Rates of convergence towards the Fréchet distribution

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Abstract

We develop Stein's method for the Fréchet distribution and apply it to compute rates of convergence in distribution of renormalized sample maxima to the Fréchet distribution.

1 Introduction

Let $X_1, X_2, \dots$ be independent random variables with common distribution function F and let $M_n = \max(X_1, X_2, \dots, X_n)$. The distribution F is in the domain of attraction of a Fréchet distribution with index $\alpha > 0$, and we write $F \in DA(\alpha)$, if there exist normalizing constants $a_n > 0$ and b_n such that

$$P\left[(M_n - b_n)/a_n \le x\right] = F(a_n x + b_n)^n \longrightarrow \Phi_\alpha(x) \text{ as } n \to \infty, \tag{1}$$

where $\Phi_\alpha(x) = \exp(-x^{-\alpha})\mathbb{I}(x \ge 0)$ is the Fréchet cumulative distribution.

Specific sufficient and necessary conditions on F for (1) to hold have been long known.

Theorem 1.1 (Gnedenko [4])**.** *Let $L(t) = -t^{-\alpha} \log F(t)(t > 0)$. Then $F \in DA(\alpha)$ if, and only if, $F(x) < 1$ for all $x < \infty$ and $L(t)$ is slowly varying at ∞.*

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There exist no Berry-Esseen type results for Fréchet convergence because rates can vary much depending on the properties of F. However, good control on the function $L(t)$ allows to determine a sequence a_n for which precise rates of convergence in (1) are readily obtained; more precisely [8, 9] prove that

$$\sup_x |F(a_n x)^n - \Phi_\alpha(x)| = O(r_n) \tag{2}$$

where both a_n and r_n are explicit quantities depending on the behavior of L, see forthcoming Theorem 3.1 for details.

In this note we use a new version of Stein's method ([1, 2, 6]) to provide explicit (fixed n) bounds on $D(M_n, G)$ between the law of M_n and the Fréchet, with $D(\cdot,\cdot)$ a probability distance. We stress the fact that our take on the Stein's method rests on new identities for the Fréchet which do not fit within the recently developed general approaches to Stein's method via diffusions [3, 11] or via the so-called density approach [5, 10].

2 Stein's method for Fréchet distribution

Fix throughout $\alpha > 0$ and write $G \sim \Phi_\alpha$. Let $\mathcal{F}(\alpha)$ be the collection of all differentiable functions on $\mathbb{R}$ such that

$$\lim_{x\to\infty} \varphi(x) e^{-x^{-\alpha}} = \lim_{x\to 0} \varphi(x) e^{-x^{-\alpha}} = 0.$$

Note that the second condition is satisfied as soon as φ is well-behaved at 0, whereas the first condition requires that we impose $\lim_{x\to\infty}\varphi(x) = 0$. We define the differential operator

$$\mathcal{T}_\alpha \varphi(x) = \varphi'(x) x^{\alpha+1} + \alpha\varphi(x) \tag{3}$$

for all $\varphi \in \mathcal{F}(\alpha)$. Straightforward computations show that (3) satisfies $E[\mathcal{T}_\alpha\varphi(G)] = 0$ for all $\varphi \in \mathcal{F}(\alpha)$ and, more generally,

$$E[f(G)\mathcal{T}_\alpha\varphi(G)] = -E\left[G^{\alpha+1} f'(G)\varphi(G)\right] \tag{4}$$

for all $\varphi \in \mathcal{F}(\alpha)$ and all f such that $|f(0)| < \infty$ and $\lim_{x\to\infty}|f(x)| < \infty$.

