

RECOVERING THE GEOMETRY OF A FLAT SPACETIME FROM BACKGROUND RADIATION

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ABSTRACT. We consider globally hyperbolic flat spacetimes in 2+1 and 3+1 dimensions, in which a uniform light signal is emitted on the r -level surface of the cosmological time for $r \rightarrow 0$. We show that the frequency of this signal, as perceived by a fixed observer, is a well-defined, bounded function which is generally not continuous. This defines a model with anisotropic background radiation that contains information about initial singularity of the spacetime. In dimension $2 + 1$, we show that this observed frequency function is stable under suitable perturbations of the spacetime, and that, under certain conditions, it contains sufficient information to recover its geometry and topology. We compute an approximation of this frequency function for a few simple examples.

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1. INTRODUCTION

1.1. Motivation. There is considerable interest in the cosmic background radiation as an indicator of the history and structure of the universe. Its anisotropy is explained by quantum fluctuations early in the history of the universe, whose classical remnants became visible when the universe became transparent to electromagnetic radiation after decoupling. Background radiation has also been used to determine the topology of the universe [14, 9, 13, 18].

In the framework of general relativity, such measurements of cosmic background radiation are described by a spacetime in which light signals are emitted near the initial singularity and received by an observer. The question to what degree the geometry of a spacetime can be reconstructed from such light signals is of interest both from the mathematics and the physics perspective. However, it has not been investigated systematically yet, even for simple examples such as constant curvature spacetimes or lower-dimensional models.

A further motivation to investigate the properties of such light signals are possible applications in 2+1 (and higher-dimensional) quantum gravity. To give a physical interpretation to a quantum theory of gravity, it is essential to relate the variables that describe the physical phase space of the theory and serve as the basic building blocks in quantization to concrete measurements that could be performed by an observer. From a mathematics perspective, this amounts to a (partial) classification of spacetimes in terms of light signals and other general relativistic quantities measured by observers or, equivalently, to reconstructing the geometry of the spacetime from such measurements.

In this article, we investigate this question for a class of simple examples, namely flat globally hyperbolic spacetimes in 3+1 and 2+1 dimensions. We consider a uniform light signal that is emitted from a hypersurface of constant cosmological time ϵ where $\epsilon \rightarrow 0$. The observer who receives the signal at a cosmological time $T > \epsilon$ is modeled by a point $p \in M$ of cosmological time T and a unit, future-oriented timelike vector v , which specifies his velocity. The frequency of the light signal measured by the observer then defines a frequency function on, respectively, S^2 and S^1 , which depends on both the spacetime M and the observer (p, v) .

If the spacetime M is conformally static, i. e. characterized by a linear holonomy representation, then the associated frequency function is isotropic, and contains no relevant information on M . However, as soon as the holonomy representation of M has a non-trivial translation component, which corresponds to a universe whose geometry changes with the cosmological time, the frequency function contains essential information about the underlying spacetime. Under certain conditions, this information allows the observer to recover the geometry and topology of M as well as his motion relative to the initial singularity.

The models under consideration in this article are unrealistic insofar as they are purely classical — we do not consider any quantum fluctuations near the initial singularity — and as their initial singularities are not of the same type as those in cosmological models. However, they allow one to investigate the mathematical properties of such light signals emitted near the initial singularity systematically for a large class of spacetimes. Moreover, it turns out that the resulting frequency functions have rich and subtle properties and contain interesting information about the underlying spacetimes. In particular, they exhibit anisotropies, which are due entirely to the classical geometry of the spacetime M or, more specifically, the initial singularity of the universal cover of M , which is a domain of dependence in Minkowski space.

The model spaces considered here could actually be more relevant than they might appear at first sight. Indeed, Mess [19] proved that *any* maximal globally hyperbolic Minkowski space of dimension 2+1 is of this form. Moreover, any globally hyperbolic flat space-like slice must embed isometrically into one of those maximal globally hyperbolic Minkowski space-times. So, if we consider for instance a universe U with a very strong curvature at small cosmological time, but suppose that this curvature decreases fast enough so that it can be considered zero outside a neighborhood Ω of the initial singularity, then $U \setminus \Omega$ will embed in a globally hyperbolic Minkowski manifold as studied here. The presence of the curvature might of course change the signal emitted close to the initial singularity but, if the curved region Ω is thin enough, it is conceivable that the model used here remains relevant.

An additional reason to investigate flat globally hyperbolic spacetimes is their role in 2+1 gravity. This theory plays an essential role as a toy model for quantum gravity in higher dimensions (see [8] and references therein) because it allows one to investigate important questions of quantum gravity in a fully quantized theory. The classification result by Mess [19] implies that any globally hyperbolic vacuum solution Einstein's equations in 2+1 dimensions is a flat globally hyperbolic spacetime of the type considered in this article and can be characterized in terms of its holonomy representation. The holonomies associated with closed curves in M are diffeomorphism invariant observables and serve as the fundamental building blocks in the quantization of the theory.

Characterizing the holonomy representation of a spacetime in terms of light signals measured by an observer thus allows one to give a physical interpretation to these variables and to model cosmological measurements. The characterization of observables of 2+1 gravity in terms of light signals has been explored to some degree in [20], but the measurements of background radiation considered in this article provide a richer and more realistic model.

1.2. The frequency function. After recalling the relevant background material on flat maximally globally hyperbolic spacetimes and their description in terms of domains of dependence in Section 2, we introduce the rescaled frequency function of a domain of dependence \tilde{M} in Section 3. The rescaled frequency function is defined in Section 3.1. It is given in terms of the limit $\epsilon \rightarrow 0$ of a uniform light signal emitted from the hypersurface of cosmological time $T = \epsilon$ of \tilde{M} and received by a free-falling observer in the spacetime. Sections 3.2 and 3.3 contain an explicit description of the frequency function for the two basic examples, namely the future of a point and the future of a spacelike line.

These two basic examples are the building blocks in the analysis of the frequency function for a general domain \tilde{M} , whose properties are investigated in Section 3.4. The central result is Proposition 3.1, which asserts that the frequency function is well-defined and locally bounded. It should be stressed that even at this point, the mathematical analysis of the frequency function is not as simple as it may appear at first sight, and some care is needed. The situation is simpler for domains \tilde{M} whose initial singularity is closed, which are investigated in Section 3.5. In this case, the associated frequency function is continuous. Note, however, that many relevant examples are not of this type.

Section 3.6 analyses the properties of the frequency functions for generic domains in 2+1 dimensions. This case is more accessible than its 3+1-dimensional counterpart due to a simple description of domains of dependence, discovered by Mess [19], in terms of a measured lamination on the hyperbolic plane. Domains of dependence, which are the universal covers of globally hyperbolic flat spacetimes, are obtained from measured geodesic laminations on closed hyperbolic surfaces. By using these results we prove (Proposition 3.6) that the frequency function is lower semi-continuous, and that its discontinuity set is meagre.

1.3. Stability. In Section 4 we investigate the stability properties of the frequency function. We analyze the variation of the frequency function under small deformations of the domain of dependence \tilde{M} and changes of the observer. Note that stability of the frequency function, at least with respect to small changes of the observer, is a minimum requirement for assigning any physical meaning to it.

Again, this question turns out to be more subtle than it appears and some care is needed in the analysis. This is illustrated in Section 4.1, where we show in a very simple example that if a domain of dependence \tilde{M} is the limit (in the Hausdorff sense) of a decreasing sequence of *finite domains* \tilde{M}_n (domains which are the intersection of the futures of a finite set of lightlike planes) the frequency function of \tilde{M} does not necessarily coincide with the limit of the frequency functions of the domains \tilde{M}_n .

With this example in mind, we introduce in Section 4.2 a notion of domain of dependence with a *flat boundary*. Important examples of this are finite domains, which always have flat boundary, and universal covers of globally hyperbolic flat spacetimes in 2+1 dimensions (Proposition 4.22). We prove (Theorem 4.13) that if a sequence of domains with flat boundary converges to a domain with flat boundary, then the limit of the frequency functions is the frequency function of the limit.

If a domain D does not have flat boundary, and if $(D_n)_{n \in \mathbb{N}}$ is sequence of domains with flat boundary converging to D , then the sequences of the frequency functions ι_n of D_n always converges to a limit frequency function ι_{lim} . This limit frequency function is not necessarily equal to the frequency function ι of D , but it is independent of the sequence $(D_n)_{n \in \mathbb{N}}$. We prove (Theorem 4.14) that frequency function ι is at least equal to the limit frequency function ι_{lim} , and at most equal to $d \iota_{lim}$, where d is the dimension of the spacetime. In particular, this shows that the example of Section 4.1 exhibits the worst possible behavior with respect to this limit, as the ratio of the frequency function of the domain to its limit frequency is the largest possible in dimension $2 + 1$.

1.4. Recovering the spacetime geometry and topology. In Section 6 we turn to the question of reconstructing the geometry and topology of a globally hyperbolic flat spacetime M from the frequency function of the background radiation as seen by an observer. We investigate this question in 2+1 dimensions, and there are two basic remarks regarding the general situation.

- Reconstructing the geometry or topology of the spacetime from the observed frequency function is only possible for observers that receive the light signal at a sufficiently large cosmological time. If the observer is too close to the initial singularity, he might see only a small part of M and could infer little from the observed background radiation.
- The observer can only determine parts of the initial singularity of the universal cover \tilde{M} of M . Therefore, there is no way for the observer to be sure, at any given time, that what he observes is really the topology of M . It could happen that M is “almost” a finite cover of a globally hyperbolic flat spacetime M' , with only a tiny difference in a part not “seen” by the observer. In this case the observer could only conclude that the spacetime is either M' or one of its finite covers.

To obtain results we make a (presumably) technical assumption, which simplifies the situation to some extent. We only consider spacetimes obtained by “grafting” a hyperbolic surface along a *rational* measured lamination, that is, a measured lamination with support on a finite set of simple closed curves. Under those hypothesis, we prove (Proposition 5.8) that the observer can reconstruct the part of the lamination corresponding to the part of the initial singularity that intersects his past. We also prove (Proposition 5.13) that the observer can reconstruct the whole geometry and topology of the spacetime in finite eigentime up to the above-mentioned problem with finite covers.

1.5. Computations for examples. In Section 6 we present explicit computations for the frequency function seen by an observer for three different globally hyperbolic flat 2+1-dimensional spacetimes. These spacetimes are chosen for their simplicity. Two are obtained by grafting a hyperbolic surface along a rational measured lamination, the third by grafting along an irrational lamination. For each of those spacetimes, we provide pictures of the frequency function as seen by an observer located at different points in the spacetime. This allows one to observe explicitly the variation of the frequency function depending on the cosmological time.

In dimension $3 + 1$, we only consider one example, described in Section 7. This is due to the fact that 3+1-dimensional globally hyperbolic flat spacetimes are much more difficult to construct than their 2 + 1-dimensional counterparts. In both cases, they are associated to first-order deformations of the flat conformal structure underlying a hyperbolic manifold. However, hyperbolic manifolds are flexible in dimension 2, while they are rigid in dimension 3. Consequently, it becomes more difficult to find an adequate deformation cocycle in dimension $3 + 1$. The example considered in Section 7.1 is due to Apanasov [2], and it has the relatively rare property of admitting several distinct deformation cocycles. We provide some pictures of the frequency function measured by an observer in a spacetime constructed from this example.

1.6. Possible extensions. In this article, we consider only flat globally hyperbolic spacetimes. However, it should be possible to perform a similar analysis for globally hyperbolic de Sitter or anti-de Sitter spacetimes, which have a similar structure, at least with respect to the geometry of their initial singularity.

2. GLOBALLY HYPERBOLIC MINKOWSKI SPACE-TIMES

2.1. Minkowski space and domain of dependences. In this section, we collect some properties of Minkowski space and refer the reader to [19, 7] for details. Minkowski space in $n + 1$ dimensions, denoted $\mathbb{R}^{1,n}$ in the following, is the manifold \mathbb{R}^{n+1} equipped with the flat Lorentzian form $\eta = -dx_0^2 + dx_1^2 + \dots + dx_n^2$, often referred to as Minkowski metric.

Isometry group. Isometries of Minkowski space are affine transformations of \mathbb{R}^{n+1} whose linear part preserves the Minkowski metric. We denote by $O(1, n)$ the group of linear transformations of \mathbb{R}^{n+1} which preserve the Minkowski metric (Lorentz group in $n + 1$ dimensions) and by $\text{Isom}(n, 1)$ the group of isometries of Minkowski space (Poincaré group in $n + 1$ dimensions). $O(n, 1)$ is a $n(n + 1)/2$ -dimensional Lie group with four connected components, and we denote by $SO^+(n, 1)$ its identity component, which contains linear orthochronous transformations with positive determinant. The dimension of $\text{Isom}(n, 1)$ is $n(n + 1)/2 + (n + 1) = (n + 1)(n + 2)/2$, and for $n \geq 3$ this group has four connected components. The identity component, denoted $\text{Isom}_0(1, n)$, contains the transformations that preserve both the orientation and the time orientation.

Flat spacetimes. It is well-known that every flat spacetime is locally modeled on Minkowski space. For globally hyperbolic flat spacetimes, a more precise result holds (see [19, 1]). For every flat spacetime M with a closed Cauchy surface, there is a discrete group of isometries $\Gamma \subset \text{Isom}_0(n, 1)$ and a convex domain $D \subset \mathbb{R}^{1,n}$ such that D is Γ -invariant and M embeds into the quotient D/Γ . The domain D is a domain of dependence, in the sense that it is the intersection of the futures of one or more lightlike planes. Domains of dependence play an essential role in this paper, and will be described in more detail below. The quotient space D/Γ is called a maximal globally hyperbolic flat space-time with compact Cauchy surfaces, for which we use the acronym MGHFC.

Hyperbolic representations. The unit timelike vectors in $\mathbb{R}^{1,n}$ form a smooth hypersurface, $H \subset \mathbb{R}^{1,n}$, which contains two connected components: the component H^+ that contains future oriented unit vectors, and H^- that contains past oriented unit vectors. Both H^+ and H^- are achronal spacelike smooth surfaces. The Minkowski metric induces a Riemannian metric of constant curvature -1 on H^+ and H^- . Equipped with this metric H^+ and H^- are isometric to the n -dimensional hyperbolic space \mathbb{H}^n . The group $SO^+(1, n)$ acts by isometries on H^+ , and it is identified with the identity component of the isometry group of H^+ . Every geodesic of H^+ is given as the intersection of H^+ with a timelike linear 2-dimensional plane.

2.2. Domains of dependence. A domain of dependence (called regular domain in [7]) D is a convex domain of $\mathbb{R}^{1,n}$ that is given as the intersection of the future (or the past) of a number of lightlike n -planes. We will exclude two limit cases: the whole space and the future of a single lightlike n -plane. In other words, we require that $\mathbb{R}^{1,n} \setminus D$ contains at least two non-parallel lightlike n -planes.

Simple examples of domains of dependence are the future of a point, or the future of a spacelike $(n-1)$ -plane, whereas the future of a spacelike n -plane is not a domain of dependence. Examples with interesting geometrical properties are the universal covers of maximal globally hyperbolic flat manifolds with compact Cauchy surface (MGHFC manifolds). Figure 1 shows two of those more complex examples, corresponding to the domains of dependence described in sections 6.1.2 and 6.1.3.

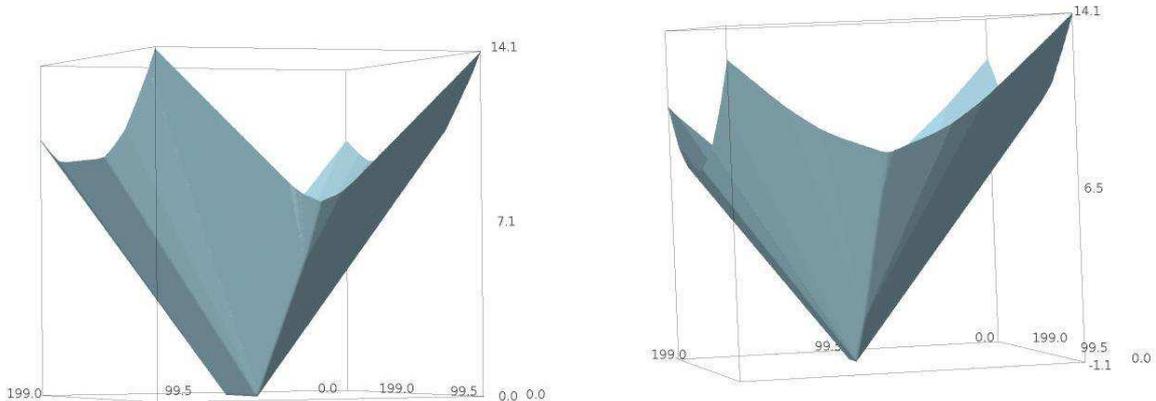


FIGURE 1. Two examples of domains of dependence

Let us recall that for any Lorentzian manifold M , the cosmological time is a function $\tau : M \rightarrow (0, +\infty]$ whose value at a point $p \in M$ is the supremum of the length of causal curves in M ending at p :

$$\tau(p) = \sup\{\ell(c) \mid c \text{ is a casual curve ending at } p\}.$$

One of the main features of domains of dependence is that their cosmological time is a regular function. This means that τ is finite-valued and $C^{1,1}$. In fact, if D is a domain of dependence and $p \in D$, there is a unique point $r = \mathbf{r}(p) \in \partial D \cap I^-(p)$ such that $\tau(p) = |p - r|$. Level surfaces $H_a = \tau^{-1}(a)$ of the cosmological time are spacelike Cauchy surfaces, and their normal vector at a point $p \in H_a$ is the vector $p - \mathbf{r}(p)$.

Example 2.1.

- If D is the future of a point r_0 , then $\mathbf{r}(p) = r_0$ for all $p \in D$, and the cosmological time $\tau(p)$ coincides with the distance of p from r_0 . In this case, the cosmological time is smooth (real analytic in fact), and the induced metric on the level surface H_a has constant curvature $-1/a^2$.

- If D is the future of a spacelike affine plane l_0 of dimension $k \leq n - 1$ then D is a domain of dependence. For $p \in D$, $\mathbf{r}(p)$ is the intersection point of l_0 with the affine subspace orthogonal to l_0 passing through p . Also in this case τ is smooth. The level surface H_a are isometric to $\mathbb{R}^k \times \mathbb{H}^{n-k}$. If $n = 2$ and $k = 1$, this implies that the metric is flat.
- If $D \subset \mathbb{R}^{n,1}$ is the future of a spacelike segment $[p_0, p_1]$, then D is divided into three regions by two timelike hyperplanes P_0, P_1 orthogonal to $[p_0, p_1]$ and passing, respectively, through p_0 and p_1 . The first region is the half-space D_0 bounded by P_0 which does not contain p_1 , the second is the half-space D_1 bounded by P_1 which does not contain p_0 , and the third is the intersection V of the other two half-spaces bounded by P_0 and by P_1 .

For $p \in D_0$, one has $\mathbf{r}(p) = p_0$, for $p \in D_1$ $\mathbf{r}(p) = p_1$ and for $p \in V$, $\mathbf{r}(p)$ is the intersection point of $[p_0, p_1]$ with the plane orthogonal to $[p_0, p_1]$ that passes through p . In this case τ is smooth outside the boundaries of the regions D_0, D_1 and V and is only $C^{1,1}$ on their boundaries. Level surfaces are divided in three regions: the regions $H_0(a) = H_a \cap D_0$ and $H_1(a) = H_a \cap D_1$ are isometric to half-spaces of constant curvature $-1/a^2$, while $B_a = H_a \cap V$ is isometric to the product of the hyperbolic space of dimension $n - 1$ with an interval of length equal to $|p_1 - p_0|$. (For $n = 2$, this is a flat strip of width $|p_1 - p_0|$).

These examples are illustrated in Figure 2.

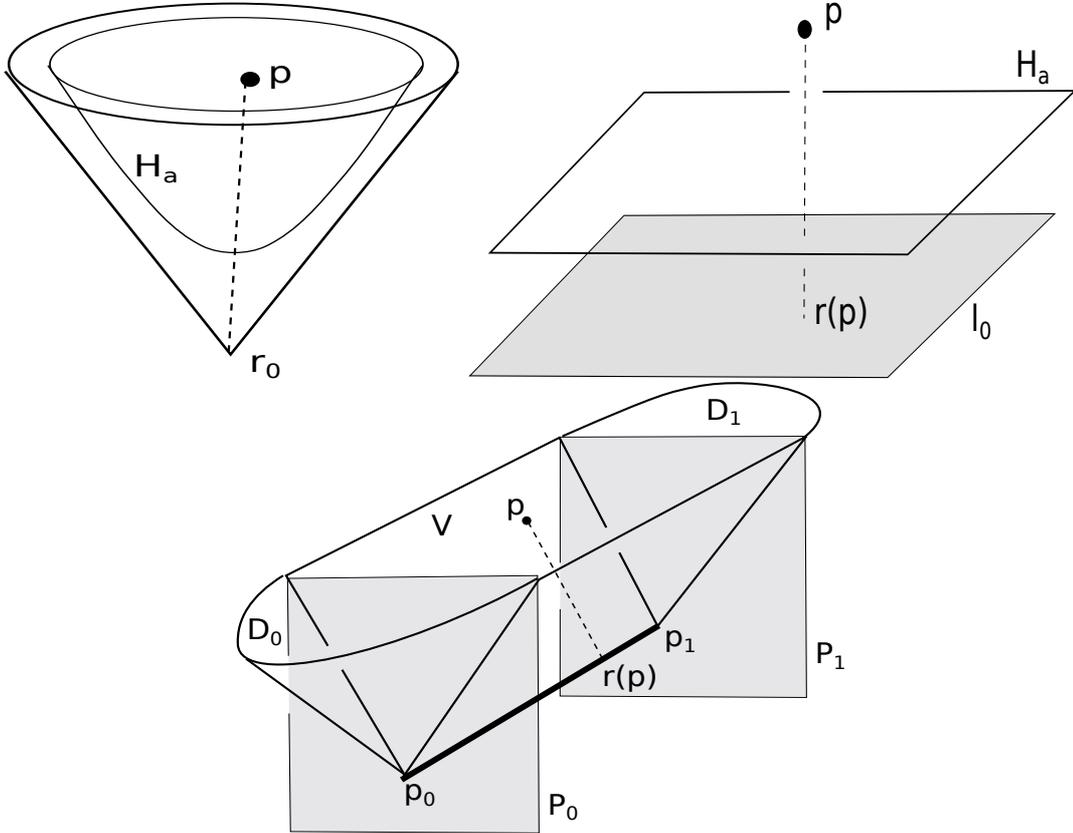


FIGURE 2. The domains in Example 2.1

2.3. The boundary of a domain of dependence and the initial singularity. In this section we recall important facts about the geometry of the boundary ∂D of a domain of dependence $D \subset \mathbb{R}^{1,n}$.

We start by summarizing a useful description of the boundary ∂D . If P is a spacelike $(n - 1)$ -dimensional plane in $\mathbb{R}^{1,n}$, the orthogonal projection $\pi : \partial D \rightarrow P$ is 1-to-1, so ∂D can be regarded as the graph of a convex function u on P . Since ∂D is achronal, u turns out to be 1-Lipschitz. More precisely, one finds that the graph of a convex function on P is the boundary of a domain of dependence if and only if $\|\text{grad}u\| = 1$ at each point where u is differentiable.

For any point $r \in \partial D$, there is at least one future directed lightlike half-line l contained in ∂D that passes through r . It is important to note that there are always points on ∂D from which at least two lightlike half-lines contained in D originate. Indeed this occurs exactly when p is in the image of the map $\mathbf{r} : D \rightarrow \partial D$ introduced in the previous section. This subset is called the *initial singularity*¹ of the domain D , and will be denoted by T . It is the smallest subset of \bar{D} such that $D = I^+(T)$.

