1, u_2, u_3, and L_1 generate the algebra of* $\mathrm{CVect}(M)$*-invariant fields in* $\mathcal{S}(M)$.

Proof. Any $\mathrm{CVect}(M)$-invariant field in $\mathcal{S}^{km}_{\ell;\nu}(M)$ is over every Darboux chart an sp_{2n+2}-invariant polynomial, Theorem 1. Hence, due to Theorem 2, it reads as a linear combination of polynomials $u_1^a u_2^b u_3^c u_4^d u_5^e L_1^f$. Invariance with respect to the second Heisenberg algebra $\tilde{\mathfrak{h}}_{n,2}$, see Section 3, allows to satisfy oneself that $d = e = 0$. Computations are straightforward (but tedious) and will not be given here. ∎

6 Invariant operators between symbol modules implemented by the same density weight

We now use Theorem 3 concerning contact-invariant tensor fields to classify specific "classical" module morphisms, i.e. intertwining operators between sp_{2n+2}-modules of symbols induced by the same density weight, see below.

We first recall the definition of two invariant operators that were basic in [FMP07].

Let (M,α) be a Pfaffian manifold. In the following, we denote the $\mathrm{CVect}(M)$-invariant tensor field $u_3 = \alpha \otimes |\Omega|^{-\mathrm{I}} \in \mathcal{S}^{00}_{1;-\mathrm{I}}(M) = \Gamma(T^*M \otimes \mathbb{F}_{-\mathrm{I}}M)$ simply by α (if more precise notation is not required in order to guard against confusion). Contact form α can then be viewed as a contraction operator

$$i_\alpha : \mathcal{S}^k_\delta(M) \to \mathcal{S}^{k-1}_{\delta-\mathrm{I}}(M),$$

where $\mathcal{S}^k_\delta(M) = \Gamma(\mathcal{S}^kTM \otimes \mathbb{F}_\delta M)$. Due to invariance of α, the vertical cotangent lift i_α of α is clearly a $\mathrm{CVect}(M)$-intertwining operator.

We also extend the contact Hamiltonian operator, see Equation (10), to the spaces $\mathcal{S}^k_\delta(\mathbb{R}^{2n+1})$ of symmetric contravariant density valued tensor fields over $\mathbb{R}^{2n+1}$. This generalized Hamiltonian

$$X : \mathcal{S}^k_\delta(\mathbb{R}^{2n+1}) \to \mathcal{S}^{k+1}_{\delta+\mathrm{I}}(\mathbb{R}^{2n+1})$$

maps $S = S(\xi)$ to

$$X(S) = X(S)(\xi) = \sum_j (\xi_{q^j}\partial_{p_j} - \xi_{p_j}\partial_{q^j})S(\xi) + \xi_t\mathcal{E}_sS(\xi) - \langle \mathcal{E}_s, \xi_s\rangle \partial_t S(\xi) + a(k,\delta)\xi_t S(\xi), \tag{20}$$

where $a(k,\delta) = 2(n+1)\delta - k$. If $k = 0$, operator X obviously coincides with the map

$$X : \mathcal{S}^0_\delta(M) \ni f \to \sigma_1(\{f, .\}) \in \mathcal{S}^1_{\delta+\mathrm{I}}(M),$$

where $\{.,.\}$ is the Lagrange bracket and where $\{f,.\} : \mathcal{S}^0_0(M) \ni g \to \{f,g\} \in \mathcal{S}^0_{\delta+\mathrm{I}}(M)$. In [FMP07], we proved the following

Proposition 2. *Operator* $X : \mathcal{S}^k_\delta(\mathbb{R}^{2n+1}) \to \mathcal{S}^{k+1}_{\delta+\mathrm{I}}(\mathbb{R}^{2n+1})$ *intertwines the* sp_{2n+2}*-action and does not commute with the* $\mathrm{CVect}(\mathbb{R}^{2n+1})$*-action, unless* $k = 0$.

Since the operators i_α and X modify the density weight of their arguments, we introduce the space

$$R_\delta = \oplus_{k\in\mathbb{N}} R^k_\delta := \oplus_{k\in\mathbb{N}} \mathcal{S}^k_{\delta+\frac{k}{n+1}}(\mathbb{R}^{2n+1}).$$

Theorem 4. *The algebra of* sp_{2n+2}*-invariant (differential) operators from* R_δ *into* R_δ *is generated by* i_α *and* X*. More precisely, the space of* sp_{2n+2}*- invariant operators from* R^ℓ_δ *into* R^k_δ *is spanned by* $\{X^m \circ i_\alpha^{\ell+m-k} : \sup(0, k-\ell) \le m \le k\}$.

Proof. In order to simplify notations, we set $M := \mathbb{R}^{2n+1}$. Let I be an sp_{2n+2}-invariant differential operator, say of order m, from R^ℓ_δ into R^k_δ. The principal symbol $\sigma_m(I)$ of I is an invariant tensor field in

$$\Gamma(\mathcal{S}^mTM \otimes \mathrm{Hom}(\mathcal{S}^\ell TM \otimes \mathbb{F}_{\delta+\frac{\ell}{n+1}}M, \mathcal{S}^kTM \otimes \mathbb{F}_{\delta+\frac{k}{n+1}}M)) \simeq \Gamma(\mathcal{S}^kTM \otimes \mathcal{S}^mTM \otimes \mathcal{S}^\ell T^*M \otimes \mathbb{F}_{\frac{k-\ell}{n+1}}M),$$

see Remark 2 below. In view of Theorem 1 and Theorem 3, we then have

$$\sigma_m(I) = \sum_{a,b,c,d\in\mathbb{N}} C_{abcd}\, u_1^a u_2^b u_3^c L_1^d \qquad (C_{abcd} \in \mathbb{R}),$$

where a, b, c, and d are subject to the conditions

$$\begin{cases} a + d &= k \\ b + d &= m \\ a + b + c &= \ell \\ d - c &= k - \ell \end{cases}.$$

This system has a unique solution $a = k-m$, $b = 0$, $c = \ell+m-k$, and $d = m$, with $\sup(0, k-\ell) \leq m \leq k$. Hence,

$$\sigma_m(I)(\eta, \xi; Y^\ell) = C\ u_1^{k-m} u_3^{\ell+m-k} L_1^m \qquad (C \in \mathbb{R}),$$

where we used conventional notations of affine symbol calculus, see Remark 2 below.

Observe now that operator $X^m \circ i_\alpha^{\ell+m-k} : R_\delta^\ell \to R_\delta^k$ is an sp_{2n+2}-invariant differential operator of order m (since X has order 1 and i_α has order 0). Thus its principal symbol reads

$$\sigma_m(X^m \circ i_\alpha^{\ell+m-k})(\eta, \xi; Y^\ell) = C_0\ u_1^{k-m} u_3^{\ell+m-k} L_1^m \qquad (C_0 \in \mathbb{R}_0).$$

It follows that the operator $I - \frac{C}{C_0} X^m \circ i_\alpha^{\ell+m-k} : R_\delta^\ell \to R_\delta^k$ is sp_{2n+2}-invariant and of order $\leq m-1$. An easy induction on the order of differentiation then yields the result. ∎

Remark 2. a. Let us mention that operator X^m is tightly related with the mth order Lagrange bracket. This observation will be further developed in a subsequent work.

b. Although it is a known result in equivariant quantization, commutativity—for differential operators between symmetric contravariant density valued tensor fields—of the principal symbol and the canonical actions of vector fields might not be obvious for all the readers. Beyond computations in local coordinates, affine symbol calculus allows to elegantly make sure of the validity of this statement. Affine symbol calculus is a non-standard computing technique. For further information we refer the interested reader to [Pon04]. Below we give the proof, via symbol calculus, of the aforementioned commutativity.

