A Stochastic Analysis of Table Tennis

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Abstract

We establish a general formula for the distribution of the score in table tennis. We use this formula to derive the probability distribution (and hence the expectation and variance) of the number of rallies necessary to achieve any given score. We use these findings to investigate the dependence of these quantities on the different parameters involved (number of points needed to win a set, number of consecutive serves, etc.), with particular focus on the rule change imposed in 2001 by the International Table Tennis Federation (ITTF). Finally we briefly indicate how our results can lead to more efficient estimation techniques of individual players' abilities.

1   Introduction.

We consider the following situation. Two players (or teams), referred to as A and B, play a sequence of *rallies* after each of which either A or B is declared winner. Every rally is initiated by a *server*—the other player is then called the *receiver*—and a point is scored (by the winner) after each rally. The players continue competing until a *match*-winner is declared, with a match being composed of several *sets* according to the (m, n, G)-scoring system, which we define as follows.

The (m, n, G)-scoring system: (i) A *set* consists of a sequence of independent rallies; the winner of a set is the first player to score n points or, in case of a *tie* at $n-1$, the first player to create a difference of two points after the tie. (ii) A *match* consists of a sequence of independent sets; the winner of a match is the first player to win G sets. (iii) The server in the first rally of the first set is determined by flipping a coin; if $G \geq 2$, the right to serve in the first rally of each subsequent set alternates between the two opponents. (iv) Within a set, the right to serve changes between the opponents after each sequence of m consecutive rallies by a server until either a set-winner is declared or a tie is reached at $(n-1, n-1)$. (v) After a tie at $(n-1, n-1)$, the right to serve alternates after each rally.

To the best of our knowledge, table tennis is the only sport currently using the (m, n, G)-scoring system, although speed badminton (a.k.a. speedminton or, before 2001, shuttleball) can nevertheless be shown to fit, up to a minor modification, within the above scoring system for $m = 3$, $n = 16$ and $G = 3$ (the change concerns the right to serve in the first rally of each subsequent set, as here the player who has lost the previous set serves first). Until 2001, table tennis was played according to the $(5, 21, G)$-scoring system, with $G = 2$ or 3. In 2001 the International Table Tennis Federation (ITTF) decided to switch to the $(2, 11, G)$-scoring system, with G either 3 or 4.

In this article we regard table tennis as a succession of identical and independent random experiments (the rallies) and analyze the properties of the random processes (score change, game duration, etc.) induced by the scoring system. As in most of the (mathematical) literature on this kind of sport we will restrict our attention to two well-known models. The first is the so-called *server model* in which it is assumed that rally-outcomes are mutually independent and are, conditionally upon the server, identically distributed. Denoting by p_a (resp., p_b) the probability that player A (resp., player B) wins a rally he/she initiates, the game is then entirely governed by the bivariate parameter $(p_a, p_b) \in (0, 1) \times (0, 1)$. The second model under investigation is the simpler *no-server model*, in which it is assumed that rallies are won with probabilities that do not depend on the server; the latter model is thus

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a particular case of the former, with parameters $p_a = 1 - p_b = p \in (0,1)$. For a discussion on the validity of these models we refer the reader to e.g. Klaassen and Magnus (2001) where is is shown, within the framework of the game of tennis, that i.i.d. models provide a good approximation to real games. We have not investigated whether their conclusions can be transposed to our framework, and reserve such empirical considerations for later publications.

While an important literature has been devoted to sports such as tennis (see, e.g., Hsi and Burich 1971, George 1973, Carter and Crews 1974, Riddle 1988 or Klaassen and Magnus 2003) or badminton (see, e.g., Phillips 1978, Simmons 1989 or the recent contributions by Percy 2009 and Paindaveine and Swan 2011), the corresponding literature on table tennis is, to say the least, scant. Indeed, to the best of our knowledge, only one paper (Schulman and Hamdan 1977) addresses this issue and proposes a solution only for $(m, n, G) = (5, 21, 2)$. This lack of interest perhaps originates in the apparent simplicity of the stochastics underlying the game of table tennis. Indeed, whereas in sports such as badminton or tennis the number of consecutive serves by any player is random, the right to serve in table tennis changes according to a deterministic rule, and thus rather simple (albeit delicate) combinatoric arguments allow to obtain all the corresponding probability distributions – see Section 2. Moreover the formulae we obtain are – see Appendix A.1 – rather lengthy and cumbersome.

This dismissive view of the problem is, in our opinion, wrong. Consequently, we tackle in this paper all possible questions of interest related to sports based on the (m, n, G)-scoring system, and solve them in full generality. The formulae we obtain are based on rally-level combinatorial arguments and allow us to derive the probability distribution (and hence the expectation and variance) of the number of rallies necessary to achieve *any* given succession of rally-outcomes. By doing so, we extend the previous contribution on the subject (Schulman and Hamdan 1977) where only the special case of the $(5, 21, 2)$-scoring system and only set-winning probabilities are computed.

The outline of the paper is as follows. In Section 2 we fix the notations, obtain the main theoretical results and derive the distributions of the score and game-duration. In Section 3

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we investigate the dependence of the quantities obtained in Section 2 on the parameters (m, n, G), with a particular attention given to the effects of the ITTF's rule change (see above) on the game-winning probabilities and durations. Some final comments and an outlook on future research are stated in Section 4, while an appendix collects the full-length formulae and their proofs.

2 Distribution of the score, set-winning probabilities and distribution of the number of rallies in a single set.

In this section we fix the notations and obtain the fundamental probabilities associated with the scoring process within a single set (see Lemma 2.2 below). We use these to derive the distribution of the scores, the set-winning probabilities as well as the distribution of the number of rallies needed to complete a set.

2.1 Notations.

Here and throughout we denote scores by couples of integers where the first entry (resp., the second entry) stands for the number of points scored by player A (resp., by player B). We will reserve the use of the notation (n, k) (resp., (k, n)) to indicate the final score in a set won by A (resp., by B) without a tie. General intermediate scores will be denoted (α, β).

We call A-set (resp., B-set) a set in which player A (resp., player B) is the first server. Note the symmetric roles played by A and B, which will often allow us to state our results in terms of A-sets only. For $C_1, C_2 \in \{A, B\}$, we denote by $p_{C_1}^{C_2}$ the probability that C_2 wins a C_1-set. Now, similarly to Schulman and Hamdan (1977), it can be shown that, for m dividing $n - 1$, $p_A^A = p_B^A$ (resp., $p_A^B = p_B^B$), i.e. the probability for each player to win a single set is not affected by the choice of the first server in the set. Consequently, the probability of player A (resp., player B) winning a match can be obtained in a straightforward manner by simple conditioning. Since the values of m chosen by international federations always satisfy the aforementioned divisibility constraint, we can (and will) restrict most of our attention on the outcome of a single A-set.

For $C \in \{A, B\}$, we denote by $E_A^{\alpha, \beta, C}$ the event associated with all sequences of $\alpha + \beta$ rallies in an A-set that gives rise to α (resp., β) points scored by player A (resp., by player B) and sees player C score the last point. The latter condition entails that we necessarily assume $\alpha > 0$ (resp., $\beta > 0$) when $C = A$ (resp., when $C = B$). We denote the corresponding probability by

$$p_A^{\alpha, \beta, C} = P[E_A^{\alpha, \beta, C}]. \tag{2.1}$$

With these notations, an A-set is won by A (resp., by B) on the score (n, k) (resp., (k, n)) with probability $p_A^{n, k, A}$ (resp., $p_A^{k, n, B}$), for $k \leq n - 2$. Obviously these quantities do not suffice to compute the probability p_A^C that C wins a set initiated by A, since we still need to account for what happens in case of a tie at $n - 1$, an event which occurs with probability $p_A^{n-1, n-1, A} + p_A^{n-1, n-1, B}$. For this we introduce the notation $p_A^{\text{tie}, A}$ (resp., $p_A^{\text{tie}, B}$) to represent the probability that A (resp., B) scores 2 more consecutive points than his/her opponent after the tie; this quantity will have to be computed differently than those defined above, since the rules governing the game after a tie are not the same as those which were applicable before this event. With these notations we get

$$p_A^A = \sum_{k=0}^{n-2} p_A^{n, k, A} + (p_A^{n-1, n-1, A} + p_A^{n-1, n-1, B}) p_A^{\text{tie}, A}, \tag{2.2}$$

and

$$p_A^B = \sum_{k=0}^{n-2} p_A^{k, n, B} + (p_A^{n-1, n-1, A} + p_A^{n-1, n-1, B}) p_A^{\text{tie}, B}. \tag{2.3}$$

The corresponding quantities for B-sets are defined by switching the roles of the two players in these definitions.

Determining (2.1) and computing $p_A^{\text{tie}, C}$ is the key to our understanding of the problem. We derive a general formula for these quantities in the following section.

