Comparison inequalities on Wiener space

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Abstract: We define a covariance-type operator on Wiener space: for F and G two random variables in the Gross-Sobolev space $D^{1,2}$ of random variables with a square-integrable Malliavin derivative, we let $\Gamma_{F,G} := \left\langle DF, -DL^{-1}G \right\rangle$ where D is the Malliavin derivative operator and L^{-1} is the pseudo-inverse of the generator of the Ornstein-Uhlenbeck semigroup. We use Γ to extend the notion of covariance and canonical metric for vectors and random fields on Wiener space, and prove corresponding non-Gaussian comparison inequalities on Wiener space, which extend the Sudakov-Fernique result on comparison of expected suprema of Gaussian fields, and the Slepian inequality for functionals of Gaussian vectors. These results are proved using a so-called smart-path method on Wiener space, and are illustrated via various examples. We also illustrate the use of the same method by proving a Sherrington-Kirkpatrick universality result for spin systems in correlated and non-stationary non-Gaussian random media.

Key words: Gaussian Processes; Malliavin calculus; Ornstein-Uhlenbeck Semigroup.

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1 Introduction

The canonical metric of a centered field G on an index set T is the square root of the quantity $\delta_G^2(s,t) = \mathbf{E}\left[(G_t - G_s)^2\right]$, $s,t \in T$. When G is Gaussian, this δ^2 characterizes much of G's distribution, and is useful in various contexts for estimating G's behavior, from its modulus of continuity, to its expected supremum; see [1] for an introduction. The canonical metric, together with the variances of G, are of course equivalent to the covariance function $Q_G(s,t) = \mathbf{E}[G_t G_s]$, which defines G's law when G is Gaussian. In this article, we concentrate on comparison results for expectations of suprema and other types of functionals, beyond the Gaussian context, by using an extension of the concepts of covariance and canonical metric on Wiener space. We introduce these concepts now. For the details of analysis on Wiener space needed for the next definitions, including the space $\mathbb{D}^{1,2}$ and the operators D and L^{-1}, see Chapter 1 in [15] or Chapter 2 in [11]. The notion of a 'separable random field' is formally defined e.g. in [2, p. 8].

Definition 1.1 *Consider an isonormal Gaussian process W defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and associated with the real separable Hilbert space $\mathfrak{H}$: recall that this means that $W = \{W(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family such that $\mathbf{E}[W(h)W(k)] = \langle h,k \rangle_{\mathfrak{H}}$. Let $\mathbb{D}^{1,2}$ be the Gross-Sobolev*

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space of random variables F with a square-integrable Malliavin derivative, i.e. such that $DF \in L^2(\Omega \times \mathfrak{H})$. We denote the generator of the associated Ornstein-Uhlenbeck operator by L. For a pair of random variables $F, G \in \mathbb{D}^{1,2}$, we define a covariance-type operator by

$$\Gamma_{F,G} := \langle DF, -DL^{-1}G\rangle_{\mathfrak{H}}. \tag{1.1}$$

Let $F = \{F_t\}_{t\in T}$ be a separable random field on an index set T, such that $F_t \in \mathbb{D}^{1,2}$ for each $t \in T$. The analogue for the operator Γ of the covariance of F is denoted by

$$\Gamma_F(s,t) := \Gamma_{F_s,F_t} = \langle D(F_t), -DL^{-1}(F_s)\rangle_{\mathfrak{H}}. \tag{1.2}$$

The analogue for Γ of the canonical metric δ^2 of F is denoted by

$$\Delta_F(s,t) := \langle D(F_t - F_s), -DL^{-1}(F_t - F_s)\rangle_{\mathfrak{H}}. \tag{1.3}$$

Remark 1.2 *(i) When $F = \{F_t\}_{t\in T}$ is in the first Wiener chaos, and hence is a centered Gaussian field, Γ_F coincides with its covariance function Q_F.*

(ii) In general, the random variable $\Delta_F(s,t)$ is not positive. However, according e.g. to [10, Proposition 3.9], one has that $\mathbf{E}[\Delta_F(s,t)|F_t - F_s] \geqslant 0$, a.s.-$\mathbf{P}$.

The extension of the concept of covariance function given above in (1.1) appeared in [3] and in [12], respectively to aid in the study of densities of random vectors and of multivariate normal approximations, both on Wiener space. Comparison results on Wiener space have, in the past, focused on concentration or Poincaré inequalities: see [20]. Recently, the scalar analogue of the covariance operator above, i.e. $\Gamma_{F,F}$, was exploited to derive sharp tail comparisons on Wiener space, in [14] and [21].

The two main types of comparison results we will investigate herein are those of Sudakov-Fernique type and those of Slepian type. See [1, 2] for details of the classical proofs.

In the basic *Sudakov-Fernique* inequality, one considers two centered separable Gaussian fields F and G on T, such that $\delta_F^2(s,t) \geqslant \delta_G^2(s,t)$ for all $s,t \in T$; then $\mathbf{E}[\sup_T F] \geqslant \mathbf{E}[\sup_T G]$. Here T can be any index set, as long as the laws of F and G can be determined by considering only countably many elements of T; this works for instance if T is a subset of Euclidean space and F and G are a.s. continuous. To try to extend this result to non-Gaussian fields with no additional machinery, for illustrative purposes, the following setup provides an easy example.

Proposition 1.3 *Let F and G be two separable fields on T, with G and $F - G$ independent, and $E[F_t] = E[G_t]$ for every $t \in T$. Then $\mathbf{E}[\sup_T F] \geqslant \mathbf{E}[\sup_T G]$.*

The proof of this proposition is elementary. Let $H = F - G$. Note that for any $t_0 \in T$, $\mathbf{E}[H(t_0)] = 0$. We may write $\mathbf{P} = \mathbf{P}_H \times \mathbf{P}_F$ with obvious notation. Thus

$$\mathbf{E}\left[\sup_T F\right] = \mathbf{E}\left[\sup_T (H+G)\right] = \mathbf{E}_G\left[\mathbf{E}_H\left[\sup_T (H+G)\right]\right]$$

where under $\mathbf{P}_H$, G is deterministic. Thus

$$\mathbf{E}\left[\sup_T F\right] \geqslant \mathbf{E}_G\left[\mathbf{E}_H\left[H(t_0) + \sup_T G\right]\right] = \mathbf{E}_G\left[\mathbf{E}_H[H(t_0)] + \sup_T G\right] = \mathbf{E}_G\left[\sup_T G\right].$$

What makes this proposition so easy to establish is the very strong joint distributional assumption on (F, G), even though we do not make any marginal distributional assumptions about F and G. Also note that in the Gaussian case, the covariance assumption on (F, G) implies that $\delta_F^2(s,t) \geqslant \delta_G^2(s,t)$, and is in fact a much stronger assumption than simply comparing these canonical metrics, so that the classical Sudakov-Fernique inequality applies handily.