Let $T \in \mathrm{Hom}(\mathcal{S}_{\delta'}^{k'}(\mathbb{R}^p), \mathcal{S}_{\delta''}^{k''}(\mathbb{R}^p))_{\mathrm{loc}}$, where subscript "loc" means that we confine ourselves to support preserving operators. We fix coordinates and call *affine symbol* $\sigma_{\mathrm{aff}}(T)$ of T, its total symbol (the highest order terms of which coincide with the principal symbol $\sigma(T)$). Hence, if T has order m, $\sigma_{\mathrm{aff}}(T) \in \Gamma(\mathcal{S}_m T\mathbb{R}^p \otimes \mathrm{Hom}(\mathcal{S}^{k'} T\mathbb{R}^p \otimes \mathbb{F}_{\delta'}\mathbb{R}^p, \mathcal{S}^{k''} T\mathbb{R}^p \otimes \mathbb{F}_{\delta''}\mathbb{R}^p))$, where $\mathcal{S}_m T\mathbb{R}^p$ is the mth order filter (of the increasing filtration) associated with the natural grading of $\mathcal{S}T\mathbb{R}^p$. It is easily checked that, for any $X \in \mathrm{Vect}(\mathbb{R}^p)$,

$$\begin{aligned}\sigma_{\mathrm{aff}}(L_X T) \quad = \quad & (X.T)(\eta; Y^{k'}) - \langle X, \eta\rangle \tau_\theta \left(T(\eta; Y^{k'})\right) - X(\theta\partial_\xi) T(\eta; Y^{k'}) \\ & + T(\eta+\theta; X(\theta\partial_\xi) Y^{k'}) + \delta''\langle X, \theta\rangle T(\eta; Y^{k'}) - \delta'\langle X, \theta\rangle T(\eta+\theta; Y^{k'}),\end{aligned} \tag{21}$$

where we used standard notations, see [Pon04] ($X.T$ denotes the derivatives of the coefficients of T, $Y^{k'} = Y \vee \ldots \vee Y$ (k' factors), $\tau_\theta\star$ is just a notation for the translation $\star(. + \theta) - \star(.)$, η symbolizes the derivatives that act on the argument—represented by $Y^{k'}$—of T, θ symbolizes the derivatives of the coefficients of X, and ξ denotes the variable of the polynomials $Y^{k'}$ and $T(\eta; Y^{k'})$). On the other hand, we have

$$\begin{aligned}L_X \sigma(T) = \quad & (X.\sigma(T))(\eta; Y^{k'}) - \langle X, \eta\rangle (\theta\partial_\eta)\sigma(T)(\eta; Y^{k'}) \\ & - X(\theta\partial_\xi)(\sigma(T)(\eta; Y^{k'})) + \sigma(T)(\eta; X(\theta\partial_\xi) Y^{k'}) + (\delta'' - \delta')\langle X, \theta\rangle \sigma(T)(\eta; Y^{k'}).\end{aligned}$$

When selecting the highest order terms in Equation (21), we see that $\sigma(L_X T) = L_X \sigma(T), \forall X \in \mathrm{Vect}(\mathbb{R}^p)$. It is a matter of common knowledge that the same result holds true for the total affine symbol σ_{aff}, if we confine ourselves to the action of affine vector fields $X \in \mathrm{AVect}(\mathbb{R}^p)$.

A similar proof is possible for differential operators T acting between tensor densities. The corresponding result has already been used earlier in this note. However, the observation that the principal symbol intertwines the actions by Lie derivatives on operators and symbols, is not true in general. It is for instance not valid for "quantum level operators" $T \in \mathrm{Hom}(\mathcal{D}_{\lambda'\mu'}^{k'}(\mathbb{R}^p), \mathcal{D}_{\lambda''\mu''}^{k''}(\mathbb{R}^p))_{\mathrm{loc}}$.

7 Casimir operator

As an application of our decomposition of the module of symbols into submodules, see [FMP07], of Section 3, and Section 6, we now prove that the Casimir operator C_δ^k of the canonical representation of sp_{2n+2} on R_δ^k (with respect to the Killing form) is diagonal. Computation of this Casimir is a challenge by itself, but, in addition, it will turn out that this operator imposes restrictions on the parameters k, k', δ, δ' of symbol modules R^k_δ and $R^{k'}_{\delta'}$ that are implemented by different density weights δ, δ' and are linked by an invariant operator.

The following upshots are well-known and mostly easily checked. The symplectic algebra $\mathrm{sp}(2n, \mathbb{C})$ is a classical simple Lie algebra of type C_n (if $n \geq 3$). Its Killing form K reads $K : \mathrm{sp}(2n, \mathbb{C}) \times \mathrm{sp}(2n, \mathbb{C}) \ni (S, S') \to 2(n+1)\,\mathrm{tr}(SS') \in \mathbb{C}$, and its classical Cartan subalgebra $C \subset \mathrm{sp}(2n, \mathbb{C})$ is $C = \{\mathrm{diag}(\Delta, -\Delta), \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n), \Delta_i \in \mathbb{C}\}$. The corresponding roots are

$$-(\nu_i + \nu_j) \quad (i \leq j \leq n), \quad \nu_i + \nu_j \quad (i \leq j \leq n), \quad \text{and} \quad \nu_i - \nu_j \quad (i, j \leq n), \tag{22}$$

where ν_k is the $\mathbb{C}$-linear form of C defined by $\nu_k(\mathrm{diag}(\Delta, -\Delta)) = \Delta_k$. If $(e_1, \ldots, e_{2n})$ denotes as above the canonical basis of $\mathbb{C}^{2n}$ and $(\epsilon^1, \ldots, \epsilon^{2n})$ the dual basis in $\mathbb{C}^{2n*}$, the respective eigenvectors are

$$\epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n},\ -\epsilon^{j+n} \otimes e_i - \epsilon^{i+n} \otimes e_j, \text{ and } -\epsilon^j \otimes e_i + \epsilon^{i+n} \otimes e_{j+n}. \tag{23}$$

We thus recover the result that the eigenspaces sp_ν associated with the above-detailed roots ν, see Equation (22), are 1-dimensional for $\nu \neq 0$. Moreover, if Λ denotes the set of roots, we have the decomposition $\mathrm{sp}(2n, \mathbb{C}) = \bigoplus_{\nu \neq 0, \nu \in \Lambda} \mathrm{sp}_\nu \oplus C$. This splitting allows computing the Killing-dual basis of basis (23), see also Equation (8).

Proposition 3. *The bases*

$$\epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n}\, (i \leq j \leq n),\ -\epsilon^{j+n} \otimes e_i - \epsilon^{i+n} \otimes e_j\, (i \leq j \leq n),\ -\epsilon^j \otimes e_i + \epsilon^{i+n} \otimes e_{j+n}\, (i, j \leq n)$$

and

$$k_{ij}(-\epsilon^{j+n} \otimes e_i - \epsilon^{i+n} \otimes e_j),\ k_{ij}(\epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n}),\ k(-\epsilon^i \otimes e_j + \epsilon^{j+n} \otimes e_{i+n})$$

of $\mathrm{sp}(2n, \mathbb{C})$ *are dual with respect to the Killing form, if and only if* $k_{ij} = -1/(4(n+1)(1+\delta_{ij}))$ *and* $k = 1/(4(n+1))$.