2.2 Distribution of the scores in a single set.

In order to give the reader a feeling for what kind of mechanics are at play, we start with the (very simple) no-server model. Here every rally is won by player A (resp., by player B)

with probability p (resp., $1-p$), irrespective of the server. The process therefore reduces to a succession of independent Bernoulli trials with success-probability p, yielding

$$p_A^{\alpha,\beta,A} = \binom{\alpha+\beta-1}{\alpha-1} p^\alpha (1-p)^\beta \text{ and } p_A^{\alpha,\beta,B} = \binom{\alpha+\beta-1}{\alpha} p^\alpha (1-p)^\beta \qquad (2.4)$$

for all $\alpha, \beta \in \mathbb{N}$. All that remains is to determine the probabilities $p_A^{\text{tie},A}$ and $p_A^{\text{tie},B}$. Clearly, the number of rallies played after a tie until either player wins the set has to be even. Let $\ell \in \mathbb{N}_0$. In order to have 2ℓ rallies after the tie, it suffices that A and B win alternatively during the first $2(\ell-1)$ rallies, and that one of the two players scores 2 successive points thereafter. This entails that the probability that A (resp., B) wins the set after 2ℓ points is given by $p^2(2p(1-p))^{\ell-1}$ (resp., by $(1-p)^2(2p(1-p))^{\ell-1}$). Hence the probability of A (resp., of B) winning the set after a tie at $n-1$ is given by

$$p_A^{\text{tie},A} = \frac{p^2}{1-2p(1-p)} \text{ and } p_A^{\text{tie},B} = \frac{(1-p)^2}{1-2p(1-p)}. \qquad (2.5)$$

Thus, from (2.2), (2.3) and (2.4), simple summation yields the game-winning probabilities p_A^A and p_A^B. Note that these quantities do not depend on the parameter m.

The case of the server model is trickier. The scoring process of A (resp., of B) is a succession of m independent Bernoulli trials with success-probability p_a (resp., $1-p_a$), followed by m independent trials with success-probability $1-p_b$ (resp., p_b), followed again by m Bernoulli trials identical to the first and so on, until the end of the set (or until the tie is reached). We aim to derive the probability of the event $E_A^{\alpha,\beta,C}$. We know that this event is equivalent to a succession of independent (albeit not identically distributed) binomial experiments with parameters (m, p_a) or $(m, 1-p_b)$. Hence all related distributions will be relatively straightforward to derive as soon as we are able to compute, for any given score, the number K, say, of complete service sequences (of m rallies) and the number $R \, (< m)$, say, of remaining serves (by either A or B) during the last service sequence. This is performed in the following lemma (which we state without proof).

Lemma 2.1 *Let $0 \le \alpha, \beta \le n-1$. Define the integers K and R as the unique solutions of*

$\alpha + \beta = Km + R$ with $R \le m-1$. Then the event $E_A^{\alpha,\beta,C}$ can be decomposed into $k_1 := \lceil K/2 \rceil$ service sequences by A and $k_2 := \lfloor K/2 \rfloor$ service sequences by B, ended by R serves of A (resp., of B) when K is even (resp., when K is odd).

Let us now consider the event $E_A^{\alpha,\beta,C}$ for $C \in \{A, B\}$. From the nature of the game it is clear that, given the score (α, β), the only random quantity left is the number of points A (or, equivalently, B) has obtained on his/her own serve. Hence it is natural to define the events $E_A^{\alpha,\beta,C}(j) \subset E_A^{\alpha,\beta,C}$ for which the scoring sequence occurs with A scoring exactly j points on his/her serve. Denoting by $p_A^{\alpha,\beta,C}(j)$ the corresponding probability, these quantities can then be derived by exploring the essentially binomial nature of the experiment, as summarized in the following lemma.

Lemma 2.2 *Let $\alpha, \beta \in \mathbb{N}$, and define K, R as in Lemma 2.1. Also, for $C \in \{A, B\}$, define δ_A^C as the indicator of the equality between A and C. Then, (i) if $R > 0$ and K is even (or $R = 0$ and K is odd),*

$$p_A^{\alpha,\beta,C}(j) = \binom{\lceil \frac{K}{2} \rceil m + R - 1}{j - \delta_A^C} p_a^j (1 - p_a)^{\lceil \frac{K}{2} \rceil m + R - j} \binom{\lfloor \frac{K}{2} \rfloor m}{\alpha - j} (1 - p_b)^{\alpha - j} p_b^{\lfloor \frac{K}{2} \rfloor m - \alpha + j},$$

where $j \in \{\max(\delta_A^C, \alpha - \lceil \frac{K}{2} \rceil m), \ldots, \min(\alpha, \lceil \frac{K}{2} \rceil m + R - 1 + \delta_A^C)\}$; (ii) if $R > 0$ and K is odd (or $R = 0$ and K is even),

$$p_A^{\alpha,\beta,C}(j) = \binom{\lceil \frac{K}{2} \rceil m}{j} p_a^j (1 - p_a)^{\lceil \frac{K}{2} \rceil m - j} \binom{\lfloor \frac{K}{2} \rfloor m + R - 1}{\alpha - j - \delta_A^C} (1 - p_b)^{\alpha - j} p_b^{\lfloor \frac{K}{2} \rfloor m + R - \alpha + j},$$

where $j \in \{\max(0, \alpha - \lfloor \frac{K}{2} \rfloor m - R + 1 - \delta_A^C), \ldots, \min(\alpha - \delta_A^C, \lceil \frac{K}{2} \rceil m)\}$.

Note that the above condensed formulae are easier to read if spelt out explicitly in each of the 8 possible cases. We therefore provide a complete version of Lemma 2.2 in the Appendix, accompanied by a formal proof. Summing up it is now a simple matter to obtain the final formulae – at least for game scores that do not involve a tie.

Theorem 2.1 *Fix $\alpha, \beta \in \mathbb{N}$, and let $p_A^{\alpha,\beta,C} := P[E_A^{\alpha,\beta,C}]$, where $E_A^{\alpha,\beta,C} := \bigcup_j E_A^{\alpha,\beta,C}(j)$ with $C \in \{A, B\}$. Then $p_A^{\alpha,\beta,A} = \sum_{j=0}^{\infty} p_A^{\alpha,\beta,A}(j)$ and $p_A^{\alpha,\beta,B} = \sum_{j=0}^{\infty} p_A^{\alpha,\beta,B}(j)$, where the probabilities*

$p_A^{\alpha,\beta,C}(j)$ *are defined in Lemma 2.2 .*

When the parameters are chosen so as to satisfy the constraint $p_a = 1 - p_b = p$, the corresponding results concur with those obtained for the no-server model. In all other cases, these results do not allow for an agreeable form so we dispense with their explicit expression. The behavior of these quantities in terms of the ruling parameters of the game is illustrated (both in the old and new scoring systems) in Figure 3. Finally regarding the winning probabilities in case of a tie, the same reasoning as for (2.5) reveals that the probability that A (resp., B) wins the set after 2ℓ points is given by $p_a(1-p_b)((1-p_a)(1-p_b)+p_ap_b)^{\ell-1}$ (resp., by $(1-p_a)p_b((1-p_a)(1-p_b)+p_ap_b)^{\ell-1})$. The probability of A (resp., of B) winning the set after a tie at $n-1$ readily follows and thus, combining Theorem 2.1 and equations (2.2) and (2.3) yields p_A^A and p_A^B, as desired. Because these probabilities are particularly cumbersome to spell out explicitly we rather choose to illustrate their behavior (both in the old and new scoring systems) numerically and graphically (see Table 1 and the different figures below).

2.3   Distribution of the number of rallies in a single set.

The results established in the previous section allow us now to investigate the distribution of the number of rallies, D say, needed to end a single set. Let S stand for the random variable recording the first server of a set, and, for $C \in \{A, B\}$ and E an event, denote by $P_C[E] := P[E \,|\, S = C]$ the probability of E conditional upon who serves first in a set. The main object of interest in this section is the mapping $d \mapsto P_A[D = d]$ for $d \in \mathbb{N}$.