If one regards D as the graph of a convex function u , lightlike lines in ∂D correspond to integral lines of the gradient of u , whereas the initial singularity corresponds to the set of points in which u is not differentiable. In many interesting cases, the shape of the initial singularity can be quite complicated. For instance, it was shown by Mess [19] that for the universal covering D of a generic (2+1)-dimensional MGHFC spacetime, the image of \mathbf{r} is a dense subset of ∂D . On the boundary ∂D , we consider the pseudo-distance defined as follows:

- given a Lipschitz arc k contained in ∂D , its velocity (defined a.e.) is not timelike. So we can define the length of k as

$$\ell(k) = \int \sqrt{\langle \dot{k}(t), \dot{k}(t) \rangle} dt.$$

- Given $r_1, r_2 \in \partial D$ the space of Lipschitz arcs $\mathcal{K}(r_1, r_2)$ joining them is not empty. So we can define

$$d_0(r_1, r_2) = \inf \{ \ell(k) \mid k \in \mathcal{K}(r_1, r_2) \}.$$

As the boundary of D contains lightlike segments (whose length is clearly 0), the pseudo-distance d is not a genuine distance, since there exist pairs of points with $d(p, q) = 0$. However, the following lemma shows that this can occur only if the images of p and q under the map \mathbf{r} coincides.

Lemma 2.2 ([7]). *If D is a domain of dependence and $p, q \in D$ then $\mathbf{r}(p) \neq \mathbf{r}(q)$ implies $d_0(\mathbf{r}(p), \mathbf{r}(q)) \neq 0$.*

In other words, this lemma states that the restriction of d_0 to the initial singularity is a distance. Let us also remark that the topology we consider on T is the one induced by the distance d_0 , which in general is different from the topology induced by Minkowski space.

Example 2.3.

- If D is the future of a point r_0 , the initial singularity contains only one point that can be identified with r_0 .
- If D is the future of an affine subspace E of dimension $k \leq (n-1)$, then the initial singularity is isometric to E .
- If D is the future of a segment in $\mathbb{R}^{2,1}$, then the initial singularity is the segment itself.
- If $D \subset \mathbb{R}^{2,1}$ is the intersection of three half-spaces bounded by lightlike planes, the initial singularity is the union of three spacelike rays starting from the intersection point of the planes. Note that in this case, the initial singularity is not a submanifold. In fact, generically the initial singularity does not have a manifold structure. The geometry of the initial singularities of domains of dependence in dimension $2+1$ is discussed in more depth in the next section.

2.4. The 2+1-dimensional case: the Mess construction. Mess [19] discovered an efficient way to construct regular domains in $\mathbb{R}^{2,1}$. This construction is general in the sense that every regular domain can be constructed in this way. It also has the major advantage that the geometrical features of the initial singularities are readily apparent.

We will describe the Mess construction in simple cases, namely for domains obtained by grafting along weighted multicurves. These simple cases are dense, in the sense that every domain of dependence can be approximated by domains of dependence obtained in this way.

Let us start from a collection of disjoint geodesics of H^+ , $L = l_1 \cup \dots \cup l_k$, and a collection of positive numbers a_1, \dots, a_k . Every geodesic is given as the intersection of H^+ with a timelike linear plane P_1, \dots, P_k . The planes P_i disconnect H^+ into a collection of regions D_1, \dots, D_h . (Note that each of them is the cone on some component of $H^+ \setminus L$).

For each plane P_i , let v_i be the vector in $\mathbb{R}^{1,2}$ characterized by the following conditions:

- it is orthogonal to P_i with respect to the Minkowski metric (in particular it is spacelike);
- its norm is equal to a_i ;
- it points to the component of $\mathbb{R}^{1,2}$ that does not contain D_1 .

¹Note that the use of the term *initial singularity* here differs from the one in the physics literature. The set of points of \bar{D} at which causal curves cannot be extended into the past is the entire boundary ∂D .

Now for any region D_j take the sum of all vectors v_i associated with to planes P_i that separate D_j from D_1 :

$$w_j = \sum_{i: P_i \text{ separates } D_1 \text{ from } D_j} v_i.$$

Translating each region D_j by the vector w_j yields a collection of disjoint domains D'_1, \dots, D'_h which are convex cones with vertices at w_j . In particular, note that if D_{j_1} is adjacent to D_{j_2} , then $w_{j_1} - w_{j_2}$ is a vector orthogonal to the plane P_i separating D_{j_1} from D_{j_2} and is of norm a_i .

In order to connect the domains D'_j we consider the domains V_i obtained as follows. If D_{j_1} and D_{j_2} are adjacent along P_i , then V_i is the region of the future of the segment $s_i = [w_{j_1}, w_{j_2}]$ bounded by the two timelike planes orthogonal to the segment s_i through its end-points.

It turns out that $D = \bigcup D'_j \cup \bigcup V_i$ is a domain of dependence. The map \mathbf{r} can be easily defined on each piece: \mathbf{r} sends all points of D'_j onto w_j , while it sends points of V_i to the segment $s_i = [w_{j_1}, w_{j_2}]$. The level surface H_a can be decomposed into different regions: the regions $H_a \cap D'_i$, which have constant curvature $-1/a^2$, and the regions $H_a \cap V_i$, which are Euclidean strips of width a_i .

The initial singularity is then given as the union of the line segments s_i and the vertices w_j . In particular, it is a graph with a vertex for every region of $H^+ \setminus L$. Two vertices w_{j_1} and w_{j_2} are connected by one edge if and only if the corresponding regions are adjacent. Combinatorially the singularity is a tree, that is, a graph which does not contain any closed loop. Notice that the length of each segment s_i is precisely a_i .

Although we summarized this construction for a finite number of geodesics, it works analogously also when L is an infinite family of disjoint geodesics that is locally finite (i.e. every compact subset of H^+ meet only a finite number of l_i).

2.5. The equivariant construction. Using the construction from the previous subsection, one can construct the universal coverings of MGHFC spacetimes different from $I^+(0)$ as follows. Take a hyperbolic surface F and consider the metric universal covering $\pi : H^+ \rightarrow F$ and covering group $\Gamma < SO^+(1, 2)$. Consider on F a disjoint collection of simple closed geodesics $c_1 \dots c_k$ and positive numbers $a_1 \dots a_k$. Then the preimage $L = \pi^{-1}(c_1 \cup \dots \cup c_k)$ is a union of infinitely many disjoint geodesics. The weight of each geodesic $\tilde{l}_i \subset L$ is the number corresponding to $\pi(\tilde{l}_i)$. As above, the geodesics \tilde{l}_i correspond to planes that cut $I^+(0)$ into infinitely many pieces D_j . By the invariance of L under the action of Γ , elements of Γ permute the regions D_j .

The construction explained in the previous subsection then produces a domain D , and Mess showed that there is an affine deformation Γ' of Γ , so that D is Γ' -invariant and the quotient is a MGHFC spacetime. Namely any $\gamma \in \Gamma$ is changed by adding a translation part of vector $w(\gamma)$ which is the sum of all vectors w_i corresponding to the planes P_i disconnecting D_1 from γD_1 .

Remark 2.4. In the example above, it can be seen that each D_j bounds infinitely many planes P_i . This implies that the vertex in the initial singularity corresponding to D_j is the end-point of infinitely many edges, or equivalently has infinite valence.

In the examples illustrated in the previous section it turns out that the initial singularity of domains of dependence in $\mathbb{R}^{2,1}$ is always a graph, and in fact a tree (possibly with vertices of infinite valence). In fact, there are more complicated examples in which the initial singularity does not have a simple graph structure, but it is always a real tree according to the following definition.

Definition 2.5. A metric space (T, d) is a **real tree** if for every $p, q \in T$ there is a unique arc $k \subset T$ joining them. Moreover k is the image of an isometric immersion $I \rightarrow T$ where I is an interval of length equal to $d(p, q)$.

Real trees are generalizations of the usual trees (which, by contrast, are often called simplicial trees). The domains of dependence whose singularity is a simplicial tree are exactly those constructed in the previous section [5, 4]. In particular, every domain of dependence with simplicial tree as initial singularity is determined by a simplicial measured geodesic lamination of H^+ , which, by definition, is a locally finite union L of disjoint geodesics l_i , each equipped with a weight $a_i > 0$.

Proposition 2.6. [5] *If D is a domain of dependence in $\mathbb{R}^{2,1}$ then its initial singularity T is a real tree. Moreover, the vertices of T are those points in ∂D at which at least three lightlike segments in ∂D originate.*

Given a point $r \in T$, let D_r be the convex hull in Minkowski space of the lightlike lines contained in ∂D which start at r . Then D_r is a convex subset of $\overline{I^+}(r)$. Notice that the dimension of D_r is 3 if and only if r is a vertex, otherwise D_r is the intersection between D and the timelike plane containing the two lightlike rays starting at r .

If τ_r is the translation which send r to 0, we denote by \mathcal{F}_r the intersection of H^+ with $\tau_r(D_r)$. Note that \mathcal{F}_r can be interpreted as the set of unit normals of the support planes of D at r . A number of consequences follow directly.

- If r is a vertex then \mathcal{F}_r is a region of H^+ bounded by disjoint geodesics.
- If r is not a vertex then \mathcal{F}_r is a complete geodesic.
- If $r \neq s$ then \mathcal{F}_r and \mathcal{F}_s have disjoint interiors. \mathcal{F}_r and \mathcal{F}_s can be disjoint, they can coincide if they are both lines, or they can meet along a boundary component.

In particular the set $L = \bigcup_{r \text{ is a vertex}} \partial \mathcal{F}_r \cup \bigcup_{r \text{ is not a vertex}} \mathcal{F}_r$ is a union of disjoint geodesics that are called the leaves of L . In general, the set L can be quite complicated. The intersection of a geodesic arc in H^+ and L can be uncountable (and sometimes a Cantor set).

The simplest case is when the singularity is a tree. In this case, the set L is the union of isolated geodesics: any compact subset of H^+ meets only a finite number of leaves. In this case, it is also evident that components of $H^+ \setminus L$ corresponds to vertices of T , whereas each leaf of L corresponds to an edge of T .

In addition to L we can construct a *transverse measure* that is the assignment of a non-negative number for any arc transverse to the leaves of L which verify some additivity conditions, see e.g. [6]. If k is an arc on H^+ that joins two points in $H^+ \setminus L$ and meets each leaf at most once (for instance if k is a geodesic segment that is not contained in any leaf), we define $\mu(k) = d_0(r_0, r_1)$ where r_0 and r_1 are the points on T such that the end-points of k are contained in \mathcal{F}_{r_0} and \mathcal{F}_{r_1} .

If the lamination is locally finite, for each leaf l there is a number $a(l)$ that coincides with the measure of any arc k transversely meeting only l . Any transverse arc k can be subdivided into a finite number of arcs k_1, \dots, k_p such that each k_i meets every leaf at most once. So we can define $\mu(k) = \sum \mu(k_i)$.

Mess [19] showed that the data (L, μ) determines D up to translation. In the simple case where the lamination is locally finite, the construction of D from (L, μ) is the one summarized in Section 2.4.

Remark 2.7. In dimension $n + 1 \geq 4$, it is no longer true that the initial singularity is a tree. In fact the geometry of the initial singularity is still not understood. In [7] a description of the singularity is given in some special cases.

2.6. Holonomies of domains of dependence and hyperbolic structures. In Section 2.1, we summarized the construction which assigns a domain of dependence to each flat MGHFC manifold. There is also a deep relation between holonomies of flat Lorentzian manifolds and first-order deformations of holonomy representations of hyperbolic manifolds, which was already used in dimension $2 + 1$ for instance in [12]. In the following, we summarize this relation, which behaves somewhat differently in dimension $2 + 1$ and in higher dimension.

Dimension 2+1. In this subsection, we recall how a flat $(2+1)$ -dimensional MGHFC manifold can be obtained from a point in Teichmüller space together with a deformation 1-cocycle. For this, note that $\mathbb{R}^{2,1}$ can be identified with the Lie algebra $sl(2, \mathbb{R})$ with its Killing metric. The canonical action of $SO(2, 1)$ on $\mathbb{R}^{2,1}$ corresponds to the adjoint action of $SL(2, \mathbb{R})$ on $sl(2, \mathbb{R})$. For each representation of $\pi_1 S$ in $SO(2, 1)$, it determines a vector bundle over S with fiber $\mathbb{R}^{2,1}$, which corresponds to the $sl(2, \mathbb{R})$ -bundle over S defined by the adjoint representation.

Proposition 2.8. [19] *Let M be a flat MGHFC manifold homeomorphic to $S \times \mathbb{R}$, where S is a closed surface of genus at least 2, and let $h : \pi_1 S \rightarrow \text{Isom}(2, 1)$ be its holonomy representation. Then h decomposes in $\text{Isom}(2, 1) = SO(2, 1) \ltimes \mathbb{R}^{2,1}$ as $h = (\rho, \tau)$ where $\rho : \pi_1 S \rightarrow SO(2, 1)$ has maximal Euler number, and $\tau : \pi_1 S \rightarrow sl(2, \mathbb{R})$ is a 1-cocycle for ρ . Conversely, any couple (ρ, τ) where $\rho : \pi_1 S \rightarrow SO(2, 1)$ has maximal Euler number and $\tau : \pi_1 S \rightarrow sl(2, \mathbb{R})$ is a 1-cocycle for ρ defines a representation of $\pi_1 S$ in $\text{Isom}(2, 1)$ which is the holonomy representation of a flat MGHFC manifold.*

One way to obtain a 1-cocycle is by considering first-order deformations of a surface group representation in $SO(2, 1)$. This is summarized in the following proposition, which allows one to construct the holonomy representation of a flat MGHFC as a first-order deformation of the holonomy representation of a hyperbolic metric on a surface.

Proposition 2.9. *Let $(\rho_t)_{t \in [0,1]}$ be a smooth one-parameter family of morphisms from $\pi_1(S)$ to $PSL(2, \mathbb{R})$. Then the map $\tau = \rho_0^{-1}(d\rho/dt)_{t=0}$ from $\pi_1(S)$ to $sl(2, \mathbb{R})$ is a 1-cocycle for ρ_0 .*

MGHFC spacetimes as first-order deformations in higher dimension. We will now consider the construction of flat MGHFC spacetimes as first-order deformations in dimension $n + 1 > 3$. For this, we consider a closed, orientable, hyperbolic n -dimensional manifold M , with fundamental group Γ . The holonomy representation of M is a homomorphism $\rho_0 : \Gamma \rightarrow SO(n, 1)$. It is rigid by Mostow's theorem, and also infinitesimally rigid, in the sense that any deformation cocycle for ρ_0 vanishes.

However, M can be considered as a totally geodesic hypersurface in a complete, non-compact hyperbolic manifold N of dimension $n + 1$. This corresponds to extending ρ_0 to a representation $\rho : \Gamma \rightarrow SO_0(n + 1, 1)$ with image in $SO_0(n, 1) \subset SO_0(n + 1, 1)$. Now consider a deformation $(\rho_t)_{t \in [0, 1]}$ of ρ . As in dimension $2 + 1$ one obtains the map

$$\rho_1 := \rho(0)^{-1} \rho'(0) : \Gamma \rightarrow o(n + 1, 1) ,$$

which is a deformation cocycle for ρ . Moreover, there is an orthogonal decomposition $o(n + 1, 1) = o(n, 1) \oplus \mathbb{R}^{n, 1}$, and we can decompose ρ_1 along this direct sum. The component in $o(n, 1)$ is a deformation cocycle for ρ_0 , so it vanishes by the infinitesimal rigidity of ρ_0 , and thus ρ_1 determines a $\mathbb{R}^{n, 1}$ -valued cocycle.

This cocycle then determines a MGHFC spacetime, with holonomy representation (ρ_0, ρ_1) considered as a homomorphism from Γ to $\text{Isom}(\mathbb{R}^{n, 1}) = SO_0(n, 1) \ltimes \mathbb{R}^{n, 1}$. Moreover, the holonomy representations of all MGHFC spacetimes can be obtained in this way, see [19, 7].

Proposition 2.10. *The MGHFC spacetimes for which the linear part of the holonomy is equal to ρ_0 are in one-to-one correspondence with the deformations of ρ .*

There is another geometrical interpretation of the deformation cocycle ρ , namely as a first-order deformation of the flat conformal structure on M underlying its hyperbolic metric. Indeed, in the situation described above, where M is considered as a totally geodesic submanifold of N , the conformal structure at infinity of N remains conformally flat under a deformation. Conversely, any first-order deformation of the conformally flat structure on M determines a deformation of its developing map in S^n and, by taking the convex hull of the complement of its image, one obtains a first-order deformation of N .

One direct consequence Proposition 2.10 is that it is much more difficult to construct examples of MGHFC spacetimes in higher dimensions than in dimension $2 + 1$. An $(n + 1)$ -dimensional MGHFC spacetime is uniquely determined by a closed n -dimensional hyperbolic manifold along with a $\mathbb{R}^{n, 1}$ -valued deformation cocycle. For $n = 2$, the latter corresponds to a tangent vector to the Teichmüller space for a surface S of given genus g , which implies that the MGHFC spacetimes homeomorphic to $S \times \mathbb{R}$ form a manifold of dimension $12g - 12$. For $n \geq 3$, finding a MGHFC spacetime homeomorphic to $M \times \mathbb{R}$ is more difficult. Any closed manifold M admits at most one hyperbolic metric g by Mostow's rigidity theorem. Finding a deformation cocycle is equivalent to finding a first-order deformation of the warped product hyperbolic metric $dt^2 + \cosh^2(t)g$ on $\mathbb{R} \times M$. For many choices of (M, g) , such a deformation cocycle does not exist. However there are also many examples where (M, g) does admit a $\mathbb{R}^{n, 1}$ -valued cocycle.

- This occurs whenever (M, g) contains a closed, totally geodesic submanifold, and the cocycle corresponds to “bending” along this totally geodesic manifold, see [17, 15]. There are many (arithmetic) examples of closed hyperbolic manifolds (in any dimension) containing a closed totally geodesic surface.
- Other examples of deformation cocycles can be found in specific cases, see e.g. [16, 2, 22].

In Section 7 we investigate the examples constructed by Apanasov in [2] and show how the frequency function encodes information on the holonomy representation and hence on the topology of the spacetime.

2.7. Reconstructing a domain of dependence from its holonomy representation. In Section 6 and 7 we compute domains of dependence which are universal covers of MGHFC spacetimes. This requires a practical way of reconstructing (to a good approximation) the shape of a domain from the holonomy representation of the MGHFC spacetimes. For this, we use another characterization of those domains due to Barbot [3].

We consider a MGHFC spacetime M of dimension $n + 1$ with fundamental group Γ . As explained in Section 2.1, the universal cover of M can be identified isometrically with a future-complete domain of dependence $D \subset \mathbb{R}^{1, n}$. The fundamental group Γ acts isometrically on D with a quotient D/Γ isometric to M . Moreover all elements of Γ except the unit element act on $\mathbb{R}^{1, n}$ as loxodromic elements.

Definition 2.11. For $g \in \Gamma$, we denote by D_g the set of points $x \in \mathbb{R}^{1, n}$ such that $g^p(x) - x$ is spacelike for all $p \in \mathbb{Z}$.

This is a simpler version of the definition at the beginning of Section 7 in [3], but both definitions are equivalent in our case because the linear part of each nontrivial element $g \in \Gamma$ is loxodromic. In $(2 + 1)$ dimensions, it is easy to give a more explicit description of the set D_g . If the linear part of g is loxodromic, then there is a unique spacelike line l_g in $\mathbb{R}^{2, 1}$ which is invariant under the action of g . It is proved in [3] that the set D_g is then equal to the union of the past and the future of l_g .

Proposition 2.12 (Barbot [3]). *The domain of dependence D is one of the two connected components of $\bigcap_{g \in \Gamma} D_g$.*

A proof can be found — in a more general setting — in Barbot's work [3], see Section 7 for the definitions and Section 10 for the statements corresponding to Proposition 2.12.

3. LIGHT EMITTED BY THE INITIAL SINGULARITY

3.1. Definitions. We consider a domain of dependence M in $(n+1)$ -dimensional Minkowski space, as described in the previous section.

An observer in free fall in M is characterized by his worldline, which is a future-oriented timelike geodesic in M . This geodesic is specified by the choice of a point $p \in M$ and a future-directed timelike unit vector $v \in \mathbb{H}^n$, where we use the identification of \mathbb{H}^n with the set of future directed timelike unit vectors H^+ from section 2. The point $p \in M$ corresponds to a given event on the worldline of the observer, while the vector v is his velocity unit vector.

We consider a uniform light signal emitted near the initial singularity of M which is received by the observer at the point $p \in M$. The quantity measured by the observer is the frequency of this light signal, which depends on the observer and on the direction in which the light is observed. We can construct this quantity as follows. The space of lightlike rays arriving at p can be identified with the set of unit spacelike vectors orthogonal to v and hence with $T_v^1 \mathbb{H}^n$, which corresponds to the $(n-1)$ -dimensional sphere S^{n-1} . We associate to each unit vector $u \in T_v^1 \mathbb{H}^n$ the ray through p with the direction given by the lightlike vector $u - v$. The basic idea is to define the (rescaled) frequency function as a function

$$\rho_{p,v} : T_v^1 \mathbb{H}^n \rightarrow \mathbb{R},$$

which is given as the renormalized limit of the functions that measure the frequency of the light emitted from the surface H_ϵ of constant cosmological time ϵ :

$$\rho_{p,v}(u) = \lim_{\epsilon \rightarrow 0} \epsilon \rho_{p,v}^\epsilon(u).$$

The functions $\rho_{p,v}^\epsilon$ are defined by the rule

$$\rho_{p,v}^\epsilon(u) = \langle v, \nu_{p,v}^\epsilon(u) \rangle$$

where $\nu_{p,v}^\epsilon(u)$ is the normal of the surface H_ϵ at the intersection point of H_ϵ with the light ray $p + \mathbb{R}(u - v)$: $\{q_\epsilon(u)\} = H_\epsilon \cap (p + \mathbb{R}(-v + u))$. It is clear that the frequency function describes a frequency shift due to the motion of the observer relative to the initial singularity and is closely related to the red-shift observed in astronomical observations.

We consider first the $(2+1)$ -dimensional case. In this situation, the frequency function of a domain that is the future of a finite spacelike tree can be understood by considering two main examples. The first is a domain that is the future of a point, i. e. a light cone, and the second is a domain which is the future of a spacelike line. We first consider these two examples and then use them as the building blocks to analyze the general situation.

3.2. Example 1: future of a point. We consider the domain of dependence D which is the future of $0 \in \mathbb{R}^{2,1}$ together with an observer in D which is given by a point $p \in D$ and a future directed timelike unit vector $v \in \mathbb{H}^2$ as shown in Figure 3. The cosmological time τ of the event p is then determined by

$$(1) \quad \langle p, p \rangle = -\tau^2.$$

We also consider the quantity δ defined by

$$(2) \quad \langle p, v \rangle = -\tau \cosh \delta,$$

which is the hyperbolic distance between v and the point p/τ (see Figure 3). and measures the discrepancy of the observer's eigentime and the cosmological time. The cosmological time coincides with the observer's eigentime up to a time translation if and only if $\delta = 0$. We can also interpret δ as the rapidity of the boost from the worldline of the observer to the geodesic through p and the origin.

For a given unit vector $u \in T_v^1 \mathbb{H}^2$ we also introduce a parameter ϕ defined by

$$(3) \quad \langle p, u \rangle = \tau \sinh \phi, \quad \phi \in [-\delta, \delta].$$

Geometrically, ϕ is the hyperbolic distance of the point p/τ from the geodesic orthogonal to u . It becomes maximal when $u \in T_v^1 \mathbb{H}^2$ points in the direction of $p/\tau \in \mathbb{H}^2$ and minimal when u points away from it.