Let us now discuss the *Slepian inequality* similarly. In the basic inequality, consider two centered Gaussian vectors F and G in $\mathbb{R}^d$, with covariance matrices (B_{ij}) and (C_{ij}). Let $f \in C^2(\mathbb{R}^d)$ with bounded partial derivatives up to order 2. Assume that for all $x \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} (B_{ij} - C_{ij}) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geqslant 0.$$

Then $\mathbf{E}[f(F)] \geqslant \mathbf{E}[f(G)]$. To obtain such a result for non-Gaussian vectors, one may again try to impose strong joint-distributional conditions to avoid marginal conditions. The following example is a good illustration. With F and G two random vectors in $\mathbb{R}^d$ and f convex on $\mathbb{R}^d$, assume that $\mathbf{E}[F] = \mathbf{E}[G]$, $\mathbf{E}|f(F)| < \infty$, $\mathbf{E}|f(G)| < \infty$, and G and $F - G$ are independent. By convexity for any $c \in \mathbb{R}^d$ we have that

$$f(F - G + c) \geqslant f(c) + \langle \nabla f(c), F - G \rangle_{\mathbb{R}^d}.$$

Hence $\mathbf{E}[f(F - G + c)] \geqslant f(c)$. By choosing $c = G$ and then taking expectations, we get $\mathbf{E}[f(F)] \geqslant \mathbf{E}[f(G)]$, i.e. the Slepian inequality conclusion holds. In other word we have the following.

Proposition 1.4 *Let F and G be two random vectors in $\mathbb{R}^d$, with G and $F - G$ independent. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Assume $\mathbf{E}[F] = \mathbf{E}[G]$, $\mathbf{E}|f(F)| < \infty$, $\mathbf{E}|f(G)| < \infty$. Then $\mathbf{E}[f(F)] \geqslant \mathbf{E}[f(G)]$.*

To avoid very strong joint law assumptions on (F, G) such as those used in the two elementary propositions above, this paper concentrates instead on exploiting some mild assumptions on the marginals of F and G, particularly imposing Malliavin differentiability as in Definition 1.1. We will see in particular that to obtain a Sudakov-Fernique inequality for highly non-Gaussian fields, one can use Δ instead of δ^2, and to get a Slepian inequality in the same setting, one can use Γ_{F_i,F_j} and Γ_{G_i,G_j} instead of $B_{i,j}$ and $C_{i,j}$ respectively. The proofs we use are based on the technique of interpolation, and on the following integration-by-parts theorem on Wiener space, which was first introduced in [10] (also see Theorem 2.9.1 in [11]): for any centered $F, G \in \mathbb{D}^{1,2}$, $\mathbf{E}[FG] = \mathbf{E}[\Gamma_{F,G}]$. This formula is particularly useful when combined with the chain rule of the Malliavin calculus, to yield that for any $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{E}\left[\Phi'(F)^2\right] < \infty$,

$$\mathbf{E}[\Phi(F) G] = \mathbf{E}\left[\Phi'(F) \Gamma_{F,G}\right]. \tag{1.4}$$

The remainder of this paper is structured as follows. In Section 2, we prove a new Sudakov-Fernique inequality for comparing suprema of random fields on Wiener space, and show how this may be applied to the supremum of the solution of a stochastic differential equation with non-linear drift, driven by fractional Brownian motion. In Section 3, we prove a Slepian-type inequality for comparing non-linear functionals of random vectors on Wiener space, and apply it to a comparison result for perturbations of Gaussian vectors, and to a concentration inequality. Finally in Section 4, we show how to extend the universality class of the Sherrington-Kirkpatrick spin system, to some random media on Wiener space with dependence and non-stationarity. All our main theorems' proofs are based on the extension to Wiener space of the so-called smart-path method using the objects identified in Definition 1.1.

2 A result of Sudakov-Fernique type

The proof of the following result is based on an extension of classical computations based on a 'smart path method' that are available in the Gaussian setting. The reader is referred to [2, p. 61] for a similar proof (originally due to S. Chatterjee, see also [7]) in the simpler Gaussian setting.

Theorem 2.1 *Let $F = \{F_t\}_{t\in T}$ and $G = \{G_t\}_{t\in T}$ be separable centered random fields on an index set T, such that $F_t, G_t \in \mathbb{D}^{1,2}$ for every $t \in T$. Their canonical metrics on Wiener space, Δ_F and Δ_G, are defined according to (1.3). Assume that $\mathbf{E}[\sup_T F] < \infty$ and $\mathbf{E}[\sup_T G] < \infty$. Assume that almost surely for all $s, t \in T$,*

$$\Delta_F(s,t) \leqslant \Delta_G(s,t). \tag{2.5}$$

Assume furthermore that almost surely for all $s, t \in T$,

$$\Gamma_{F_s,G_t} = 0. \tag{2.6}$$

Then

$$\mathbf{E}\left[\sup_{t\in T} F_t\right] \leqslant \mathbf{E}\left[\sup_{t\in T} G_t\right].$$

Remark 2.2 *If (F, G) is jointly Gaussian, one can assume that both processes belong to the first Wiener chaos, and then*

$$\langle D(F_t - F_s), -DL^{-1}(F_t - F_s)\rangle_{\mathfrak{H}} = E[(F_t - F_s)^2],$$

and similarly for G. The orthogonality condition (2.6) is then equivalent to independence. As such, Theorem 2.1 extends the classical Sudakov-Fernique inequality, as stated e.g. in Vitale [22, Theorem 1] in the case $|T| < \infty$.

Corollary 2.3 *When G belongs to the first Wiener chaos (in particular, G is Gaussian), then $\Delta_G(s,t) = \delta_G^2(s,t)$ is G's (non-random) canonical metric, and the conclusion of Theorem 2.1 continues to hold without Assumption (2.6).*

Proof. Since Δ_G is non-random, the Gaussian process G in this corollary can be defined on any probability space, and thus we can assume that G is independent of F, and therefore that Assumption (2.6) holds. ■

Proof of Theorem 2.1.
Step 1: Approximation. For each $n > 0$, let T_n be a finite subset of T such that $T_n \subset T_{n+1}$ and T_n increases to a countable subset of T on which the laws of F and G are determined (for instance, if $T = \mathbb{R}_+$ and F and G are continuous, we may choose for T_n the set of dyadics of order n). By separability, as $n \to \infty$,

$$\sup_{t\in T_n} F_t \overset{\text{a.s.}}{\to} \sup_{t\in T} F_t \quad \text{and} \quad \sup_{t\in T_n} G_t \overset{\text{a.s.}}{\to} \sup_{t\in T} G_t$$

and, since the convergence is monotone, we also have that as $n \to \infty$,

$$\mathbf{E}\left[\sup_{t\in T_n} F_t\right] \to \mathbf{E}\left[\sup_{t\in T} F_t\right] \quad \text{and} \quad \mathbf{E}\left[\sup_{t\in T_n} F_t\right] \to \mathbf{E}\left[\sup_{t\in T} F_t\right].$$

Therefore, we assume without loss of generality in the remainder of the proof that $T = \{1, 2, \dots, d\}$ is finite.