Proof. Remember first that if N is a nilpotent subalgebra of a complex Lie algebra L, and if $\nu_1, \nu_2 \in \Lambda$ are roots of N, such that $\nu_2 \neq -\nu_1$, then the corresponding eigenspaces L_{ν_1} and L_{ν_2} are orthogonal with respect to the Killing form K of L. Further, the basis $-\epsilon^i \otimes e_i + \epsilon^{i+n} \otimes e_{i+n}$ $(i \in \{1, \ldots, n\})$ of C is orthogonal with respect to K. Hence, it suffices to compute K on each pair of nonorthogonal vectors. For instance, we have

$$\begin{aligned}
&k_{ij} K(-\epsilon^{j+n} \otimes e_i - \epsilon^{i+n} \otimes e_j, \epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n}) \\
&= -2(n+1) k_{ij}\, \mathrm{tr}((\epsilon^{j+n} \otimes e_i + \epsilon^{i+n} \otimes e_j)(\epsilon^j \otimes e_{i+n} + \epsilon^i \otimes e_{j+n})) \\
&= -2(n+1) k_{ij}\, \mathrm{tr}(\delta_{ij} \epsilon^i \otimes e_i + \epsilon^i \otimes e_i + \epsilon^j \otimes e_j + \delta_{ij} \epsilon^i \otimes e_i) \\
&= 1.
\end{aligned}$$

The result follows. ∎

Remarks.

- All the matrices used above are actually real matrices. The result on Killing-dual bases still holds true for $\mathrm{sp}(2n, \mathbb{R})$ ($\mathrm{sp}(2n, \mathbb{R})$ is a split real form of $\mathrm{sp}(2n, \mathbb{C})$, the Killing form of $\mathrm{sp}(2n, \mathbb{R})$ is the restriction of the Killing form of $\mathrm{sp}(2n, \mathbb{C})$).

- If read through $J^{-1} \circ \mathcal{X} : (\mathrm{Pol}^2(\mathbb{R}^{2n}), \{.,.\}_\Pi) \to (\mathrm{sp}(2n, \mathbb{R}), [.,.]_\circ)$, see Equation (11), Proposition 3 states that the bases

  $$p_i p_j \ (i \leq j \leq n),\ q^i q^j \ (i \leq j \leq n),\ p_j q^i \ (i, j \leq n)$$

  and

  $$k_{ij}\, q^i q^j \ (i \leq j \leq n),\ k_{ij}\, p_i p_j \ (i \leq j \leq n),\ k\, p_i q^j \ (i, j \leq n)$$

  of $\mathrm{Pol}^2(\mathbb{R}^{2n})$ are Killing-dual.

- The preceding result, written for space $\mathbb{R}^{2n+2}$ (coordinates: $(p_1, \ldots, p_n, q^1, \ldots, q^n; t, \tau)$) and read through Lie algebra isomorphism $X \circ \chi : (\mathrm{Pol}^2(\mathbb{R}^{2n+2}_+), \{.,.\}_\Pi) \to (\mathrm{sp}_{2n+2}, [.,.])$, see Equations (9) and (10), shows that the bases

$$\begin{gathered} X_{p_ip_j}\ (i \le j \le n),\ X_{tp_i}\ (i \in \{1, \ldots, n\}),\ X_{t^2}; \\ X_{q^iq^j}\ (i \le j \le n),\ X_{q^i}\ (i \in \{1, \ldots, n\}),\ X_1; \\ X_{p_jq^i}\ (i, j \le n),\ X_{tq^i}\ (i \in \{1, \ldots, n\}),\ X_{p_i}\ (i \in \{1, \ldots, n\}),\ X_t \end{gathered} \tag{24}$$

  and

$$\begin{gathered} k_{ij}\, X_{q^iq^j}\ (i \le j \le n),\ -k\, X_{q^i}\ (i \in \{1, \ldots, n\}),\ -k/2\, X_1; \\ k_{ij}\, X_{p_ip_j}\ (i \le j \le n),\ -k\, X_{tp_i}\ (i \in \{1, \ldots, n\}),\ -k/2\, X_{t^2}; \\ k\, X_{p_iq^j}\ (i, j \le n),\ k\, X_{p_i}\ (i \in \{1, \ldots, n\}),\ k\, X_{tq^i}\ (i \in \{1, \ldots, n\}),\ k\, X_t, \end{gathered} \tag{25}$$

  with $k_{ij} = -1/(4(n+2)(1+\delta_{ij}))$ and $k = 1/(4(n+2))$, are bases of the algebra sp_{2n+2} of infinitesimal projective contact transformations, which are dual with respect to the Killing form. Observe that the first basis is the basis computed in Section 3 and that both bases are explicitly known, see Equations (12), (13), (14), and (15).

We already mentioned that action (16) of $X \in \mathrm{Vect}(\mathbb{R}^{2n+1})$ on $P \in \mathcal{S}_\delta(\mathbb{R}^{2n+1})$ has the explicit form

$$L_X P = X(P) - \partial_j X^i \xi_i \partial_{\xi_j} P + \delta\, \partial_i X^i\, P, \tag{26}$$

see Equation (18). Remark that in this section we denote the base coordinates by $(p_1, \ldots, p_n, q^1, \ldots, q^n, t)$ and the fiber coordinates by $(\xi_{p_1}, \ldots, \xi_{p_n}, \xi_{q^1}, \ldots, \xi_{q^n}, \xi_t)$. Moreover, we took an interest in the Casimir operator C^k_δ of the preceding action of sp_{2n+2} on $R^k_\delta = \mathcal{S}^k_{\delta+k/(n+1)}(\mathbb{R}^{2n+1})$, so that the weight in Equation (26) must be modified accordingly. The actions on R^k_δ of the dual bases (24) and (25) are now straightforwardly obtained:

$$\begin{array}{lll} L_{X_1} & = & -2\partial_t, \\ L_{X_{p_i}} & = & \partial_{q^i} - p_i\partial_t + \xi_t\partial_{\xi_{p_i}}, \\ L_{X_{q^i}} & = & -\partial_{p_i} - q^i\partial_t + \xi_t\partial_{\xi_{q^i}}, \\ L_{X_t} & = & -\mathcal{E}_s - 2t\partial_t + \mathcal{E}_{\xi_s} + 2\xi_t\partial_{\xi_t} - 2((n+1)\delta + k), \\ L_{X_{p_ip_j}} & = & p_j\partial_{q^i} + p_i\partial_{q^j} - \xi_{q^i}\partial_{\xi_{p_j}} - \xi_{q^j}\partial_{\xi_{p_i}}, \\ L_{X_{q^iq^j}} & = & -q^j\partial_{p_i} - q^i\partial_{p_j} + \xi_{p_i}\partial_{\xi_{q^j}} + \xi_{p_j}\partial_{\xi_{q^i}}, \\ L_{X_{p_jq^i}} & = & q^i\partial_{q^j} - p_j\partial_{p_i} + \xi_{p_i}\partial_{\xi_{p_j}} - \xi_{q^j}\partial_{\xi_{q^i}}, \\ L_{X_{tp_i}} & = & t(\partial_{q^i} - p_i\partial_t) - p_i\mathcal{E}_s - \xi_{q^i}\partial_{\xi_t} + p_i\mathcal{E}_\xi + \mathcal{E}(\xi)\partial_{\xi_{p_i}} - 2((n+1)\delta + k)p_i, \\ L_{X_{tq^i}} & = & -t(\partial_{p_i} + q^i\partial_t) - q^i\mathcal{E}_s + \xi_{p_i}\partial_{\xi_t} + q^i\mathcal{E}_\xi + \mathcal{E}(\xi)\partial_{\xi_{q^i}} - 2((n+1)\delta + k)q^i, \\ L_{X_{t^2}} & = & -2t\mathcal{E} + 2t\mathcal{E}_\xi + 2\mathcal{E}(\xi)\partial_{\xi_t} - 4((n+1)\delta + k)t. \end{array} \tag{27}$$

In these equations, $\mathcal{E}$ is the Euler field of $\mathbb{R}^{2n+1}$, $\mathcal{E}_s$ is its spatial part, $\mathcal{E}_\xi$ is the Euler field with respect to the fiber coordinates, $\mathcal{E}_\xi = \xi_{p_i}\partial_{\xi_{p_i}} + \xi_{q^i}\partial_{\xi_{q^i}} + \xi_t\partial_{\xi_t}$, $\mathcal{E}_{\xi_s}$ denotes the spatial part of $\mathcal{E}_\xi$, and $\mathcal{E}(\xi)$ is the contraction of $\mathcal{E}$ and $\xi = \xi_{p_i}dp_i + \xi_{q^i}dq^i + \xi_t dt$.