Obviously, because a point is scored on every rally, the number of rallies needed to reach a score (α, β) is equal to $\alpha + \beta$. Hence $P_A[D = d] = 0$ for all $d \leq n - 1$ and $P_A[D = d] = p_A^{n,d-n,A} + p_A^{d-n,n,B}$ for all $n \leq d \leq 2(n - 1)$. Finally, for $d \geq 2n - 1$, such length can only be achieved through the occurrence of a tie so that $P_A[D = d] = 0$ if $d = 2n - 1$ and otherwise $P_A[D = d] = (p_A^{n-1,n-1,A} + p_A^{n-1,n-1,B})p_{\text{tie}}^{d,n}$, where the latter quantity gives the probability that there are exactly $d - 2(n - 1)$ rallies after the tie. All that remains is therefore to compute $p_{\text{tie}}^{d,n}$. For this first note that $p_{\text{tie}}^{d,n} = 0$ if $d - 2(n - 1)$ is
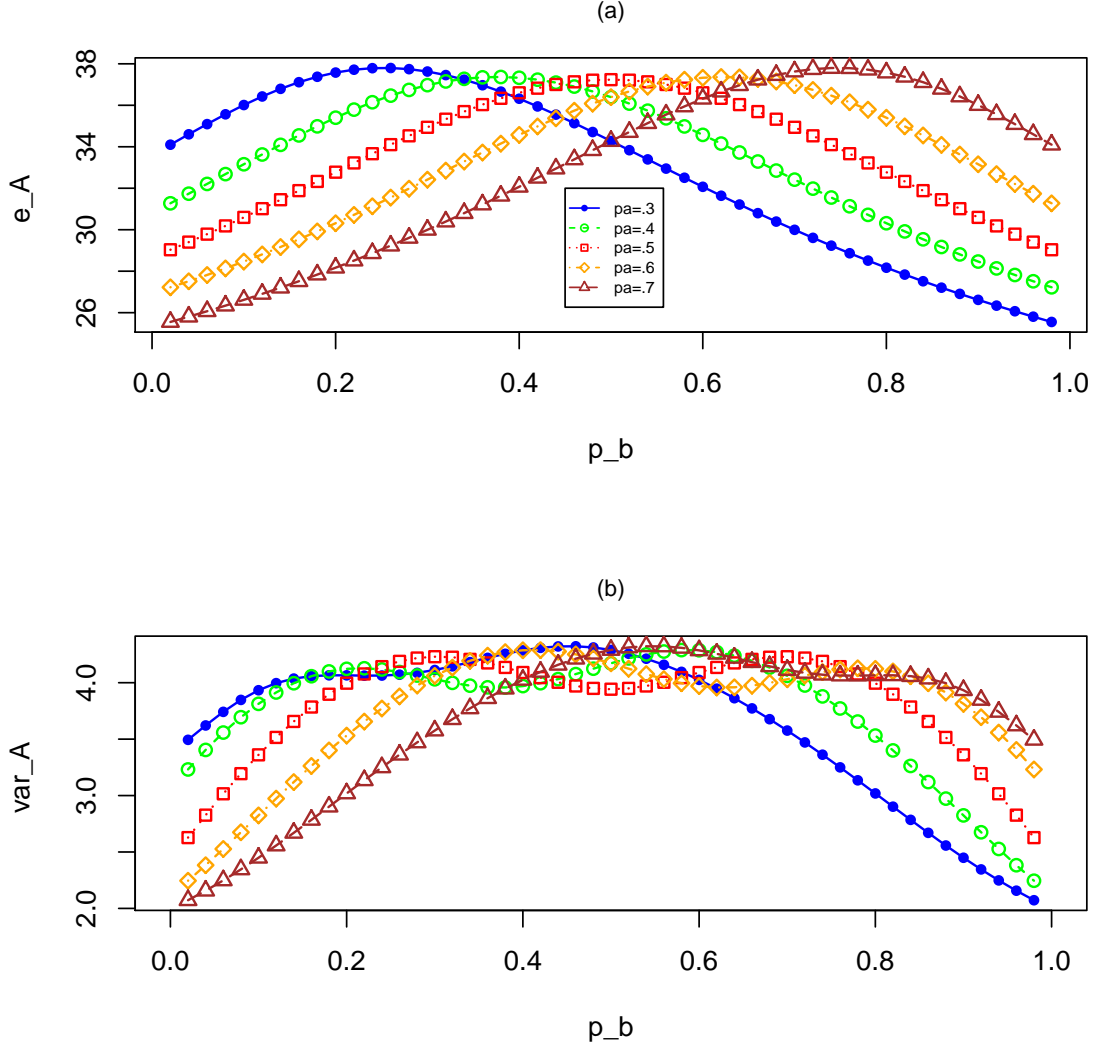
Figure 1: Both subfigures refer to an A-set played under the $(5, 21, 1)$-scoring system for different values of p_a, as a function of p_b. In (a) we plot the evolution of the expectation of the duration D unconditional on the winner of the set. In (b) we plot the evolution of the variance of the duration D unconditional on the winner of the set. In both plots the parameter p_a takes values $(0.3, 0.4, 0.5, 0.6, 0.7)$; the legend is the same in (b) as in (a).

odd. Next it is a simple matter of applying the same arguments as in Section 2.2 to obtain

$$p_{\text{tie}}^{d,n} = (p_a(1 - p_b) + (1 - p_a)p_b)((1 - p_a)(1 - p_b) + p_a p_b)^{\ell-1} \text{ with } 2\ell = d - 2(n - 1). \text{ The}$$

distribution of D (as well as its moments) readily follows, as illustrated in Table 1, Figure 1 and Figure 4.

The evolution of the expectation and variance of the number of rallies D in an A-set unconditional on the winner of that set are displayed in Figure 1. (Note that the effect of a variation of n in one set played according to the (m, n, G)-rule will be a straightforward rescaling. Plots are therefore not provided.) The expectation curves are quite smooth and unimodal around $p_a = p_b$, while the variance curves are also smooth but exhibit a slightly bimodal nature around $p_a = p_b$. These findings are in accordance with the intuition that players of approximately the same strength play longer. Moreover, when the strengths are only slightly different, the variance becomes biggest, as in that case there may be both tight and uneven sets. Finally note that when $p_b \to 0$ or $p_b \to 1$ the expected duration is not equal to 21 as one might think at first sight but is rather around 25. This evidently follows from the fact that p_a is confined between .3 and .7, and hence when player A serves the outcomes remain random.

3   Comparing scoring systems in table tennis.

In this section, we discuss the influence of the parameters (m, n, G) in terms of score distributions, durations and match-winning probabilities. Our main interest is to discuss the effects of the switch from the "old" $(5, 21, G)$-scoring system to the "new" $(2, 11, G)$-scoring system.

We first stress how little influence the choice of m has on the different distributions, as long as m is chosen so as to satisfy the above described divisibility constraints. This is illustrated in Figure 2, where the ratio between the expected durations (Figure 2 a) and standard deviations (Figure 2 b) under the $(5, 11, 1)$ and the $(2, 11, 1)$ rules are reported. Aside from the fact that the different curves cross the horizontal line $y = 1$ at each couple (p_a, p_b) such that $p_a + p_b = 1$, we draw the reader's attention to how little variation these quantities endure. This encourages to disregard the role of m in the future, and concentrate on the dependence on the parameters n, G and (p_a, p_b).

Table 1 shows the probability that A wins a set, in which he/she serves first, both for the old and new scoring system, as well as the expected duration of such a set in both scoring
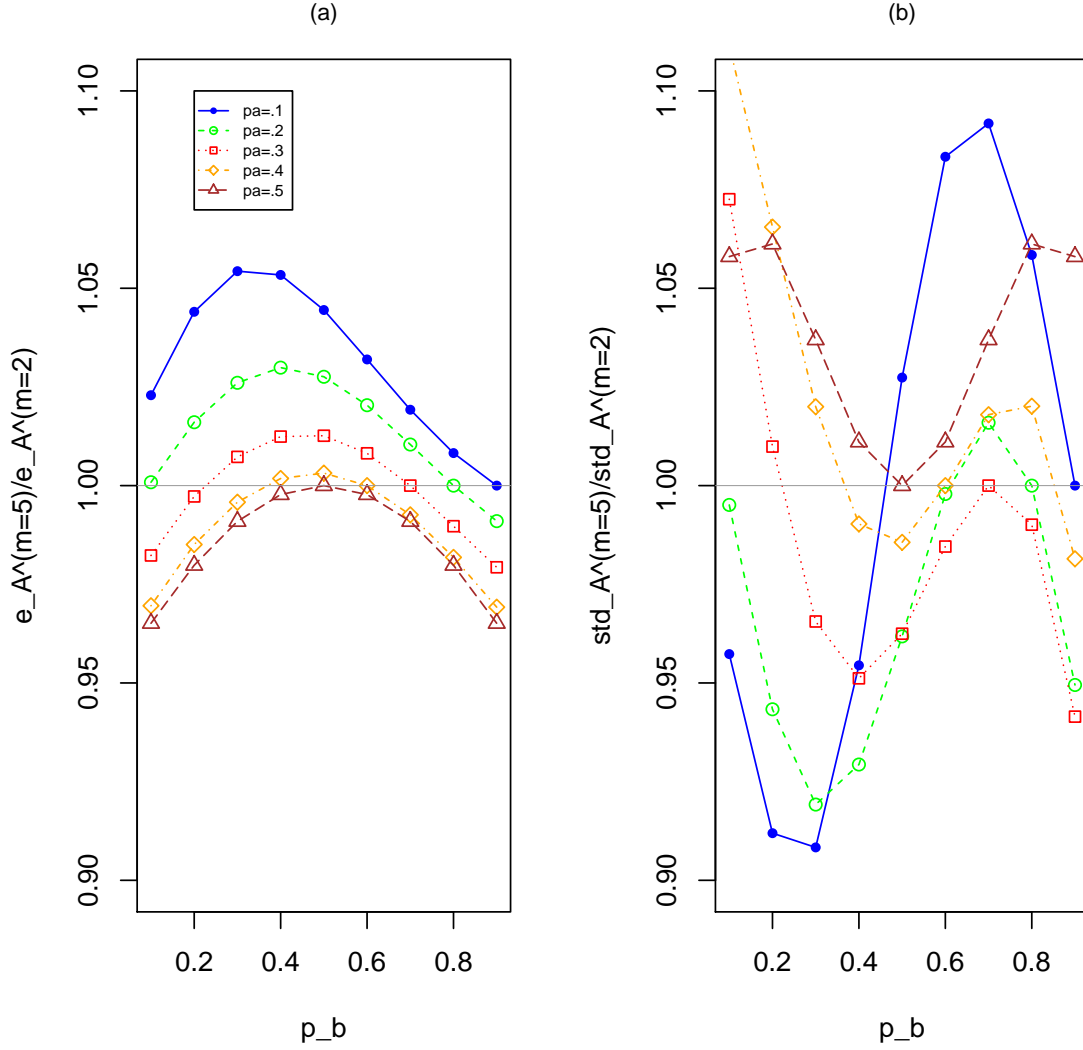
Figure 2: Both plots refer to an A-set played with $n = 11$ and $G = 1$ for different values of p_a, as a function of p_b. In (a) we plot the ratio between the expected durations under the $(2, 11, 1)$ rule and that under the $(5, 11, 1)$ rule. In (b) we plot the ratio between the standard deviations. In both plots the parameter p_a takes values $(0.3, 0.4, 0.5, 0.6, 0.7)$; the legend is the same in (b) as in (a).