Step 2: calculation. Fix $\beta > 0$, and consider, for any $t \in [0, 1]$,

$$\varphi(t) = \frac{1}{\beta} \mathbf{E}\left[\log\left(\sum_{i=1}^{d} e^{\beta(\sqrt{1-t}G_i + \sqrt{t}F_i)}\right)\right].$$

Let us differentiate φ with respect to $t \in (0, 1)$. We get

$$\varphi'(t) = \frac{1}{2}\sum_{i=1}^{d} \mathbf{E}\left[\left(\frac{1}{\sqrt{t}}F_i - \frac{1}{\sqrt{1-t}}G_i\right) h_{t,\beta,i}(F, G)\right], \tag{2.7}$$

where, for $x, y \in \mathbb{R}^d$, $i = 1, \dots, d$, $t \in (0, 1)$ and $\beta > 0$, we set

$$h_{t,\beta,i}(x, y) = \frac{e^{\beta(\sqrt{1-t}y_i + \sqrt{t}x_i)}}{\sum_{j=1}^{d} e^{\beta(\sqrt{1-t}y_j + \sqrt{t}x_j)}}.$$

Using the integration-by-parts formula (1.4) in (2.7) yields

$$\begin{aligned}
&\varphi'(t)\\
&= \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{1}{\sqrt{t}}\mathbf{E}\left[\frac{\partial h_{t,\beta,i}}{\partial x_j}(F, G)\Gamma_{F_j,F_i}\right] - \frac{1}{\sqrt{1-t}}\mathbf{E}\left[\frac{\partial h_{t,\beta,i}}{\partial y_j}(F, G)\Gamma_{G_j,G_i}\right]\right)\\
&+ \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{1}{\sqrt{t}}\mathbf{E}\left[\frac{\partial h_{t,\beta,i}}{\partial x_j}(F, G)\Gamma_{G_j,F_i}\right] - \frac{1}{\sqrt{1-t}}\mathbf{E}\left[\frac{\partial h_{t,\beta,i}}{\partial y_j}(F, G)\Gamma_{F_j,G_i}\right]\right).
\end{aligned} \tag{2.8}$$

The orthogonality assumption (2.6) implies that all the terms in the last line of (2.8) are zero. For $i \neq j$, we have

$$\begin{aligned}
\frac{\partial h_{t,\beta,i}}{\partial x_i}(x, y) &= \beta\sqrt{t}\big(h_{t,\beta,i}(x, y) - h_{t,\beta,i}(x, y)^2\big)\\
\frac{\partial h_{t,\beta,i}}{\partial x_j}(x, y) &= -\beta\sqrt{t}\, h_{t,\beta,i}(x, y)h_{t,\beta,j}(x, y)\\
\frac{\partial h_{t,\beta,i}}{\partial y_i}(x, y) &= \beta\sqrt{1-t}\big(h_{t,\beta,i}(x, y) - h_{t,\beta,i}(x, y)^2\big)\\
\frac{\partial h_{t,\beta,i}}{\partial y_j}(x, y) &= -\beta\sqrt{1-t}\, h_{t,\beta,i}(x, y)h_{t,\beta,j}(x, y).
\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi'(t) &= \frac{\beta}{2}\sum_{i}\mathbf{E}\Big[h_{t,\beta,i}(F, G)(1 - h_{t,\beta,i}(F, G))\big(\Gamma_{F_i,F_i} - \Gamma_{G_i,G_i}\big)\Big]\\
&\quad - \frac{\beta}{2}\sum_{i\neq j}\mathbf{E}\Big[h_{t,\beta,i}(F, G)h_{t,\beta,j}(F, G)\big(\Gamma_{F_i,F_j} - \Gamma_{G_i,G_j}\big)\Big]\\
&= \frac{\beta}{2}\sum_{i}\mathbf{E}\Big[h_{t,\beta,i}(F, G)\big(\Gamma_{F_i,F_i} - \Gamma_{G_i,G_i}\big)\Big]\\
&\quad - \frac{\beta}{2}\sum_{i,j}\mathbf{E}\Big[h_{t,\beta,i}(F, G)h_{t,\beta,j}(F, G)\big(\Gamma_{F_i,F_j} - \Gamma_{G_i,G_j}\big)\Big].
\end{aligned}$$

But $\sum_{i=1}^d h_{t,\beta,i}(F,G)=1$, hence $\varphi'(t)$ is given by

$$\frac{\beta}{4}\sum_{i,j=1}^d \mathbf{E}\Big[h_{t,\beta,i}(F,G)h_{t,\beta,j}(F,G)\big(\Delta_F\,(i,j)-\Delta_G\,(i,j)\,\big)\Big].$$

Step 3: estimation and conclusion. We observe that $h_{t,\beta,i}(F,G)>0$ for all i. Moreover, by assumption (2.5) we get $\varphi'(t)\leqslant 0$ for all t, implying in turn that $\varphi(0)\geqslant\varphi(1)$, that is

$$\frac{1}{\beta}\mathbf{E}\left[\log\left(\sum_{i=1}^d e^{\beta F_i}\right)\right]\leqslant\frac{1}{\beta}\mathbf{E}\left[\log\left(\sum_{i=1}^d e^{\beta G_i}\right)\right]$$

for any $\beta>0$. But

$$\max_{1\leqslant i\leqslant d}F_i=\frac{1}{\beta}\log\left(e^{\beta\times\max\limits_{1\leqslant i\leqslant d}F_i}\right)\leqslant\frac{1}{\beta}\log\left(\sum_{i=1}^d e^{\beta F_i}\right)\leqslant\frac{\log d}{\beta}+\max_{1\leqslant i\leqslant d}F_i,$$

and the same with G instead of F. Therefore

$$\mathbf{E}\left[\max_{1\leqslant i\leqslant d}F_i\right]\leqslant\mathbf{E}\left[\frac{1}{\beta}\log\left(\sum_{i=1}^d e^{\beta F_i}\right)\right]\leqslant\mathbf{E}\left[\frac{1}{\beta}\log\left(\sum_{i=1}^d e^{\beta G_i}\right)\right]\leqslant\frac{\log d}{\beta}+\mathbf{E}\left[\max_{1\leqslant i\leqslant d}G_i\right],$$

and the desired conclusion follows by letting β goes to infinity. ■

We now give an example of application of Theorem 2.1, to a problem of current interest in stochastic analysis.