Following the custom in Stein's method we consider Stein equations of the form

$$h(x) - Eh(G) = \mathcal{T}_\alpha\varphi(x) \tag{5}$$

for $h(x)$ some function such that $E|h(G)| < \infty$. We write the solution of (5) as $\varphi_h := \mathcal{T}_\alpha^{-1} h$. By definition of $\mathcal{T}_\alpha$ these are given by

$$\varphi_h(x) = e^{x^{-\alpha}} \int_0^x (h(y) - E[h(G)])\, y^{-\alpha-1} e^{-y^{-\alpha}} dy \tag{6}$$

or, equivalently,

$$\varphi_h(x) = e^{x^{-\alpha}} \int_x^\infty \left(E\left[h(G)\right] - h(y)\right) y^{-\alpha-1} e^{-y^{-\alpha}} dy \tag{7}$$

for all $x \geq 0$. By l'Hospital's rule, this function satisfies $\lim_{x\to 0} \varphi_h(x) = \lim_{x\to 0} \left(h(x) - E\left[h(G)\right]\right) = (h(0) - Eh(G))/\alpha$. Also we use the fact that $E|h(G)| < \infty$ to deduce that $\lim_{x\to\infty} \varphi_h(x) = 0$. Hence, in particular, this φ_h belong to $\mathcal{F}(\alpha)$ under reasonable assumptions on the function h. More precise estimates will be given in the sequel, as the need arises.

Next take $W_n = (M_n - b_n)/a_n$ with a_n, b_n and M_n as in the Introduction. Suppose that F has interval support I with closure $\bar{I} = [a, b]$ and let f_n be the probability density function of W_n; further suppose that f_n is differentiable on I except perhaps in a finite number of points and define

$$\rho_n(x) = (\log f_n(x))' = a_n \left((n-1) \frac{f(a_n x + b_n)}{F(a_n x + b_n)} + \frac{f'(a_n x + b_n)}{f(a_n x + b_n)} \right) \tag{8}$$

(the score function of W_n).

Let $\mathcal{F}(f_n)$ be the collection of all differentiable functions on $\mathbb{R}$ such that $\lim_{x\to a^+} \varphi(x) x^{\alpha+1} f_n(x) = \lim_{x\to b^-} \varphi(x) x^{\alpha+1} f_n(x) = 0$ and define the differential operator

$$\mathcal{T}_n \varphi(x) = \varphi'(x) x^{\alpha+1} + \varphi(x) x^\alpha \left(\alpha + 1 + x\rho_n(x)\right), \tag{9}$$

for all $\varphi \in \mathcal{F}(f_n)$. Then straightforward computations show that (9) satisfies

$$E\left[\mathcal{T}_n \varphi(W_n)\right] = 0 \tag{10}$$

for all $\varphi \in \mathcal{F}(f_n)$.

Our next result is an immediate consequence of the above identities.

Lemma 2.1. *Let W_n and G be as above, and let $\varphi_h = \mathcal{T}_\alpha^{-1} h$. For all h such that $\varphi \in \mathcal{F}(f_n)$ we have*

$$Eh(W_n) - Eh(G) = \alpha E\left[\varphi_h(W_n)\left(1 - W_n^\alpha \left(\frac{\alpha+1}{\alpha} + \frac{W_n}{\alpha}\rho_n(W_n)\right)\right)\right] \tag{11}$$

for all h.

Proof. Taking φ solution of (5), we use (10) to deduce

$$Eh(W_n) - Eh(G) = E\left[\mathcal{T}_\alpha \varphi(W_n)\right] = E\left[\mathcal{T}_\alpha \varphi(W_n)\right] - E\left[\mathcal{T}_n \varphi(W_n)\right].$$

Conclusion follows by definition of $\mathcal{T}_\alpha$ and $\mathcal{T}_n$. □

Example 2.1. *The standard example of convergence towards the Fréchet is the maximum of Pareto random variables. Taking* $F(x) = (1-x^{-\alpha})\mathbb{I}(x \geq 1)$ *and fix* $a_n = n^{1/\alpha}$, $b_n = 0$. *The cdf and pdf of* $W_n = M_n/n^{1/\alpha}$ *have support* $[n^{-1/\alpha}, +\infty)$ *and are*

$$F_n(x) = \left(1 - \frac{x^{-\alpha}}{n}\right)^n \text{ and } f_n(x) = \alpha x^{-\alpha-1}\left(1 - \frac{x^{-\alpha}}{n}\right)^{n-1}, \tag{12}$$