Table 1: Probability and Expected Duration

p_a	0.1	0.2	0.3	0.4	p_b 0.5	0.6	0.7	0.8	0.9
p_a	Probability that A wins a set when A serves first with $m=5$ and $n=21$
0.1	0.5000	0.1657	0.0404	0.0072	0.0008	0	0	0	0
0.2	0.8342	0.5000	0.2190	0.0702	0.0159	0.0023	0.0002	0	0
0.3	0.9595	0.7809	0.5000	0.2430	0.0863	0.0209	0.0030	0.0002	0
0.4	0.9928	0.9297	0.7569	0.5000	0.2530	0.0914	0.0209	0.0023	0
0.5	0.9991	0.9840	0.9136	0.7469	0.5000	0.2530	0.0863	0.0159	0.0008
0.6	0.9999	0.9976	0.9790	0.9085	0.7469	0.5000	0.2430	0.0702	0.0072
0.7	0.9999	0.9998	0.9969	0.9790	0.9136	0.7569	0.5000	0.2190	0.0404
0.8	1	0.9999	0.9998	0.9976	0.9840	0.9297	0.7809	0.5000	0.0166
0.9	1	1	0.9999	0.9999	0.9991	0.9928	0.9595	0.8342	0.5000
p_a	Probability that A wins a set when A serves first with $m=2$ and $n=11$
0.1	0.5000	0.2300	0.0951	0.0343	0.0103	0.0024	0.0003	0	0
0.2	0.7699	0.5000	0.2819	0.1376	0.0568	0.0189	0.0046	0.0006	0
0.3	0.9048	0.7180	0.5000	0.3035	0.1576	0.0673	0.0219	0.0046	0.0003
0.4	0.9656	0.8623	0.6964	0.5000	0.3121	0.1635	0.0673	0.0189	0.0024
0.5	0.9896	0.9431	0.8423	0.6878	0.5000	0.3121	0.1576	0.0568	0.0103
0.6	0.9975	0.9810	0.9326	0.8364	0.6878	0.5000	0.3035	0.1376	0.0343
0.7	0.9996	0.9953	0.9780	0.9326	0.8423	0.6964	0.5000	0.2819	0.9513
0.8	0.9999	0.9993	0.9953	0.9810	0.9431	0.8623	0.7180	0.5000	0.2300
0.9	0.9999	0.9999	0.9996	0.9975	0.9896	0.9656	0.9048	0.7699	0.5000
p_a	Expected duration of a set when A serves first with $m=5$ and $n=21$
0.1	40.1616	36.5982	33.7171	31.3842	29.3013	27.4362	25.8432	24.5067	23.3333
0.2	38.6311	38.2747	36.3666	34.0094	31.7268	29.6526	27.8216	26.2500	24.9032
0.3	36.0125	37.5782	37.6279	36.3163	34.2761	32.0696	29.9949	28.1681	26.6186
0.4	33.1567	35.4021	36.9720	37.3272	36.3937	34.5748	32.4159	30.3135	28.4663
0.5	30.5940	32.7743	34.9382	36.6022	37.2359	36.6022	34.9382	32.7743	30.5940
0.6	28.4663	30.3135	32.4159	34.5748	36.3937	37.3272	36.9720	35.4021	33.1567
0.7	26.6186	28.1681	29.9949	32.0696	34.2761	36.3163	37.6279	37.5782	36.0125
0.8	24.9032	26.2500	27.8216	29.6526	31.7268	34.0094	36.3666	38.2747	38.6311
0.9	23.3333	24.5068	25.8433	27.4362	29.3013	31.3843	33.7172	36.5982	40.1616
p_a	Expected duration of a set when A serves first with $m=2$ and $n=11$
0.1	22.5402	19.9063	18.0171	16.5767	15.4215	14.4575	13.6267	12.8901	12.2222
0.2	20.4333	20.0069	19.0096	17.8241	16.6473	15.5657	14.6029	13.7513	12.9904
0.3	18.7138	19.3163	19.2465	18.6612	17.7620	16.7301	15.6955	14.7315	13.8635
0.4	17.2440	18.2349	18.8266	18.9228	18.5459	17.8052	16.8511	15.8300	14.8509
0.5	15.9686	17.0222	17.9547	18.5985	18.8285	18.5985	17.9547	17.0222	15.9686
0.6	14.8509	15.8300	16.8511	17.8052	18.5459	18.9228	18.8266	18.2349	17.2440
0.7	13.8635	14.7315	15.6955	16.7301	17.7620	18.6612	19.2465	19.3163	18.7138
0.8	12.9904	13.7513	14.6029	15.5657	16.6473	17.8241	19.0096	20.0069	20.4333
0.9	12.2222	12.8901	13.6267	14.4575	15.4215	16.5767	18.0171	19.9063	22.5402

systems. The two first parts in Table 1 list the values of p_A^A for all possible combinations of couples (p_a, p_b) varying respectively from 0.1 to 0.9 with a step size of 0.1. A simple inspection of the probabilities indicates that the stronger one of the two players inevitably wins the set, which is a confirmation of the common intuition. The main difference between the two scoring systems lies in the fact that on the one hand, when $p_a > p_b$ (values under the diagonal), player A has a higher chance to win the set in the old system than in the new one, whereas on the other hand, when $p_a < p_b$ (values over the diagonal), player A's (small) chances to win the set are higher in the new system. This can be explained as follows: the more points are necessary to win the set, the more important the strengths p_a and p_b become, and the "victory against the odds" is therefore less probable in the old scoring system than in the new one. We also draw the reader's attention to the fact that $p_A^A(p_a, p_b) = p_A^A(1 - p_b, 1 - p_a)$ and $p_A^A(p_a, p_b) + p_A^A(p_b, p_a) = 1$ (with $p_A^A(x, y)$ standing for the probability that player A wins an A-set under the conditions $p_a = x$ and $p_b = y$, $x, y \in (0, 1)$). These phenomena are discussed in Schulman and Hamdan (1977), and will not be further elaborated on here.

The two last parts of Table 1 show the expected duration of a set in which player A serves first. The sets with the largest expected durations are those on the diagonal, where both players are roughly of similar strength. Clearly the duration of a set is smaller in the new scoring sytem than in the older one. More interesting is the fact that the duration of a set in the old scoring system is always roughly double that in the new scoring system or, more precisely, that we have $e_A^{A,old}/e_A^{A,new} \approx 21/11$. This seemingly linear relationship between the average length of a set and the number of points played is easily confirmed by further computations, as shown in Table 2. Likewise, from Table 2, we see that the same multiplicative factor (under a square root) appears when considering the standard deviation of D. The same conclusions hold true when passing from a single set to a full match. From this we deduce that in order for the ITTF, for instance, to derive a scoring triple (m_2, n_2, G_2) which *does not* change the average length of the game when switching from the original triple (m_1, n_1, G_1), it is advisable to choose the number of sets G_2 in such a way that $G_1 n_1 = G_2 n_2$. If, for instance, $n_1 = 21$ and $G_1 = 3$, then the choice of $n_2 = 11$

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Table 2: Ratio of expected durations and standard deviations in the no-server model

n_2/n_1	0.1	0.2	0.3	0.4	0.5
	Ratio of expectations				
$21/11 = 1.9090$	1.9000	1.9058	1.9297	1.9623	1.9776
$31/21 = 1.4761$	1.4670	1.4697	1.4797	1.4963	1.5052
$41/31 = 1.3225$	1.3149	1.3172	1.3221	1.3326	1.3392
$31/11 = 2.8181$	2.8231	2.8276	2.8728	2.9419	2.9769
$41/11 = 3.7272$	3.7482	3.7527	3.8151	3.9260	3.9867
	Ratio of Standard Deviations				
$\sqrt{21/11} = 1.3816$	1.3437	1.3037	1.3426	1.2975	1.2356
$\sqrt{31/21} = 1.2149$	1.1789	1.2171	1.2527	1.2356	1.1754
$\sqrt{41/31} = 1.1500$	1.1553	1.1500	1.1790	1.1863	1.1349
$\sqrt{31/11} = 1.6787$	1.5490	1.6080	1.6819	1.5944	1.4524
$\sqrt{41/11} = 1.9306$	1.8476	1.8529	1.9749	1.8816	1.6484

and $G_2 = 6$ would influence the expected durations less than $G_2 = 3$ or $G_2 = 4$. The choice of $G_2 = 3$ or 4 nevertheless ensures a difference that is negligible (see Figures 3 and 6 and 7 where different choices of triples are compared in the no-server model).