When combining Equations (24), (25), and (27), we get the explicit form of Casimir operator C^k_δ. A direct computation and simplification of C^k_δ are possible, but almost inextricable. In the sequel, we provide a much more economic method based upon Theorem 4.

Set $\mathfrak{C}_k = \{-p/(2(n+1)) : p = 0, 1, \ldots, 2k-2\}$.

Theorem 5. *The Casimir operator C^k_δ of the Lie algebra sp_{2n+2} of infinitesimal projective contact transformations, with respect to its Killing form, and for its canonical action (26) on $R^k_\delta = \mathcal{S}^k_{\delta+k/(n+1)}(\mathbb{R}^{2n+1})$, is given by*

$$C^k_\delta = \frac{1}{n+2}(c(k,\delta)\,\mathrm{id} + X \circ i_\alpha),$$

where

$$c(k,\delta) = (n+1)^2\delta^2 - (n+1)^2\delta + k(n+1)\delta + \frac{k^2-k}{2}$$

and where X (resp. i_α) is the generalized Hamiltonian (resp. the vertical cotangent lift of α) defined in Section 6.

Let us first recall two results obtained in [FMP07].

Proposition 4. *In R^k_δ and for $\ell \in \mathbb{N}_0$, we have*

$$i_\alpha \circ X^\ell = X^\ell \circ i_\alpha + r(\ell,k)X^{\ell-1},$$

where $r(\ell,k) = -\frac{\ell}{2}(2(n+1)\delta + 2k + \ell - 1)$.

Theorem 6. *If $\delta \notin \mathfrak{C}_k$, then*

$$R^k_\delta = \bigoplus_{m=0}^{k} R^{k,\ell}_\delta := \bigoplus_{\ell=0}^{k} X^\ell(R^{k-\ell}_\delta \cap \ker i_\alpha).$$

The last upshot, which extends splitting (7), is the main result of [FMP07] and has actually been proved in view of the present application.

Proof of Theorem 5. The proof consists of four stages.

First suppose that $\delta \notin \mathfrak{C}_k$.
1. Casimir operator C^k_δ reads

$$\begin{aligned} C^k_\delta &= -\tfrac{1}{8(n+2)}(L_{X_1} \circ L_{X_{t^2}} + L_{X_{t^2}} \circ L_{X_1}) + \tfrac{1}{4(n+2)}(L_{X_t})^2 - \tfrac{1}{4(n+2)}\textstyle\sum_i(L_{X_{q^i}} \circ L_{X_{tp_i}} + L_{X_{tp_i}} \circ L_{X_{q^i}}) \\ &+ \tfrac{1}{4(n+2)}\textstyle\sum_i(L_{X_{p_i}} \circ L_{X_{tq^i}} + L_{X_{tq^i}} \circ L_{X_{p_i}}) - \tfrac{1}{8(n+2)}\textstyle\sum_i(L_{X_{q^{i2}}} \circ L_{X_{p_i^2}} + L_{X_{p_i^2}} \circ L_{X_{q^{i2}}}) \\ &- \tfrac{1}{4(n+2)}\textstyle\sum_{i<j}(L_{X_{q^iq^j}} \circ L_{X_{p_ip_j}} + L_{X_{p_ip_j}} \circ L_{X_{q^iq^j}}) + \tfrac{1}{4(n+2)}\textstyle\sum_i\sum_j L_{X_{p_iq^j}} \circ L_{X_{p_jq^i}}. \end{aligned} \tag{28}$$

Since $X \circ \chi : (\mathrm{Pol}^2(\mathbb{R}^{2n+2}_+), \{.,.\}_\Pi) \to (\mathrm{sp}_{2n+2}, [.,.])$ and $L : (\mathrm{sp}_{2n+2}, [.,.]) \to (\mathrm{End}(R^k_\delta), [.,.]_\circ)$ are Lie algebra homomorphisms, and $\{\tau^2, t^2\}_\Pi = -4t\tau$, we have $[L_{X_1}, L_{X_{t^2}}]_\circ = -4L_{X_t}$. Hence, the first term of the RHS of Equation (28) is equal to $-1/(4(n+2))(L_{X_{t^2}} \circ L_{X_1} - 2L_{X_t})$. When using similarly the Poisson brackets $\{\tau q^i, tp_i\}_\Pi = -p_iq^i - t\tau$, $\{\tau p_i, tq^i\}_\Pi = -p_iq^i + t\tau$, $\{q^{i2}, p_i^2\}_\Pi = -4p_iq^i$, $\{q^iq^j, p_ip_j\}_\Pi = -p_iq^i - p_jq^j$ $(i \neq j)$, we finally get

$$\begin{aligned} C^k_\delta &= -\tfrac{1}{4(n+2)}L_{X_{t^2}} \circ L_{X_1} + \tfrac{1}{4(n+2)}(L_{X_t})^2 + \tfrac{1}{2(n+2)}\textstyle\sum_i(L_{X_{tq^i}} \circ L_{X_{p_i}} - L_{X_{tp_i}} \circ L_{X_{q^i}}) \\ &- \tfrac{1}{4(n+2)}\textstyle\sum_i L_{X_{p_i^2}} \circ L_{X_{q^{i2}}} - \tfrac{1}{2(n+2)}\textstyle\sum_{i<j} L_{X_{p_ip_j}} \circ L_{X_{q^iq^j}} \\ &+ \tfrac{1}{4(n+2)}\textstyle\sum_i\sum_j L_{X_{p_iq^j}} \circ L_{X_{p_jq^i}} + \tfrac{1}{2}\tfrac{n+1}{n+2}L_{X_t} + \tfrac{n+1}{4(n+2)}\textstyle\sum_i L_{X_{p_iq^i}}. \end{aligned} \tag{29}$$

2. Theorem 4 entails that

$$C^k_\delta = \sum_{m=0}^{\inf(2,k)} c^k_{\delta,m}\, X^m \circ i^m_\alpha \quad (c^k_{\delta,m} \in \mathbb{R}), \tag{30}$$

where we have used the fact that Casimir C^k_δ is a second order differential operator, see Equation (26). Observe also that it follows from Proposition 4 that $R^{k,\ell}_\delta = X^\ell(R^{k-\ell}_\delta \cap \ker i_\alpha)$, $\ell \in \{0,\ldots,k\}$, is an eigenspace of C^k_δ with eigenvalue

$$\varepsilon^{k,\ell}_\delta = \sum_{m=0}^{\inf(2,\ell)} c^k_{\delta,m}\Pi^\ell_{i=\ell-m+1} r(i, k-\ell). \tag{31}$$

In particular,

$$\begin{gathered} C^k_\delta|_{R^{k,0}_\delta} = c^k_{\delta,0}\,\mathrm{id}, C^k_\delta|_{R^{k,1}_\delta} = (c^k_{\delta,0} + r(1,k-1)c^k_{\delta,1})\,\mathrm{id}, \\ C^k_\delta|_{R^{k,2}_\delta} = (c^k_{\delta,0} + r(2,k-2)c^k_{\delta,1} + r(1,k-2)r(2,k-2)c^k_{\delta,2})\,\mathrm{id}. \end{gathered} \tag{32}$$