2.1 Example: supremum of an SDE driven by fBm

Let B^H be a fractional Brownian motion with Hurst index $H>1/2$, let $b:\mathbb{R}\to\mathbb{R}$ be increasing and Lipschitz (in particular, $b'\geqslant 0$ almost everywhere), and let $x_0\in\mathbb{R}$. We consider the process $F=(F_t)_{t\in[0,T]}$ defined as the unique solution to

$$F_t=x_0+B_t^H+\int_0^t b(F_s)ds. \tag{2.9}$$

(For more details about this equation, we refer the reader to [16].) It is well-known (see e.g. [17] or [13]) that, for any $t\in(0,T]$, we have that $F_t\in\mathbb{D}^{1,2}$ with

$$D_uF_t=\mathbf{1}_{[0,t]}(u)\,\exp\left(\int_u^t b'(F_w)dw\right). \tag{2.10}$$

Fix $t > s \geqslant 0$. By combining (2.10) with a calculation technique described e.g. in [14, Proposition 3.7] based on the so-called Mehler formula, we get

$$\begin{aligned}
&\Delta_F(s,t) \qquad (2.11)\\
&= H(2H-1)\widehat{E}\Big\{\int_0^\infty e^{-z}\Big[\int_{[0,s]^2}\left(e^{\int_u^t b'(F_w)dw} - e^{\int_u^s b'(F_w)dw}\right) \\
&\qquad\qquad \times \left(e^{\int_v^t b'(F_w^{(z)})dw} - e^{\int_v^s b'(F_w^{(z)})dw}\right)|u-v|^{2H-2}dudv \\
&+ \int_{[0,s]\times[s,t]}\left(e^{\int_u^t b'(F_w)dw} - e^{\int_u^s b'(F_w)dw}\right)e^{\int_v^t b'(F_w^{(z)})dw}|u-v|^{2H-2}dudv \\
&+ \int_{[s,t]\times[0,s]} e^{\int_u^t b'(F_w)dw}\left(e^{\int_v^t b'(F_w^{(z)})dw} - e^{\int_v^s b'(F_w^{(z)})dw}\right)|u-v|^{2H-2}dudv \\
&+ \int_{[s,t]^2} e^{\int_u^t b'(F_w)dw + \int_v^t b'(F_w^{(z)})dw}|u-v|^{2H-2}dudv\Big]dz\Big\}.
\end{aligned}$$

Here, $F^{(z)}$ means the solution to (2.9), but when B^H is replaced by the new fractional Brownian motion $e^{-z}B^H + \sqrt{1-e^{-2z}}\widehat{B}^H$, for $\widehat{B}^H$ an independent copy of B^H, and $\widehat{E}$ is the mathematical expectation with respect to $\widehat{B}^H$ only. Because $b' \geqslant 0$, we see that

$$\begin{aligned}
\exp\left\{\int_u^t b'(F_w)dw\right\} - \exp\left\{\int_u^s b'(F_w)dw\right\} &\geqslant 0 \quad \text{for any } 0 \leqslant u \leqslant s < t, \\
\exp\left\{\int_v^t b'(F_w^{(z)})dw\right\} - \exp\left\{\int_v^s b'(F_w^{(z)})dw\right\} &\geqslant 0 \quad \text{for any } 0 \leqslant v \leqslant s < t, \\
\exp\left\{\int_u^t b'(F_w)dw + \int_v^t b'(F_w^{(z)})dw\right\} &\geqslant 1 \quad \text{for any } s \leqslant u, v \leqslant t.
\end{aligned}$$

In particular, $\Delta_F(s,t) \geqslant H(2H-1)\int_{[s,t]^2}|u-v|^{2H-2}dudv = |t-s|^{2H}$. We recognize $|t-s|^{2H}$ as the squared canonical metric of fractional Brownian motion, and we deduce from Theorem 2.1 (observe that it is not a loss of generality to have assumed that $s < t$) that

$$E\left[\max_{t\in[0,T]}\left(F_t - E[F_t]\right)\right] \geqslant E\left[\max_{t\in[0,T]} B_t^H\right].$$

Also note that by the same calculation as above, the inequality in the conclusion is reversed if b is decreasing.

3 A result of Slepian type

In Section 2, we investigated the ability to compare suprema of random vectors and fields based on covariances and the Wiener-space extensions of the concept of covariance in Definition 1.1. In this section, we show that these extensions also apply to functionals beyond the supremum, under appropriate convexity assumptions.

Theorem 3.1 *Let F, G be two centered rv's in $\mathbb{D}^{1,2}(\mathbb{R}^d)$, in other words, assume that for every $i = 1, 2, \cdots, d$, $F_i \in \mathbb{D}^{1,2}$ and $G_i \in \mathbb{D}^{1,2}$ and $\mathbf{E}[F_i] = \mathbf{E}[G_i] = 0$. Let also $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2-function. We define the $d \times d$ random "covariance"-type matrix*

$$\Gamma^F = \left\{\Gamma_{ij}^F := \Gamma_{F_i,F_j} : i, j = 1, \cdots, d\right\}$$

for F, according to (1.1), and similarly for Γ^G. We assume that $\mathbf{E}\left[\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\sqrt{1-t}G+\sqrt{t}F)\right|\right]$ is finite for every $i,j = 1,\cdots,d$ and $t \in [0,1]$, that $\Gamma_{F_i,G_j} = 0$ for any i,j and that for all $x \in \mathbb{R}^d$, almost surely,

$$\sum_{i,j=1}^{d} \left(\Gamma_{ij}^F - \Gamma_{ij}^G\right) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geqslant 0. \tag{3.12}$$

Then $\mathbf{E}[f(F)] \geqslant \mathbf{E}[f(G)]$.

Remark 3.2 *If F and G are Gaussian, then Γ^F and Γ^G are the covariance matrices of F and G a.s., and we recover the classical Slepian inequality, see e.g. [19], or the paragraph in the Introduction preceding Proposition 1.4.*

Corollary 3.3 *If F is Gaussian (but not necessarily G), then the conclusion of Theorem 3.1 holds without any information on the joint law of (F,G), except for assuming that if F and G are independent, then $\mathbf{E}\left[\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\sqrt{1-t}G+\sqrt{t}F)\right|\right]$ is finite for every $i,j = 1,\cdots,d$ and $t \in [0,1]$.*

Proof of Theorem 3.1. For $t \in [0,1]$, set

$$\varphi(t) = E[f(\sqrt{1-t}G+\sqrt{t}F)].$$

We have

$$\varphi'(t) = \frac{1}{2}\sum_{i=1}^{d}\left(\frac{1}{\sqrt{t}}E\left[\frac{\partial f}{\partial x_i}(\sqrt{1-t}G+\sqrt{t}F)F_i\right] - \frac{1}{\sqrt{1-t}}E\left[\frac{\partial f}{\partial x_i}(\sqrt{1-t}G+\sqrt{t}F)G_i\right]\right).$$

By using the integrating-by-parts formula (1.4), we get the following extension of a classical identity due to Piterbarg [18]:

$$\begin{aligned}
\varphi'(t) &= \frac{1}{2}\sum_{i,j=1}^{d} E\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\sqrt{1-t}G+\sqrt{t}F)\big(\langle DF_j, -DL^{-1}F_i\rangle_{\mathfrak{H}} - \langle DG_j, -DL^{-1}G_i\rangle_{\mathfrak{H}}\big)\right] \\
&= \frac{1}{2}\sum_{i,j=1}^{d} E\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\sqrt{1-t}G+\sqrt{t}F)\left(\Gamma_{ij}^F - \Gamma_{ij}^G\right)\right].
\end{aligned}$$

As a consequence, $\varphi'(t) \geqslant 0$ (resp. $\leqslant$), implying in turn that $\varphi(1) \geqslant \varphi(0)$ (resp. $\leqslant$), which is the desired conclusion. ∎

Proof of the corollary. When F is Gaussian, Γ^F is almost surely deterministic, and we may thus assume that F and G are defined on the same probability space and are independent. The finiteness of $\mathbf{E}\left[\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\sqrt{1-t}G+\sqrt{t}F)\right|\right]$ can then be assumed to hold, and the theorem applies. ∎

3.1 Example: perturbation of a Gaussian vector

Here we present an example of how to perturb an arbitrary Gaussian vector $G \in \mathbb{R}^d$ using a functional on Wiener space to guarantee that for any function f with non-negative (resp. non-positive) second derivatives, $f(G)$ sees its expectation increase (resp. decrease) with the perturbation. It is sufficient for the perturbation to be based on variables that are positively "correlated" to G, in a sense defined using the covariance operator Γ of Definition 1.1. Let C be the covariance matrix of G.