Figure 3 presents, for $n = 11$ and $n = 21$ respectively, the score distribution associated with $(p_a, p_b) = (.7, .5), (.6, .5), (.5, .5), (.4, .5)$ and $(.3, .5)$. Subplots (a) and (c) contain the probabilities that player A wins an A-set in which the opponent scores a total of k points (for instance, in the new scoring system, $k = 0, \ldots, 9$ corresponds to a victory of A on the score $(11, 0), \ldots, (11, 9)$, and in case of a tie, for which $k \geq 10$, player A wins on scores of the form $(k+2, k)$) and the subplots (b) and (d) give the probability for B winning an A-set where this time player A scores k points. It appears that these distributions are highly sensitive to the values of (p_a, p_b); obviously, this extends to the corresponding match-winning probabilities. In the $(5, 21, G)$-scoring system, p_A^A varies between .09 and .91 and p_A^B between .91 and .09, when, for fixed $p_b = .5$, p_a ranges from .3 to .7. For the other scoring system, p_A^A takes values between .16 and .84, while p_A^B ranges from .84 to .16. This confirms our previous findings that the difference in strengths is more important in the old than in the new system. The following three comments hold for both scoring systems: (i) as A (resp., B) is getting stronger with respect to B (resp., A), his/her winning probability dramatically increases, (ii) whatever the value of p_a, white-washes seldom occur, and (iii) the score distributions are not monotone in p_a. Finally note that subfigures (a) and (b), as well as subfigures (c) and (d), display the same curves with the symbols indicating points in reverse order; this is
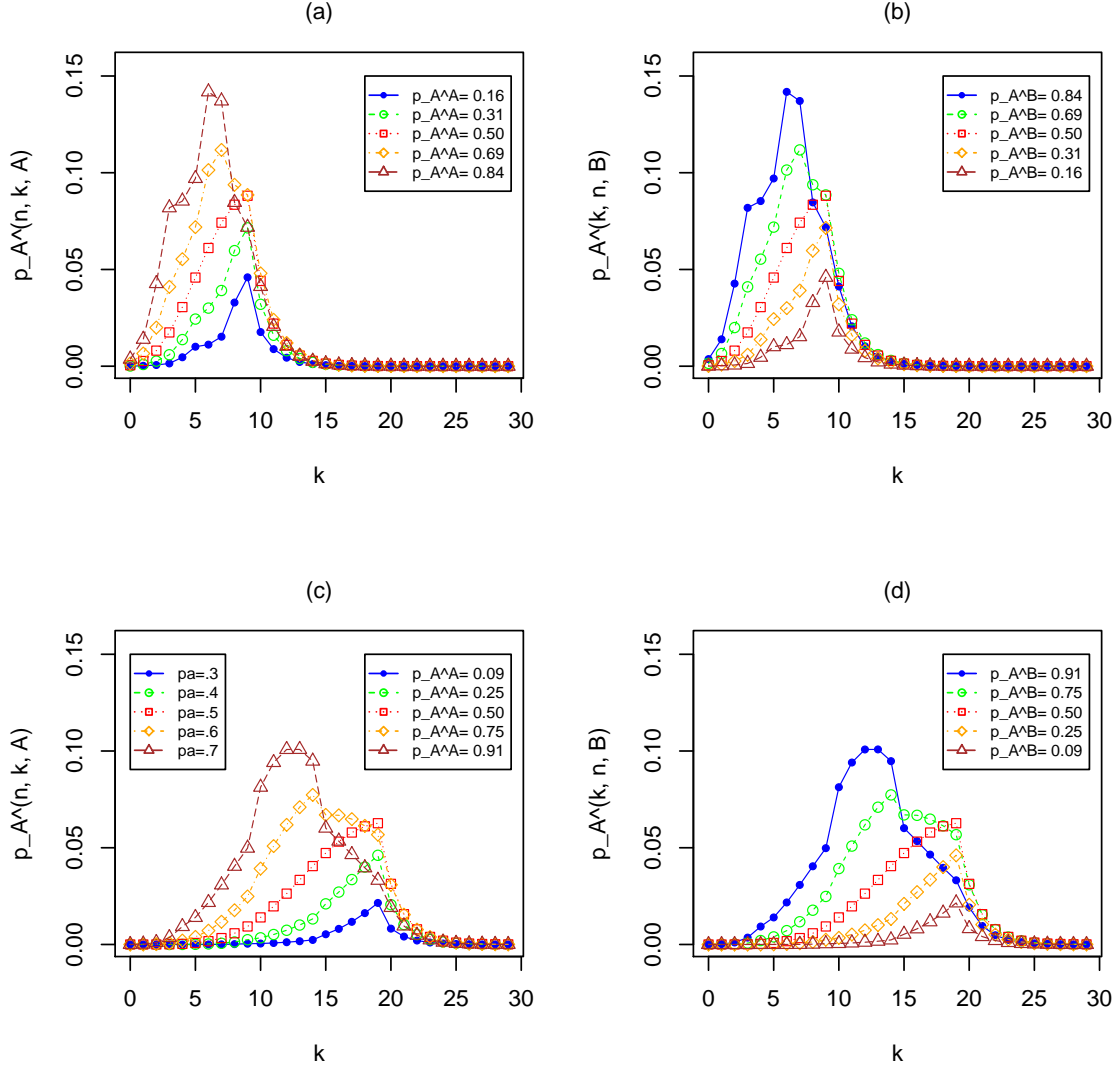
Figure 3: In all plots we fix $p_b = .5$ and take $p_a \in (0.3, 0.4, 0.5, 0.6, 0.7)$ (see Subfigure (c) for the legend). Subfigures (a) and (b) refer to an A-set played under the $(2, 11, 1)$-scoring system. Subfigure (a): for $(p_a, p_b) = (.7, .5), (.6, .5), (.5, .5), (.4, .5)$ and $(.3, .5)$, probabilities $p_A^{n,k,A}$ that player A wins the set on the score (n, k). Subfigure (b): the corresponding values for victories of B. Subfigures (c) and (d) refer to an A-set played under the $(5, 21, 1)$-scoring system. Subfigure (c): for $(p_a, p_b) = (.7, .5), (.6, .5), (.5, .5), (.4, .5)$ and $(.3, .5)$, probabilities $p_A^{n,k,A}$ that player A wins the set on the score (n, k). Subfigure (d): the corresponding values for victories of B. The different set-winning probabilities p_A^C for $C \in \{A, B\}$ and each couple of values (p_a, p_b) are given in each subfigure.
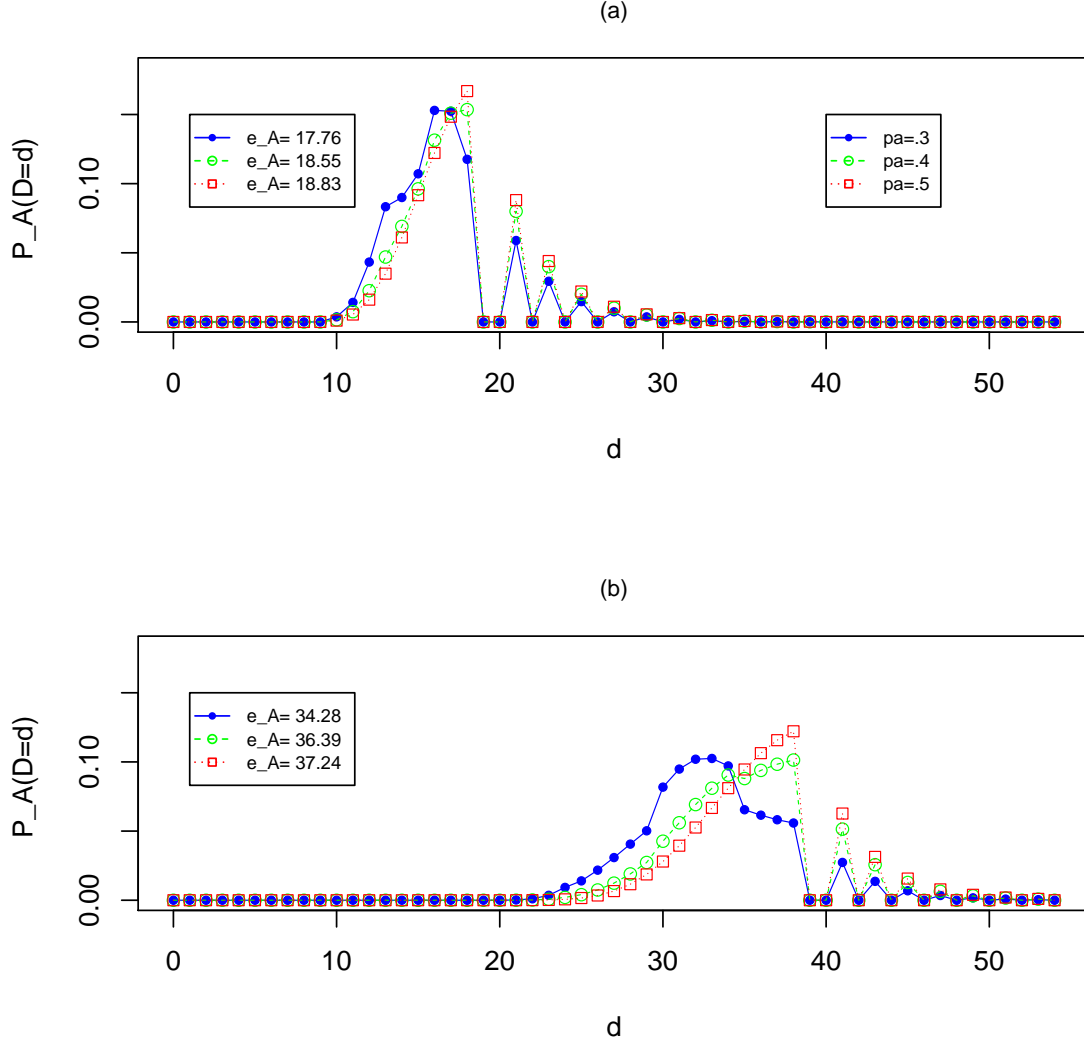
Figure 4: Both plots refer to an A-set played with $p_b = .5$. For $(p_a, p_b) = (.5, .5), (.4, .5)$ and $(.3, .5)$ they report the probabilities that the number of rallies D needed to finish the set takes value d unconditional on the winner of the set. Plot (a) shows the probabilities and expectations for the $(2, 11, 1)$-scoring system and plot (b) for the $(5, 21, 1)$-scoring system.