Let now $P^k_{\delta_a}$ be the polynomial $(p_1\xi_t + \xi_{q^1})^k$ viewed as an element of $\mathcal{S}^k_{\delta+\frac{a}{n+1}}(\mathbb{R}^{2n+1})$. Since, for any $S \in R^k_\delta$, the contraction $i_\alpha S \in R^{k-1}_\delta$ is given by

$$(i_\alpha S)(\xi) = \frac{1}{2}\left(\sum_j (p_j \partial_{\xi_{q^j}} - q^j \partial_{\xi_{p_j}}) - \partial_{\xi_t}\right) S(\xi),$$

see Section 6, it is clear that $P^k_{\delta_k} \in R^{k,0}_\delta$. Moreover, it is easily checked, see Equation (20), that

$$X(P^{k-1}_{\delta_{k-1}}) = (2(n+1)\delta + 2k - 2))\xi_t P^{k-1}_{\delta_k} = -2r(1, k-1)\xi_t P^{k-1}_{\delta_k} \in R^{k,1}_\delta$$

and that

$$X^2(P^{k-2}_{\delta_{k-2}}) = (2(n+1)\delta + 2k - 4)(2(n+1)\delta + 2k - 3)\xi_t^2 P^{k-2}_{\delta_k} = 2r(1, k-2)r(2, k-2)\xi_t^2 P^{k-2}_{\delta_k} \in R^{k,2}_\delta,$$

where the coefficients in the RHS of the two last equations do not vanish, since $\delta \notin \mathfrak{C}_k$. As generalized Hamiltonian X intertwines the sp_{2n+2}-action, see Proposition 2, we also have $C^k_\delta X(P^{k-1}_{\delta_{k-1}}) = XC^{k-1}_\delta(P^{k-1}_{\delta_{k-1}})$. If we now apply Equation (32) to both sides, we get $(c^k_{\delta,0} + r(1, k-1)c^k_{\delta,1})X(P^{k-1}_{\delta_{k-1}}) = c^{k-1}_{\delta,0} X(P^{k-1}_{\delta_{k-1}})$. Thus,

$$c^k_{\delta,1} = \frac{c^{k-1}_{\delta,0} - c^k_{\delta,0}}{r(1, k-1)}. \tag{33}$$

When proceeding analogously for $X^2(P^{k-2}_{\delta_{k-2}})$, we obtain

$$c^k_{\delta,2} = \frac{r(1, k-1)(c^{k-2}_{\delta,0} - c^k_{\delta,0}) - r(2, k-2)(c^{k-1}_{\delta,0} - c^k_{\delta,0})}{r(1, k-1)r(1, k-2)r(2, k-2)}. \tag{34}$$

3. Hence, Casimir operator C^k_δ is completely known, see Equations (30), (33), and (34), if we find $c^k_{\delta,0}$. In this effect, we use Equation (32) for $P^k_{\delta_k} \in R^{k,0}_\delta$, and compute the LHS by means of Equation (29). Straightforward (and even fairly short) computations allow checking the contributions of the successive terms T_1–T_8 of the RHS of (29).

$$T_1 P^k_{\delta_k} = 0, T_2 P^k_{\delta_k} = \tfrac{1}{4(n+2)}(2(n+1)\delta + k)^2 P^k_{\delta_k}, T_3 P^k_{\delta_k} = 0, T_4 P^k_{\delta_k} = \tfrac{k}{n+2} P^k_{\delta_k}, T_5 P^k_{\delta_k} = \tfrac{k(n-1)}{2(n+2)} P^k_{\delta_k},$$
$$T_6 P^k_{\delta_k} = \tfrac{k(n-1+k)}{4(n+2)} P^k_{\delta_k}, T_7 P^k_{\delta_k} = -\tfrac{1}{2}\tfrac{n+1}{n+2}(2(n+1)\delta + k) P^k_{\delta_k}, T_8 P^k_{\delta_k} = -\tfrac{k(n+1)}{4(n+2)} P^k_{\delta_k}.$$

When summing up these terms, we get

$$c^k_{\delta,0} = \frac{1}{n+2}((n+1)^2\delta^2 - (n+1)^2\delta + k(n+1)\delta + (k^2 - k)/2), \tag{35}$$

and when substituting in Equations (33) and (34), we obtain

$$c^k_{\delta,1} = \frac{1}{n+2} \text{ and } c^k_{\delta,2} = 0. \tag{36}$$

4. For every δ, operator C^k_δ is a member of the finite dimensional space of differential operators of order at most two, with polynomial coefficients of degree at most four, in view of equations (27) and (29). Still using these equations, we see that C^k_δ depends in a polynomial way on δ. Since this operator coincides with the desired expression for every $\delta \in \mathbb{R} \setminus \mathfrak{C}_k$, it is equal to this expression for every δ. ∎

Proposition 5. *If $\delta \notin \mathfrak{C}_k$, space R^k_δ is the direct sum of the eigenspaces $R^{k,\ell}_\delta$, $\ell \in \{0, \ldots, k\}$, of Casimir operator C^k_δ. The corresponding eigenvalues are $\varepsilon^{k,\ell}_\delta = 1/(n+2)(c(k,\delta) + r(\ell, k-\ell))$, see Theorem 5 and Proposition 4, and eigenvalues $\varepsilon^{k,\ell}_\delta$ associated with different ℓ cannot coincide.*

Proof. The first assertion and the values of the $\varepsilon^{k,\ell}_\delta$, $\ell \in \{0, \ldots, k\}$, are direct consequences of Theorem 6 and Equations (31), (35), and (36). Assume now that, for $\ell_1 \neq \ell_2$, we have $\varepsilon^{k,\ell_1}_\delta = \varepsilon^{k,\ell_2}_\delta$. This means that $r(\ell_1, k-\ell_1) = r(\ell_2, k-\ell_2)$, i.e. that $(\ell_1 - \ell_2)(2(n+1)\delta + 2k - (\ell_1 + \ell_2 + 1)) = 0$. As $2 \leq \ell_1 + \ell_2 + 1 \leq 2k$, the last result is possible only if $\delta \in \mathfrak{C}_k$. ∎

8 Invariant operators between symbol modules implemented by different density weights

In this section, we investigate invariant operators $T : R^k_\delta \to R^{k'}_{\delta'}$ between symbol spaces implemented by different weights δ and δ'.

Let $T : R^k_\delta \to R^{k'}_{\delta'}$ be an sp_{2n+2}-invariant operator and assume for a moment that $\delta \notin \mathfrak{C}_k$, $\delta' \notin \mathfrak{C}_{k'}$. It then easily follows from $TC^k_\delta = C^{k'}_{\delta'}T$ that, for any eigenspace $R^{k,\ell}_\delta$, $\ell \in \{0, \ldots, k\}$, the restriction T_ℓ of T to $R^{k,\ell}_\delta$, either vanishes, or is an sp_{2n+2}-invariant operator from $R^{k,\ell}_\delta$ into an eigenspace $R^{k',\ell'}_{\delta'}$, $\ell' \in \{0, \ldots, k'\}$, where ℓ' verifies the equation $\varepsilon^{k',\ell'}_{\delta'} = \varepsilon^{k,\ell}_\delta$. A short computation shows that this condition reads

$$\begin{aligned} &2(n+1)^2(\delta+\delta'-1)(\delta-\delta') + (k+k'-1)(k-k') + 2(n+1)(k\delta - k'\delta') \\ &-2(n+1)(\ell\delta - \ell'\delta') - 2(\ell k - \ell' k') + (\ell+\ell'+1)(\ell-\ell') = 0 \quad . \end{aligned} \tag{37}$$

On the one hand, the appearance of this Diophantine-type equation [let us recall that Diophantine equations are indeterminate polynomial equations with integer variables, that Y. Matiyasevich's solution of Hilbert's 10th problem shows that there is no algorithm that allows solving arbitrary Diophantine systems, and that one of the most celebrated results in this field is of course A. Wiles' conclusion concerning Fermat's equation $x^n + y^n = z^n$, $n > 2$] is the shadow of the intricacy of the investigated problem and points out that the entire classification of all the invariant operators $T : R^k_\delta \to R^{k'}_{\delta'}$, $\delta \neq \delta'$, cannot be given in the frame of this work.