We denote by $t_\epsilon \in \mathbb{R}^+$ the parameter that characterizes the intersection point $q_\epsilon(u)$ of the light ray $p + \mathbb{R}(u - v)$ with the surface H_ϵ of constant cosmological time ϵ . As the latter is the set of points

$$H_\epsilon = \{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = -\epsilon^2\},$$

the parameter t_ϵ is characterized uniquely as the positive solution of the equation

$$\langle p + t_\epsilon(-v + u), p + t_\epsilon(-v + u) \rangle = -\epsilon^2.$$

Inserting the parameters δ and ϕ defined in (2) and (3) and solving the equation for t_ϵ , we obtain

$$t_\epsilon(u) = \frac{\tau^2 - \epsilon^2}{2\tau(\cosh \delta + \sinh \phi(u))}.$$

The unit normal vector $\nu_{p,v}^\epsilon$ in the intersection point of $p + \mathbb{R}(u - v)$ and H_ϵ is given by

$$\nu_{p,v}^\epsilon(u) = \frac{1}{\epsilon}(p + t_\epsilon(-v + u)),$$

and the function $\rho_{p,v}^\epsilon$ by

$$\rho_{p,v}^\epsilon(u) = - \left\langle v, \frac{p + t_\epsilon(-v + u)}{\epsilon} \right\rangle.$$

A direct computation then shows that the rescaled frequency function then takes the form

$$\rho_{p,v}(u) = \frac{\tau}{2(\cosh \delta + \sinh \phi(u))}.$$

Using the fact that the function ϕ takes values $\phi(u) \in [-\delta, \delta]$, one finds that the maximum and minimum frequency are given by

$$(4) \quad \rho_{p,v}^{max} = \frac{\tau}{2}e^\delta, \quad \rho_{p,v}^{min} = \frac{\tau}{2}e^{-\delta}.$$

These equations show that the frequency function $\rho_{p,v}$ allows one to re-construct the cosmological time $\tau(p)$ of the observer at the reception of the light signal and the discrepancy between his eigentime and the cosmological time, which is given by δ . Moreover, by determining the direction of the maximum, the observer can deduce the direction of $p/\tau \in \mathbb{H}^2$.

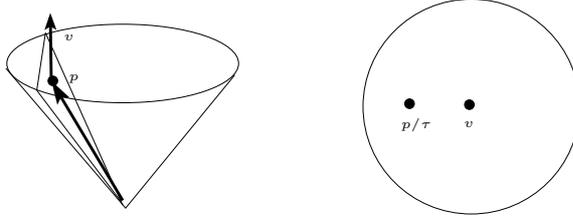


FIGURE 3. Description of an observer for Example 1.

3.3. Example 2: future of a spacelike line. We consider the domain of dependence D which is the future of a spacelike line $\mathbb{R}e$ in $\mathbb{R}^{2,1}$ together with an observer given by a point $p \in D$ and a timelike future directed timelike unit vector $v \in \mathbb{H}^2$. Using the symmetry of the system under translations in the direction of the line and Lorentz transformations with the line as their axis, we can parameterize the data for the observer as follows

$$(5) \quad e = (0, 0, 1) \quad p = \tau(\cosh \delta, \sinh \delta, 0) \quad v = (\cosh \xi, 0, \sinh \xi).$$

If we denote by $x_0 \in \mathbb{H}^2$ the timelike unit vector corresponding to the shortest line in $\mathbb{R}^{2,1}$ from $\mathbb{R}e$ to p , then ξ is the hyperbolic distance of $v \in \mathbb{H}^2$ from from the geodesic through x_0 that is orthogonal to $\mathbb{R}e$. The parameter δ is the hyperbolic distance from x_0 to the projection of v onto this geodesic, as shown in Figure 4.

The rescaled frequency function is defined as in Example 1. For $u \in T_v^1\mathbb{H}^2$, we have

$$(6) \quad \rho_{p,v}(u) = \lim_{\epsilon \rightarrow 0} \epsilon \rho_{p,v}^\epsilon(u) \quad \rho_{p,v}^\epsilon(u) = \langle v, \nu_{p,v}^\epsilon(u) \rangle,$$

where $\nu_{p,v}^\epsilon(u)$ is the unit normal vector to the constant cosmological time surface H_ϵ at the intersection point $\{q_\epsilon(u)\} = H_\epsilon \cap (p + \mathbb{R}(u - v))$. As the constant cosmological time surface H_ϵ is of the form

$$(7) \quad H_\epsilon = \{x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle - \langle x, e \rangle^2 = -\epsilon^2\},$$

the intersection point $q_\epsilon(u)$ is given by the equations

$$(8) \quad q_\epsilon(u) = p + t_\epsilon(u - v), \quad \langle p + t_\epsilon(u - v), p + t_\epsilon(u - v) \rangle - \langle p + t_\epsilon(u - v), e \rangle^2 = -\epsilon^2.$$

Using the parameterization (5), in particular the identities $\langle u - v, u - v \rangle = 0$, $\langle p, e \rangle = 0$, we obtain a quadratic equation in t_ϵ

$$(9) \quad t_\epsilon^2 \langle e, u - v \rangle^2 - 2t_\epsilon \langle p, u - v \rangle + \tau^2 - \epsilon^2 = 0$$

with solutions

$$(10) \quad t_\epsilon^\pm = \frac{\langle p, u - v \rangle \pm \sqrt{\langle p, u - v \rangle^2 - (\tau^2 - \epsilon^2)\langle e, u - v \rangle^2}}{\langle e, u - v \rangle^2}.$$

In the limit $\epsilon \rightarrow 0$ this reduces to

$$(11) \quad t_\pm = \frac{\langle p, u - v \rangle \pm \sqrt{\langle p, u - v \rangle^2 - \tau^2\langle e, u - v \rangle^2}}{\langle e, u - v \rangle^2},$$

and the rescaled frequency function is given by

$$(12) \quad \rho = -\langle v, p \rangle - t_\epsilon(1 - \langle u - v, e \rangle \langle v, e \rangle).$$

To obtain a concrete parametrization for ρ , we parameterize the unit vector $u \in T_v^1(\mathbb{H}^2)$ as

$$(13) \quad u = \cos\theta(\sinh\xi, 0, \cosh\xi) + \sin\theta(0, 1, 0).$$

This implies

$$\begin{aligned} \langle v, p \rangle &= -\tau \cosh\delta \cosh\xi, & \langle v, e \rangle &= \sinh\xi, \\ \langle p, u - v \rangle &= -\tau \cosh\delta(\cos\theta \sinh\xi - \cosh\xi) + \tau \sinh\delta \sin\theta, \\ \langle e, u - v \rangle &= \cos\theta \cosh\xi - \sinh\xi, \end{aligned}$$

and the expression for t_\pm becomes

$$(14) \quad t_\pm = \tau e^{\mp\delta} \frac{\cosh\xi - \sinh\xi \cos\theta \mp \sin\theta}{(\cosh\xi \cos\theta - \sinh\xi)^2}.$$

If we introduce an ‘‘angle variable’’ θ_ξ defined by

$$(15) \quad \tan\frac{\theta_\xi}{2} = e^\xi \quad \text{with } \theta_\xi \in [0, \pi/2],$$

then we obtain

$$(16) \quad t_\pm = \frac{\tau e^{\mp\delta} \sin\theta_\xi}{2 \sin^2\left(\frac{\theta \pm \theta_\xi}{2}\right)}.$$

Note that $t_\pm \geq 0$ and that t_\pm diverges for $\theta = \mp\theta_\xi$. The two cases for t_\pm, ρ_\pm correspond to the intersection points of the ray $p + \mathbb{R}(u - v)$ with the two lightlike planes Q_\pm containing $\mathbb{R}e$. The relevant intersection point is the one that is closer to p , i.e. the one with $t = \min(t_\pm)$. From (14) it follows that this is the one associated with t_+ if

$$(17) \quad \left(\frac{\sin\left(\frac{\theta - \theta_\xi}{2}\right)}{\sin\left(\frac{\theta + \theta_\xi}{2}\right)} \right)^2 \leq e^{2\delta}$$

and the one for t_- otherwise. For the associated frequency functions, we obtain

$$\rho_\pm(\theta) = \frac{\tau}{2} \cosh\xi \left(e^{\pm\delta} - e^{\mp\delta} \left(\frac{\cosh\xi - \sinh\xi \cos\theta \mp \sin\theta}{\cosh\xi \cos\theta - \sinh\xi} \right)^2 \right) = \frac{\tau}{2} \cosh\xi \left(e^{\pm\delta} - e^{\mp\delta} \left(\frac{\sin\left(\frac{\theta \mp \theta_\xi}{2}\right)}{\sin\left(\frac{\theta \pm \theta_\xi}{2}\right)} \right)^2 \right).$$

Clearly, $\rho_+(\theta) \geq 0$ if and only if (17) is satisfied, and $\rho_-(\theta) \geq 0$ otherwise. The frequency function is therefore given by

$$(18) \quad \rho_{p,v}(\theta) = \max(\rho_+(\theta), \rho_-(\theta)) = \begin{cases} \rho_+(\theta) & \text{for } \left(\frac{\sin\left(\frac{\theta - \theta_\xi}{2}\right)}{\sin\left(\frac{\theta + \theta_\xi}{2}\right)} \right)^2 \leq e^{2\delta} \\ \rho_-(\theta) & \text{for } \left(\frac{\sin\left(\frac{\theta - \theta_\xi}{2}\right)}{\sin\left(\frac{\theta + \theta_\xi}{2}\right)} \right)^2 \geq e^{2\delta}. \end{cases}$$

The frequency function has local maxima in

$$(19) \quad \phi_\pm^{max} = \pm\theta_\xi,$$

where it takes the values

$$(20) \quad \rho_\pm^{max} = \frac{\tau}{2} \cosh\xi e^{\pm\delta},$$

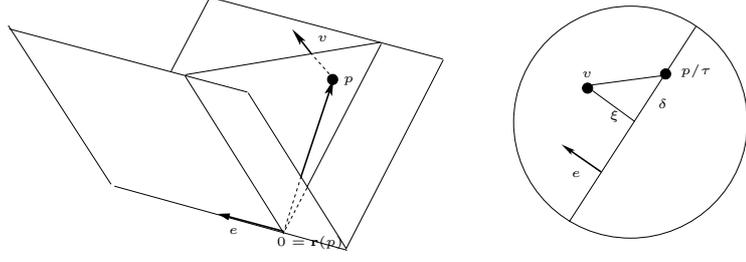


FIGURE 4. Description of an observer for Example 2.

and it vanishes if and only if

$$(21) \quad \left(\frac{\sin\left(\frac{\theta - \theta_\xi}{2}\right)}{\sin\left(\frac{\theta + \theta_\xi}{2}\right)} \right)^2 = e^{2\delta}.$$

This corresponds to $t_+ = t_-$ or, equivalently,

$$(22) \quad \langle p - \tau e, u - v \rangle = 0.$$

This condition is satisfied if and only if light ray $p + \mathbb{R}(u - v)$ intersects the line $\mathbb{R} \cdot e$, which, for each observer, happens for exactly two values of θ .

The observer can therefore extract all relevant information from the function $\rho_{p,v}(\theta)$. He can determine the cosmological time τ at the reception of the light signal, his position relative to the line, which is given by δ , and his velocity relative to the line, which is given by ξ . The development of the measured frequency function in terms of the eigentime of a moving observer is given by the dependence of his cosmological time and the parameter δ on his eigentime. For an observer with a worldline specified by $p \in D$ and $v \in \mathbb{H}^2$, his position at an eigentime t after the event p is given by $p' = p + tv$. This implies that the cosmological time of p' and the associated parameter δ are given by

$$(23) \quad \tau(t) = \sqrt{\tau^2 + t^2 - 2\langle p, v \rangle} = \sqrt{\tau^2 + t^2 + \tau \cosh \delta \cosh \xi}, \quad \coth \delta(t) = \coth \delta + t \frac{\cosh \xi}{\cosh \delta}.$$

The time development of τ with the eigentime corresponds to an overall rescaling of the frequency function. The time development of δ changes the relation between its constant and its angle-dependent part.

3.4. The rescaled density for a general domain. The results in the previous subsections allow one to construct the rescaled frequency function for a domain which can be obtained from the light cone by the Mess construction [19], i. e. via grafting along a weighted multicurve. To show that the rescaled density is well-defined also for the case of a general geodesic lamination, one has to prove that the limit $\lim_{\epsilon \rightarrow 0} \epsilon \rho_{p,v}^\epsilon$ exists for these domains and for a general observer (p, v) . As we will see, in general the problem is more subtle than it appears and some care is needed to pass to the limit.

Let D be a generic domain of dependence in $\mathbb{R}^{n,1}$. Fix an observer in D by specifying a point $p \in D$ and a future directed timelike unit vector v . To analyze the behavior of the frequency function, it is convenient to express the function $\epsilon \rho_{p,v}(\epsilon)$ as the composite of two functions $q_\epsilon : T_v^1 \mathbb{H}^2 \rightarrow H_\epsilon$ and $\iota_\epsilon : H_\epsilon \rightarrow \mathbb{R}$, where $q_\epsilon(e)$ is the intersection of the light ray $p + \mathbb{R} \cdot (e - v)$ with the level surface of the cosmological time $H_\epsilon = \tau^{-1}(\epsilon)$, and

$$\iota_\epsilon(q) = -\epsilon \langle \nu_\epsilon(q), v \rangle,$$

where $\nu(q)$ denotes the unit normal vector of H_ϵ in q . It is clear that the maps $q_\epsilon : T_v^1 \mathbb{H}^2 \rightarrow H_\epsilon$ converge to a map $q_0 : T_v^1 \mathbb{H}^2 \rightarrow \partial D$ as $\epsilon \rightarrow 0$. The idea is to show that the maps $\iota_\epsilon : H_\epsilon \rightarrow \mathbb{R}$ converge to a function $\iota : \partial D \rightarrow \mathbb{R}$ as $\epsilon \rightarrow 0$. Note, however, that functions $\iota_\epsilon : H_\epsilon \rightarrow \mathbb{R}$ are defined on different domains, so we need to make this statement more precise.

Let P_v be the hyperplane in $\mathbb{R}^{n,1}$ orthogonal to v . By [19, 7], the surfaces H_ϵ can be realized as the graphs of convex functions $u_\epsilon : P_v \rightarrow \mathbb{R}$. More precisely, the level surfaces of the cosmological time can be identified with the set of points $H_\epsilon = \{q = x + u_\epsilon(x)v \mid x \in P_v\}$. As $\epsilon \rightarrow 0$, the functions u_ϵ converge to a convex function u_0 , whose graph is the boundary ∂D .

Thus, there is a natural identification between P_v and H_ϵ given by $x \mapsto x + u_\epsilon(x)v$. In particular, we may consider the functions ι_ϵ as functions defined on P_v . The following result then shows that the frequency function can be defined pointwise on the boundary of any domain of dependence.

Proposition 3.1. *For a fixed $x \in P_v$ the function $\epsilon \mapsto \iota_\epsilon(x)$ is increasing. Moreover the function*

$$\iota(x) = \lim_{\epsilon \rightarrow 0} \iota_\epsilon(x) = \inf_{\epsilon} \iota_\epsilon(x)$$

is finite-valued at each point and locally bounded.

The proof of this proposition will be based on the following technical lemma.

Lemma 3.2. *The square of the cosmological time τ^2 is convex along each timelike line.*

Proof. Take $r \in \mathbb{R}^{n,1}$ and consider the cosmological time function on $I^+(r)$ which is defined by

$$\tau_r(p) = \sqrt{-\langle p - r, p - r \rangle}$$

It is clear that the restriction of τ_r^2 along every line $p + \mathbb{R} \cdot w$ is a degree two polynomial function of the affine parameter with leading coefficient $-\langle w, w \rangle$. In particular, the function τ_r is convex along all timelike directions.

Given a point $q \in D$, let $r = \mathbf{r}(q)$ be the corresponding point on the singularity. Then $I^+(r) \subset D$ and on $I^+(r)$ we have $\tau \geq \tau_r$. Moreover, the cosmological time of q and its gradient are given by $\tau(q) = \tau_r(q)$ and $\text{grad}\tau(q) = \text{grad}\tau_r(q) = \frac{1}{\tau}(q - r)$.

Take a timelike vector w and consider the functions $f(t) = \tau^2(q + tw)$ and $g(t) = \tau_r^2(q + tw)$. They are $C^{1,1}$ -functions, which coincide with their derivatives at $t = 0$. As $f(t) \geq g(t)$ we deduce that if f'' exists in 0 then $f''(0) \geq g''(0) > 0$. Thus $f''(t) > 0$ for all t for which f'' exists. Since f' is Lipschitz, for $s < t$

$$f'(t) - f'(s) = \int_s^t f''(x) dx > 0,$$

and hence f is convex. □

Proof of Proposition 3.1. Let D be a domain with cosmological time function τ . It then follows from the results in [7] that for all points $q \in H(\epsilon)$ one has $\text{grad}\tau(q) = -\nu_\epsilon(q)$. This implies $-\epsilon\nu_\epsilon = \tau\text{grad}\tau = \frac{1}{2}\text{grad}(\tau^2)$, and we deduce

$$\iota_\epsilon(x) = \frac{1}{2} \langle \text{grad}\tau^2(x + u_\epsilon(x)v), v \rangle.$$

For a given point $x \in P_v$, we can consider the restriction of τ^2 to the vertical line $x + \mathbb{R} \cdot v$

$$f_x(s) = \tau^2(x + sv),$$

which is a convex function of s by Lemma 3.2. As we have

$$\iota_\epsilon(x) = \frac{1}{2} (f_x)'(u_\epsilon(x)),$$

the monotonicity of ι_ϵ then follows from the monotonicity of $(f_x)'$. □

Proposition 3.1 allows us to define the rescaled frequency function for a domain D and an observer (p, v) as the map

$$\rho_{(p,v)} : T_v^1 \mathbb{H}^2 \rightarrow \mathbb{R}_{\geq 0}, \quad \rho_\epsilon(e) = \rho_{(p,v)}(e) = \iota_\epsilon(q_0(e)).$$

Note, however, that it is in general not true that $\rho_\epsilon(e) \rightarrow \rho(e)$ as $\epsilon \rightarrow 0$, since the convergence $\iota_\epsilon \rightarrow \iota$ is not necessarily uniform. In the next section, we will investigate the regularity of ι and show that the convergence of ρ_ϵ to ρ holds for generic observers and generic directions.

3.5. Domains with a closed singularity in dimension $2 + 1$. To analyze the convergence properties of the rescaled frequency functions ρ_ϵ , we first consider the case where the initial singularity T is a closed subset of ∂D . In this situation, the frequency function simplifies considerably. Note, however, that this condition never holds for the universal covering of MGHFC spacetimes, as will be proved in the next section. Nevertheless, the results for this case are useful to compute the frequency function for domains of dependence that are the intersection of a finite number of half-spaces.

The simplifications in the case of a closed initial singularity arise from the fact that there is an extension of the map \mathbf{r} on the boundary of D , based on the following geometric idea. For each point $q_0 \in \partial D$, there is a lightlike ray R through q_0 which is contained in ∂D . We will suppose that the lightlike ray R is maximal with respect to inclusion. The ray R can always be extended to infinity in the future, but it has a past endpoint $r_0 \in \partial D$. The ray R is unique unless $q_0 = r_0$, which implies that the point r_0 is uniquely determined by q_0 .

This defines a natural map $\mathbf{r}_0 : \partial D \rightarrow \partial D$, and it follows directly from its definition that $\mathbf{r}_0 \circ \mathbf{r}_0 = \mathbf{r}_0$. The image of \mathbf{r}_0 is called the *extended singularity* and denoted by \hat{T} . It contains all points which are the past endpoint of a maximal lightlike ray contained in ∂D . In particular, the initial singularity T is contained in \hat{T} . Note that, unless \hat{T} is closed, the map \mathbf{r}_0 cannot be continuous. We will see in the next section that this non-continuity occurs in many interesting and relevant examples.

Proposition 3.3. *Consider the function $I_v : \partial D \rightarrow \mathbb{R}_{\geq 0}$ defined by*

$$I_v(q_0) = \langle q_0 - \mathbf{r}_0(q_0), v \rangle .$$

If the singularity is closed in ∂D then $T = \hat{T}$ and ι_ϵ uniformly converges to I_v . As a consequence, the function $\iota_v = I_v$ is continuous.

Proof. For any point $x \in P_v$ we denote by $q_\epsilon(x)$ the point $x + u_\epsilon(x)v \in H_\epsilon$ and by $r_\epsilon(x)$ be the projection of $q_\epsilon(x)$ on the initial singularity. The results in [7] imply that these points are related by the following equation

$$(24) \quad q_\epsilon(x) = r_\epsilon(x) + \epsilon v_\epsilon(x) .$$

Take now any sequence $x_n \in P_v$ that converges to x and $\epsilon_n \rightarrow 0$. Then the associated sequence $q_\epsilon(x)$ converges to $q_0(x) = x + u_0(x)v$, and we obtain

$$\iota_{\epsilon_n}(x_n) = -\langle q_{\epsilon_n}(x_n) - r_{\epsilon_n}(x_n), v \rangle .$$

To prove that ι_ϵ converges uniformly to I_v , it is then sufficient to check that $r_n = r_{\epsilon_n}(x_n)$ converges to $\mathbf{r}_0(q_0)$. For this, note that the sequence r_n is contained in a compact subset of the boundary ∂D and hence has a subsequence $(r_{n_k})_{k \in \mathbb{N}}$ which converges to a point $r_0 \in \partial D$. By the assumption on T , the point r_0 is also contained in T . We prove in the next paragraph that $r_0 = \mathbf{r}(q_0(x))$. The uniqueness of the limit implies that the whole sequence r_n converges to r_0 .

That $r_0 = \mathbf{r}_0(q_0)$ can be established as follows. The sequence of timelike vectors $q_{n_k} - r_{n_k}$ converges to $q_0 - r_0$. This implies that $q_0 - r_0$ is not spacelike. As ∂D is an achronal surface, it must be lightlike and the lightlike ray $R = r_0 + \mathbb{R}_{\geq 0} \cdot (q_0 - r_0)$ is contained in ∂D . Since r_0 is on the singularity, we obtain that the ray R is maximal so that $r_0 = \mathbf{r}(q_0)$. This also shows that the image of \mathbf{r}_0 is contained in T . \square

Remark 3.4. In the general case, we cannot conclude because the limit point r_0 may not be on the singularity. However, it is always true that the ray $q_0 + \mathbb{R} \cdot (q_0 - r_0)$ is contained in ∂D , so it is contained in the maximal lightlike ray through q_0 . In other words the point r_0 lies on the segment $[q_0, \mathbf{r}_0(q_0)]$. This shows that in general

$$\limsup \iota_{\epsilon_n}(x_n) \leq I_v(x)$$

for any sequence $\epsilon_n \rightarrow 0$ and $x_n \rightarrow x$. In particular, it implies $\iota_v(x) \leq I_v(x)$ and hence that ι_v is zero on the initial singularity.

3.6. Generic domains in dimension $2+1$. Although at a first sight, the hypothesis of Proposition 3.3 could appear to hold generally, this is not the case. Indeed, if D is the universal covering of a MGHFC spacetime whose holonomy representation is not linear, then the condition cannot be satisfied.

Remark 3.5. [19] Let D be the universal covering of a MGHFC spacetime. Then T is never closed in ∂D .

The following proposition describes the regularity properties of the functions ι_v and I_v for in general domains in $2+1$ dimensions. The result is that at generic points, these functions are continuous and coincide.