respectively. The score function is

$$\rho_n(x) = -\frac{\alpha+1}{x} + \frac{n-1}{n}\frac{\alpha}{x^{\alpha+1}}\left(1 - \frac{x^{-\alpha}}{n}\right)^{-1}$$

so that, from (11),

$$Eh(W_n) - Eh(G) = \alpha E\left[\varphi_h(W_n)\left(1 - \frac{n-1}{n}\left(1 - \frac{W_n^{-\alpha}}{n}\right)^{-1}\right)\right]$$

with $\varphi_h = \mathcal{T}_\alpha^{-1}h$. *We readily compute*

$$E\left[\left|1 - \frac{n-1}{n}\left(1 - \frac{W_n^{-\alpha}}{n}\right)^{-1}\right|\right] = \frac{2}{n-1}\left(1 - \frac{1}{n}\right)^n.$$

Finally, for $h(u) = \mathbb{I}(u \leq x)$, *the solutions* $\varphi_h(u)$ *satisfy* $\|\varphi_h\| \leq 1/\alpha$ *for all* t *(this is proved in the forthcoming Lemma 3.1). Hence*

$$\sup_{x\in\mathbb{R}} |P(W_n \leq x) - P(G \leq x)| \leq \frac{2e^{-1}}{n-1}.$$

3 Uniform rates of convergence towards Fréchet distribution

In light of (11) we have $\sup_{x\in\mathbb{R}} |F^n(a_n x) - \Phi_\alpha(x)| \leq \Delta_n$ with

$$\Delta_n = E\left|1 - W_n^\alpha\left(\frac{\alpha+1}{\alpha} + \frac{W_n}{\alpha}\rho_n(W_n)\right)\right|.$$

Hence, if

$$\rho_n(x) = \frac{\alpha}{x^{\alpha+1}}O(1+r_n) - \frac{\alpha+1}{x}O(1)$$

then we have convergence towards the Fréchet. This last condition is equivalent to

$$f_n(x) = x^{-\alpha-1} e^{-x^{-\alpha}} O(1+r_n) = \phi_\alpha(x) O(1+r_n).$$

with $r_n \to 0$, and thus a very transparent explanation of why $\Delta_n \to 0$ implies convergence towards the Fréchet.

More generally, suppose as in [8, 9] that Gnedenko's Theorem 1.1 is satisfied with $L(t)$ slowly varying with remainder g, that is take

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = O(g(x)). \tag{13}$$

Let $b_n = 0$ and a_n be such that $-\log F(a_n) \le n^{-1} \le -\log F(a_n^-)$. Then [9, equation (2.2)]

$$-n \log F(a_n) = 1 + O(g(a_n))$$

so that, as in [9, page 602],

$$\begin{aligned} -n \log F(a_n x) &= -n \frac{\log F(a_n x)}{\log F(a_n)} \log F(a_n) \\ &= x^{-\alpha} \frac{L(a_n x)}{L(a_n)} (1 + O(g(a_n))) \\ &= x^{-\alpha} (1 + O(g(a_n))) \end{aligned} \tag{14}$$

and thus

$$F^n(a_n x) - \Phi_\alpha(x) = O(r_n)$$

with $r_n = g(a_n)$. Hence the question of pointwise convergence of the law of M_n/a_n towards G is settled.

There remains the problem of determining rates at which the convergence takes place; this problem is tackled and solved in [9] under further assumptions on L. Here we use Lemma 2.1 to obtain show that no further assumptions are unnecessary. Our proof is, also, elementary.

Theorem 3.1. *Suppose that F satisfies (13). Let $b_n = 0$ and a_n be such that $-\log F(a_n) \le n^{-1} \le -\log F(a_n^-)$ and define $W_n = M_n/a_n$. Then*

$$Eh(W_n) - Eh(G) = -\alpha E\left[\mathcal{T}_\alpha^{-1} h(W_n)\right] O(g(a_n)) \tag{15}$$

for all h such that $E|h(W_n)| < \infty$ and $E|h(G)| < \infty$.