On the other hand, Equation (37) entails for instance that $1, \delta, \delta', \delta^2, \delta\delta', \delta'^2$ are linearly independent over the field $\mathbb{Q}$ of rational numbers, which is known to be a very strong condition on δ and δ'.

We explain below that any invariant operator $T : R^k_\delta \to R^{k'}_{\delta'}$ has a "trace" in the family of invariant operators $T : R^k_\delta \cap \ker i_\alpha \to R^{k'}_{\delta'} \cap \ker i_\alpha$, we provide a complete description of this family, and conclude that the number of possible values of δ and δ' is definitely very limited.

8.1 Reduction of the problem and contact-affine invariant operators

Let
$$T : R^k_\delta \to R^{k'}_{\delta'}$$
be a nontrivial sp_{2n+2}-invariant differential operator. If $\delta \notin \mathfrak{C}_k$, the source space R^k_δ is graded by the subspaces $R^{k,\ell}_\delta = X^\ell(R^{k-\ell}_\delta \cap \ker i_\alpha)$, $\ell \in \{0, \ldots, k\}$, whereas the target space $R^{k'}_{\delta'}$ is filtered by the increasing subspaces $R^{k'}_{\delta',l} := R^{k'}_{\delta'} \cap \ker i^l_\alpha$, $l \in \{0, \ldots, k'+1\}$, where $R^{k'}_{\delta',0} = 0$ and $R^{k'}_{\delta',k'+1} = R^{k'}_{\delta'}$, see [FMP07]. Hence, there exist $\ell \in \{0, \ldots, k\}$ and $l \in \{1, \ldots, k'+1\}$, such that T_ℓ is a nonvanishing sp_{2n+2}-invariant differential operator $T_\ell : R^{k,\ell}_\delta \to R^{k'}_{\delta',l}$, whose image $\mathrm{im}\, T_\ell$ is not contained in $R^{k'}_{\delta',l-1}$. Eventually, $T_{l\ell} := i^{l-1}_\alpha \circ T \circ X^\ell$ is a nonvanishing sp_{2n+2}-invariant differential operator

$$T_{l\ell} : R^{k-\ell}_\delta \cap \ker i_\alpha \to R^{k'-l+1}_{\delta'} \cap \ker i_\alpha. \tag{38}$$

Any sp_{2n+2}-invariant operator is of course $\mathrm{CVect}(\mathbb{R}^{2n+1}) \cap \mathrm{AVect}(\mathbb{R}^{2n+1})$-invariant. The following proposition describes the algebra of contact-affine invariant operators.

Proposition 6. *The associative algebra of contact-affine invariant differential operators is generated by the identity map and the operators*

- $D : R^k_\delta \to R^{k+1}_\delta$, $P \mapsto D(P)(\xi) := \sum_i (\xi_{q^i}\partial_{p_i} - \xi_{p_i}\partial_{q^i})P(\xi) + \xi_t \mathcal{E}_s P(\xi) - \langle \mathcal{E}_s, \xi_s\rangle \partial_t P(\xi)$
- $\mathrm{div} : R^k_\delta \to R^{k-1}_{\delta+1/(n+1)}, P \mapsto \mathrm{div}(P)(\xi) := \sum_i \partial_{q^i}\partial_{\xi_{q^i}} P(\xi) + \sum_i \partial_{p_i}\partial_{\xi_{p_i}} P(\xi) + \partial_t\partial_{\xi_t} P(\xi)$
- $i_\alpha : R^k_\delta \to R^{k-1}_\delta, P \mapsto i_\alpha P$
- $R_1 : R^k_\delta \to R^{k+1}_\delta, P \mapsto R_1(P)(\xi) := \xi_t P(\xi)$

- $R_2 : R^k_\delta \to R^k_{\delta+1/(n+1)}, P \mapsto R_2(P)(\xi) := \partial_t P(\xi)$

Moreover, every contact-affine invariant differential operator defined on R^k_δ is a linear combination of operators of the form

$$R_2^a \circ R_1^b \circ D^c \circ \operatorname{div}^d \circ i_\alpha^e \quad (a, b, c, d, e \in \mathbb{N}).$$

Proof. As the affine symbol map σ_{aff} provides a $\mathrm{CVect}(\mathbb{R}^{2n+1}) \cap \mathrm{AVect}(\mathbb{R}^{2n+1})$-module isomorphism from differential operators onto symbols, see Remark 2.b., Section 6, it suffices to use Theorem 2. The possibility of writing the generators in the announced order, is a consequence of the next proposition, which is easily checked by direct computation. ∎

Proposition 7. *The following commutation relations hold true on R^k_δ:*

1. $[D, \operatorname{div}] = -R_1 \circ \operatorname{div} + (2n+k)R_2$
   $[D, i_\alpha] = (k/2)\operatorname{id} + R_1 \circ i_\alpha$
   $[D, R_1] = 0$
   $[D, R_2] = 0$
2. $[\operatorname{div}, i_\alpha] = 0$
   $[\operatorname{div}, R_1] = R_2$
   $[\operatorname{div}, R_2] = 0$
3. $[i_\alpha, R_1] = -(1/2)\operatorname{id}$
   $[i_\alpha, R_2] = 0$
4. $[R_1, R_2] = 0$

Observe now that, since there is an invariant projector p^k_δ from R^k_δ to $R^k_\delta \cap \ker i_\alpha$, see [FMP07], every invariant differential operator I defined on $R^k_\delta \cap \ker i_\alpha$ is canonically the restriction of an invariant differential operator $I \circ p^k_\delta$ defined on R^k_δ. Therefore, any contact-affine invariant differential operator on $R^k_\delta \cap \ker i_\alpha$ is a linear combination of operators of shape

$$R_2^a \circ R_1^b \circ D^c \circ \operatorname{div}^d \quad (a, b, c, d \in \mathbb{N}). \tag{39}$$

Eventually, in view of Equations (38) and (39), any sp_{2n+2}-invariant differential operator $T : R^k_\delta \to R^{k'}_{\delta'}$ induces an sp_{2n+2}-*invariant differential operator* $T : R^k_\delta \cap \ker i_\alpha \to R^{k'}_{\delta'} \cap \ker i_\alpha$, *which is a combination of terms of the type* $R_2^a \circ R_1^b \circ D^c \circ \operatorname{div}^d$, $a, b, c, d \in \mathbb{N}$. In the sequel, we refer to these invariants as *basic invariants* and provide a complete description, which is valid in whole generality, i.e. for regular and for singular weights.

8.2 Contact-projective invariant operators

We first give two examples of basic sp_{2n+2}-invariant differential operators.

1. For any $k \in \mathbb{N}$ and any $\ell \in \{0, \ldots, k\}$, the operator

$$\operatorname{div}^\ell : R^k_\delta \to R^{k-\ell}_{\delta+\ell/(n+1)} \tag{40}$$

   is, for $\ell \neq 0$, sl_{2n+2}-invariant (and thus sp_{2n+2}-invariant), if

$$\delta = \frac{2n+1-\ell}{2(n+1)}.$$

   For details pertaining to the projective embedding sl_{2n+2} of $\mathrm{sl}(2n+2, \mathbb{R})$ into $\mathrm{Vect}(\mathbb{R}^{2n+1})$, see Section 3, for the proof of the sl_{2n+2}-invariance of div^ℓ, see [Lec00]. The commutation relation $[i_\alpha, \operatorname{div}] = 0$ implies that the restriction of this operator to $R^k_\delta \cap \ker i_\alpha$ is valued in $R^{k-\ell}_{\delta+\ell/(n+1)} \cap \ker i_\alpha$.