We may assume that for every $i = 1, \dots, d$, $G_i = I_1(g_i)$ where the g_i's are such that $\langle g_i, g_j \rangle_{\mathfrak{H}} = C_{i,j}$. Fix integers $n_1, \dots, n_d \geqslant 1$, let $f_{i,k}$ $i = 1, \dots, d$, $k = 1, \dots, n_d$, be a sequence of elements of H such that $\langle f_{i,k}, g_j \rangle_{\mathfrak{H}} \geqslant 0$ and $\langle f_{i,k}, f_{j,l} \rangle_{\mathfrak{H}} \geqslant 0$ for all i, j, k, l, and let $\Phi_i : \mathbb{R}^{n_i} \to \mathbb{R}$, $i = 1, \dots, d$, be a sequence of C^1-functions such that $\frac{\partial \Phi_i}{\partial x_k} \geqslant 0$ for all k (each Φ_i is increasing w.r.t. every component). For $i = 1, \dots, d$, we set

$$F_i = G_i + \Phi_i\big(I_1(f_{i,1}), \dots, I_1(f_{i,n_i})\big).$$

Our assumptions are simply saying that all the Gaussian pairs $(G_j, I_1(f_{i,k}))$ are non-negatively correlated, as are all the Gaussian pairs $(I_1(f_{i,k}), I_1(f_{j,\ell}))$. For any $i, j = 1, \dots, d$, we compute

$$\begin{aligned} DF_i &= g_i + \sum_{k=1}^{n_i} \frac{\partial \Phi_i}{\partial x_k}(I_1(f_{i,1}), \dots, I_1(f_{i,n_i})) f_{i,k} \\ P_z DF_j &= g_j + \sum_{l=1}^{n_j} \widehat{E}\left[\frac{\partial \Phi_j}{\partial x_l}(I_1^{(z)}(f_{j,1}), \dots, I_1^{(z)}(f_{j,n_j}))\right] f_{j,l}, \end{aligned}$$

where $I_1^{(z)}$ means that the Wiener integral is taken with respect to $W^{(z)} = e^{-z}W + \sqrt{1-e^{-2z}}\widehat{W}$ instead of W, for $\widehat{W}$ an independent copy of W, and where $\widehat{E}$ is the mathematical expectation with respect to $\widehat{W}$ only. Therefore, using the Mehler-formula representation of DL^{-1} (see [14]),

$$\begin{aligned} &\Gamma_{i,j} := \Gamma_{F_i,F_j} = \int_0^\infty e^{-z} \langle DF_i, P_z DF_j \rangle_{\mathfrak{H}} dz \\ &= C_{i,j} + \sum_{k=1}^{n_i} \frac{\partial \Phi_i}{\partial x_k}(I_1(f_{i,1}), \dots, I_1(f_{i,n_i})) \langle f_{i,k}, g_j \rangle_{\mathfrak{H}} \\ &+ \sum_{l=1}^{n_j} \langle f_{j,l}, g_i \rangle_{\mathfrak{H}} \int_0^\infty e^{-z} \widehat{E}\left[\frac{\partial \Phi_j}{\partial x_l}(I_1^{(z)}(f_{j,1}), \dots, I_1^{(z)}(f_{j,n_j}))\right] dz \\ &+ \sum_{k=1}^{n_i} \sum_{l=1}^{n_j} \langle f_{i,k}, f_{j,l} \rangle_{\mathfrak{H}} \frac{\partial \Phi_i}{\partial x_k}(I_1(f_{i,1}), \dots, I_1(f_{i,n_i})) \int_0^\infty e^{-z} \widehat{E}\left[\frac{\partial \Phi_j}{\partial x_l}(I_1^{(z)}(f_{j,1}), \dots, I_1^{(z)}(f_{j,n_j}))\right] dz. \end{aligned}$$

Using the assumptions, we see that $\Gamma_{i,j} \geqslant \langle g_i, g_j \rangle_{\mathfrak{H}}$ for all $i, j = 1, \dots, d$. Hence, for all C^2-function $\Psi : \mathbb{R}^d \to \mathbb{R}$ such that $\frac{\partial^2 \Psi}{\partial x_i \partial x_j}(x) \geqslant 0$ (resp. $\leqslant$), condition (3.12) is in order, so that $E[\Psi(F)] \geqslant E[\Psi(G)]$ (resp. $\leqslant$) by virtue of Theorem 3.1.

3.2 Example: a concentration inequality

Next we encounter an application of Theorem 3.1 to compare distributions of non-Gaussian vectors to Gaussian distributions.

Corollary 3.4 *Let $F = (F_1, \dots, F_d) \in \mathbb{R}^d$ be such that $F_i \in \mathbb{D}^{1,2}$ and $E[F_i] = 0$ for every i, and define $\Gamma = \big\{\Gamma_{ij} := \Gamma_{F_i,F_j} : i, j = 1, \cdots, d\big\}$, according to (1.1). Let C be a deterministic non-negative definite $d \times d$ matrix such that, almost surely, $C - \Gamma$ is non-negative definite. Then, with*

$\|C\|_{op}$ *the operator norm of* C*, for any* $x_1, \ldots, x_d \geqslant 0$*, we have*

$$P[F_1 \geqslant x_1, \ldots, F_d \geqslant x_d] \leqslant \exp\left\{-\frac{x_1^2 + \ldots + x_d^2}{2\|C\|_{op}}\right\}.$$

Proof. For any $\theta \in \mathbb{R}_+^d$, we can write

$$P[F_1 \geqslant x_1, \ldots, F_d \geqslant x_d] \leqslant P\big[\langle \theta, F\rangle_{\mathbb{R}^d} \geqslant \langle \theta, x\rangle_{\mathbb{R}^d}\big] \leqslant e^{-\langle \theta, x\rangle_{\mathbb{R}^d}} E[e^{\langle \theta, F\rangle_{\mathbb{R}^d}}].$$

Let $f : x \mapsto e^{\langle \theta, x\rangle_{\mathbb{R}^d}}$. This is a C^2 function with $\frac{\partial^2 f}{\partial x_i \partial x_j} = \theta_i \theta_j f$.