Proof. Starting from (14) we write, on the one hand,

$$a_n \frac{f(a_n x)}{F(a_n x)} = \frac{\alpha x^{-\alpha-1}}{n}\,(1+O(r_n))$$

and, on the other hand,

$$a_n \frac{f'(a_n x)}{f(a_n x)} = -\frac{\alpha+1}{x} + \alpha \frac{x^{-\alpha-1}}{n}\,(1+O(r_n))$$

so that

$$\rho_n(W_n) = \alpha W_n^{-\alpha-1}\,(1+O(r_n)) - \frac{\alpha+1}{W_n}.$$

Consequently

$$1 - W_n^{\alpha}\left(\frac{\alpha+1}{\alpha} + \frac{W_n}{\alpha}\rho_n(W_n)\right) = -\alpha O(r_n).$$

Plugging this expression into (11) we get (15). □

Let $\mathcal{H}$ be any class of functions and define the probability distance $d_{\mathcal{H}}(X,Y) = \sup_{h\in\mathcal{H}} |Eh(X) - Eh(Y)|$. Taking suprema on each side of (11) we get

$$d_{\mathcal{H}}(W_n, G) = \kappa_{\mathcal{H}} O(r_n) \tag{16}$$

for $\kappa_{\mathcal{H}} = \alpha \sup_{h\in\mathcal{H}} E\left|\mathcal{T}_\alpha^{-1} h\right|$. The following Lemma then settles the question of convergence in Kolmogorov distance.

Lemma 3.1. *If $h(x) = \mathbb{I}(x \le t)$ and $\varphi_h = \mathcal{T}_\alpha^{-1} h$ then $\|\varphi_h\|_\infty \le 1/\alpha$.*

Proof. With $h(x) = \mathbb{I}(x \le t)$ we apply (6) to get

$$\alpha\varphi_h(x) = \begin{cases} 1 - e^{-t^{-\alpha}} & \text{if } x \le t, \\ e^{-t^{-\alpha}}(e^{x^{-\alpha}} - 1) & \text{if } x > t. \end{cases}$$

We observe that $e^{-t^{-\alpha}}(e^{x^{-\alpha}} - 1) < (1 - e^{-t^{-\alpha}})$ if $x > t$ so that

$$|\varphi_h(x)| \le (1 - e^{-t^{-\alpha}})/\alpha$$

for all $x \ge 0$. The function $1 - e^{-t^{-\alpha}}$ is strictly decreasing for $t \in (0,\infty)$. Thus, $\|\varphi_h\|_\infty \le 1/\alpha$ for all t. □

Corollary 3.1. *As $n \to \infty$ we have*

$$\sup_x |F^n(a_n x) - \Phi_\alpha(x)| = O(r_n).$$

A local limit version of Corollary 3.1 (uniform convergence of the respective pdfs) was first obtained in [7]. In our notations it suffices to take $h_u(x) = \delta(x = u)$ (a Dirac delta) in (16) to get

$$f_n(u) - \phi_\alpha(u) = -\alpha E\left[\varphi_u(W_n)\right] O(r_n).$$

The problem remains to understand the quantity

$$E\left[\varphi_u(W_n)\right] = \phi_\alpha(u)\left(\int_u^\infty e^{w^{-\alpha}} f_n(w)dw - 1\right) \tag{17}$$

in terms of $u \in \mathbb{R}$. Still under the assumption (13) we can apply the same tools as in the proof of Theorem 3.1 to deduce

$$E\left[\varphi_u(W_n)\right] = \phi_\alpha(u)\left(\frac{1 - (O(r_n) + 1)e^{-O(r_n)u^{-\alpha}}}{O(r_n)}\right).$$

This quantity, as a function of u, is bounded uniformly by some constant depending on α. Hence we obtain the following improvement on [7].

Corollary 3.2. *Still under Assumption A we have*

$$\sup_{u\in\mathbb{R}} |f_n(u) - \phi_\alpha(u)| \le O(r_n)\kappa_\alpha$$

with $\kappa_\alpha = \sup_{u>0} \sup_n E\left[\varphi_u(W_n)\right]$.

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