2. For any $r \in \mathbb{N}$, the operator

$$X^r : R^k_\delta \to R^{k+r}_\delta, \tag{41}$$

where $X : R^\kappa_\delta \to R^{\kappa+1}_\delta$ is defined by $X = D + (2(n+1)\delta + \kappa)R_1$, see Section 6 and Proposition 6, is an sp_{2n+2}-invariant differential operator. The commutation relation $[i_\alpha, X^r] = -(r/2)(2(n+1)\delta + 2k + r - 1)X^{r-1}$, which is valid on R^k_δ, see [FMP07], entails that the restriction of X^r, $r \neq 0$, to $R^k_\delta \cap \ker i_\alpha$ is valued in $R^{k+r}_\delta \cap \ker i_\alpha$, if and only if

$$\delta = \frac{1 - 2k - r}{2(n+1)}.$$

It is easy to see that, for this special value of δ, operator X^r coincides on R^k_δ with

$$X^r = \Pi^r_{j=1}(D - (k + j - 1)R_1),$$

where, due to commutation relation $[D, R_1] = 0$, the RHS factors may be ordered arbitrarily. Let us also immediately mention, that, whatever the value of δ, the RHS composite operator reads on R^k_δ

$$\Pi^r_{j=1}(D - (k + j - 1)R_1) = (D - R_1 \circ \mathcal{E}_\xi)^r, \tag{42}$$

where $\mathcal{E}_\xi$ denotes the Euler field with respect to $\xi \in \mathbb{R}^{2n+1}$, and that it follows from Proposition 7 that this operator maps $R^k_\delta \cap \ker i_\alpha$ into $R^{k+r}_\delta \cap \ker i_\alpha$.

In order to classify the basic invariant operators, we now prove a series of propositions, which specify the form of these operators and impose restrictions on the involved parameters.

Proposition 8. *If T is a nonvanishing sp_{2n+2}-invariant differential operator from $R^k_\delta \cap \ker i_\alpha$ to $R^{k'}_{\delta'}$, then $\delta' = \delta + \ell/(n+1)$ for some $\ell \in \{0, \ldots, k\}$ and $k' = k - \ell + r$ for some $r \in \mathbb{N}$. Moreover, operator T reads*

$$T = D^r \circ \mathrm{div}^\ell + \sum_{u=1}^{r} C_u\, R_1^u \circ D^{r-u} \circ \mathrm{div}^\ell, \tag{43}$$

where the C_u are real constants.

Proof. The considered invariant operator has the form $T = \sum C\ R_2^a \circ R_1^b \circ D^c \circ \mathrm{div}^d$, $C \in \mathbb{R}$, $a, b, c, d \in \mathbb{N}$ (R_2, R_1, D, and div are differential operators of homogeneous order 1, 0, 1, and 1, respectively). Its principal symbol $\sigma(T)$ is a member of the algebra of sp_{2n+2}-invariant tensor fields, which is generated by u_1, u_2, u_3, and L_1 (that correspond to $\mathrm{div} : R^k_\delta \to R^{k-1}_{\delta+1/(n+1)}$, $\mathrm{id} : R^k_\delta \to R^k_\delta$, $i_\alpha : R^k_\delta \to R^{k-1}_\delta$, and $D : R^k_\delta \to R^{k+1}_\delta$, respectively), see Theorem 1 and Theorem 3. Hence, up to multiplication by a nonvanishing constant, the highest order term of T is $D^r \circ \mathrm{div}^\ell : R^k_\delta \to R^{k-\ell+r}_{\delta+\ell/(n+1)}$, for some $\ell \in \{0, \ldots, k\}$ and some $r \in \mathbb{N}$. As the target space of T is now fixed, the lower order terms have necessarily the announced form. ∎

The next proposition takes into account the inclusion in $\ker i_\alpha$ of the target space of the invariant.

Proposition 9. *If T is a nonvanishing sp_{2n+2}-invariant differential operator from $R^k_\delta \cap \ker i_\alpha$ to $R^{k-\ell+r}_{\delta+\ell/(n+1)} \cap \ker i_\alpha$, $\ell \in \{0, \ldots, k\}$, $r \in \mathbb{N}$, then T is necessarily a scalar multiple of*

$$T = (D - R_1 \circ \mathcal{E}_\xi)^r \circ \mathrm{div}^\ell\,. \tag{44}$$

Proof. Any operator of the type (43) can be rewritten in the form

$$T = \sum_{u=0}^{r} b_u\, R_1^u \circ \Pi^{r-u}_{j=1}(D - (k - \ell + j - 1)R_1) \circ \mathrm{div}^\ell \quad (b_u \in \mathbb{R}).$$

This claim just amounts to a triangular system of equations in the unknowns b_u. Let now $P \in R^k_\delta \cap \ker i_\alpha$. According to the above comments on the inclusion of the operator image in $\ker i_\alpha$, see afore-detailed Examples 1 and 2, we have

$$\Pi^{r-u}_{j=1}(D - (k - \ell + j - 1)R_1)(\mathrm{div}^\ell P) \in R^{k-\ell+r-u}_{\delta+\ell/(n+1)} \cap \ker i_\alpha.$$

Since $i_\alpha^r R_1^u = (-1)^r\, 2^{-r} u(u-1)\dots(u-r+1)R_1^{u-r}$ on $\ker i_\alpha$, the computation of $i_\alpha^r(T(P))$ leads to $b_r = 0$, whereas that of $i_\alpha^{r-1}(T(P)),\dots,i_\alpha(T(P))$ gives $b_{r-1}=0,\dots,b_1=0$. The result then follows from Equation (42). ∎

The Casimir operator promptly provides a first condition on k, δ, ℓ, r.

Proposition 10. *If there exists a nontrivial* sp_{2n+2}*-invariant operator from* $R^k_\delta \cap \ker i_\alpha$ *to* $R^{k-\ell+r}_{\delta+\ell/(n+1)} \cap \ker i_\alpha$*, the parameters* k*,* δ*,* ℓ*, and* r *verify*

$$2(n+1)(\ell+r)\delta = \ell - \ell^2 + 2\ell n + r - 2kr - r^2.$$

Proof. According to Theorem 5 and the subsequent remark on the extension of this upshot to all weights, critical or not, the restriction of the Casimir operator C^k_δ to $R^k_\delta \cap \ker i_\alpha$ is given by $C^k_\delta = c(k,\delta)/(n+2)\,\mathrm{id}$. As any sp_{2n+2}-invariant operator intertwines the Casimir operator, we get

$$c(k,\delta) = c(k-\ell+r, \delta+\ell/(n+1)),$$

and the result follows. ∎

Existence of a nonvanishing invariant operator imposes further restrictions on the parameters.

Proposition 11. *If there exists a nontrivial* sp_{2n+2}*-invariant operator from* $R^k_\delta \cap \ker i_\alpha$ *to* $R^{k-\ell+r}_{\delta+\ell/(n+1)} \cap \ker i_\alpha$*, then either* $r=0$ *or* $2(n+1)\delta + 2k + r - 1 = 0$*.*

Proof. It suffices to write the invariance property for the symbol $e^{p_1}\frac{(\xi_{p_1} - q^1\xi_t)^k}{k!}$, $k \in \mathbb{N}$, and vector field X_{tq^1}, see Equation (15). Computations are straightforward but tedious and will not be reproduced here. ∎

We are now prepared to explain the main result of this section.