due to the symmetric roles played by the two players.

In Figure 4 we plot, in the setup of an A-set and for $(p_a, p_b) = (.5, .5), (.4, .5)$ and $(.3, .5)$ (we omit the couples $(.7, .5)$ and $(.6, .5)$ as the symmetry of the situation implies a confounding with the plots for $(.3, .5)$ and $(.4, .5)$, respectively), the unconditional (on the winner of the set) probabilities that the number of rallies D equals $d \in [0, 60]$ with respect

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to that value d. The expectations of D in each case are indicated as well. Subfigure (a) corresponds to the new scoring system, and subfigure (b) to the old one. In the old scoring system the probabilities are more dispersed than in the new scoring system; this is a direct consequence of the larger number of points needed to play until either player wins the set. For both plots, the peaks that occur after the main peak are due to the fact that a set can only end after an even number of rallies in case of a tie. The curve whose points are indicated by squares is the one that takes the highest values, which confirms the intuition that two players of the same strength take more time to battle out the set and that the scores are quite tight (see also the related expected values $e_A = 18.83$ in the new scoring system and $e_A = 37.24$ in the old one).

The dependence of the set-winning probabilities on (p_a, p_b) is of primary importance. In what follows, we will investigate this dependence visually by comparing both scoring systems in the no-server model ($p = p_a = 1 - p_b$). The results are illustrated in Figure 5. Figure 5 (a) shows the probability that player A wins an A-set as a function of his/her strength p. The dashed curve gives the probability for the new scoring system and the dotted curve for the old scoring system. The plot supports the claim that, for most values of p, the choice of the scoring system does not influence the set-winning probabilities. The only slight differences appear within $[.3, .45]$ and $[.55, .7]$, where the new scoring system quicker equalizes both players, which is in agreement with our previous findings. The duration plots simply confirm what common-sense already tells us. Indeed, we see in Figure 5 (b) that the expectation of D is, uniformly in $p \in (0, 1)$, smaller for the new scoring system than for the old one; this is of course a direct consequence of the dramatic reduction of the number of points necessary to win the set. The same conclusion happens to be true for the standard deviation of D (the old scoring system varies between 3.26 and 4.29, the new scoring system between 2.42 and 3.22), as illustrated in Figure 5 (c). The twin-peak shape of both standard error curves is most certainly due to the quasi-equality in rally-winning probabilities between both players which leads to both tighter final scores and more uneven scores. Finally note that both curves of Figures 5 (b) and (c) are symmetric about $p = .5$, as expected.

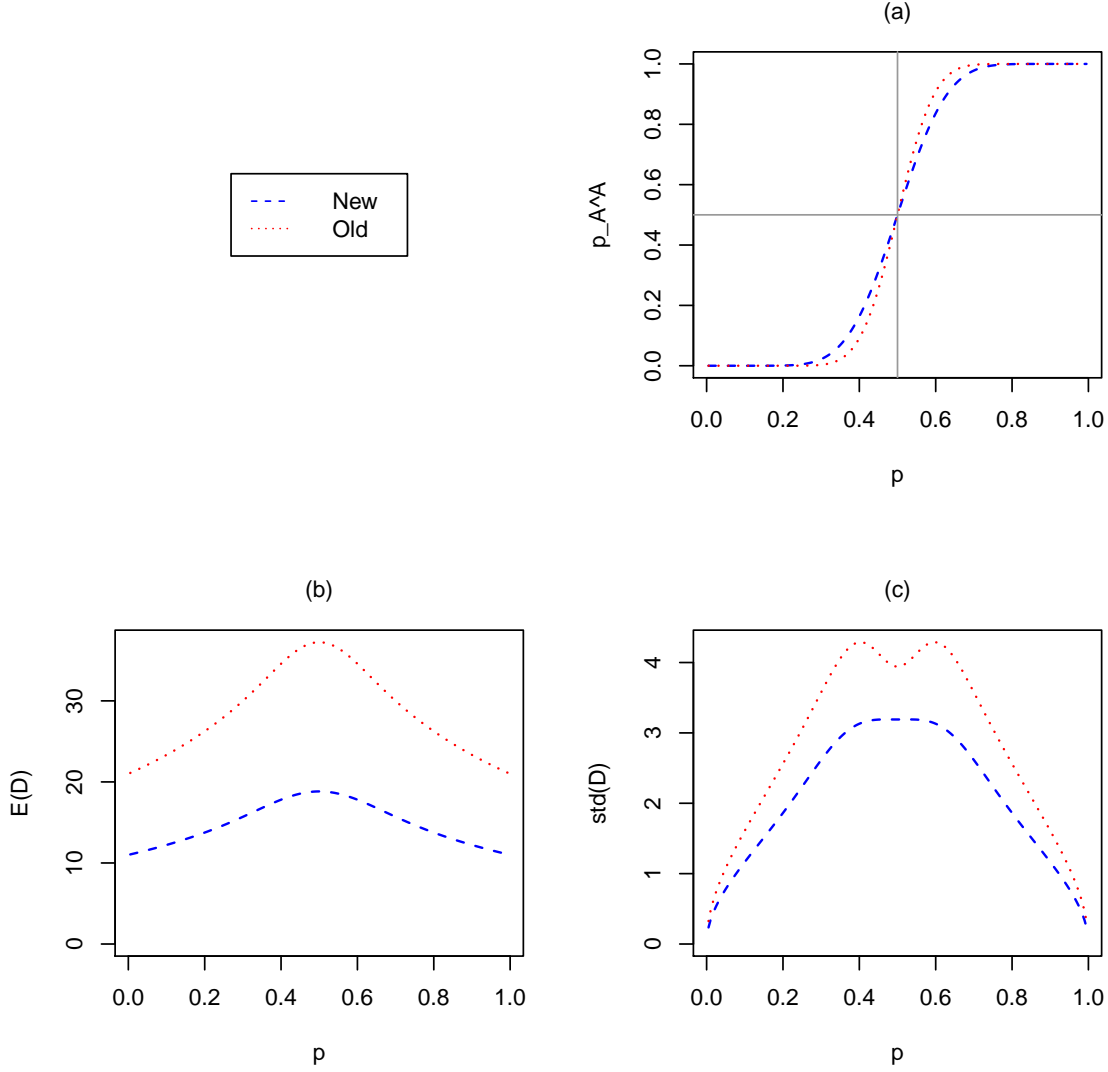We conduct, in Figure 6, a similar comparison between both scoring systems at match