We first need to check the integrability assumption on f in Theorem 3.1. This is equivalent to $E[e^{\langle \theta, F\rangle_{\mathbb{R}^d}}] < \infty$. To prove this integrability, we compute

$$\Gamma_{\langle \theta, F\rangle, \langle \theta, F\rangle} = \sum_{i,j} \theta_i \theta_j \Gamma_{ij},$$

and we note by the positivity of $C - \Gamma$ that this is bounded above almost surely by the non-random positive constant $K := \sum_{i,j} \theta_i \theta_j C_{ij}$. This implies (see for instance [21]) that $P\left[\langle \theta, F\rangle / K > x\right] \leqslant \Phi(x)$ where Φ is the standard normal tail. The finiteness of $E[e^{\langle \theta, F\rangle_{\mathbb{R}^d}}]$ follows immediately.

Next, by the positivity of $C - \Gamma$,

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\left(\Gamma_{ij} - C_{ij}\right) = f(x) \sum_{i,j} \theta_i \theta_j \left(\Gamma_{ij} - C_{ij}\right) \leqslant 0.$$

This is condition (3.12), so that Theorem 3.1 implies that $E[e^{\langle \theta, F\rangle_{\mathbb{R}^d}}] \leqslant E[e^{\langle \theta, G\rangle_{\mathbb{R}^d}}]$ with G a centered Gaussian vector with covariance matrix C. Therefore, since $E[e^{\langle \theta, G\rangle_{\mathbb{R}^d}}] = e^{\frac{1}{2}\langle \theta, C\theta\rangle_{\mathbb{R}^d}}$, we have

$$P[F_1 \geqslant x_1, \ldots, F_d \geqslant x_d] \leqslant e^{-\langle \theta, x\rangle_{\mathbb{R}^d} + \frac{1}{2}\langle \theta, C\theta\rangle_{\mathbb{R}^d}} \leqslant e^{-\langle \theta, x\rangle_{\mathbb{R}^d} \frac{1}{2}\|C\|_{op}\|\theta\|^2_{\mathbb{R}^d}}.$$

The desired conclusion follows by choosing $\theta = x/\|C\|_{op}$, which represents the optimal choice. ■

4 Universality of the Sherrington-Kirkpatrick model with correlated media

Let N be a positive integer, and let $S_N = \{-1, 1\}^N$, which represents the set of all possible configurations of the spins of particles sitting at the integer positions from 1 to N. A parameter $\beta > 0$ is interpreted as the system's inverse temperature. Denote by $d\sigma$ the uniform probability measure on S_N, i.e. such that for every $\sigma \in S_N$, the mass of $\{\sigma\}$ is 2^{-N}. For any Hamiltonian H defined on S_N, we can define a probability measure P_N^H via $P_N^H(d\sigma) = d\sigma \exp(-\beta H(\sigma)) / Z_N^H$ where Z_N^H is a normalizing constant. Therefore,

$$Z_N^H = 2^{-N} \sum_{\sigma \in S_N} \exp\left(-\beta H(\sigma)\right). \tag{4.13}$$

The measure P_N^H is the distribution of the system's spins under the influence of the Hamiltonian H. The classical Sherrington-Kirkpatrick (SK, for short) model for spin systems is a random probability measure in which the Hamiltonian is random, because of the presence of an external random field $J = \{J_{i,j} : i, j = 1, \cdots, N; i > j\}$ where the random variables $J_{i,j}$ are IID standard normal (and for notational convenience we assume the matrix J is defined as being symmetric), and $H = H_N$ is given by

$$H_N(\sigma) := \frac{1}{\sqrt{2N}} \sum_{i \neq j} \sigma_i \sigma_j J_{i,j}. \tag{4.14}$$

The fact that the $J_{i,j}$'s are IID implies that there is no geometry in the spin system. Indeed, in the sense of distributions w.r.t. the law of J, the interactions between the sites $\{1, \cdots, N\}$ implied by the definition of P_N^H do not distinguish between how far apart the sites are. Such a model is usually called "mean-field", for this lack of geometry. The centered Gaussian character of the external field J is also an important element in the SK model's definition, particularly because it implies a behavior for H_N of order $\sqrt{N}$, which can be observed for instance by computing the variance of $H_N(\sigma)$ w.r.t. J for any fixed spin configuration σ: it equals $N - 1$. A quantity of importance in the study of the behavior of the measure P_N^H is its partition function, or free energy, the scalar Z_N^H in (4.13). In particular, one would like to prove that it has an almost-sure Lyapunov exponent, namely, a.s. the following limit exists and is finite:

$$p(\beta) := \lim_{N \to \infty} \frac{1}{N} \log Z_N^H. \tag{4.15}$$

A proof strategy was defined by Guerra and Toninelli [8]. In this classical case, the limit, which we denote by $p_{SK}(\beta)$, is also known as the Parisi formula (see [9] and [5, page 251]). A universality result, where the Gaussian assumption can be dropped in favor of requiring only three moments for J, with the same Parisi formula for the limit of the normalized log free energy, was established in [6].

In the theorem below, we show that the existence and finiteness of $p(\beta)$, and its equality with $p_{SK}(\beta)$, extends to external fields J on Wiener space which contain some non-stationarity and some dependence. Our proof's idea is to use the same smart-path techniques on Wiener space used in the proofs of Theorems 2.1 and 3.1, and compare Z_N^H with the free energy of a spin system with IID media J^*. As explained in more detail in Remark 4.2 below, Condition (ii) in the theorem is designed to allow for correlations in J, while Condition (iii) implies that the two random media have some asymptotic proximity in law.

Theorem 4.1 *Let $J = \{J_{i,j} : 1 \leqslant j < i\}$ and $J^* = \left\{J^*_{i,j} : 1 \leqslant j < i\right\}$ be two families of centered r.v.'s in $\mathbb{D}^{1,2}$ such that*

(i) $\left\{J^*_{i,j} : 1 \leqslant j < i\right\}$ *are IID with variance 1 and $\Gamma_{J^*_{i,j}, J^*_{k,\ell}} = 0$ for all $(i,j) \neq (k,\ell)$,*

(ii) $\sum_{1 \leqslant j < i \leqslant N} \mathbf{E}\left[\left|\Gamma_{J_{i,j}, J_{k,\ell}}\right|\right] = o\left(N^2\right)$,

(iii) $\sum_{1 \leqslant j < i \leqslant N} \mathbf{E}\left[\left|\Gamma_{J_{i,j}, J_{i,j}} - \Gamma_{J^*_{i,j}, J^*_{i,j}}\right|\right] = o\left(N^2\right)$,

(iv) $\Gamma_{J_{i,j}, J^*_{k,\ell}} = \Gamma_{J^*_{i,j}, J_{k,\ell}} = 0$ *for all i, j, k, ℓ.*