Proposition 3.6. *The following properties hold for the functions ι_v and I_v :*

- *The function ι_v is upper semicontinuous.*
- *The set of discontinuity points of ι_v is meagre.*
- *The function I_v is upper semicontinuous.*

Proof. The first property holds since f is the supremum of a family of continuous functions. Moreover, as it is the limit of continuous functions, by a classical result of Lebesgue, its discontinuity points form a meagre set.

Let us prove that I_v is upper semi-continuous. For this, take a sequence of points $x_n \in P_v$ that converges to x . Up to passing to a subsequence we may assume that $\limsup I_v(x_n) = \lim I_v(x_n)$. If $\lim I_v(x_n) = 0$, then clearly $I_v(x) \geq \limsup I_v(x_n)$.

Let us treat the case where $\lim I_v(x_n) > 0$. The sequence of points $q_n = x_n + u(x_n)v \in \partial D$ converges to $q = x + u(x)v \in \partial D$. By the assumption on $\lim I_v(x_n)$ we have that $\mathbf{r}_0(q_n) \neq q_n$ for n sufficiently large. Consider the sequence of lightlike rays R_n containing q_n and $\mathbf{r}_0(q_n)$. Up to passing to a subsequence, we may assume that it converges to a lightlike ray R through q . The sequence $(\mathbf{r}_0(q_n))_{n \in \mathbb{N}}$ converges to the past endpoint of R . Since R is contained in ∂D we deduce that $r_1 = \lim \mathbf{r}_0(q_n)$ is a point on the segment $[\mathbf{r}_0(q), q]$ and we have

$$\lim_{n \rightarrow +\infty} I_v(x_n) = -\langle q - r_1, v \rangle \leq \langle q - \mathbf{r}_0(q), v \rangle = I_v(q) .$$

\square

4. STABILITY OF THE FREQUENCY FUNCTION

In this section we investigate the stability of the frequency function. Given a sequence of domains of dependence D_n that converges to D and an observer (p, v) in D , then (p, v) is also an observer in D_n for n sufficiently large. Let ρ_v^n be the frequency function for D_n as seen by the observer (p, v) and ρ the associated frequency function for D . We will investigate under which conditions the frequency functions ρ_v^n converge to the frequency function ρ .

We will first show by a counterexample in subsection 4.1 that in general there is no convergence even in the weak sense. However, we identify a subclass of domains of dependence, called domains of dependence with a flat boundary, which includes the interesting examples. In subsection 4.2 we prove that the convergence holds for these domains. In subsection 4.5 we will then prove that universal coverings of MGHFC spacetimes in dimension 2+1 are contained in this class. As in the previous subsection, it is advantageous to work with the frequency functions ι_v and ι_v^n introduced there.

4.1. An example. We fix coordinates x_0, x_1, x_2 on $\mathbb{R}^{2,1}$, so that the Minkowski metric takes the form $-dx_0^2 + dx_1^2 + dx_2^2$ and consider the timelike vector $v = (1, 0, 0)$. Let P be the horizontal plane at height equal to 1. Then the intersection of P with the cone $I^+(0)$ is a circle C of radius 1. Let C_k be the regular polygon with k edges tangent to C . Clearly, C_k converges to C in the Hausdorff sense as $k \rightarrow +\infty$.

Now observe that for each edge of C_k the plane that contains 0 and this edge is lightlike, since it is tangent to $I^+(0)$. Let D_k be the intersection of the future of the k lightlike planes containing 0 and the edges of C_k . Then D_k is a domain of dependence and converges to D for $k \rightarrow \infty$ on compact subsets.

We denote by ι_v^k and ι_v , respectively, the frequency function of D_k and D with respect to the observer (p, v) and prove that ι_v^k does not converge to ι_v on a set of positive measure. Regarding L^1 -functions as continuous functionals on the set of continuous functions with compact support, we will show that ι_v^k weakly converges to $\frac{1}{2}\iota_v$. In other words, for every continuous function ϕ with compact support we have

$$\int_{P_v} \phi \iota_v^k dV \rightarrow \frac{1}{2} \int_{P_v} \phi \iota_v dV$$

where dV is the area measure of the horizontal plane P_v . In particular, in this example ι_v^k does not converge to ι_v even in this weak sense.

The computation of ι_v can be performed as follows. The initial singularity of $I^+(0)$ reduces to the point 0, and hence is a closed subset. It turns out that, for every point $x \in P_v$ the frequency function ι_v is given by $\iota_v(x) = -\langle x + \|x\|v, v \rangle = \|x\|$.

Consider now the domain D_k . The initial singularity of D_k is the set of spacelike lines joining 0 to the vertices of the polygon C_k . So when we project on the plane P_v , the initial singularity appears as the union of k rays s_1, \dots, s_k starting from 0, so that the angle between s_j and s_{j+1} is $2\pi/k$. Let $f_k : P_v \rightarrow \mathbb{R}$ the function whose graph is the boundary of D_k . On the region P_j of P_v that is bounded by s_j and s_{j+1} , the function f_k is differentiable, and the gradient of f_k is a unit vector whose angle with s_j is equal to π/k . The integral lines of the gradient are parallel lines that form an angle equal to π/k with both s_j and s_{j+1} . If we denote by $r(x)$ the intersection point of the line through x with the singularity, then the frequency function is given by $\iota_v^k(x) = f_k(x) - f_k(r(x)) = \|x - r(x)\|$.

Let now s'_j be the unique line of this foliation which starts at 0. Clearly, it is the bisector of P_j . If x is on the right of s'_j then $r(x) \in s_j$. If x is on the left of s'_j then $r(x) \in s_{j+1}$. We now consider the set

$$E_j^k = \{x \in P_j \mid \iota_v^k(x) \leq \|x\|/2\}.$$

For each point $x \in P_j$, consider the triangle with vertices at x , $r(x)$ and 0. Note that $\iota_v^k(x)$ is the length of the edge joining x to $r(x)$. The sine formula of Euclidean triangles then shows that

$$\iota_v^k(x) = \|x\| \frac{\sin \phi}{\sin(\pi/k)}$$

where ϕ is the angle at the vertex 0 in the above triangle. Thus, let ϕ_k be such that $\sin \phi_k = \frac{1}{2} \sin(\pi/k)$. Let s''_j and s'''_j be the rays in P_j forming an angle ϕ_k respectively with s_j and s_{j+1} . Then, E_j^k is the union of two sectors bounded respectively by s_j and s''_j and by s_{j+1} and s'''_j .

By the concavity of the function \sin in $[0, \pi/2]$ we have $\phi_k > \pi/2k$, so for any radius R the area of $E_j^k \cap B(0, R)$ is bigger than $\frac{1}{2}$ times the area of $P_j \cap B(0, R)$. Now let us consider the set

$$E^k = \{x \in P_v \mid \iota_v^k(x) \leq \|x\|/2\} = \bigcup E_j^k.$$

The area of $E^k \cap B(0, R)$ is the sum of the areas of the $E_j^k \cap B(0, R)$, so that

$$\text{Area}(E^k \cap B(0, R)) \geq \frac{1}{2} \sum \text{Area}(P_j) = \frac{\pi R^2}{2}$$

and, consequently,

$$\int_{B(0, R)} (\iota_v - \iota_v^k) dV \geq \int_{B(0, R) \cap E^k} (\iota_v - \iota_v^k) dV \geq \int_{B(0, R) \cap E^k} \frac{\|x\|}{2} dV .$$

As E_k is a cone from the origin and the function $x \rightarrow \|x\|/2$ depends only on the distance from the origin, it follows that

$$\int_{B(0, R)} \iota_v - \iota_v^k \geq \frac{1}{2} \int_{B(0, R)} \frac{\|x\|}{2} = \pi R^2/4 .$$

In particular, this shows that ι_v^k does not converge weakly to ι_v .

Proposition 4.1. *The sequence ι_v^k weakly converges to $\iota_v/2$ in $L_{loc}^1(P_v)$.*

Proof. Note that since $\iota_v^k(x) \leq \|x\|$, up to passing to a subsequence, there is a weak limit in $L_{loc}^1(P_v)$, say J . We will prove that $J = \iota_v/2$. This is sufficient to deduce that the whole sequence converges to $\iota_v/2$.

For this, we consider the following sequence $\omega_k = *du_k$ of 1-forms on P_v , where u_k is the function whose graph is ∂D_k and $*$ is the Hodge operator. Note that ω_k is a L^∞ 1-form defined in the complement of the singularity.

As $du_k \rightarrow du$ at every differentiable point of u and $\|du_k\| \leq 1$, by the Dominated Convergence Theorem we have that $du_k \rightarrow du$ strongly in $L_{loc}^1(P_v)$ as $k \rightarrow +\infty$. This implies that $\omega_k \rightarrow \omega = *du$ strongly in $L_{loc}^1(P_v)$.

We claim that for any compact supported smooth function f the following formula holds:

$$(25) \quad \int_{P_v} df \wedge J\omega = \frac{1}{2} \int_{P_v} df \wedge \iota_v \omega .$$

Since $df \wedge \omega = \frac{\partial f}{\partial \rho} dV$, this formula implies

$$\int_{P_v} \frac{\partial f}{\partial \rho} (\iota_v/2 - J) dV = 0,$$

and by a simple density argument we conclude that $J = \frac{1}{2} \iota_v$.

To prove the claim, first note that

$$\int_{P_v} df \wedge J\omega = \lim \int_{P_v} df \wedge \iota_v^k \omega_k .$$

Now, note that on each region P_j of the complement of the singularity, $\iota_v^k \omega_k$ is a smooth 1-form, and its differential is equal to

$$\begin{aligned} d(\iota_v^k \omega_k) &= d\iota_v^k \wedge \omega_k + \iota_v^k d\omega_k \\ &= d\iota_v^k \wedge \omega_k + \iota_v^k d(*du_k) \\ &= d\iota_v^k \wedge \omega_k + \iota_v^k \Delta u_k dV . \end{aligned}$$

Since the function u_k is affine on P_j , the last term vanishes. Moreover, we have $d\iota_v^k \wedge \omega_k = d\iota_v^k(\text{grad} u_k) dV = dV$, since, on the integral lines of the gradient of u_k , ι_v^k is an affine function with derivative equal to 1.

Using the fact that $\iota_v^k \wedge \omega_k$ vanishes on the singularity, we obtain

$$\int_{P_j} df \wedge J\omega = - \int_{P_j} f dV,$$

which implies $\int_{P_v} df \wedge \iota_v^k \omega_k = - \int_{P_v} f dV$, and we conclude that

$$(26) \quad \int_{P_v} df \wedge J\omega = \lim \int_{P_v} df \wedge \iota_v^k \omega_k = - \int_{P_v} f dV .$$

On the other hand note that on $P_v \setminus \{0\}$ we have the identity

$$d(\iota_v \wedge \omega) = d\iota_v \wedge \omega + \iota_v \wedge d\omega .$$

Now as before $d\iota_v \wedge \omega = dV$, but $\iota_v d\omega = \iota_v \Delta u dV = \frac{\iota_v}{u} dV = dV$, where the last equality holds since $\iota_v = u$ in this example. So if B_ϵ is the disk centered at 0 with radius ϵ we have

$$\int_{P_k \setminus B_\epsilon} df \wedge \iota_v \omega = - \int_{P_k \setminus B_\epsilon} 2f dV - \int_{\partial B_\epsilon} f \iota_v \omega .$$

As ω is bounded, the last term vanishes as $\epsilon \rightarrow 0$ so we deduce

$$\int_{P_k} df \wedge \iota_v \omega = - \int_{P_k} 2f dV.$$

Equation (25) then follows by comparing this equation with (26). \square

Remark 4.2. Proposition 4.1 makes it clear that the reason why ι_v^k does not converge to ι is the fact that Δu is not concentrated on the singularity, whereas Δu_k vanishes outside the singularity. This remark will lead us below to introduce the notion of domain of dependence with flat boundary, where this problem is excluded. We will then prove (Theorem 4.13) that, for domains of dependence with flat boundary, the convergence does hold. We will then show in Section 4.4 that, without this hypothesis, although the sequence ι_v^k do not converge to ι_v , it does have a limit, and this limit differs only by bounded factor (attained in the example presented above) from ι_v .

We conclude this section with a simple remark. In the example above we have seen a sequence of domains of dependence D_n which converges to a domain D , but for which the corresponding sequence of frequency functions ι_v^n does not converge to the frequency ι_v of D . However, in it is clear that

$$\iota_v(x) \geq \limsup_{n \rightarrow +\infty} \iota_v^n(x)$$

This estimate holds in general and is a consequence of two facts:

- The frequency functions ι_ϵ^n of the surfaces of $H_\epsilon \subset D_n$ converge to the frequency function of the $H_\epsilon \subset D$.
- The frequency function of any domain is the infimum of the frequency functions of its surfaces H_ϵ of constant cosmological time.

We include a proof of the first statement for the sake of completeness.

Proposition 4.3. *Let D_k be a sequence of domain of dependence converging to a domain D . Denote by ι_v^k the frequency function of D and by ι_v the frequency function of D . Then for every $x \in P_v$ we have $\iota_v(x) \geq \limsup_{k \rightarrow +\infty} \iota_v^k(x)$.*

Proof. Let us fix ϵ . Denote by ι_ϵ^k the frequency function of the level surface $H_\epsilon^k = \tau_k^{-1}(\epsilon)$ of the cosmological time of D_k . By [7], we know that the sequence of surfaces H_ϵ^k converges to the level surface $H_\epsilon = \tau^{-1}(\epsilon)$ of D as $k \rightarrow +\infty$. This means that the function $u_\epsilon^k : P_v \rightarrow \mathbb{R}$, whose graph is H_ϵ^k , converges as $k \rightarrow \infty$ to the function $u_\epsilon : P_v \rightarrow \mathbb{R}$ which defines H_ϵ . By convexity, $\text{grad} u_\epsilon^k(x)$ converges to $\text{grad} u_\epsilon(x)$. As ι_ϵ^k is given by

$$\iota_\epsilon^k(x) = \frac{\epsilon}{\sqrt{1 - \|\text{grad} u_\epsilon^k\|^2}},$$

it follows that $\iota_\epsilon^k(x) \rightarrow \iota_\epsilon(x)$ as $k \rightarrow +\infty$. Now note that $\iota_v^k(x) \leq \iota_\epsilon^k(x)$ for every k . So passing to the lim sup we obtain

$$\limsup \iota_v^k(x) \leq \iota_\epsilon(x)$$

and by taking the infimum over ϵ

$$\limsup \iota_v^k(x) \leq \iota_v(x) .$$

\square

4.2. Domains of dependence with flat boundary. Let P be a lightlike plane in $\mathbb{R}^{n,1}$ and denote by g the degenerate metric on P induced by the Minkowski metric. We note that P is foliated by lightlike lines which are parallel to the kernel of g and denote by P/L be the space of leaves.

Lemma 4.4. *P/L is equipped with a flat metric \hat{g} that makes it isometric to \mathbb{R}^n such that g is the pull-back of \hat{g} by the natural projection $P \rightarrow P/L$*

Lemma 4.4 implies that there is a natural $(n-1)$ -form ω_P on P defined as the pull-back of the area form of g . This form has the following characterization. If S is any spacelike compact hypersurface in S oriented by a future-oriented transverse direction, its area is equal to the integral of ω_P on S .

Now, given a domain of dependence D we consider the 1-form ω on ∂D defined in the complement of the singularity. If p is not on the singularity and P is the unique support plane at x , then $\omega_x = \omega_P$.

Regarding ∂D as the graph of a function u on some fixed spacelike plane P_v , it turns out that the identification between P_v and ∂D is differentiable at each point where u is differentiable, so in the complement of the singularities. We can therefore express ω as a form on P_v .

Lemma 4.5. *$\omega = *du$ where $*$ is the Hodge star operator of P_v .*

Note in particular that ω is a L^∞ -form. Moreover, since u is convex, the differential of ω defined as a distribution on \mathbb{R}^n is in fact a positive locally finite Radon measure. More precisely, we have the following result.

Lemma 4.6. $d\omega = \Delta u$, where Δu is a positive Radon measure.

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of smooth convex functions converging to u . Then $d * du_n = \Delta u_n \rightarrow \Delta u$ as distributions. Since the u_n are convex, the Δu_n are positive, so their distribution limit Δu is a positive distribution. Therefore (essentially by the Riesz representation theorem) it is a Radon measure. \square

Definition 4.7. The boundary of a domain D is called *flat*, if $d\omega$ is a measure concentrated on the singularity.

Let us recall that a Radon measure μ on P_v is concentrated on A if $\mu(P_v \setminus A) = 0$. It is not difficult to check that if the boundary of D is locally given as the union of a finite number of lightlike planes, then it is flat. In Section 4.5 we will see that a domain of dependence in $\mathbb{R}^{2,1}$ that is the universal cover of a MGHC flat spacetime of genus $g \geq 2$ always has flat boundary. The choice of terminology is due to the following lemma.

Lemma 4.8. *Let D be a domain of dependence with a flat boundary. Suppose that A is an open subset of ∂D which does not meet the singularity. Then A is contained in a lightlike hyperplane.*

Proof. As $\text{Hess}(u)$ is a measure with values in the positive definite quadratic forms, one finds that if $\Delta u = 0$, then $\text{Hess}(u)$ is zero on A . This implies that $-u$ is a convex function too, so u is an affine function, i. e. the graph of $u|_A$ is a plane. Since it is contained in the boundary of D , this plane must be lightlike. \square

Remark 4.9. If D is a domain of dependence obtained as the intersection of the future of a locally finite family of lightlike planes in $\mathbb{R}^{n,1}$, then its boundary is clearly flat according to the previous definition. Moreover, Lemma 4.8 shows that if D is a domain with flat boundary and the initial singularity is closed in D , then D is the intersection of the future of a locally finite family of lightlike planes.

However, in the next subsection we will show some interesting examples of domains of dependence with flat boundary which are not of polyhedral type. This depends on the fact that in those examples the initial singularity is not a closed subset. In fact, in many cases the singularity is dense.

If D is a domain of dependence with flat boundary, the following proposition holds. This proposition will be the key ingredient in the proof of the stability of the frequency functions.

Proposition 4.10. *Let v be a timelike unit vector. If D is a domain of dependence with flat boundary then for every compactly supported smooth function f on P_v the following identity holds:*

$$\int_{P_v} df \wedge (\iota_v \omega) = - \int_{P_v} f dV .$$

The proof of the proposition is based on the following lemma, which is also valid if D does not have flat boundary.

Lemma 4.11. *Let $u_\epsilon : P_v \rightarrow \mathbb{R}$ be the function whose graph is the level surface H_ϵ and let ν_ϵ be the frequency function for H_ϵ . Then at every point where $\text{grad}u_\epsilon$ is differentiable we have*

$$(27) \quad \begin{aligned} 0 < \langle \text{grad}\nu_\epsilon, \text{grad}u_\epsilon \rangle &\leq 1 , \\ \nu_\epsilon \Delta(u_\epsilon) &\leq n - \|\text{grad}u_\epsilon\|^2 . \end{aligned}$$

Proof. Since the vector $\text{grad}u_\epsilon + v$ is orthogonal to the surface H_ϵ , the unit normal future-oriented vector is obtained by normalizing it as

$$\nu_\epsilon = \frac{1}{\sqrt{1 - \|\text{grad}u_\epsilon\|^2}} (\text{grad}u_\epsilon + v),$$

which implies

$$\nu_\epsilon = -\epsilon \langle \nu_\epsilon, v \rangle = \frac{\epsilon}{\sqrt{1 - \|\text{grad}u_\epsilon\|^2}} .$$

Thus at every point where $\|\text{grad}u_\epsilon\|$ is differentiable we have

$$\langle \text{grad}\nu_\epsilon, \text{grad}u_\epsilon \rangle = \frac{\epsilon}{(1 - \|\text{grad}u_\epsilon\|^2)^{3/2}} \langle \text{Hess}(u_\epsilon) \text{grad}u_\epsilon, \text{grad}u_\epsilon \rangle ,$$

which is positive by the convexity of u_ϵ .

To prove the estimate from above we use a comparison argument. Consider the retraction of the point $q = x + u_\epsilon(x)v$ on the singularity, say $r = \mathbf{r}(q)$. Note that on $I^+(r) \cap D$ the function $f(p) = \sqrt{-\langle p - r, p - r \rangle}$

is less than than the cosmological time. It follows that the level surface $H' = f^{-1}(\epsilon)$ is contained in the closure of the future of H_ϵ . Moreover, since $\tau(q) = f(q) = \epsilon$ those surfaces are tangent at the point q .

Note that H' is the graph of the function

$$h : P_v \rightarrow \mathbb{R}, \quad h(x) = c + \sqrt{\|x - \bar{r}\|^2 + \epsilon^2},$$

where $c \in \mathbb{R}$ and $\bar{r} \in P_v$ are determined by the orthogonal decomposition $r = \bar{r} + cv$.

From the fact that H' is tangent to H at q and that it is contained in its epigraph, one deduces

- $u_\epsilon(q) = h(q)$,
- $\text{grad}u_\epsilon(q) = \text{grad}h(q)$,
- $\text{Hess}u_\epsilon(q) \leq \text{Hess}h(q)$,

where the last inequality implies that the difference is a positive definite matrix. In particular, we find that at the point q

$$\langle \text{grad}\iota_\epsilon, \text{grad}u_\epsilon \rangle \leq \frac{\epsilon}{(1 - \|\text{grad}u_\epsilon\|^2)^{3/2}} \langle \text{Hess}(h)\text{grad}h, \text{grad}h \rangle .$$

Now an explicit computation shows that

$$\text{grad}h = \frac{1}{h-c}(x - \bar{r}), \quad \text{Hess}h = \frac{1}{h-c}(Id - \text{grad}h \otimes \text{grad}h),$$

which implies that, still at the point q , the following inequalities hold

$$\begin{aligned} \langle \text{grad}(\iota_\epsilon), \text{grad}(u_\epsilon) \rangle &\leq \frac{\epsilon}{(1 - \|\text{grad}(u_\epsilon)\|^2)^{3/2}(h-c)} (\|\text{grad}(h)\|^2 - \|\text{grad}(h)\|^4) \\ &\leq \frac{\epsilon}{(1 - \|\text{grad}(u_\epsilon)\|^2)^{3/2}(u-c)} (\|\text{grad}(u_\epsilon)\|^2 - \|\text{grad}(u_\epsilon)\|^4) \end{aligned}$$

Using $\|\text{grad}u_\epsilon\| < 1$ and the identities

$$u_\epsilon(q) - c = -\langle q - \mathbf{r}(q), v \rangle = \iota_\epsilon(q) = \frac{\epsilon}{\sqrt{1 - \|\text{grad}(u_\epsilon)\|^2}},$$

we obtain

$$\langle \text{grad}\iota_\epsilon, \text{grad}u_\epsilon \rangle \leq \|\text{grad}u_\epsilon\|^2 \leq 1.$$

To prove (28), it is then sufficient to note that

$$\Delta u_\epsilon(q) \leq \Delta h = \frac{1}{\iota_\epsilon(q)}(n - \|\text{grad}u_\epsilon\|^2) .$$

□

Proof of Proposition 4.10. Let ι_ϵ be the frequency function of the surface H_ϵ , let $u_\epsilon : P_v \rightarrow \mathbb{R}$ be the $C^{1,1}$ -function whose graph is the surface H_ϵ , and set $\omega_\epsilon = *du_\epsilon$. Then $\omega_\epsilon \rightarrow \omega$ in L^1_{loc} . Moreover ω_ϵ is a Lipschitz form. Analogously, we find that ι_ϵ is a Lipschitz function since

$$\iota_\epsilon(x) = \frac{1}{\sqrt{1 - \|\text{grad}u_\epsilon\|^2}}.$$

It follows that for every smooth function with compact support f we have

$$(28) \quad \int_{P_v} df \wedge (\iota_\epsilon \omega_\epsilon) = - \int_{P_v} f d\iota_\epsilon \wedge \omega_\epsilon - \int_{P_v} f \iota_\epsilon \Delta u_\epsilon dV .$$

As $\iota_\epsilon \searrow \iota$ pointwise and f has compact support, there exists a constant C such that $|f\iota_\epsilon| < C$ for $\epsilon < 1$. It then follows by the Dominated Convergence Theorem that

$$\int_{P_v} f \iota_\epsilon \Delta u \rightarrow \int_{P_v} f \iota \Delta u = 0 ,$$

where the last equality holds because ι is zero on the singularity and Δu is concentrated there.