Figure 5: As a function of $p = p_a = 1 - p_b$ (that is, in the no-server model), subfigure (a) contains the probabilities p_A^A (dotted line) that player A wins an A-set in the $(5, 21, 1)$-scoring system, along with the probabilities p_A^A (dashed line) that player A wins an A-set in the $(2, 11, 1)$-scoring system. Subfigures (b) and (c) show the expectation and the standard deviation of the number of rallies needed to complete the corresponding set, unconditional on the winner.
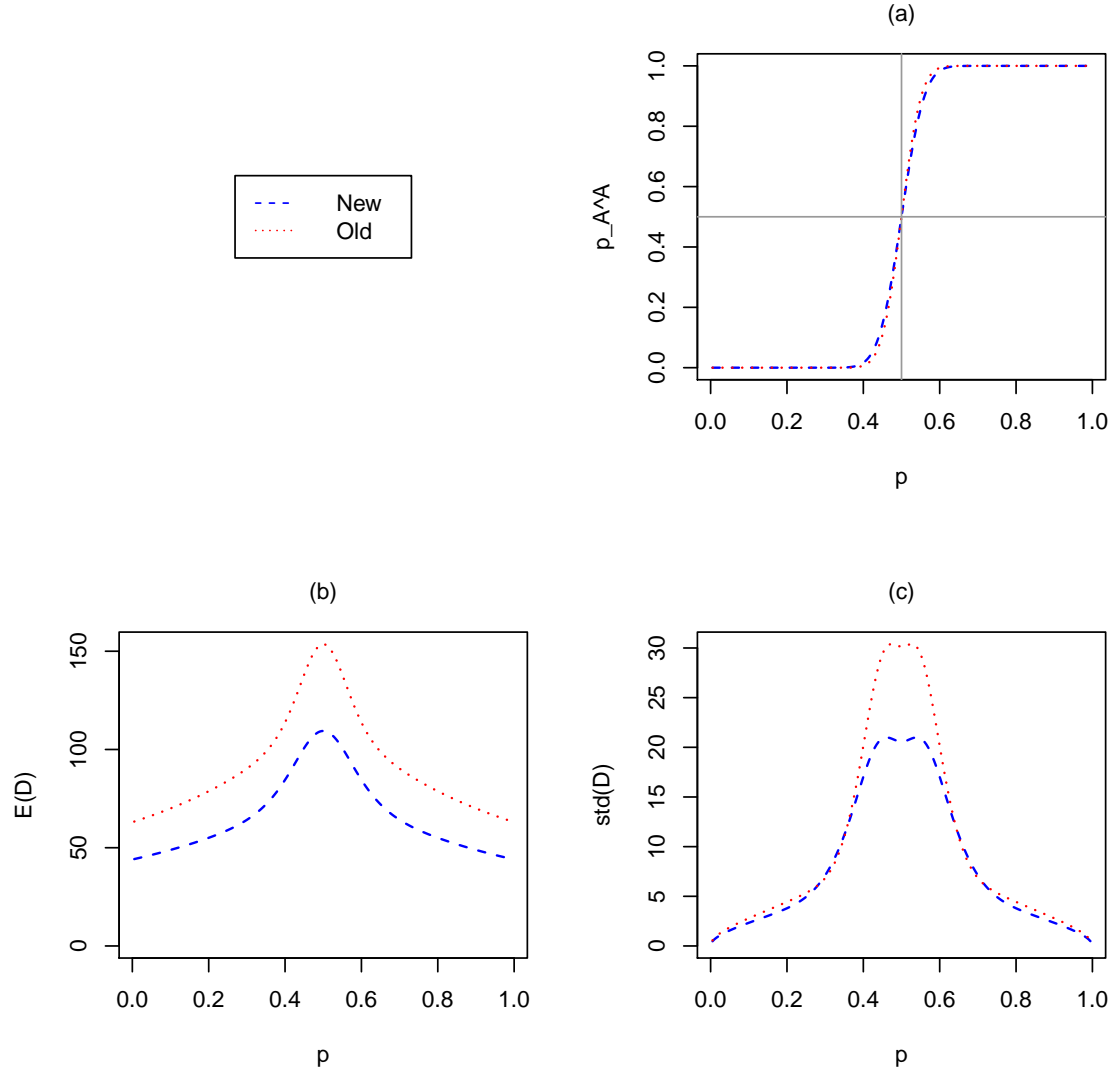
Figure 6: Comparison: a match

As a function of $p = p_a = 1 - p_b$ (that is, in the no-server model), subfigure (a) contains the probabilities p_A^A (dotted line) that player A wins an A-match in the $(5, 21, 3)$-scoring system, along with the probabilities p_A^A (dashed line) that player A wins an A-match in the $(2, 11, 4)$-scoring system. Subfigures (b) and (c) show the expectation and the standard deviation of the number of rallies needed to complete the corresponding match, unconditional on the winner.
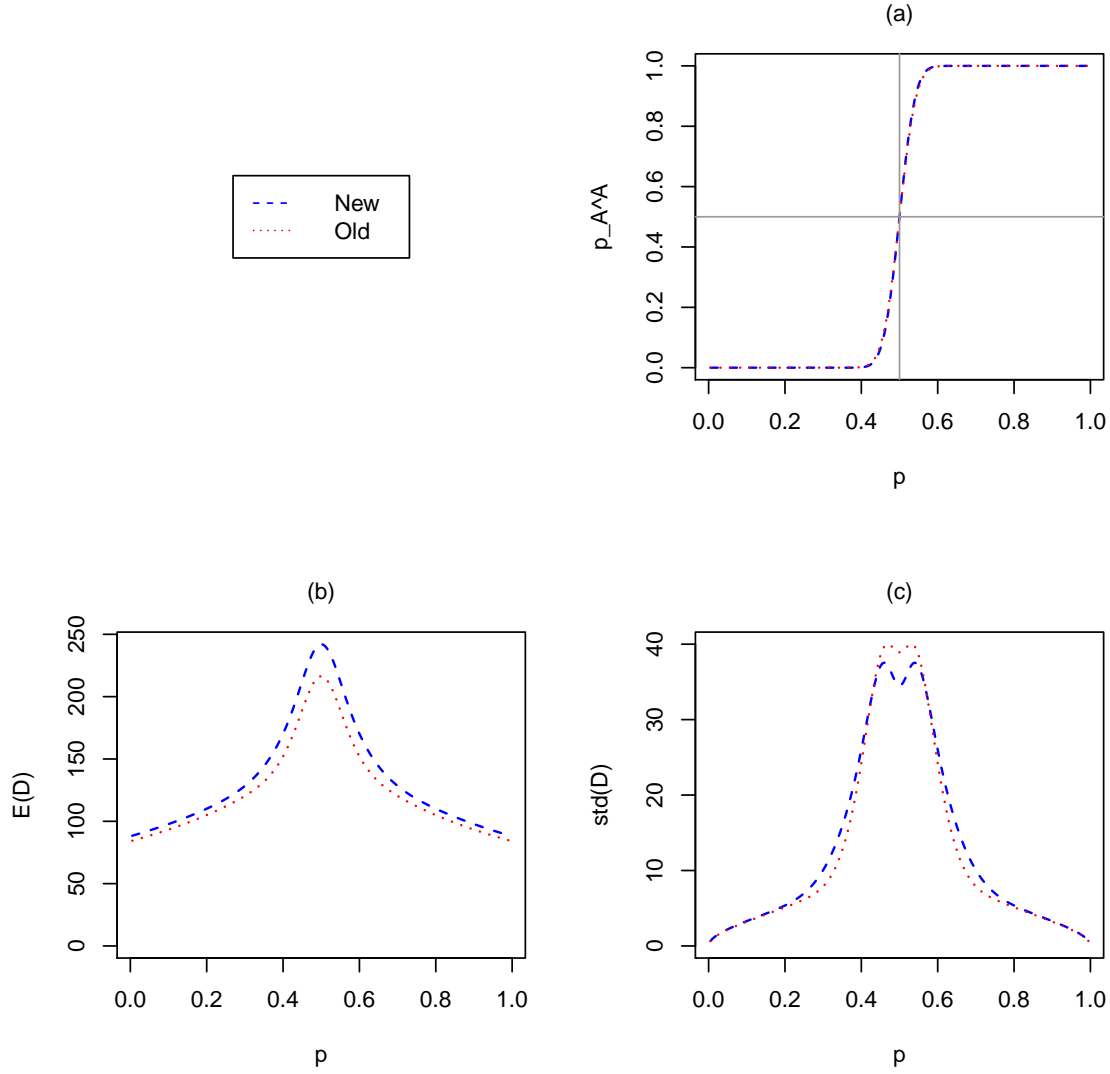
Figure 7: Comparison: a match

As a function of $p = p_a = 1 - p_b$ (that is, in the no-server model), subfigure (a) contains the probabilities p_A^A (dotted line) that player A wins an A-match in the $(5, 21, 4)$-scoring system, along with the probabilities p_A^A (dashed line) that player A wins an A-match in the $(2, 11, 8)$-scoring system. Subfigures (b) and (c) show the expectation and the standard deviation of the number of rallies needed to complete the corresponding match, unconditional on the winner.

level. More precisely, we compare a $(5, 21, 3)$-scoring system (dotted curve) with a $(2, 11, 4)$-scoring system (dashed curve). Subfigure (a) of Figure 6 (to be compared with the corresponding subfigure in Figure 5) indicates that, for any fixed p, the higher number of sets needed to achieve victory in the new scoring system ensures that match-winning probabilities remain roughly the same before and after the rule change. Subfigures (b) and (c), i.e. the expectation and standard deviation plots, convey the same message as for a single set, except that here the two curves of the standard deviation cross at some points. Note that the standard deviation is in general smaller in the new scoring system, hence the change has made the length of the match more predictable. Finally in Figure 7 we show, for the sake of illustration, how choosing G_{new} and G_{old} such that $n_{\mathrm{new}} G_{\mathrm{new}} \approx n_{\mathrm{old}} G_{\mathrm{old}}$ ensures – as is intuitively clear – that both the probabilities and the expectations remain largely unchanged when switching from $(m_{\mathrm{old}}, n_{\mathrm{old}}, G_{\mathrm{old}})$ to $(m_{\mathrm{new}}, n_{\mathrm{new}}, G_{\mathrm{new}})$.