Let Z_N^H be the free energy relative to J, as in (4.13), (4.14). We have $\lim_{N \to \infty} N^{-1} \log Z_N^H = p_{SK}(\beta)$ in probability. If moreover there exists $\varepsilon > 0$ such that

(v) $\sup_{i,j} \mathbf{E}\left[\left|\Gamma_{J_{i,j};J_{i,j}}\right|^{1+\varepsilon}\right] =: M < \infty$,

then the convergence holds almost surely; more specifically, for any $\delta < 2^{-1}\varepsilon/(1+\varepsilon)$*, as* $N \to \infty$*, a.s.*

$$\frac{1}{N} \log Z_N^H = p_{SK}(\beta) + o(N^{-\delta}).$$

Remarks 4.2

1. *The model in the theorem is the classical SK model (where J is IID standard normal) as soon as* $\Gamma_{J_{i,j},J_{i,j}} \equiv 1$ *a.s.*

2. *The classical universality result of Carmona and Hu in [6] assumes that J is IID and has three moments. Here we do away with the IID assumption for J, comparing it to an IID J^* with two moments, obtaining new SK-universality classes.*

3. *Condition (ii) above is a way to control the correlations of J. For instance, it is satisfied as soon as* $\mathbf{E}\left[\left|\Gamma_{J_{i,j},J_{k,\ell}}\right|\right] \leqslant (|i-k| + |j-\ell|)^{-r}$ *for* $r > 2$*. Since by formula (1.4),* $\mathbf{E}\left[\left|\Gamma_{J_{i,j},J_{k,\ell}}\right|\right] \geqslant \left|\mathbf{E}\left[\Gamma_{J_{i,j},J_{k,\ell}}\right]\right| = \left|\mathbf{E}\left[J_{i,j}J_{k,\ell}\right]\right|$*, this implies a corresponding decorrelation rate.*

4. *Condition (iii) in this corollary can be understood as a kind of Cesaro-type convergence in distribution. For illustrative purposes, consider the case where the comparison is with the SK model: we have* $\Gamma_{J^*_{i,j},J^*_{i,j}} \equiv 1$*, and the interpretation of Condition (iii) can be made more precise. Indeed, by Theorem 5.3.1 in [11], this type of convergence roughly leads to convergence of $J_{i,j}$ to a standard normal as i and/or $j \to \infty$ with N.*

Proof of Theorem 4.1:

Step 1: a generic result. We begin by showing a precursor result for convergence in probability, for a generic situation. Assume that J and J^* satisfy merely (ii), (iii), and (iv). We will show that for any $f \in C^2(\mathbb{R})$ with $\|f'\|_\infty \leqslant 1$ and $\|f''\|_\infty \leqslant 1$,

$$\left|\mathbf{E}\left[f(\frac{1}{N}\log Z_N^{*H})\right] - \mathbf{E}\left[f(\frac{1}{N}\log Z_N^{H})\right]\right| = o(1). \tag{4.16}$$

We compactify the notation by reindexing the set $\{i,j : i > j; i,j = 1,\cdots,N\}$ as the set $\{1,2,\cdots,\bar{N}\}$ where $\bar{N} := N(N-1)/2$, with a bijection mapping each $n = 1,\cdots,\bar{N}$ to a pair (i,j), using any fixed bijection, with $\bar{J}_n := J_{i,j}$, $\bar{J}^*_n := J^*_{i,j}$, and $\tau_n := \sigma_i\sigma_j$, with P_σ the uniform probability measure on S_N, so that each r.v. τ_n under P_σ is dominated by 1. We use $\bar{J}$ and $\bar{J}^*$ to denote the corresponding $\bar{N}$-dimensional random vectors.

Fix $\gamma > 0$, $c \in [0,1]$ and f as above. We define for any vector $u \in \mathbb{R}^{\bar{N}}$, and $t \in [0,1]$,

$$Z(\gamma,u) := E_\sigma\left[\exp\left(\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n\right)\right],$$
$$\varphi(t) := \mathbf{E}[f(c\log Z(\gamma,\sqrt{t}\bar{J}^* + \sqrt{1-t}\bar{J}))].$$

For $i=1,\dots,\bar{N}$ and $u\in\mathbb{R}^{\bar{N}}$, we define

$$h_i(u) := \frac{E_\sigma[\tau_i e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]}{E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]} f'(c\log E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]).$$

We compute that for any $i,j=1,\dots,\bar{N}$, we have $\frac{\partial h_i}{\partial u_j}(u)=\gamma\, S_{i,j}(u)$ where

$$\begin{aligned} S_{i,j}(u) &:= \left(\frac{E_\sigma[\tau_i\tau_j e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]}{E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]} - \frac{E_\sigma[\tau_i e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]E_\sigma[\tau_j e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]}{E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]^2}\right) f'(c\log E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]) \\ &\quad + c\,\frac{E_\sigma[\tau_i e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]E_\sigma[\tau_j e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]}{E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]^2} f''(c\times\log E_\sigma[e^{\gamma\sum_{n=1}^{\bar{N}}\tau_n u_n}]). \end{aligned}$$

Notice that since c,τ_i,f', and f'' are all dominated by 1, we get $|S_{i,j}(u)|\leqslant 3$. Using the chain rule of standard calculus,

$$\varphi'(t) = \frac{c\,\gamma}{2}\sum_{i=1}^{\bar{N}}\left\{\frac{1}{\sqrt{t}}\mathbf{E}[\bar{J}_i^* h_i(\sqrt{t}\bar{J}^*+\sqrt{1-t}\bar{J})] - \frac{1}{\sqrt{1-t}}\mathbf{E}[\bar{J}_i h_i(\sqrt{t}\bar{J}^*+\sqrt{1-t}\bar{J})]\right\}.$$

Now using the integration-by-parts formula on Wiener space (1.4), and Condition (iv), this computes as

$$\begin{aligned} \varphi'(t) &= \frac{c\,\gamma^2}{2}\sum_{i=1}^{\bar{N}}\mathbf{E}\left[S_{i,i}(\sqrt{t}\bar{J}^*+\sqrt{1-t}\bar{J})(\Gamma_{\bar{J}_i^*,\bar{J}_i^*}-\Gamma_{\bar{J}_i,\bar{J}_i})\right] \\ &\quad + \frac{c\,\gamma^2}{2}\sum_{1\leqslant i\neq j\leqslant\bar{N}}\mathbf{E}\left[S_{i,j}(\sqrt{t}\bar{J}^*+\sqrt{1-t}\bar{J})\Gamma_{\bar{J}_i,\bar{J}_j}\right]. \end{aligned}$$

The boundedness of $|S_{i,j}(u)|$ by 3 yields, by integrating over $t\in[0,1]$, that

$$\begin{aligned} &\left|\mathbf{E}\left[f(c\log Z_{\bar{N}}(\gamma,\bar{J}^*))\right]-\mathbf{E}\left[f(c\log Z_{\bar{N}}(\gamma,\bar{J}))\right]\right| = \left|\int_0^1\varphi'(t)dt\right| \\ &\leqslant \frac{3c\,\gamma^2}{2}\sum_{i=1}^{\bar{N}}\mathbf{E}\left[\left|\Gamma_{\bar{J}_i^*,\bar{J}_i^*}-\Gamma_{\bar{J}_i,\bar{J}_i}\right|\right] + \frac{3c\,\gamma^2}{2}\sum_{1\leqslant i\neq j\leqslant\bar{N}}\mathbf{E}\left[\left|\Gamma_{\bar{J}_i,\bar{J}_j}\right|\right]. \end{aligned}$$

By Conditions (ii) and (iii), replacing γ by $\beta/\sqrt{N}$ and c by $1/N$, with $\bar{N}=N(N-1)/2$, relation (4.16) follows.