On the other hand, we have $\Delta u_\epsilon dV \rightarrow \Delta u dV$ as measures, which implies

$$\left| \int f \iota_\epsilon \Delta u_\epsilon dV - \int f \iota \Delta u \right| < C \left| \int_K \Delta u_\epsilon dV - \int_K \Delta u \right| \rightarrow 0 .$$

As ι vanishes on the singularity, it follows that the last term on the right-hand side of (28) converges to 0. To conclude, it is sufficient to show that

$$\int_{P_v} f d\iota_\epsilon \wedge \omega_\epsilon \rightarrow \int_{P_v} f dV .$$

Now the 2-form $dl_\epsilon \wedge \omega_\epsilon$ is equal to $dl_\epsilon(\text{grad}u_\epsilon)dV$. So it is sufficient to prove that $g_\epsilon = \langle \text{grad}l_\epsilon, \text{grad}u_\epsilon \rangle$ weakly converges to 1 in L^1_{loc} .

First we consider the case where D is the intersection of the futures of a finite number of lightlike hyperplanes – we will describe this situation by saying that D is “finite”. Then the singularity is a finite tree and, in particular, it is closed. In this case, given a point $x \in P_v$, we denote by $\mathbf{r}_0(x)$ the starting point of the lightlike ray through $q(x)$. Then the restriction of the map \mathbf{r}_0 on each region E of $\partial D \setminus T$ is a smooth projection and satisfies

$$\iota_v(x) = -\langle q(x) - \mathbf{r}_0(x), v \rangle .$$

A simple computation shows that $\text{grad}\iota_v = \text{grad}u$ on E , which implies

$$(29) \quad \langle \text{grad}\iota_v, \text{grad}u \rangle = 1 .$$

On the other hand, if $\mathbf{r}_0(x)$ lies in the interior of a segment e of T , it is not difficult to check that $\text{grad}\iota_\epsilon(x) \rightarrow \text{grad}\iota(x)$. Indeed consider the domain \hat{D} defined as the future of the spacelike line which contains the segment e . Note that $\hat{D} \supset D$, and e is contained in the singularity of \hat{D} . Thus if $\hat{\mathbf{r}} : \hat{D} \rightarrow \hat{e}$ denotes the retraction on the singularity, then that $\hat{\mathbf{r}}^{-1}(e) = \mathbf{r}^{-1}(e) = U$, and the cosmological time of D coincides with the cosmological time of \hat{D} on U . Then the frequency function ι_v and ι_ϵ around q can be computed by considering the domain \hat{D} instead of the domain D . In that case an explicit computation shows that the convergence to (29) holds.

In particular, the function g_ϵ converges to 1 almost everywhere. Since Lemma 4.11 shows that g_ϵ is bounded by 1, the Dominated Convergence Theorem implies that g_ϵ converges strongly to 1 in L^1_{loc} . So the proposition is valid whenever D is finite.

Consider now the general case. By Lemma 4.11, we have $g_\epsilon < 1$ at every point. So for every sequence $\epsilon_n \rightarrow 0$, up to passing to a subsequence, we can take the weak limit in L^1_{loc} . That is, there exists a function g such that

$$\int fg dV = \lim_{n \rightarrow +\infty} \int fg_{\epsilon_n} dV ,$$

and $\|g\|_{L^\infty} \leq 1$. Note in particular that for every smooth function with compact support we obtain

$$\int df \wedge \iota\omega = - \int fg dV$$

Clearly, the same formula holds for any function f which is the limit in the Sobolev space $W^{1,1}(K)$ of smooth functions with compact support.

To prove that $g = 1$ we use an approximation argument. Let D_k be a sequence of finite domains of dependence converging to D , and let $u_k : P_v \rightarrow \mathbb{R}$ be the functions whose graph is ∂D_k . Consider a smooth function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ which is decreasing and such that its support is contained in $[0, M]$, and define

$$f_k(x) = \phi(u_k(x)) , \quad f(x) = \phi(u(x)) .$$

Then the functions f_k are C^1 with compact support and are limits of smooth functions with compact support. They satisfy

$$df_k = \phi'(u_k)du_k , \quad df = \phi'(u)du .$$

As (29) holds for finite domains we obtain for every k

$$\int -\phi'(u_k)\iota_k dV = \int \phi(u_k)dV ,$$

and with the inequality $\phi' \leq 0$ we deduce

$$\int |\phi'(u_k)|\iota_k dV = \int \phi(u_k)dV .$$

As $u_k \rightarrow u$ uniformly on compact subsets and $\phi(u_k)$ is zero outside a compact region, which is independent of k , we obtain

$$(30) \quad \int |\phi'(u_k)|\iota_k dV \rightarrow \int \phi(u)dV .$$

On the other hand, Fatou's Lemma implies

$$(31) \quad \lim \int |\phi'(u_k)|\iota_k dV \leq \int \limsup(|\phi'(u_k)|\iota_k)dV .$$

Note that by Proposition 4.3 the right-hand side is less than $\int |\phi'(u)|\iota dV = \int \phi(u)gdV$. Comparing (30) and (31) yields

$$\int |\phi'(u)|\iota dV \leq \int |\phi'(u)|gdV ,$$

and since $g(x) \leq 1$ almost everywhere, it follows that $g(x) = 1$ almost everywhere. \square

Remark 4.12. In other words, Proposition 4.10 states that $\iota_v \omega$ is a primitive of the volume form of P_v in a distributional sense.

4.3. Stability of the frequency function for domains with flat boundary. We can state now the main stability theorem of this section.

Theorem 4.13. *Let D_k be a sequence of regular domains converging to D . If the boundaries of D_k and D are all flat then $\iota_v^k \rightarrow \iota_v$ strongly in $L^1_{loc}(P_v)$.*

Proof. Consider the convex functions $u_k : P_v \rightarrow \mathbb{R}$ whose graphs are identified with ∂D_k . Note that the sequence u_k converges to u uniformly on compact subsets. Moreover, since du_k converges to du almost everywhere and $\|du_k\| < 1$, it follows from the Dominated Convergence Theorem that $du_k \rightarrow du$ in $L^1(P_v)$, which implies $\omega_k \rightarrow \omega$ in $L^1(P_v)$.

We may assume that $u_k > 0$ for every k . By this assumption, $\iota_v^k(x) < u_k(x)$ for every x . So the functions ι_v^k are uniformly bounded on compact subsets. Consequently, there exists a weak limit J of the sequence ι_v^k , that is a L^1 function J with

$$(32) \quad \int f \iota_v^k dV \rightarrow \int f J dV$$

for every compactly supported continuous function f .

Now we claim that for every compactly supported smooth function f we have

$$(33) \quad \int_{P_v} df \wedge (J\omega) = - \int_{P_v} f dV = \int_{P_v} df \wedge (\iota_v \omega) .$$

From this claim it then follows that $J\omega = \iota_v \omega$ almost everywhere and hence $J = \iota_v$.

To prove the claim, first note that the left-hand side of (33) can be rewritten as

$$(34) \quad \int_{P_v} df \wedge (J\omega) = \int_{P_v} (J - \iota_v^k) df \wedge \omega + \int_{P_v} \iota_v^k df \wedge (\omega - \omega_k) + \int_{P_v} df \wedge (\iota_v^k \omega_k) .$$

By Proposition 4.10, the last term is equal to the right-hand side of (33) and hence independent of k .

As ω_k converges to ω in $L^1_{loc}(P_v)$, the second term in (34) vanishes as $k \rightarrow +\infty$. (Note that the functions $\iota_v^k df$ are uniformly bounded in L^∞_{loc}).

Finally by the density of continuous functions in L^2_{loc} , J is the weak limit of ι_v^k in L^2_{loc} , that is, (32) holds for every L^2 -function defined on some open subset with compact closure. In particular, the first term on the right in (34) also vanishes as $k \rightarrow +\infty$ (indeed $df \wedge \omega = g dV$ for some compactly supported L^∞ -function g). So letting k go to $+\infty$ in (34), we deduce that (33) holds, and the claim is proved.

To show that ι_v is a strong limit of ι_v^k we use the following remark:

$$\forall x \in \partial D : \iota_v(x) \geq \limsup \iota_v^k(x) .$$

This holds due to the following: if r_k is the maximal lightlike ray in ∂D_k that contains the point $x + u_k(x)v$, then the sequence r_k converges to a lightlike ray r contained in ∂D , which contains the point $x + u(x)v$. So the same argument as in the proof of Proposition 4.3 may be used.

Let us now fix an open subset $A \subset P_v$ with compact closure. Then there is a constant $M > 0$ such that ι_v and ι_v^k are bounded by M on A . This implies that the functions $-(\iota_v^k)^2$ are uniformly bounded and allows one to apply Fatou's Lemma. It follows that

$$\int_A -(\iota_v)^2 \leq \int_A -\limsup (\iota_v^k)^2 \leq \liminf \left(- \int_A (\iota_v^k)^2 \right)$$

which implies

$$\|\iota_v\|_{L^2(A)} \geq \limsup \|\iota_v^k\|_{L^2(A)} .$$

As the sequence ι_v^k converges weakly to ι_v in $L^2(A)$, this estimate implies the strong convergence in $L^2(A)$. The L^2 -convergence on a compact set then implies the L^1 -convergence. \square

4.4. Uniform bounds for domains with non-flat boundaries. We have seen in Section 4.3 that if D is a domain of dependence with flat boundary, and $(D_k)_{k \in \mathbb{N}}$ is a sequence of domains of dependence with flat boundaries converging to D , then the frequency function of D_k converges to the frequency function of D . Suppose now that D is any domain of dependence (not necessarily with flat boundary) and that $(D_k)_{k \in \mathbb{N}}$ is a sequence of domains with flat boundaries converging to D . We know (Proposition 4.3) that the frequency function of D is at least the limsup of the frequency functions of the domains D_k , but the example in Section 4.1 shows that equality does not always hold. We will now see that the opposite inequality does hold, albeit with a multiplicative constant.

This result is important in view of the numerical computations in the following sections, which allow one to visualize the frequency function measured by an observer in an MGHFC manifold of dimension 3+1. These computations are done by approximating the corresponding domain of dependence by a sequence of finite domains with flat boundaries. However it might happen that the limit domain does not have flat boundary. The computed frequency function then coincides with the limit of the frequency functions of the finite domains (with flat boundary), and can differ from the actual frequency function of the limit domain. However, Theorem 4.14 then ensures that the actual frequency function is at least equal to the computed limit frequency, and that it is at most three times this computed frequency function.

Theorem 4.14. *Let $D \subset \mathbb{R}^{n,1}$ be a domain of dependence, and let $(D_k)_{k \in \mathbb{N}}$ be a sequence of domains of dependence with flat boundaries converging to D . Let ι_v be the frequency function of the boundary of D with respect to a unit timelike direction v , considered as a function on P_v , and let ι_v^k be the frequency function of D_k . Then the sequence $(\iota_v^k)_{k \in \mathbb{N}}$ converges in L^1_{loc} to a limit ι_{lim} , and*

$$\iota_{lim} \leq \iota_v \leq n \iota_{lim} .$$

In Section 4.5, we will show that all domains D that arise as universal covers of 2+1-dimensional MGHFC spacetimes have flat boundaries. Consequently, in that case the inequalities in statement of the theorem can be improved to an equality. In higher dimensions, it appears unlikely that the boundary of the universal cover of a MGHFC manifold always has a flat boundary. It is conceivable that for such domains, the inequality can be improved to $\iota_{lim} \leq \iota_v \leq (n-1)\iota_{lim}$, but we do not pursue this question further here.

Proof. We proceed as in the proof of Theorem 4.13 and only indicate the steps that differ from that proof. As already noted, Lemma 4.11 still holds, but differences occur in the proofs of Proposition 4.10 and of Theorem 4.13. As we want to obtain inequalities on ι_v we consider a positive test function f . Equation (28) still holds. However, the inequality is weakened to

$$0 \leq \int_{P_v} f \iota_v \Delta u \leq (n-1) \int_{P_v} f dV .$$

The last inequality descends by the estimate (28), when one takes the limit $\epsilon \rightarrow 0$ and uses that $\Delta u_\epsilon \rightarrow \Delta u$ as measure and that $\iota_\epsilon \searrow \iota_v$. Following the proof of Proposition 4.10, we then obtain the inequality

$$-n \int_{P_v} f dV \leq \int_{P_v} df \wedge (\iota_v \omega) \leq - \int_{P_v} f dV .$$

In the proof of Theorem 4.13, Equation (33) is therefore replaced by

$$\int_{P_v} df \wedge (J\omega) = - \int_{P_v} f dV , \quad -n \int_{P_v} f dV \leq \int_{P_v} df \wedge (\iota_v \omega) \leq - \int_{P_v} f dV .$$

The rest of the proof of Theorem 4.13 goes through and leads to the statement. \square

4.5. Universal coverings of MGHFC spacetimes in dimension 2+1. Let M be a 2+1-dimensional MGHC flat spacetime of genus $g \geq 2$. We know that the universal covering of M is a domain of dependence $D \subset \mathbb{R}^{2,1}$. We will assume in the following that M is not a Fuchsian spacetime. This means that D is not the future of a point, or, equivalently, that the holonomy representation of M is not conjugate to a linear representation in $SO(2,1)$. In this subsection we will prove that the boundary of D is flat. This is a consequence of the following geometric property of the boundary of D .

Proposition 4.15. *Identify the boundary $\partial \mathbb{H}^2$ with the set of lightlike directions in $\mathbb{R}^{2,1}$ and let D^* be the subset of $\partial \mathbb{H}^2$ consisting of lightlike directions parallel to lightlike rays contained in ∂D . Then the set D^* has Lebesgue measure zero in $\partial \mathbb{H}^2$.*

We know by the work of Mess [19] that the linear part of the holonomy representation of M defines a Fuchsian group Γ , which determines a hyperbolic surface $S = \mathbb{H}^2/\Gamma$. Moreover, there is a measured geodesic lamination

λ on S such that M is obtained by a Lorentzian grafting on the Minkowski cone of S . Denote by $\tilde{\lambda}$ the lifting of λ to the universal covering \mathbb{H}^2 . We say that a point $\xi \in \partial\mathbb{H}^2$ is *nested* for the lamination λ if for some point $v \in \mathbb{H}^2$, the intersection of the ray joining v to ξ with $\tilde{\lambda}$ is $+\infty$.

Lemma 4.16. *If ξ is a nested point for λ then*

- *the point ξ is not the end-point of any leaf of $\tilde{\lambda}$,*
- *the intersection of any ray ending at ξ with $\tilde{\lambda}$ is $+\infty$.*

Proof. Let us consider the upper half-plane model of \mathbb{H}^2 . Without loss of generality we may assume that $\xi = \infty$. Suppose there is a leaf l of $\tilde{\lambda}$ ending at ξ , and take any compact ray r_0 joining a point $v \in \mathbb{H}^2$ to l . Now any sub-arc of the ray $[v, \xi)$ can be deformed through a family of transverse arcs to a subarc of r_0 . This implies that the intersection of any subarc of $[v, \xi)$ with $\tilde{\lambda}$ is uniformly bounded by the intersection of r_0 with $\tilde{\lambda}$. This proves that ξ is not nested.

For the second statement, consider a point $v_0 \in \mathbb{H}^2$ such that the intersection of $[v_0, \xi)$ with $\tilde{\lambda}$ is $+\infty$. Take a family of leaves l_n meeting $[v_0, \xi)$ at a point $v_n \rightarrow \xi$. With the first statement, it is easy to check that l_n bounds a neighborhood U_n of ξ , and that $\{U_n\}$ is a fundamental family of neighborhoods of ξ .

In particular there is a leaf, say l_1 , cutting both $[v_0, \xi)$ at a point v_1 and $[w, \xi)$ at a point w_1 . Then every leaf of $\tilde{\lambda}$ cutting $[v_1, \xi)$ must cut also $[w_1, \xi)$. This implies that the intersection of $[w, \xi)$ with $\tilde{\lambda}$ is bigger than the intersection of $[v_1, \xi)$ with $\tilde{\lambda}$, which is clearly infinite. \square

We will see that the set D^* does not contain any nested points, so the proof of the proposition is obtained from the following lemma.

Lemma 4.17. *Almost all points in $\partial\mathbb{H}^2$ are nested for λ .*

Proof. The proof is based on the ergodicity property of the geodesic flow on S . For (x, v) in the unit tangent bundle of S let $r(x, v)$ the geodesic ray $\{\exp_x(tv) | T \geq 0\}$. Consider now the following subset of $T^1(S)$:

$$B_n = \{(x, v) \in T^1(S) | \iota(r(x, v), \lambda) < n\} .$$

We claim that B_n is a set of measure zero for the Liouville measure.

Before proving the claim, let us show how the claim proves the statement. Indeed we get that the measure of the set $B = \bigcup B_n$ is zero. Let $\tilde{B} \subset T^1\mathbb{H}^2$ be the lifting of B on the universal covering. By definition we have that \tilde{B} is made of pairs (x, v) such that the endpoint of the ray $\exp_x(tv)$ is not nested. In particular, if E is the complement in $\partial\mathbb{H}^2$ of nested points, the Liouville measure of B can be computed as

$$\int_K \mu_x(E) dA ,$$

where K is a fundamental region and μ_x is the visual measure from x . As the measure of B is zero, it immediately follows that E is a set of measure zero.

It remains to prove the claim. Let ϕ_t denote the geodesic flow on T^1S . Clearly we have

$$\phi_t(B_n) \subset B_n .$$

More precisely, $t < s$ implies $\phi_t(B_n) \subset \phi_s(B_n)$. It follows that $\hat{B}_n = \bigcup_{t>0} \phi_t(B) = \bigcup_{k \in \mathbb{N}} \phi_k(B_n)$ is a subset invariant by the geodesic flow. Moreover its Liouville measure is equal to

$$\mu(\hat{B}_n) = \inf_k \mu(\phi_k(B_n)) = \mu(B_n)$$

where the last equality holds because μ is invariant by the geodesic flow.

By the ergodicity of the flow, we have either $\mu(B_n) = 0$ or $\mu(T^1S \setminus B_n) = 0$. In order to prove that the latter is not true, it is sufficient to prove that the complement of B_n contains a non-empty open subset.

First note that if (x, v) corresponds to a closed geodesic which intersects λ , then the intersection of λ with the ray $\exp_x(tv)$ is $+\infty$. In particular $(x, v) \notin B_n$.

Moving x on the ray, we may assume that it is not on λ . Now take $M > 0$ so that the intersection of λ with the segment $r = \{\exp(tv) | t \in [0, M]\}$ is bigger than $2n$ and $\exp_x(Mv)$ is not on the lamination.

We want to show that a neighborhood of (x, v) is contained in the complement of B_n . Indeed if (x_k, v_k) converges to (x, v) , then the intersection of λ with the segment $r_k = \{\exp_{x_k}(tv_k) | t \in [0, M]\}$ converges to the intersection of λ with r . So for k sufficiently large, (x_k, v_k) does not lie on B_n . \square

In order to relate nested points with points in D^* we need the following technical lemma from Lorentzian geometry.

Lemma 4.18. *If R is a lightlike ray contained in ∂D which is maximal with respect to the inclusion, then there is a sequence of points r_n on the singularity T which converges to a point on R , and a sequence of spacelike support planes P_n at r_n which converges to the lightlike plane containing R .*

Proof. Let v be any future oriented timelike vector. Let q be a point of r and consider the segment of points $q_\epsilon = q + \epsilon v$ for $\epsilon \in [0, 1]$ and the path on Σ given by $r_\epsilon = \mathbf{r}(q_\epsilon)$.

Note that this path is contained in the closure of $D \cap I^-(q_1)$, which is a compact region of $\mathbb{R}^{2,1}$. Thus there exists a sequence $\epsilon_n \rightarrow 0$ such that r_{ϵ_n} converges to some point \bar{r} .

We claim that \bar{r} is contained in R . In order to prove the claim, note that sequence of vectors $q_{\epsilon_n} - r_{\epsilon_n}$ converges to $q - \bar{r}$. Since they are timelike, their limit cannot be spacelike. But ∂D being achronal forces $q - \bar{r}$ to be lightlike and the segment $[\bar{r}, q]$ to be contained in ∂D . As the lightlike plane P containing R is a support plane for D , it follows that $[\bar{r}, q]$ is contained in this plane, so in particular is on R .

To construct the sequence of lightlike support planes, it is sufficient to set P_n to be the plane orthogonal to $q_{\epsilon_n} - r_{\epsilon_n}$ passing through r_{ϵ_n} . \square

We are now ready to prove Proposition 4.15.

Proof of Proposition 4.15. We will prove that if $\xi \in D^*$ then ξ is not nested. Assume by contradiction that ξ is nested, and let R be a ray parallel to ξ . By Lemma 4.18, there exists a sequence of points r_n on the singularity, converging to a point on R and a sequence of spacelike support planes P_n converging to the lightlike support plane containing R . Let u_n be the unit timelike vector orthogonal to P_n . Clearly we have that $u_n \rightarrow \xi$ in \mathbb{H}^2 .

By Mess' construction [19], we have

$$r_n - r_0 = \int_{c_n} w_n(x) d\mu_\lambda,$$

where c_n is the segment joining u_0 to u_n and $w_n(x) \in \mathbb{R}^{2,1}$ is defined to be 0 if x is not in the support of λ and is the unit tangent vector at x orthogonal to the leaf through x and pointing towards u_n otherwise.

As by the hypothesis ξ is nested, the ray r joining u_0 to ξ transversely meets the lamination. In particular, by changing u_0 to a point on $r \cap \lambda$ we may assume that u_0 is on the lamination. Let e be the unit vector at u_0 orthogonal to the leaf l_0 through u_0 and pointing towards ξ . We claim that if ξ is nested then

$$\langle r_n - r_0, e \rangle \rightarrow +\infty,$$

which contradicts the assumption that the sequence r_n converges in Minkowski space.

First note that since $u_n \rightarrow \xi$, we may assume that u_n is on the half-plane bounded by l_0 and containing ξ .

Thus if l is a leaf that intersects $c(u_0, u_n)$, l disconnects l_0 from ξ , and the scalar product of vectors e and $w(x)$ is positive. Since the corresponding geodesics are disjoint, the reverse of Schwarz inequality holds, that is, $\langle w(x), e \rangle > 1$. This implies

$$\langle r_n - r_0, e \rangle \geq \iota(\tilde{\lambda}, c_n).$$

Let us prove that $\iota(\tilde{\lambda}, c_n) \rightarrow +\infty$. The reason is that for every leaf l of $\tilde{\lambda}$ cutting the ray $[u_0, \xi]$, u_n is definitively contained in the region bounded by l containing ξ . So for every point x on the segment $[u_0, \xi]$, for n sufficiently large, we have $\iota(\nu, c_n) \geq \iota(\nu, [u_0, x])$.