Theorem 7. *The basic* sp_{2n+2}*-invariant operators from* $R^k_\delta \cap \ker i_\alpha$ *to* $R^{k'}_{\delta'} \cap \ker i_\alpha$ *are the scalar multiples of* div^ℓ*,* $\ell \in \{0,\dots,k\}$*, and* X^r*,* $r \in \mathbb{N}$*.*

Proof. If $T \neq 0$ denotes such an operator, it follows from Propositions 8 and 9 that $\delta' = \delta + \ell/(n+1)$, $\ell \in \{0,\dots,k\}$, and $k' = k - \ell + r$, $r \in \mathbb{N}$, and that $T = C(D - R_1 \circ \mathcal{E}_\xi)^r \circ \mathrm{div}^\ell$, $C \in \mathbb{R}_0$. Further, Proposition 10 yields

$$2(n+1)(\ell+r)\delta = \ell - \ell^2 + 2\ell n + r - 2kr - r^2, \tag{45}$$

whereas Proposition 11 entails that $r=0$ or $2(n+1)\delta + 2k + r - 1 = 0$.

If $r=0$, Equation (45) shows that, either $\ell = 0$ and $T = C\,\mathrm{id}$, $C \in \mathbb{R}_0$, or $\delta = (2n+1-\ell)/(2(n+1))$ and $T = C\,\mathrm{div}^\ell$, $C \in \mathbb{R}_0$ (which is then actually a basic invariant).

Otherwise, we have $\delta = (1-2k-r)/(2(n+1))$ and Equation (45) implies that $\ell(2k+r+2n-\ell) = 0$, so that $\ell = 0$, as the second factor is nonnegative, and $T = CX^r$, $C \in \mathbb{R}_0$ (which is a basic invariant). ∎

References

[Arn89] Arnold V I, *Matematicheskie metody klassicheskoi mekhaniki*, Nauka, Moscow (1974), *Mathematical methods of classical Mechanics*, Springer-Verlag, Berlin (1989)

[AG89] Arnold V I, Givental A B, *Symplectic Geometry*, in : Current problems in mathematics, fundamental directions, Vol.**4**, pp 5-139, WINITI, Moscow (1986); English transl. Encyclopaedia of Mathematical Sciences, Vol.**4**, Springer Verlag, Berlin (1989)

[BHMP02] Boniver F, Hansoul S, Mathonet P, Poncin N, *Equivariant symbol calculus for differential operators acting on forms*, Lett. Math. Phys., **62**(3) (2002), pp 219-232

[BM01] Boniver F, Mathonet P, *Maximal subalgebras of vector fields for equivariant quantizations*, J. Math. Phys., **42**(2) (2001), pp 582-589

[BM06] Boniver F, Mathonet P, *IFFT-equivariant quantizations*, J. Geom. Phys., **56** (2006), pp 712-730

[Bor02] Bordemann M, *Sur l'existence d'une prescription d'ordre naturelle projectivement invariante*, arXiv:math.DG/0208171

[DLO99] Duval C, Lecomte P, Ovsienko V, *Conformally equivariant quantization: existence and uniqueness*, Ann. Inst. Fourier, **49**(6) (1999), pp 1999-2029

[DO01] Duval C, Ovsienko V, *Projectively equivariant quantization and symbol calculus: noncommutative hypergeometric functions*, Lett. Math. Phys., **57**(1) (2001), pp 61-67

[FMP07] Frégier Y, Mathonet P, Poncin N, *Decomposition of symmetric tensor fields in the presence of a flat contact projective structure*, arXiv:math/0703922

[GIMPU04] Grabowski J, Iglesias D, Marrero J C, Padron E, Urbanski P, *Poisson-Jacobi reduction of homogeneous tensors*, J. of phys. A : math. gen., **37** no20 (2004), pp 5383-5399

[GM03] Grabowski J, Marmo G, *The graded Jacobi algebras and (co)homology*, J. Phys. A: Math. Gen., **36** (2003), pp 161-181

[Gra06] Grabowski J, *Local Lie algebra determines the base*, to appear in "From Geometry to Quantum Mechanics, in Honor of Hideki Omor", Y.Maeda et al., eds., Progress in Math. **252**, Birkhaeuser (2007)

[GP07] Grabowski J, Poncin N, *On quantum and classical Poisson algebras*, Banach Center Publ., **76** (2007); activity SMR 1665, ICTP, Trieste (http://poisson.zetamu.com)

[Han06] Hansoul S, *Existence of natural and projectively equivariant quantizations*, arXiv:math.DG/0601518

[Kir76] Kirillov A, *Local Lie algebras*, Uspekhi Mat. Nauk, **31** (1976), pp 57-76

[KMS93] Kolář I, Michor P W, Slovák J, *Natural operations in differential geometry*, Springer-Verlag, Berlin, Heidelberg, New York, (1993)

[LMT96] Lecomte P, Mathonet P, Tousset E, *Comparison of some modules of the Lie algebra of vector fields*, Indag. Math., **7**(4) (1996), pp 461-471

[LO99] Lecomte P, Ovsienko V, *Projectively equivariant symbol calculus*, Lett. Math. Phys., **49**(3)(1999), pp 173-196

[Lec00] Lecomte P, *On the cohomology of* $\mathrm{sl}(m+1,\mathbb{R})$ *acting on differential operators and* $\mathrm{sl}(m+1,\mathbb{R})$*-equivariant symbol*, Indag. Math., **11**(1) (2000), pp 95-114

[Lec01] Lecomte P, *Towards projectively equivariant quantization*, Progr. Theoret. Phys. Suppl., **144** (2001), pp 125-132

[Lic78] Lichnerowicz A, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. Pures Appl., **IX** Ser. 57 (1978), pp 453-488

[LLMP99] León M, López B, Marréro J C, Padron E, *Lichnerowicz-Jacobi cohomology and homology of Jacobi manifolds: modular class and duality*, arXiv:math/9910079

[Mat99,1] Mathonet P, *Intertwining operators between some spaces of differential operators on a manifold*, Comm. Alg., **27**(2) (1999), pp 755-776

[Mat99,2] Mathonet P, *Invariant bidifferential operators on tensor densities over a contact manifold*, Lett. Math. Phys., **48**(3) (1999) pp 251-261

[MR05] Mathonet P, Radoux F, *Natural and projectively equivariant quantizations by means of Cartan connections*, Lett. Math. Phys., **72**(3) (2005), pp 183-196

[NR67] Nijenhuis A, Richardson R, *Deformations of Lie algebra structures*, J. Math. Mech., **171** (1967), pp 89-106

[OO93] Ovsienko V Yu, Ovsienko O D, *Projective structures and infinite-dimensional Lie algebras associated with a contact manifold*, Advances in Soviet Mathematics, **17** (1993)

[OR92] Ovsienko V, Roger C, *Deformations of Poisson brackets and extensions of Lie algebras of contact vector fields*, Uspekhi Mat. Nauk, **47**(6) (1992), pp 141-194, Russian Math. Surveys **47**(6) (1992), pp 135-191

[Ovs05] Ovsienko V, *Vector fields in presence of a contact structure*, preprint, arXiv:math.DG/0511499v1

[Pon04] Poncin N, Equivariant Operators between some Modules of the Lie Algebra of Vector Fields, Comm. Alg., **32**(7) (2004), pp 2559-2572

[Skr87] Skryabin S M, *The regular Lie rings of derivations of commutative rings*, preprint WINITI 4403-W87 (1987)

[Wey46] Weyl H, *The classical groups, their invariants and representations*, Princeton Math. Series, Princeton (1946)