Step 2: Convergences. In this step we assume for the moment that $\lim_{N\to\infty}N^{-1}\log Z_N^{*H}=p_{SK}(\beta)$ holds in probability. This convergence is established below in Step 3. Combining this convergence and relation (4.16), we get that $N^{-1}\log Z_N^H$ converges in distribution, and thus in probability, to $p_{SK}(\beta)$, which is the first conclusion of the theorem. To establish the second conclusion, i.e. the almost-sure convergence, let

$$F_N := \frac{1}{N}\log Z_N^H - \frac{1}{N}\mathbf{E}\left[\log Z_N^H\right].$$

By the chain rule of Malliavin calculus, and using the notation E_N^H for expectations of functions of the configuration σ under the polymer measure defined by

$$P_N^H(\{\sigma\}) = \frac{1}{2^N} \frac{\exp(-\beta\ H_N(\sigma))}{\sum_{\sigma\in S_N} \exp(-\beta\ H_N(\sigma))},$$

we compute

$$\begin{aligned} DF_N &= \frac{1}{N}\frac{1}{Z_N^H}\left(-\beta 2^{-N}\right)\sum_{\sigma\in S_N} \exp(-\beta\ H_N(\sigma))\, DH_N(\sigma) \\ &= \frac{-\beta}{N} E_N^H\left[DH_N(\sigma)\right]. \end{aligned}$$

Now, using the intermediary of the Mehler formula (see, e.g., [14, Proposition 3.7]), it is easy to check that we can express

$$\Gamma_{F_N,F_N} = \frac{\beta^2}{N^2} E_N^H \otimes \tilde{E}_N^H\left[\Gamma_{H_N(\sigma),H_N(\tilde{\sigma})}\right]$$

where for fixed random medium J, under $P_N^H \otimes \tilde{P}_N^H$, $(\sigma, \tilde{\sigma})$ are two independent copies of σ under the polymer measure P_N^H. We compute for any $\sigma, \sigma' \in S_N$,

$$\Gamma_{H_N(\sigma),H_N(\sigma')} = \frac{2}{N}\sum_{1\leqslant j<i\leqslant N} \Gamma_{J_{i,j},J_{i,j}}\sigma_i\sigma_i'\sigma_j\sigma_j'.$$

Since $|\sigma_i| = 1$ for any $\sigma \in S_N$, we get

$$|\Gamma_{F_N,F_N}| \leqslant \frac{2\beta^2}{N^3}\sum_{1\leqslant j<i\leqslant N} \left|\Gamma_{J_{i,j},J_{i,j}}\right|. \tag{4.17}$$

By Assumption (v), $\mathbf{E}\left[\left|\Gamma_{J_{i,j},J_{i,j}}\right|^{1+\varepsilon}\right]$ is uniformly bounded by M. Therefore, using Jensen's inequality for the uniform measure on the set $\{i, j = 1, \cdots, N;\ i > j\}$ and the power function $|x|^{1+\varepsilon}$,

$$\begin{aligned} \mathbf{E}\left[\left|\Gamma_{F_N,F_N}\right|^{1+\varepsilon}\right] &\leqslant \left(\frac{2\beta^2}{N^3}\right)^{1+\varepsilon} (N(N-1)/2)^{1+\varepsilon}\, \frac{2}{N(N-1)}\sum_{1\leqslant j<i\leqslant N} \mathbf{E}\left[\left|\Gamma_{J_{i,j},J_{i,j}}\right|^{1+\varepsilon}\right] \\ &\leqslant M\beta^{2+2\varepsilon}N^{-1-\varepsilon}. \end{aligned}$$

We now need a Poincaré-type inequality on Wiener space relative to the operator Γ, which is recorded and proved below in Lemma 4.3: applying this lemma with $F = F_N$ and $p = 2 + 2\varepsilon$ yields

$$\mathbf{E}\left[|F_N|^{2+2\varepsilon}\right] \leqslant (1 + 2\varepsilon)^{1+\varepsilon} M\beta^{2+2\varepsilon}N^{-1-\varepsilon}.$$

A standard application of the Borel-Cantelli lemma via Chebyshev's inequality yields that for any $\delta < 2^{-1}\varepsilon/(1 + \varepsilon)$, almost surely, $F_N = o(N^{-\delta})$, as announced in the theorem.

Step 3: Conclusion. To finish the proof of the theorem, we only need to show that $\lim_{N\to\infty} N^{-1}\log Z_N^{*H} = p_{SK}(\beta)$ holds in probability. The universality result of Carmona and Hu as stated in [6] shows that this convergence holds if we assumed in addition that $J_{i,j}^*$ had a finite third moment. However, an inspection of their proof reveals that the convergence holds in probability without the third moment condition: one may use a computation similar to the calculation in Step 1 above, to establish this; the details are omitted. ■

Lemma 4.3 *For any centered $F \in \mathbf{D}^{1,2}$, and any $p \geqslant 2$,*

$$\mathbf{E}\left[|F|^p\right] \leqslant (p-1)^{p/2}\,\mathbf{E}\left[|\Gamma_{F,F}|^{p/2}\right].$$

Proof. For $p = 2$, by relation (1.4), the inequality holds almost as an equality (one has $\mathbf{E}[F^2] = \mathbf{E}[\Gamma_{F,F}] \leqslant \mathbf{E}[|\Gamma_{F,F}|]$). Therefore we assume $p > 2$. With the notation $G(x) = \mathrm{sgn}(x)\,|x|^{p-1}$, and thus $G'(x) = (p-1)\,\mathrm{sgn}(x)\,|x|^{p-2}$, and $G(F_N) \in \mathbf{D}^{1,2}$ with $D(G(F)) = (p-1)\,\mathrm{sgn}(F)\,|F|^{p-2}\,DF$, we have, using again (1.4),

$$\mathbf{E}\left[|F|^p\right] = \mathbf{E}\left[FG(F)\right] = (p-1)\,\mathbf{E}\left[\mathrm{sgn}(F)\,|F|^{p-2}\,\Gamma_{F,F}\right].$$

Now invoking Hölder's inequality we get

$$\mathbf{E}\left[|F|^p\right] \leqslant (p-1)\,\mathbf{E}\left[|\Gamma_{F,F}|^{p/2}\right]^{2/p}\mathbf{E}\left[|F|^p\right]^{1-2/p}.$$

The lemma follows immediately. ■

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