Since we are assuming that $\nu([u_0, \xi]) = +\infty$, we can choose x so that $\iota(\nu, [u_0, x])$ is arbitrarily big, so the conclusion follows. \square

Let us fix a unit timelike vector v , and let $u : P_v \rightarrow D$ be the convex function whose graph is the boundary of D . Note that if e is a unit vector in P_v , then $e + v$ is a lightlike vector. In this way, the unit circle S^1 in P_v is identified to ∂H^2 by the map $e \mapsto [e + v]$. Under this identification, the subset D^* corresponds to the image of the map

$$\delta : P_v \setminus T \rightarrow S^1, \quad \delta(x) = \text{grad}(u)(x).$$

Fix a unit vector e in P_v and take linear orthogonal coordinates (x, y) on P_v such that $\partial_x = e$, and consider the restriction of the function u on each line parallel to e . That is, for $y \in \mathbb{R}$, let $u_y : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $u_y(x) = u(x, y)$.

Note that whenever (x, y) does not correspond to a point on T , then u_y is differentiable at x and

$$(u_y)'(x) = \langle \text{grad}u(x, y), e \rangle$$

By Proposition 4.15, at those points, the derivative takes value in a subset of zero measure of \mathbb{R} .

Now to prove that the boundary of D is flat we will proceed in three steps.

- Step 1. We will prove that for a generic choice of the vector e , if u_y is differentiable at x , then (x, y) does not correspond to a point on T . In particular, the derivative $(u_y)'$ takes value in a subset of zero measure of \mathbb{R} .
- Step 2. We will use this fact to show that for every y , the measure $(u_y)''$ is atomic with support on $T \cap R_y$.
- Step 3. Using a disintegration formula for $\partial_{xx}u$ in terms of the family of measures $(u_y)''$ we conclude that this measure $\partial_{xx}u$ is concentrated on T .

Lemma 4.19. *There is a subset A of S^1 such that:*

- *If $e \in A$ then, for every $y \in \mathbb{R}$, the points x where u_y is differentiable are exactly the points such that u is differentiable at (x, y) . Moreover, at those points,*

$$(u_y)' = \langle \text{gradu}(x, y), e \rangle .$$

- *The measure of $S^1 \setminus A$ is zero.*

Proof. Let A be the set formed by vectors e such that the geodesic in \mathbb{H}^2 starting from v with direction e does not meet any leaf of λ orthogonally. We will prove that A fulfills the requirements of the statement.

First let us prove that the only differentiable points of u_y correspond to differentiable points of u . By contradiction suppose that u_y is differentiable at a point x so that (x, y) corresponds to a point on the singularity. Up to translation we may suppose that $x = y = u(x, y) = 0$.

As the point 0 is on the singularity, there are two lightlike planes P_1, P_2 through 0 , which are support planes for D . Let V be the vertical plane containing e and v . Note that $P_1 \cap V$ and $P_2 \cap V$ are support lines for $\partial D \cap V$ at the point 0 . As $\partial D \cap V$ corresponds to the graph of u_y , and we are assuming that u_y is differentiable at $x = 0$, those support lines must coincide, $P_1 \cap V = P_2 \cap V$. This implies that V must contain the line $r = P_1 \cap P_2$. Note, however, that this line is spacelike, and its dual geodesic in \mathbb{H}^2 is a leaf of $l \in \lambda$. On the other hand $V \cap \mathbb{H}^2$ is the geodesic g starting from v with direction e , so the condition implies that g meets orthogonally l , contradicting the choice of e .

It remains to show that the complement of A in S^1 is a set of measure zero. Note that if $e \in A$ then $-e$ is also in A , so we may regard A as a subset of the projective line $P(P_v)$.

We will argue as follows. For any geodesic l of \mathbb{H}^2 , let $e(l)$ be the unit tangent vector at v such that the geodesic $\exp_v(te(l))$ hits orthogonally l . Note that $e(l)$ is defined up to the sign, so it should be considered more properly as an element of $P(P_v)$. The complement of A is the set of unit vectors $e(l)$ where l is a leaf of the lamination λ .

Let us now fix any ray r starting from v and define

$$E_r = \{e(l) \mid l \text{ is a leaf of } \lambda \text{ hitting } r\} .$$

Note that if r_n is a dense subset of the ray from v we clearly have

$$\bigcup_n E_{r_n} = P(P_v) \setminus A .$$

So in order to argue that the measure of the complement of A is zero, it is sufficient to show that E_r has measure zero.

Now on $r \cap \lambda$ we may define a vector field w by taking for $w(x)$ to be the unique vector orthogonal to the leaf l through x . Note that $e(l)$ coincides with the orthogonal projection of $w(x)$ on P_v up to renormalization. By a classical result [11], the field w can then be extended to a Lipschitz vector field, still denoted w , on the whole line. In particular, we obtain a map

$$\hat{e} : r \rightarrow P(P_v)$$

by defining $\hat{e}(x)$ to be the projective class of $w(x)$. It is not difficult to show that this map is locally Lipschitz and, by definition, $E_r = \hat{e}(r \cap \lambda)$. As the measure of $r \cap \lambda$ is zero, this concludes the proof. \square

Lemma 4.19 concludes the proof of step 1. In particular, note that if e is on the set A , then for every $y \in \mathbb{R}$, the derivative function u_y takes value on the set

$$\{\langle \text{gradu}(x, y), e \rangle\}$$

which by Proposition 4.15 has measure zero. The proof of step 2 is then based on the following simple lemma on convex functions.

Lemma 4.20. *Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Suppose that the measure of the image $u' : \mathbb{R} \rightarrow \mathbb{R}$ is zero. Then u'' is an atomic measure, and its support coincides with the set of discontinuity of u' .*

Proof. By the standard theory of convex functions the measure u'' can be split as the sum of a measure μ without atoms and an atomic part, say ν , with $\nu = \sum_k a_k \delta_{q_k}$, where δ_{q_k} is the Dirac measure concentrated on q_k and $\sum a_k$ is an absolutely convergent series.

Now we claim that the measure of the image of $u'([a, b])$ is equal to $\mu([a, b])$. Indeed, note that u is not differentiable exactly on the points $\{q_n | n \in \mathbb{N}\}$. Moreover at every point there exists the left derivative and the right derivative that can be expressed as follows. Assume that 0 is a differentiable point of u , then

$$u'_l(x) = u'(0) + \phi(x) + \sum_{q_n \in [0, x]} a_n, \quad u'_r(x) = u'(0) + \phi(x) + \sum_{q_n \in [0, x]} a_n,$$

where we put $\phi(x) = \mu([0, x])$.

If u is differentiable at some point x , then the values that u' takes on the interval $[0, x]$ can be described as

$$u'([0, x]) = [u'(0), u'(x)] \setminus \bigcup_{q_n \in [0, x]} I_n$$

where $I_n = [u'_l(q_n), u'_r(q_n)]$ is the interval between the left and right derivative at q_n , which are pairwise disjoint.

So the measure of this set $u'([0, x])$ is given by

$$\phi(x) = \sum_{q_n \in [0, x]} a_n - \sum_{q_n \in [0, x]} a_n = \phi(x).$$

By the assumption, $\phi(x) = 0$ for any x , so $\mu = 0$. □

Finally, in order to prove step 3 we need the following disintegration result of the measure $\partial_{xx}^2 u$ in terms of the measure $(u_y)''$.

Lemma 4.21. *Let x, y be coordinates on \mathbb{R}^2 and consider a convex function u . For every $y \in \mathbb{R}^2$ denote by u_y the convex function $x \mapsto u(x, y)$. If $\partial_{xx} u$ denotes the second derivative of u along the x axis (which is a Radon measure on \mathbb{R}^2) and $(u_y)''$ denotes the second derivative of u_y (which is a Radon measure on \mathbb{R}) then the following formula holds*

$$\int_{\mathbb{R}^2} f(x, y) \partial_{xx} u = \int_{\mathbb{R}} dy \int_{\mathbb{R}} f(x, y) (u_y)''$$

for every bounded Borel function f with compact support.

The proof of this analytical Lemma can be found in [21] (Theorem 1.3 formula (1.31)) for the wider class of bounded Hessian functions. We are ready now to prove that the boundary of D is flat.

Proposition 4.22. *If D is the universal covering of a MGHC flat spacetime of dimension $2 + 1$, then its boundary is flat.*

Proof. We will prove that $\text{Hess}(u)$, considered as a matrix-valued measure on P_v , is supported on T .

Indeed it is sufficient to prove that there are three independent directions e_1, e_2, e_3 such that $D_{e_i, e_i}^2(u)$ is a measure supported on T . As the subset A from Lemma 4.19 is dense, it is sufficient to prove that $D_{e, e}^2(u)$ is supported on T for $e \in A$.

If x, y are the Cartesian coordinates on P_v such that $e = \partial_x$, then $D_{e, e}^2(u)$ coincides with $\partial_{xx}^2(u)$. So we have to prove that if f is a measurable function which is zero on T , then

$$\int f \partial_{xx}^2(u) = 0.$$

We may compute the integral above using Lemma 4.21, which yields

$$\int f \partial_{xx}^2(u) = \int dy \int (f_y) (u_y)''.$$

Now, by Lemma 4.20 and Lemma 4.19, $(u_y)''$ is supported on the discontinuity of u'_y which corresponds to points x such that (x, y) is on the projection of the singularity. It follows that f_y is zero on the support of $(u_y)''$, and hence the integral is zero. □

5. THE FREQUENCY FUNCTION FOR SPACETIMES CONSTRUCTED FROM MEASURED GEODESIC LAMINATIONS

The aim of this section is to understand to what extent and under what conditions an observer in a $2 + 1$ -dimensional domain of dependence can reconstruct the geometry and topology of the ambient space from his observation — either at one time or over a fixed time interval — from the frequency function of the signal emitted by the initial singularity. In particular, we investigate this question for a domain of dependence, which is the universal cover of a (non-Fuchsian) MGHFC spacetime M containing a closed Cauchy surface of genus g .

A basic remark, which somewhat complicates the statements and the analysis below, is that the observer can only “see” the universal cover of M , so he can in no way distinguish M from any of its finite covers. In other words, the observer can only determine the largest discrete subgroup of $\text{Isom}(\mathbb{R}^{2,1})$ compatible with the signal she observes. Moreover, he can only be certain to have determined correctly the fundamental group of M if he knows the genus of S , since otherwise it remains possible that his spacetime is topologically a finite cover of M , with a flat metric which is “almost” lifted from a flat metric on M , with only a small change in a region not visible by her.

In Section 5.1 we study the relationship between the frequency function measured by an observer and her cosmological time (see Proposition 5.6). Then in Section 5.2, we show (see Proposition 5.13) that an observer in the universal cover of a non-Fuchsian MGHFC Minkowski spacetime can reconstruct in finite eigentime the geometry and topology of the space, if the genus of the Cauchy surface is known to him. In Section 5.3, we briefly explain how those arguments can be adapted to higher dimensions.

5.1. Estimating the cosmological time from the frequency function. We consider a domain D and an observer in D given by a point $p \in D$ and a future directed timelike unit vector $v \in \mathbb{H}^2$. We consider the associated rescaled frequency function

$$\rho_{p,v}^D : S^1 \rightarrow \mathbb{R}_0^+$$

defined as in the previous section. We will be mostly interested in the case where D is the universal cover of a spacetime constructed by grafting along a measured geodesic lamination. In this case, the observer will see a division of the circle into intervals, on which the frequency function behaves like that of a spacelike line, and intervals in which the frequency function behaves like the one of a light cone. The former correspond to the edges of the singular tree of D , the latter to its vertices. It follows from the results in the previous sections that the frequency function is analytic on the segments of the circle that correspond to the edges, while it is generally not analytic on the segments that correspond to the vertices. In general, the segments of the circle that correspond to vertices of the singular tree form a Cantor set.

We define the *maximum frequency* as

$$\rho_{p,v}^{D,max} = \sup_{u \in S^1} \rho_{p,v}^D(u)$$

To understand its properties, we consider again our two main examples.

Example 5.1. Consider a cone $D = I^+(q)$ and an observer characterized by $p \in I^+(q)$, $v \in \mathbb{H}^2$. Then

$$\rho_{p,v}^{D,max} = \frac{T(p)}{2} e^\delta,$$

where $T(p) = |p - q|$ is the cosmological time of the observer and $\delta = d_{\mathbb{H}^2}(v, \text{grad}_p T)$ is the hyperbolic distance between v and $\text{grad}_p T$. This follows from Equation (4).

Example 5.2. Consider the future $D = I^+(l)$ of a spacelike line l and an observer with $p \in I^+(l)$, $v \in \mathbb{H}^2$. It follows from Equation (20) that

$$\rho_{p,v}^{D,max} = \frac{T(p)}{2} e^\delta \cosh \xi,$$

where $T(p)$ is the cosmological time of p and δ, ξ are defined as follows. Denote by \tilde{l} the geodesic in \mathbb{H} that is stabilized by the $PSL(2, \mathbb{R})$ element that fixes the direction of l . Then $\text{grad}_p T$ defined a point on \tilde{l} , δ is the hyperbolic distance of v and \tilde{l} and ξ the distance of the projection of v on \tilde{l} from the point in \mathbb{H}^2 defined by $\text{grad}_p T$.

We will now determine an estimate on the cosmological time. The central ingredient is the following proposition.

Proposition 5.3. *The maximum frequency is given by*

$$\rho_{p,v}^{D,max} = \sup\{\langle v, y - x \rangle : x, y \in I^-(p) \cap D, y - x \text{ future directed and lightlike}\}.$$

Proof. By definition, we have

$$\rho_{p,v}^{D,max} \leq \sup\{\langle v, y - x \rangle : x, y \in I^-(p) \cap D, y - x \text{ future directed and lightlike}\} .$$

To show the opposite inequality, we choose sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ with $x_n, y_n \in I^-(p) \cap D$ that satisfy

$$(35) \quad \lim_{n \rightarrow \infty} \langle v, y_n - x_n \rangle = \sup\{\langle v, y - x \rangle : x, y \in I^-(p) \cap D, y - x \text{ future directed and lightlike}\} .$$

As the intersection $I^-(p) \cap \bar{D}$ is compact, there exist convergent subsequences $(x_{n_k})_{k \in \mathbb{N}}, (y_{n_k})_{k \in \mathbb{N}}$ with limits $x_{n_k} \rightarrow \bar{x} \in I^-(p) \cap \bar{D}, y_{n_k} \rightarrow \bar{y} \in I^-(p) \cap \bar{D}$.

If the segment $[\bar{x}, \bar{y}]$ is extensible, i. e. if there exist $\bar{x}', \bar{y}' \in I^-(p) \cap \bar{D}$ with $[\bar{x}, \bar{y}] \subset [\bar{x}', \bar{y}']$, then we obtain a contradiction to (35). Therefore $[\bar{x}, \bar{y}]$ is inextensible, \bar{x} lies on the tree and $\bar{y} \in \partial D \cap I^-(p)$. This implies that there exists a $\theta \in S^1$ such that the past directed lightlike ray starting at p that is defined by θ intersects ∂D in \bar{y} and $\langle v, \bar{y} - \bar{x} \rangle = \rho_{p,v}^D(\theta)$. \square

An immediate consequence is that if an observer moves along a timelike geodesic then the maximum frequency is increasing with time. More generally, Proposition 5.3 allows us to give estimates for the maximum frequency of domains that are contained in each other.

Corollary 5.4. *Let D, D' be domains with $p \in D \subset D'$. Then for all $v \in \mathbb{H}^2$:*

$$\rho_{p,v}^{D,max} \leq \rho_{p,v}^{D',max} .$$

In particular, we can estimate the maximum frequency for any domain.

Corollary 5.5. *For a domain D and any observer characterized by $p \in D$ and $v \in \mathbb{H}^2$, the following inequalities hold:*

$$(36) \quad \rho_{p,v}^{D,max} \geq \sup_{q \in I^-(p) \cap D} \rho_{p,v}^{I^+(q),max} , \quad \rho_{p,v}^{D,max} \leq \inf_{\substack{l \text{ spacelike line} \\ D \subset I^+(l)}} \rho_{p,v}^{I^+(l),max} .$$

By applying this corollary to a general domain and using the results of Examples 5.1 and 5.2 we obtain the following statement.

Proposition 5.6. *Let D be a domain with an observer characterized by $p \in D$ and $v \in \mathbb{H}^2$. Then the following inequalities relate the maximum frequency and the cosmological time:*

$$(37) \quad \frac{T(p)}{2} \leq \rho_{p,v}^{D,max} \leq T(p) \cosh d_{\mathbb{H}}(v, \text{grad}_p T) .$$

Proof. From the first inequality in Corollary 5.5 and Example 5.1 we have

$$\rho_{p,v}^{D,max} \geq \sup_{q \in I^-(p) \cap D} \rho_{p,v}^{I^+(q),max} = \frac{1}{2} \sup_{q \in I^-(p) \cap D} |p - q| e^{d_{\mathbb{H}}(v, \frac{p-q}{|p-q|})} \geq \frac{1}{2} \sup_{q \in I^-(p) \cap D} |p - q| .$$

By definition of the cosmological time, $T(p) = \sup_{q \in I^-(p) \cap D} |p - q|$, which proves the first inequality.

To prove the second inequality, we use the fact that for any lightlike vector ξ and any two future-directed timelike unit vectors u, v , we have

$$(38) \quad |\langle v, \xi \rangle| \leq 2|\langle u, v \rangle||\langle u, \xi \rangle| .$$

This can be seen as follows: after applying suitable elements of $SO(2,1)^+$, we can suppose that the vectors ξ, u, v are given by

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} \cosh \alpha \\ \sinh \alpha \\ 0 \end{pmatrix} \quad \xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{with} \quad a^2 = b^2 + c^2 .$$

This yields

$$|\langle u, \xi \rangle| = |a| , \quad |\langle v, \xi \rangle| = |a \cosh \alpha - b \sinh \alpha| \leq 2|a| \cosh \alpha , \quad |\langle u, v \rangle| = \cosh \alpha ,$$

and proves (38). By combining (38) with Proposition 5.3, we obtain for all $p \in D$ and $u, v \in \mathbb{H}^2$

$$\rho_{p,v}^{D,max} \leq 2|\langle u, v \rangle| \rho_{p,u}^{D,max} .$$

For $u = \text{grad}_p T$ this yields

$$\rho_{p,v}^{D,max} \leq 2 \cosh d_{\mathbb{H}}(v, \text{grad}_p(T)) \rho_{p, \text{grad}_p T}^{D,max} .$$

For the future of a spacelike line, we have from Example 5.2

$$\rho_{p, \text{grad}_p T}^{I^+(l),max} = \frac{T(p)}{2} ,$$

because the parameters δ, ξ in Example 5.2 vanish. This implies together with Corollary 5.5 that

$$\rho_{p,v}^{D,max} \leq 2 \cosh d_{\mathbb{H}}(v, \text{grad}_p(T)) \inf_{\substack{l \text{ spacelike line} \\ D \subset I^+(l)}} \rho_{p,\text{grad}_p T}^{I^+(l),max}.$$

By definition of the domain D there exists a point q in the tree with $T(p) = |p - q|$ and two lightlike support planes that contain q . Let \tilde{l} be the line obtained by intersecting these support planes. Then the cosmological time $\tilde{T}(p)$ of p with respect to $I^+(\tilde{l})$ and its gradient $\text{grad}_p \tilde{T}$ at p coincides with the cosmological time $T(p)$ with respect to D and its gradient $\text{grad}_p T$. This implies

$$\inf_{\substack{l \text{ spacelike line} \\ D \subset I^+(l)}} \rho_{p,\text{grad}_p T}^{I^+(l),max} = \rho_{p,\text{grad}_p T}^{I^+(\tilde{l}),max} = \frac{T(p)}{2}$$

and proves the claim. \square

5.2. Reconstructing the holonomy from the frequency function. From the frequency function on the circular segments, the observer can reconstruct the relevant data (position of the edges and vertices, his geodesic distance from the edge segments and vertices), but only for *the pieces of the singular tree he sees*. If the observer is very close to the singularity, he will only see a single edge of the tree and the picture will look like the one for a line. With time, he moves away from the tree and more and more intervals corresponding to the edges and vertices of the tree will appear. In the limit where his eigentime and his cosmological time go to infinity, he will see the image of the whole tree.

This implies that the observer can reconstruct the domain (up to a global Poincaré transformation) from his measurements if he waits infinitely long. From his observations, he can reconstruct the edges of the singular tree, and — if the spacetime is obtained by Lorentzian grafting on a closed hyperbolic surface — the action of the fundamental group on the tree. This amounts to recovering the underlying measured geodesic lamination.

If the spacetime corresponds to a grafted genus g surface *and* the observer knows the associated Fuchsian group (i. e. the linear part of the holonomy), he can construct the *complete domain* in finite eigentime. We will consider below to what extent the observer can reconstruct the geometry and topology of the spacetime without knowing the linear part of the holonomy.

We now concentrate on the case, where M is a maximal flat globally hyperbolic space-time with closed Cauchy surface S of genus $g \geq 2$. The universal covering of M is then isometric to a regular domain D . More precisely, there is a subgroup G of $Isom(\mathbb{R}^{2,1})$ such that D is invariant under the action of G and $M = D/G$. Let Γ be the subgroup of $SO^+(2,1)$ consisting of the $SO^+(2,1)$ components of elements of G . It is known (see [19]) that Γ is a discrete subgroup of $SO^+(2,1)$ and that \mathbb{H}^2/Γ is a surface diffeomorphic to S . Moreover the measured geodesic lamination $\tilde{\lambda}$ dual to the initial singularity of D is invariant under the action of Γ and induces a measured geodesic lamination λ on \mathbb{H}^2/Γ .

The main result we present in this section (Proposition 5.13) states that if M is not a conformally static space-time (which would correspond to the empty lamination) and if its initial singularity is on a simplicial tree, then an observer can construct in finite time a finite set of elements of $SO^+(2,1)$ which generates a finite extension of Γ . In other words, we will prove the result only when the lamination λ is rational (that is, its support is a disjoint union of closed curves). We believe that the result could hold also for a general lamination. However, in that case some technical issues arise which make the analysis more complex, and we prefer to focus on the simpler case where λ is rational.

The central idea is to consider the isotropy group Γ_0 of $\tilde{\lambda}$

$$\Gamma_0 = \{\gamma \in SO^+(2,1) \mid \gamma(\tilde{\lambda}) = \tilde{\lambda}\}.$$

It is clear that Γ_0 is a discrete subgroup of $SO^+(2,1)$ containing Γ . The quotient \mathbb{H}^2/Γ_0 is a surface, possibly with singular points, which arise from points in \mathbb{H}^2 that are fixed by an element of Γ_0 . There is a natural projection map

$$\pi : \mathbb{H}^2/\Gamma \rightarrow \mathbb{H}^2/\Gamma_0$$

which is a finite covering. In particular, the index of Γ in Γ_0 is equal to the cardinality of the fibers of π and, consequently, is finite.

Remark 5.7. Any element of Γ_0 is the linear part of an affine transformation that preserves the regular domain D . So elements of Γ_0 are the linear parts of the elements in the isotropy group G_0 of D

$$G_0 = \{g \in Isom(\mathbb{R}^{2,1}) \mid g(D) = D\}.$$

It should be noted that in principle there are many subgroups G' of G_0 (of finite index) such that D/G' is a MGH spacetime with compact Cauchy surface. Clearly the frequency function measured by an observer in M is equal to the frequency function of some observer in such spacetimes. This suggests that an observer cannot precisely determine the group G (or Γ), but only the group Γ_0 . It should also be noted that in the generic case, $\Gamma = \Gamma_0$ and it does not contain proper cocompact subgroups.

We say that a leaf l of $\tilde{\lambda}$ is *seen* by an observer (p, v) if the intersection of D with the support plane orthogonal to some point on l intersects $I^-(p)$. Note that if $x, y \in l$, then the intersection of D with the support plane orthogonal to x is equal to the intersection of D with the support plane orthogonal to y .

In the following proposition (and therefore in the final result of this section) we restrict attention to a lamination $\tilde{\lambda}$ with a simplicial dual tree, although it appears quite likely that the proposition holds for general laminations.