4   Final comments.

This paper provides a *complete* probabilistic description of games played according to the (m, n, G)-scoring system. It complements and extends the previous contribution by Schulman and Hamdan (1977). The formulae provided, although cumbersome, can easily be implemented for numerical purposes, and yield some striking illustrations. The results of our paper can be useful, in practice, to the sport community (e.g. the ITTF) and to TV broadcasting programs, as they allow to analyze any (m, n, G)-scoring system. Our findings have allowed us to perform an in-depth comparison of the old and new scoring system in table tennis. If the aims behind the system change were (i) a better control of the length of a match and (ii) an increase in the potential number of crucial points without influencing the relative strengths of the players too much (i.e. without changing the winning probabilities), then our results confirm that these goals are achieved.

We also mention how our results also allow for constructing elegant estimation procedures of the respective strengths of the players. Obviously disposing of the full formulae permits to work out maximum likelihood estimators for p_a, p_b. These will necessarily be obtained

by numerical maximization on basis of the formulae in Lemma 2.2. There is, however, a more elegant (and efficient) way to perform such estimation, which only requires retaining one more information from a given encounter between two players, namely the number of points scored by player A (resp., player B) on his/her own serve and on the serve of his/her opponent. Indeed with this information one can show that the maximum likelihood estimator for p_a, say, is given by the ratio of the number of points won by A on his/her serve and the total number of his/her serves. Based on Bradley-Terry paired comparison methods, these estimators will allow for constructing interesting new ranking methods within round-robin tournaments. This will be the subject of a future publication.

Finally, since our results of Lemma 2.2 are valid for general intermediate scores (α, β), an extension to the case where the initial score differs from $(0,0)$ is readily implemented. This naturally paves the way to a more dynamic analysis of a match, *à la* Klaassen and Magnus (2003), in which the winning probabilities and duration are sequentially estimated throughout the course of a given match.

A   Appendix.

A.1   Complete formulae.

Lemma A.1 *Fix $m \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}$, and define $K, R \in \mathbb{N}$ via the Euclidian division $\alpha + \beta = Km + R$. Then we have*

  1. if $R = 0$ and K is even, then

$$p_A^{\alpha, \beta, A}(x) = \binom{\frac{K}{2}m}{x} p_a^x (1 - p_a)^{\frac{K}{2}m - x} \binom{\frac{K}{2}m - 1}{\alpha - x - 1} (1 - p_b)^{\alpha - x} p_b^{\frac{K}{2}m - \alpha + x},$$

where $x \in \{\max(0, \alpha - \frac{K}{2}m), \ldots, \min(\alpha - 1, \frac{K}{2}m)\}$, and

$$p_A^{\alpha,\beta,B}(x) = \binom{\frac{K}{2}m}{x} p_a^x (1-p_a)^{\frac{K}{2}m-x} \binom{\frac{K}{2}m-1}{\alpha-x} (1-p_b)^{\alpha-x} p_b^{\frac{K}{2}m-\alpha+x},$$

where $x \in \{\max(0, \alpha - \frac{K}{2}m + 1), \ldots, \min(\alpha, \frac{K}{2}m)\}$.

2. *if $R = 0$ and K is odd, then*

$$p_A^{\alpha,\beta,A}(x) = \binom{\lceil\frac{K}{2}\rceil m - 1}{x-1} p_a^x (1-p_a)^{\lceil\frac{K}{2}\rceil m-x} \binom{\lfloor\frac{K}{2}\rfloor m}{\alpha-x} (1-p_b)^{\alpha-x} p_b^{\lfloor\frac{K}{2}\rfloor m-\alpha+x},$$

where $x \in \{\max(1, \alpha - \lfloor\frac{K}{2}\rfloor m), \ldots, \min(\alpha, \lceil\frac{K}{2}\rceil m)\}$, and

$$p_A^{\alpha,\beta,B}(x) = \binom{\lceil\frac{K}{2}\rceil m - 1}{x} p_a^x (1-p_a)^{\lceil\frac{K}{2}\rceil m-x} \binom{\lfloor\frac{K}{2}\rfloor m}{\alpha-x} (1-p_b)^{\alpha-x} p_b^{\lfloor\frac{K}{2}\rfloor m-\alpha+x},$$

where $x \in \{\max(0, \alpha - \lfloor\frac{K}{2}\rfloor m), \ldots, \min(\alpha, \lceil\frac{K}{2}\rceil m - 1)\}$.

3. *if $R > 0$ and K is even, then*

$$p_A^{\alpha,\beta,A}(x) = \binom{\frac{K}{2}m + R - 1}{x-1} p_a^x (1-p_a)^{\frac{K}{2}m+R-x} \binom{\frac{K}{2}m}{\alpha-x} (1-p_b)^{\alpha-x} p_b^{\frac{K}{2}m-\alpha+x},$$

where $x \in \{\max(1, \alpha - \frac{K}{2}m), \ldots, \min(\alpha, \frac{K}{2}m + R)\}$, and

$$p_A^{\alpha,\beta,B}(x) = \binom{\frac{K}{2}m + R - 1}{x} p_a^x (1-p_a)^{\frac{K}{2}m+R-x} \binom{\frac{K}{2}m}{\alpha-x} (1-p_b)^{\alpha-x} p_b^{\frac{K}{2}m-\alpha+x},$$

where $x \in \{\max(0, \alpha - \frac{K}{2}m), \ldots, \min(\alpha, \frac{K}{2}m + R - 1)\}$.

4. *if $R > 0$ and K is odd, then*

$$p_A^{\alpha,\beta,A}(x) = \binom{\lceil\frac{K}{2}\rceil m}{x} p_a^x (1-p_a)^{\lceil\frac{K}{2}\rceil m-x} \binom{\lfloor\frac{K}{2}\rfloor m + R - 1}{\alpha-x-1} (1-p_b)^{\alpha-x} p_b^{\lfloor\frac{K}{2}\rfloor m+R-\alpha+x},$$

where $x \in \{\max(0, \alpha - \lfloor \frac{K}{2} \rfloor m - R), \ldots, \min(\alpha - 1, \lceil \frac{K}{2} \rceil m)\}$, and

$$p_A^{\alpha,\beta,B}(x) = \binom{\lceil \frac{K}{2} \rceil m}{x} p_a^x (1 - p_a)^{\lceil \frac{K}{2} \rceil m - x} \binom{\lfloor \frac{K}{2} \rfloor m + R - 1}{\alpha - x} (1 - p_b)^{\alpha - x} p_b^{\lfloor \frac{K}{2} \rfloor m + R - \alpha + x},$$

where $x \in \{\max(0, \alpha - \lfloor \frac{K}{2} \rfloor m - R + 1), \ldots, \min(\alpha, \lceil \frac{K}{2} \rceil m)\}$.

A.2 Proof of the complete formulae.

Since all the formulae follow a very similar pattern, we only provide formal proofs for two of the eight expressions, namely for $p_A^{\alpha,\beta,A}(x)$ when $R = 0$ and K is even and for $p_A^{\alpha,\beta,B}(x)$ when $R > 0$ and K is odd.

Suppose α and β are such that $\alpha + \beta = Km$ with K even, and player A scores the last point. In this setting, the last point is served by player B, hence only the outcome of $\frac{K}{2} m - 1$ serves of player B are random, while all the $\frac{K}{2} m$ serves of player A are random. Thanks to the independence of the rallies, we may consider the serves of player A and player B separately. Moreover, as explained in Section 2, each sequence of serves corresponds to a binomial distribution, with parameters $(\frac{K}{2} m, p_a)$ for player A serving and parameters $(\frac{K}{2} m - 1, 1 - p_b)$ for player B serving, where we adopt each time the point of view of player A. Since player A scores x points on his/her own serve, he/she necessarily wins $\alpha - x$ rallies initiated by player B. The last point being considered apart, we are left with $\alpha - x - 1$ successes for the binomial distribution with parameters $(\frac{K}{2} m - 1, 1 - p_b)$. Combining all these facts yields the announced formula, and it remains to establish the domain of the possible values of x.

As player B serves exactly $\frac{K}{2} m$ times, it is clear that, whenever $\alpha > \frac{K}{2} m$, necessarily player A has to score at least $\alpha - \frac{K}{2} m$ points on his/her own serve. Otherwise, he/she may very well score only on player B's serve. These two observations readily yield the lower bound for x. Regarding the upper bound, as player A scores at least one point not served by himself/herself, he/she cannot score more than $\alpha - 1$ points on his/her own serve. This directly yields the upper bound, as player A is limited to $\frac{K}{2} m$ serves.

Suppose now that $\alpha + \beta = Km + R$ with K odd and $R > 0$, and player B scores the last

point. In this case, player B also serves the last point; would R be zero, then the oddness of K would give the last serve to player A. With this said, player A serves exactly $\lceil \frac{K}{2} \rceil m$ points, while player B initiates $\lfloor \frac{K}{2} \rfloor m + R$ exchanges. The formula for $p_A^{\alpha,\beta,B}(x)$ and the corresponding lower and upper bound for x follow along the same lines as for $p_A^{\alpha,\beta,A}(x)$, which concludes the proof. $\qquad\square$

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