Proposition 5.8. *Suppose that the dual tree of the lamination $\tilde{\lambda}$ is simplicial. Then the frequency function of an observer (p, v) allows one to reconstruct the sublamination $\tilde{\lambda}_{(p,v)}$ consisting of the leaves of $\tilde{\lambda}$ seen by (p, v)*

Proof. As mentioned above, the frequency function seen by the observer can be split into frequency functions of different regions which correspond, respectively, to the edges and to the vertices of the singular tree. In the regions corresponding to the edges, the frequency function is analytic and behaves as in Example 2 in Section 3.2. It is shown there that knowing the frequency function on an open subset of S^1 is sufficient to determine the positions of the edges, and therefore the leaves of the lamination $\tilde{\lambda}_{(p,v)}$. \square

Let us fix a point $x_0 \in \mathbb{H}^2$, and denote by B_d the ball in \mathbb{H}^2 centered at x_0 with radius d . For simplicity, suppose that the point x_0 does not lie in a leaf of $\tilde{\lambda}$. We denote by $\tilde{\lambda}_d$ the sublamination of $\tilde{\lambda}$ made of leaves that intersects B_d :

$$\tilde{\lambda}_d = \bigcup_{\substack{l \text{ leaf of } \tilde{\lambda} \\ l \cap B_d \neq \emptyset}} l.$$

Lemma 5.9. *For any $d > 0$ there is a time T such that for $t \geq T$ the observer $(p + tv, v)$ sees all the leaves in $\tilde{\lambda}_d$, or, equivalently, $\tilde{\lambda}_d \subset \tilde{\lambda}_{p+tv,v}$*

Proof. There is a compact subset K of ∂D such that if $x \in B_d$ then the support plane orthogonal to x intersects ∂D in K . Since $I^-(p + tv) \cap D$ is an increasing sequence of open subsets that cover D , there is a constant T such that $I^-(p + tv)$ contains K for $t \geq T$. By definition, we then have $\tilde{\lambda}_{p+tv} \subset \tilde{\lambda}_d$, and the conclusion follows. \square

We now consider the elements of $SO^+(2, 1)$ that send leaves of $\tilde{\lambda}_d$ either out of B_d or to other leaves of $\tilde{\lambda}_d$:

$$\Gamma_d = \{\gamma \in SO^+(2, 1) \mid \gamma(\tilde{\lambda}_d) \cap B_d \subset \tilde{\lambda}_d\}.$$

It is easy to check that $\Gamma_0 = \bigcap_{d>0} \Gamma_d$. Note that Γ_d is not discrete. Indeed, transformations γ such that $\gamma(\tilde{\lambda}_d) \cap B_p = \emptyset$ form an open subset of $SO^+(2, 1)$ that is contained in Γ_d . On the other hand, we will prove that the intersection of a neighborhood of the identity with Γ_d is discrete and that this neighborhood can be chosen arbitrarily large, by choosing d sufficiently large.

Lemma 5.10. *For any compact neighborhood H of the identity in $SO^+(2, 1)$, there is a constant d such that $\Gamma_d \cap H$ is finite.*

Sublemma 5.11. *For any $a > 0$ and $d > 0$ there is a finite number of strata F of $H^2 \setminus \tilde{\lambda}$ such that $F \cap B_d$ contains a point at distance exactly a from ∂F .*

Proof. By contradiction, suppose there are countable many strata F_n as in the Lemma, and denote by $x_n \in F_n \cap B_d$ the points such that $d(x_n, \partial F_n) = a$.

Up to passing to a subsequence, we can suppose that $x_n \rightarrow x$. If x does not lie in the lamination, then x_n definitively lies in the stratum F through x , so $F_n = F$, and this contradicts the assumption on F_n .

If x lies on $\tilde{\lambda}$, then $d(x_n, \partial F_n) = d(x_n, \tilde{\lambda}) \rightarrow 0$, which contradicts the assumption that this distance is a constant larger than 0. \square

Proof of Lemma 5.10. Let d_0 be a fixed number such that B_{d_0} intersects two leaves l_1 and l_2 on the boundary of the stratum F_0 through x_0 .

By the compactness of H , there is a constant $r > 0$ such that $d_{\mathbb{H}^2}(x, \gamma(x)) < r$ for any $x \in B_{d_0}$ and $\gamma \in H$. Note that $\gamma(l_i)$ intersects B_d with $d = d_0 + r$.

If $\gamma \in H \cap \Gamma_d$ with $d > d_0 + r$, then γ sends l_i to some leaves c_1 and c_2 of $\tilde{\lambda}_d$. Clearly, $\gamma(x_0)$ lies in a stratum bounded by c_1 and c_2 , and the distance between $\gamma(x_0)$ and c_1 is the same as the distance between x_0 and l_1 (say $a > 0$). On the other hand, by Sublemma 5.11, there are finitely many strata F_1, \dots, F_N of $\tilde{\lambda}_d$ such that $F_i \cap B_d$ contains a point at distance a from ∂F_i . Moreover, the boundary of each F_i intersects B_d into a finite number of segments.

In particular, there are a finite number of leaves t_1, \dots, t_M such that every γ in $H \cap \Gamma_d$ sends l_i to one of the leaves t_i . However, for two pairs of geodesics (l_1, l_2) and (t_1, t_2) in \mathbb{H}^2 , there is at most one isometry sending l_i to t_i . This implies that $H \cap \Gamma_d$ contains at most 2^M elements. \square

Let us now fix an observer (p, v) . It then follows from Lemma 5.9 and Lemma 5.10 that for any compact subset $H \subset SO^+(2, 1)$ and d sufficiently large, there is a time $T = T(H, d)$ such that the observer at proper time $t > T$ can list the elements of $\Gamma_d \cap H$. On the other hand, in principle, an observer would have to wait an infinite amount of time to determine if a given element in Γ_{d_0} lies also in Γ_0 . Indeed, this amounts to determining whether such an element lies also in all Γ_d for $d > d_0$. The following lemma ensures that this is not the case and that the observer can be sure after a finite amount of time that elements of $\Gamma_d \cap H$ also lie in Γ_0 .

Lemma 5.12. *For any compact subset $H \subset SO^+(2, 1)$ there is a constant d such that $\gamma \in H \cap \Gamma_d$ implies $\gamma \in \Gamma_0$.*

Proof. By contradiction, suppose that there is a diverging sequence d_n and a sequence $\gamma_n \in H$ such that $\gamma_n \in \Gamma_{d_n} \cap H$, but $\gamma_n \notin \Gamma_0$.

Up to passing to a subsequence, we may suppose that γ_n converges to γ_∞ . On the other hand, by Lemma 5.10, we can choose n_0 big enough so that $\Gamma_{d_{n_0}} \cap H$ is a finite set. Now $\gamma_n \in \Gamma_{d_{n_0}} \cap H$ for $n \geq n_0$, and since it is a convergent sequence we have that $\gamma_n = \gamma_\infty$ for $n \geq n_1$. This implies that $\gamma_\infty \in \Gamma_0$ and contradicts the assumption on the sequence. \square

We can now state and prove the main result of this section.

Proposition 5.13. *Let (p, v) be an observer in a domain of dependence D which is the universal cover of a Minkowski spacetime obtained by Lorentzian grafting of a closed hyperbolic surface along a rational measured lamination. Then the observer can construct a finite set of generators of Γ_0 in finite time.*

Proof. As Γ_0 is finitely generated, there is a compact subset $H \subset SO^+(2, 1)$ such that $H \cap \Gamma_0$ is a set of generators of Γ_0 . By Proposition 5.8, Lemma 5.10 and Lemma 5.12, the observer can detect elements of $H \cap \Gamma_0$ in a finite time. \square

5.3. Higher dimensions. In sections 5.1 and 5.2 we focussed on flat spacetimes of dimension $2 + 1$. It appears possible to give an analogous analysis in dimension $3 + 1$ or higher. However, proving the results is more involved, since the structure of MGHFC spacetimes is less well understood and their description is more complicated than in dimension $2 + 1$.

In dimension $3 + 1$, MGHFC spacetimes can still be constructed from a hyperbolic metric on a 3-manifolds along with a “geodesic foliation”. Those foliations, however, are different from those occurring on hyperbolic surfaces, since they have two-dimensional leaves which can possibly meet along one-dimensional strata.

Still, it appears plausible that the same conclusions can be reached as in dimension $2 + 1$ for observers in a domain of dependence which is the universal cover of a MGHFC spacetime in dimension $3 + 1$:

- If the linear part of the holonomy is known, the observer can reconstruct the complete holonomy in finite time.
- The observer can determine in finite time the part of the initial singularity corresponding to the part of space he “sees”, which is increasing with time.

6. EXAMPLES IN $2+1$ DIMENSIONS

In this section, we determine explicitly the frequency function measured by an observer in a few simple examples of $2+1$ -dimensional domains of dependence.

6.1. Explicit holonomies. We consider below three examples, one based on a reflection in the edges of a hyperbolic quadrilateral and two based on a hyperbolic punctured torus. In the first example involving the punctured torus, the translation part of the holonomy corresponds to a measured lamination with support on a closed curve. In the second, the support of the measured lamination is more complicated.

6.1.1. *Example 1: A hyperbolic reflection group.* In the first example, we consider a group $\Gamma_{\pi/3}$ which is generated by the reflections on the edges of a quadrilateral with angles $\pi/3$. (This angle condition completely determines a presentation of the group, see e.g. [10].) The holonomy representation $\rho_t : \Gamma_{\pi/3} \rightarrow \text{Isom}(\mathbb{R}^{1,2})$ depends on a real parameter t .

We describe first the linear part $\rho_t^l : \Gamma_{\pi/3} \rightarrow O(1,2)$. The construction is based on a quadrilateral p with vertices v_1, \dots, v_4 . Consider the hyperbolic plane as a quadric in the Minkowski space $\mathbb{R}^{2,1}$, and let w_1, w_2, w_3, w_4 be the unit spacelike vectors which are orthogonal to the oriented plane through 0 containing the geodesic segments (v_1, v_2) , (v_2, v_3) , (v_3, v_4) and (v_4, v_1) . The cosine of the exterior angle of p at v_i is then equal to the scalar product between w_{i-1} and w_i , so that p has interior angles equal to $\pi/3$ if and only if $\langle w_i, w_{i+1} \rangle = -1/2$ for all $i \in \mathbb{Z}/4\mathbb{Z}$.

The fact that those scalar products are equal means that (w_1, w_2, w_3, w_4) is a rhombus in the de Sitter plane. In particular it is invariant under the symmetry with respect to a timelike line in $\mathbb{R}^{2,1}$, corresponding to a point $o \in H^2$ which is the midpoint of both, (v_1, v_3) and (v_2, v_4) .

Choosing a coordinate system compatible with this symmetry, we can write the w_i as

$$w_1 = (\sinh(t), \cosh(t), 0), w_2 = (\sinh(t'), 0, \cosh(t')), w_3 = (\sinh(t), -\cosh(t), 0), w_4 = (\sinh(t'), 0, -\cosh(t'))$$

with t, t' satisfying the condition $\sinh(t)\sinh(t') = 1/2$. This yields a representation $\rho_t : \Gamma \rightarrow O(2,1)$ which can be described as follows. Γ is generated by the elements a_1, \dots, a_4 corresponding to the reflections in the edges of p , with the relations

$$a_i^2 = 1, \quad (a_i a_{i+1})^3 = 1$$

for all $i \in \mathbb{Z}/4\mathbb{Z}$. The representation ρ sends a_i to the reflection in (v_i, v_{i+1}) , that is

$$\rho(a_i)(x) = x - 2\langle x, w_i \rangle w_i.$$

The quotient of \mathbb{H}^2 by $\rho(\Gamma)$ is an orbifold. The subgroup Γ_2 of Γ of elements $\gamma \in \Gamma$ for which $\rho(\gamma)$ is orientation-preserving has index two, and the quotient of \mathbb{H}^2 by $\rho(\Gamma_2)$ is a surface.

There is a unique choice of a deformation cocycle associated to ρ_t , which is obtained by varying t . It can be written as $\tau_t = \rho_t^{-1} d\rho_t/dt$. So we obtain a one-parameter family of domains of dependence parametrized by t . In the following, we mainly consider the simplest case, where $t = t_0 = \sinh^{-1}(1/\sqrt{2})$, so that $t' = t_0$.

6.1.2. *Example 2: A punctured torus with a rational measured lamination.* Another simple example can be constructed, by choosing as the linear part of the holonomy the holonomy representation of a hyperbolic punctured torus.

Then the group Γ is the free group generated by two elements a, b . We consider the situation with an extra symmetry, corresponding to the condition that the images of a, b by the linear part ρ of the holonomy are hyperbolic translations with orthogonal axes. The translation lengths of $\rho(a)$ and $\rho(b)$ can then be written as $2t_a, 2t_b$, subject to the conditions that $\sinh(t_a)\sinh(t_b) = 1$. This corresponds to the condition that the image of the commutator of a, b is parabolic.

In the computations below, we choose, somewhat arbitrarily, the parameters $t_a = \sinh^{-1}(2), t_b = \sinh^{-1}(1/2)$. We also choose the translation component of the holonomy as the cocycle τ corresponding, through the relation explained in Section 2.5, to a closed curve corresponding to b , with weight 1.

The domain of dependence obtained in this way is shown on the left in Figure 1. To compute this image as well as frequency function and “distances” to the boundary in other domains of dependence – see figures 6 and 7 below — we use the description of a domain of dependence as an intersection of half-spaces bounded by lightlike planes from Section 2.7. The image is computed by taking a ball of radius 6 in Γ , for the distance defined by the choice of generators described above, computing for each element of Γ in this ball the corresponding axis and lightlike hyperplanes, and then determining their intersection.

6.1.3. *Example 3: A punctured torus with an irrational lamination.* In this example, the linear part of the holonomy is the same as in the previous one. However, the translation part of the holonomy is given by a cocycle corresponding to a measured lamination which does not have its support on a closed curve. The corresponding domain of dependence is given on the right in Figure 1.

6.2. **Results.** Figure 5 presents (in green) the frequency function measured by all directions by an observer in the $2+1$ -dimensional flat spacetime described in Section 6.1.1. More precisely, the observer is located above the origin at time distances 10, 30 and 50 to achieve the most “readable” results. The Euclidean length of the lightlike segments from the observer to the boundary of the domain are drawn in blue. The frequency function becomes more complex as the cosmological time of the observer increases.

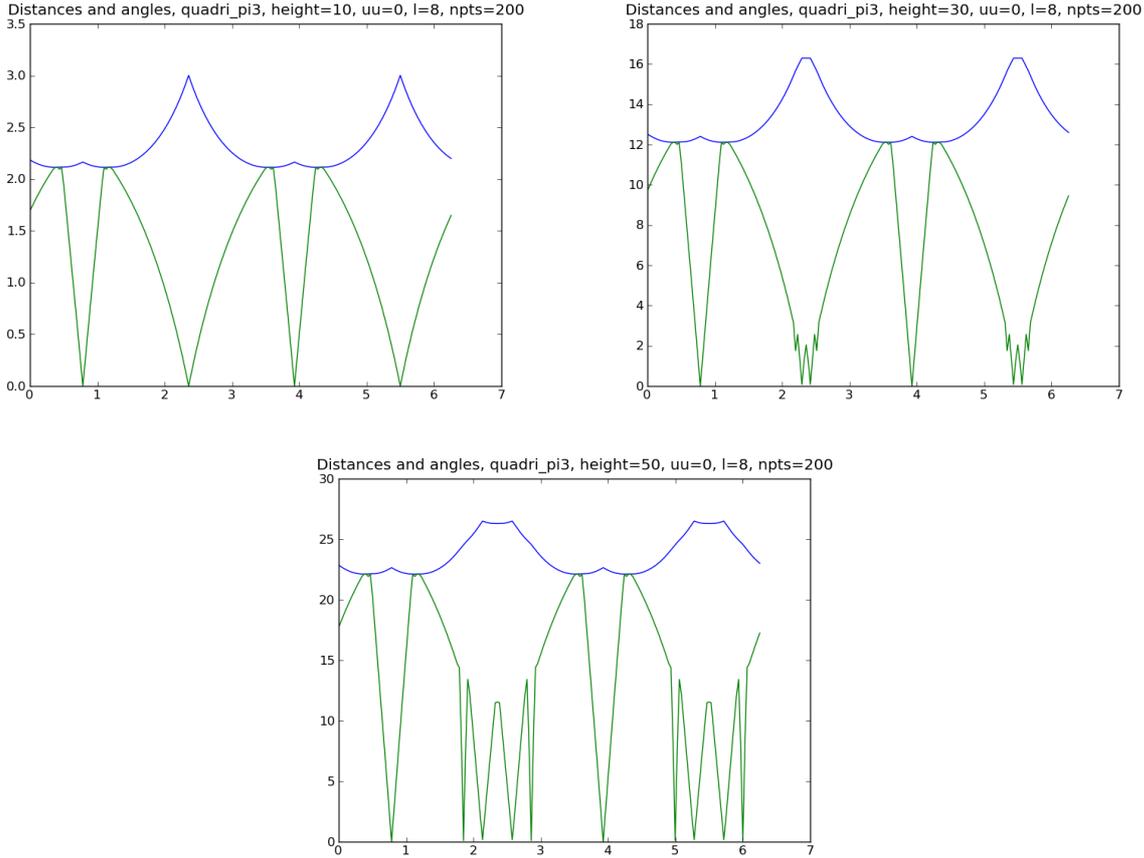


FIGURE 5. The frequency function (green) and Euclidean distance to the boundary (blue) in a domain of dependence based on a quadrilateral with angles $\pi/3$ with a rational lamination, seen from increasing distance from the initial singularity.

Figure 6 is the analogue of Figure 5 for the 2 + 1-dimensional flat spacetime described in Section 6.1.2. More precisely, the observer is located above the origin at time distances 1, 5, 10 and 30. Figure 7 shows the analogous results, for the domain of dependence obtained from an irrational lamination, described in Section 6.1.3.

The computation of both the Euclidean distance and the frequency function are made for an approximation of the domain of dependence, as explained above. It follows from Section 4 that the frequency function computed in this way is not reliable as a continuous function, but only — possibly at least — as a L^1 function, due to Theorem 4.13.

Those graphs should be considered as preliminary results, since, even for this relatively simple setting, the computations needed to obtain the results are quite involved relative to our programming capabilities and computing equipment. It is possible that heavier computations — in particular, computing a better approximation of the domain of dependence by using a larger subset of the fundamental group — could lead to notably different results. However, it is already apparent in those pictures, that the frequency function behaves in a very non-smooth way, as explained in Section 3 and Section 4.

7. EXAMPLES IN 3+1 DIMENSIONS

In this section, we consider an example of a domain of dependence in 3+1 dimensions and show that the light emitted from the initial singularity and received by an observer contains rich information on its geometry and topology. In the first part, we describe the domain of dependence in 3+1 dimensions, based on a construction of Apanasov [2]. The second part contains some images of the light emitted by the initial singularity, as seen by an observer.

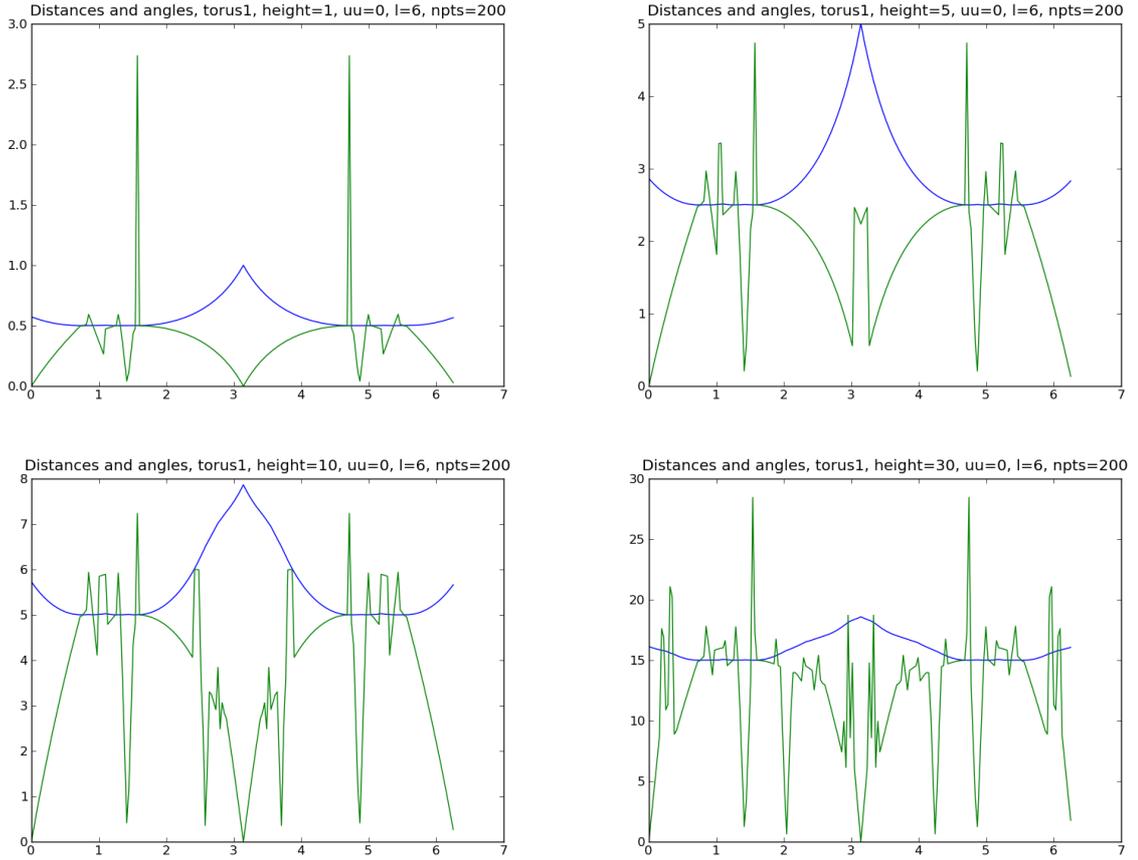


FIGURE 6. The frequency function (green) and Euclidean distance to the boundary (blue) in a domain of dependence based on a punctured torus with a rational lamination, seen from increasing distance from the initial singularity.

7.1. An explicit example. We consider a particularly interesting example of domain of dependence, which is used in computations below. The remarkable property of this example is that, for only one linear part of the holonomy, there is a four-dimensional space of possible translation components.

7.1.1. The construction of the group. The following example is essentially due to Apanasov [2]. It is a discrete group Γ of $Isom(\mathbb{H}^3)$ generated by 8 reflections, so that the quotient \mathbb{H}^3/Γ is a (non-orientable) orbifold of finite volume.

On $S_\infty^2 = \mathbb{C} \cup \{\infty\}$ we consider the following circles

- (1) C_k with center at $z_k = \sqrt{3}e^{i\frac{k\pi}{3}}$ and radius 1.
- (2) C with center at 0 and radius 1.
- (3) C' with center at 0 and radius 2.

It can be shown easily that the configuration of such circles is the one shown in the picture. Moreover the angle formed by any two circles in the list (that meet each other) is $\pi/3$.

We consider the planes P_k, P, P' in \mathbb{H}^3 that bound at infinity the circles C_k, C and C' . We denote by γ_k the reflection along P_k , by γ the reflection along P and by γ' the reflection along P' .

Proposition 7.1. *The group Γ generated by $\gamma_k, \gamma, \gamma'$ is a discrete group in $Isom(\mathbb{H}^3)$ and the quotient \mathbb{H}^3/Γ is a non-orientable orbifold of finite volume.*

A fundamental region for the action of Γ can be obtained as follows. Denote for any k by E_k the exterior of the plane P_k , e. g. the region of $\mathbb{H}^3 \setminus P$ that contains ∞ . Analogously, denote by E the exterior of P and by I'

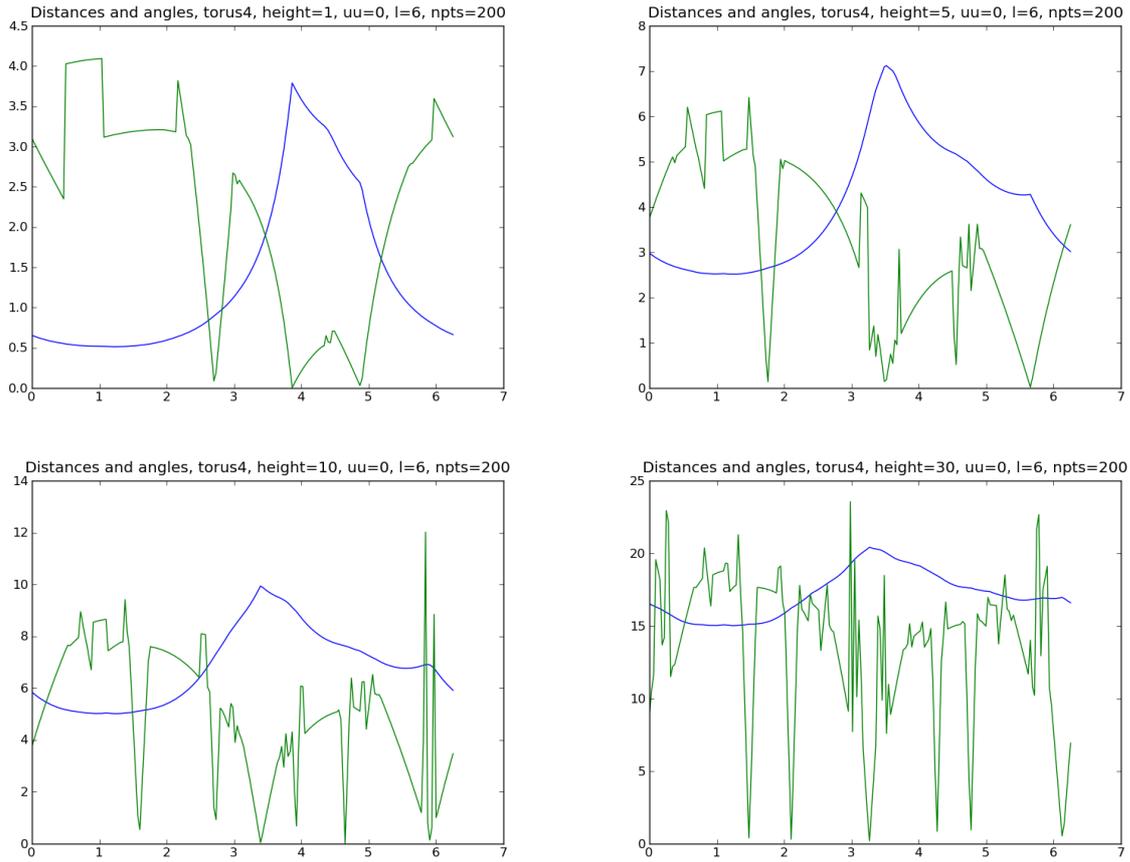


FIGURE 7. The frequency function (green) and Euclidean distance to the boundary (blue) in a domain of dependence based on a punctured torus with an irrational lamination, seen from increasing distance from the initial singularity.

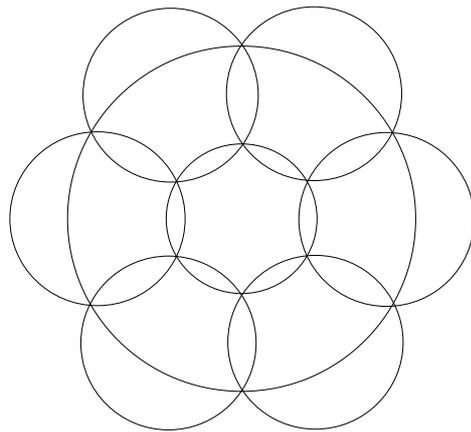


FIGURE 8. Construction of the linear part of the holonomy

the interior region bounded by P' . Then a fundamental region for Γ is given by

$$K = \bigcap_{k=0}^5 E_k \cap E \cap I'.$$

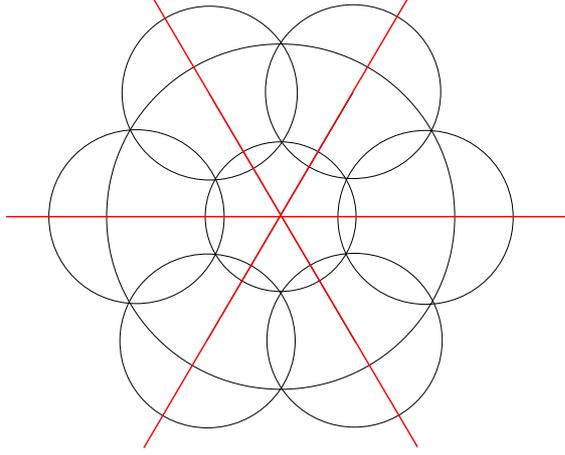


FIGURE 9. Construction of the translation cocycle

As we are interested in $\mathbb{R}^{3,1}$ -valued cocycles, we need to determine explicitly the matrices in $O(3,1)^+$ corresponding to the transformations $\gamma_k, \gamma, \gamma'$.

In $\mathbb{R}^{3,1}$ we consider coordinates x_0, x_1, x_2, x_3 so that the Minkowski metric takes the form $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. Given two real numbers v_0, v_1 and a complex number $z = x + iy$, we denote by (v_0, v_1, z) the point (v_0, v_1, x, y) in $\mathbb{R}^{3,1}$.

We have to fix explicitly an isometry between the half-space model of \mathbb{H}^3 (denoted by Π here) and the hyperboloid model denoted by \mathbb{H}^3 . Such an isometry $\phi : \Pi \rightarrow \mathbb{H}^3$ is given by

$$\phi(i) = (1, 0, 0, 0) \quad \phi_{*,i}(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}) = (0, c, a, b).$$

With this choice, the plane P_k is given by $P_k \mathbb{H}^3 \cap v_k^\perp$, where

$$v_k = (3/2, 1/2, \sqrt{3}e^{i\frac{k\pi}{3}})$$

is a unit vector. Analogously, we obtain $P = \mathbb{H}^3 \cap v^\perp$ and $P' = \mathbb{H}^3 \cap (v')^\perp$ where

$$v = (0, 1, 0, 0) \quad v' = (3/4, 5/4, 0, 0).$$

The associated transformations in $O^+(3,1)$ then take the form

$$\gamma_k(x) = x - 2\langle x, v_k \rangle v_k, \quad \gamma(x) = x - 2\langle x, v \rangle v \quad \gamma'(x) = x - 2\langle x, v' \rangle v'.$$

7.1.2. The construction of the cocycle. In $S_\infty^2 = \mathbb{C} \cup \{\infty\}$ we consider the three lines through 0 passing through the centers of the circles C_k and denote by W be the union of the planes in \mathbb{H}^3 bounding these lines. Clearly, W is the union of 6 half-planes which meet along the geodesic l_0 joining 0 to ∞ . We denote these half-planes by $W_0, W_1, W_2, W_3, W_4, W_5$ where the indices correspond to the ones of the circles in the obvious way. Let $w_k \in \mathbb{R}^{3,1}$ be the vector orthogonal to W_k and pointing towards W_{k+1} . A direct computation then shows that w_k is given by

$$w_k = (0, 0, ie^{i\frac{k\pi}{3}}).$$

Note that $w_{k+3} = -w_k$, where the index k is considered mod 6.

Now the Γ -orbit of W is a branched-surface in \mathbb{H}^3 . In particular, the sets $\hat{W} = \Gamma \cdot W$ and $\hat{W}_0 = \Gamma \cdot l_0$ have the following properties:

- \hat{W}_0 is a disjoint union of geodesics.
- Every connected component of $\hat{W} \setminus \hat{W}_0$ is a convex polygon (with infinitely many edges) and every edge is an element of \hat{W}_0 .
- There are exactly six faces up to the action of Γ . Indeed, let F_k be the face of \hat{W} bounding l_0 and contained in W_k . Then the orbits of F_0, \dots, F_5 are disjoint and cover \hat{W} .
- Every connected component of $\mathbb{H}^3 \setminus \hat{W}$ is a convex polyhedron.

These properties can be proved by considering the intersection of W with K and using the fact that W is orthogonal to the faces of K .

We can then use a general construction explained in [7] to obtain non-trivial cocycles. Given six numbers a_i such that

$$(39) \quad \sum_{i=0}^5 a_i w_i = 0,$$

we obtain a cocycle via the following prescription. We associate to the face F_i the number a_i . In this way a number $a(F)$ is associated to every face F by requiring that $a(\alpha(F)) = a(F)$ for every $\alpha \in \Gamma$.

Then we fix a basepoint x_0 in \mathbb{H}^3 that does not lie in \hat{W} . Given a transformation $\alpha \in \Gamma$ we construct a vector in $\mathbb{R}^{3,1}$ in the following way: we take any path c joining x_0 to $\alpha(x_0)$ and avoiding \hat{W}_0 . The path c intersects some faces F^1, \dots, F^n . We consider the unit vector $w^j \in \mathbb{R}^{3,1}$ orthogonal to F^j and pointing towards $\alpha(x_0)$ and set

$$\tau(\alpha) = \sum_{j=1}^n a(F^j) w^j.$$

It can be easily checked that

- $\tau(\alpha)$ does not depend on the path c (this essentially follows from (39)).
- τ is a $\mathbb{R}^{3,1}$ -valued cocycle
- changing the basepoint changes τ by a coboundary.

Let $H \subset \mathbb{R}^6$ be the subspace of solutions of (39), which is of dimension $\dim H = 4$. From [7] we have that the map

$$H \rightarrow H^1(\Gamma, \mathbb{R}^{3,1})$$

is injective. This map can be computed explicitly. More precisely, we fix the base point x_0 in the region of K between W_0 and W_1 .

Given a set of numbers $a_0, a_1, a_2, a_3, a_4, a_5$ that satisfy (39) we compute the corresponding cocycle τ evaluated on the generators. This yields

$$\begin{aligned} \tau(\gamma) &= \tau(\gamma') = \tau(\gamma_0) = \tau(\gamma_1) = 0 \\ \tau(\gamma_2) &= w_1 - \gamma_2 w_1 = 3a_1 v_2 \\ \tau(\gamma_3) &= w_1 - \gamma_3 w_1 + w_2 - \gamma_3 w_2 = 3(a_1 + a_2) v_3 \\ \tau(\gamma_4) &= -w_0 + \gamma_4 w_0 - w_5 + \gamma_4 w_5 = -3(a_0 + a_5) v_4 \\ \tau(\gamma_5) &= -w_0 + \gamma_5 w_0 = -3a_0 v_5. \end{aligned}$$

7.2. Computations. The computations of the frequency functions were limited by the speed of the available computers, and more complete computations could provide better results. As in dimension $2+1$, we constructed a domain of dependence in $\mathbb{R}^{3,1}$, invariant under the group actions described above via the construction in Section 2.7. However, the computations are much heavier in dimension $3+1$, so we only considered the elements of the fundamental group in a ball of radius 4.

Although we only considered one linear part of the holonomy — the one in Section 7.1.1 — we worked with two deformation cycles, one corresponding to weights $(1, 0, 0, 1, 0, 0)$ as described in Section 7.1.2, the other to the weights $(1, 1/2, 0, 1, 1/2, 0)$. In both cases, the observer was located at the point of coordinates $(50, 0, 0, 0)$. This is a somewhat arbitrary choice, made after trying different possibilities, which leads to interesting pictures.

The frequency function measured by the observer for the first choice of cocycle is presented in Figure 10, with different colors encoding different values of the frequency.

It should be noted that those results are less certain than those obtained in dimension $2+1$. This is due to the fact that we compute the limit frequency function for a decreasing sequence of finite domains of dependence approximating the domain under examination. In dimension $2+1$, Theorem 4.13 and Proposition 4.22 indicate that the limit frequency function is the frequency function of the limit, if the limit is the universal cover of a MGHFC spacetime. However, in dimension $3+1$, we only know by Theorem 4.14 that the frequency function of the limit is at least equal to the limit frequency function, and at most equal to three times the limit frequency function. So the frequency functions computed here are the limit frequency (which is a well-defined notion for any domain of dependence, see Theorem 4.14) which differs from the “real” frequency function by a factor at most three.

The limit frequency function computed for the second choice of cocycle is depicted in Figure 11. It is apparent how the less symmetric cocycle leads to a distortion in the picture. The symmetry of degree six, which is present in the linear part of the holonomy, is readily apparent in Figure 10. In Figure 11 it remains visible, but with differences in the size of the corresponding parts of the picture.

Even for this fairly simple example, it would be interesting to perform more powerful and complete computations, for instance by computing the domain of dependence with all elements of the fundamental group in a

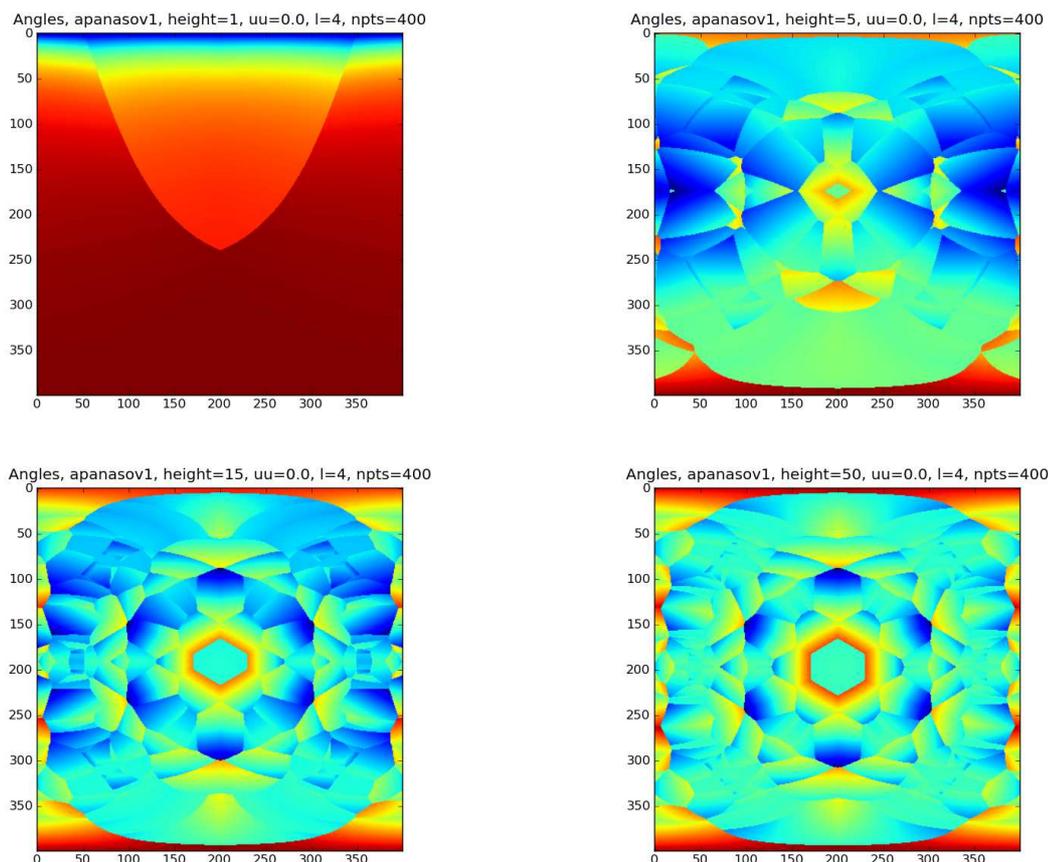


FIGURE 10. Computed frequency function, with translation coefficients $(1, 0, 0, 1, 0, 0)$, for observers at an increasing distance from the initial singularity

ball of radius larger than 4. It is conceivable that one would obtain somewhat different pictures. Additionally, the picture should vary with the position of the observer. It should be simpler for an observer close to the initial singularity, but become increasingly complex as the observer moves away from it.

REFERENCES

1. Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker, *Notes on: “Lorentz spacetimes of constant curvature”* [*Geom. Dedicata* **126** (2007), 3–45; *mr2328921*] by G. Mess, *Geom. Dedicata* **126** (2007), 47–70. MR MR2328922
2. Boris N. Apanasov, *Deformations of conformal structures on hyperbolic manifolds*, *J. Differential Geom.* **35** (1992), no. 1, 1–20. MR 1152224 (92k:57042)
3. Thierry Barbot, *Globally hyperbolic flat space-times*, *J. Geom. Phys.* **53** (2005), no. 2, 123–165. MR 2110829 (2006d:53085)
4. Riccardo Benedetti and Francesco Bonsante, *Canonical Wick rotations in 3-dimensional gravity*, *Memoirs of the American Mathematical Society* **198** (2009), 164pp, [math.DG/0508485](https://arxiv.org/abs/math/0508485).
5. Riccardo Benedetti and Enore Guadagnini, *Geometric cone surfaces and $(2 + 1)$ -gravity coupled to particles*, *Nuclear Phys. B* **588** (2000), no. 1-2, 436–450. MR MR1787158 (2001g:83094)
6. Francis Bonahon, *Geodesic laminations on surfaces*, *Laminations and foliations in dynamics, geometry and topology* (Stony Brook, NY, 1998), *Contemp. Math.*, vol. 269, Amer. Math. Soc., Providence, RI, 2001, pp. 1–37. MR 1810534 (2001m:57023)
7. Francesco Bonsante, *Flat spacetimes with compact hyperbolic Cauchy surfaces*, *J. Differential Geom.* **69** (2005), no. 3, 441–521. MR MR2170277 (2006h:53068)
8. Steven Carlip, *Quantum gravity in 2 + 1 dimensions*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 1998. MR 1637718 (99e:83026)
9. Neil J Cornish, David N Spergel, and Glenn D Starkman, *Measuring the topology of the universe*, *Proceedings of the National Academy of Sciences* **95** (1998), no. 1, 82–84.
10. Michael W. Davis, *The geometry and topology of Coxeter groups*, *London Mathematical Society Monographs Series*, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474 (2008k:20091)

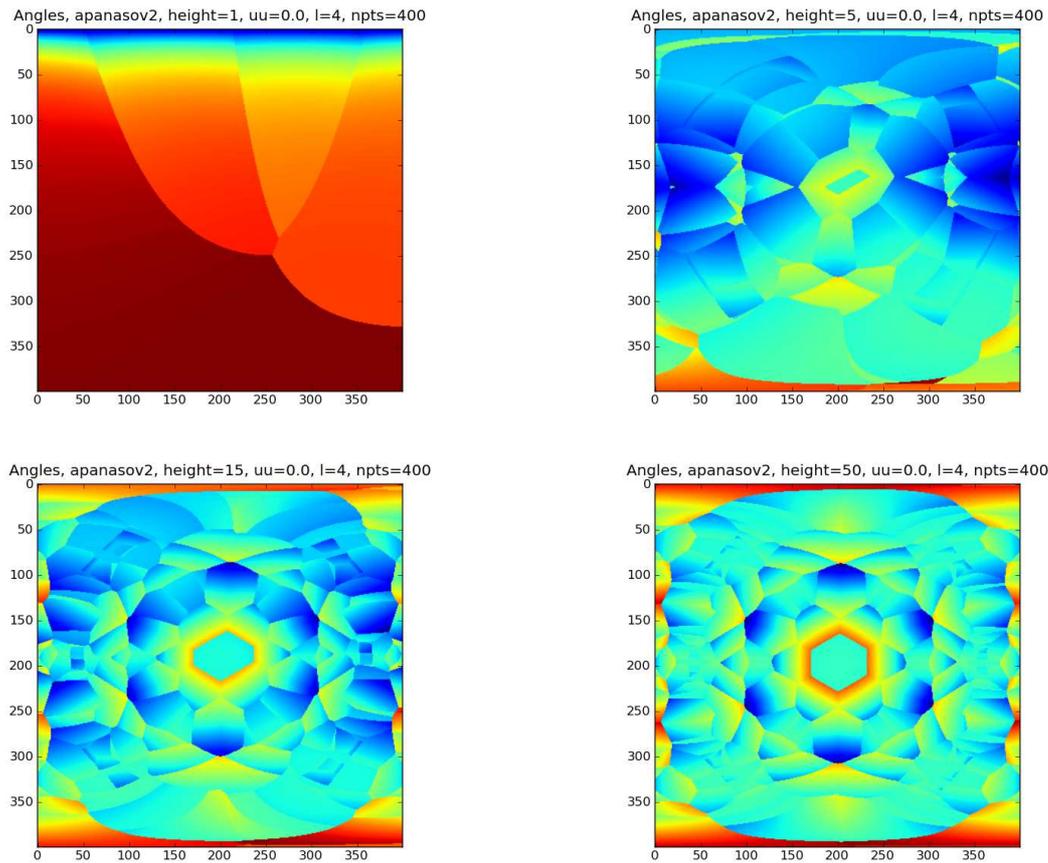


FIGURE 11. Computed frequency function, with translation coefficients $(1, 1/2, 0, 1, 1/2, 0)$, for observers at an increasing distance from the initial singularity

11. D. B. A. Epstein and A. Marden, *Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces*, Analytical and geometric aspects of hyperbolic space (D. B. A. Epstein, ed.), L.M.S. Lecture Note Series, vol. 111, Cambridge University Press, 1986.
12. William M. Goldman and Gregory A. Margulis, *Flat Lorentz 3-manifolds and cocompact Fuchsian groups*, Crystallographic groups and their generalizations (Kortrijk, 1999), Contemp. Math., vol. 262, Amer. Math. Soc., Providence, RI, 2000, pp. 135–145. MR 1796129 (2001m:53124)
13. J. Richard Gott, III, *Topology and the universe*, Classical Quantum Gravity **15** (1998), no. 9, 2719–2731, Topology of the Universe Conference (Cleveland, OH, 1997). MR 1649670 (99g:83084)
14. J. Richard Gott, III, Adrian L. Melott, and Mark Dickinson, *The sponge-like topology of large-scale structure in the universe*, Astrophys. J. **306** (1986), no. 2, part 1, 341–357. MR 861880 (87k:85005)
15. Dennis Johnson and John J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., vol. 67, Birkhäuser Boston, Boston, MA, 1987, pp. 48–106. MR 900823 (88j:22010)
16. Michael Kapovich, *Deformations of representations of discrete subgroups of $SO(3, 1)$* , Math. Ann. **299** (1994), no. 2, 341–354. MR 1275772 (95d:57010)
17. Christos Kourouniotis, *Deformations of hyperbolic structures*, Math. Proc. Cambridge Philos. Soc. **98** (1985), no. 2, 247–261. MR 795891 (87g:32022)
18. J.P. Luminet, J.R. Weeks, A. Riazuelo, R. Lehoucq, and J.P. Uzan, *Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background*, Nature **425** (2003), no. 6958, 593–595.
19. Geoffrey Mess, *Lorentz spacetimes of constant curvature*, Geom. Dedicata **126** (2007), 3–45. MR MR2328921
20. C. Meusburger, *Cosmological measurements, time and observables in $(2 + 1)$ -dimensional gravity*, Classical Quantum Gravity **26** (2009), no. 5, 055006, 32. MR 2486312 (2010f:83078)
21. Giuseppe Savaré and Franco Tomarelli, *Superposition and chain rule for bounded hessian functions*, Adv. Math. **140** (1998), 237–281.
22. Kevin P. Scannell, *Infinitesimal deformations of some $SO(3, 1)$ lattices*, Pacific J. Math. **194** (2000), no. 2, 455–464. MR 1760793 (2001c:57018)

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