Multidimensional semicircular limits on the free Wigner chaos

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Abstract: We show that, for sequences of vectors of multiple Wigner integrals with respect to a free Brownian motion, componentwise convergence to semicircular is equivalent to joint convergence. This result extends to the free probability setting some findings by Peccati and Tudor (2005), and represents a multidimensional counterpart of a limit theorem inside the free Wigner chaos established by Kemp, Nourdin, Peccati and Speicher (2011).

Key words: Convergence in Distribution; Fourth Moment Condition; Free Brownian Motion; Free Probability; Multidimensional Limit Theorems; Semicircular Law; Wigner Chaos.

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1. Introduction

Let $W = \{W_t : t \geq 0\}$ be a one-dimensional standard Brownian motion (living on some probability space $(\Omega, \mathscr{F}, P)$). For every $n \geq 1$ and every real-valued, symmetric and square-integrable function $f \in L^2(\mathbb{R}_+^n)$, we denote by $I^W(f)$ the multiple Wiener-Itô integral of f, with respect to W. Random variables of this type compose the so-called nth *Wiener chaos* associated with f. In an infinite-dimensional setting, the concept of Wiener chaos plays the same role as that of the Hermite polynomials for the one-dimensional Gaussian distribution, and represents one of the staples of modern Gaussian analysis (see e.g. [5, 10, 13, 15] for an introduction to these topics).

In recent years, many efforts have been made in order to characterize Central Limit Theorems (CLTs) – that is, limit theorems involving convergence in distribution to a Gaussian element – for random variables living inside a Wiener chaos. The following statement gathers the main findings of [14] (Part 1) and [16] (Part 2), and provides a complete characterization of (both one- and multi-dimensional) CLTs on the Wiener chaos.

Theorem 1.1 (See [14, 16]). (A) *Let $F_k = I^W(f_k)$, $k \geq 1$, be a sequence of multiple integrals of order $n \geq 2$, such that $E[F_k^2] \to 1$. Then, the following two assertions are equivalent, as $k \to \infty$:* **(i)** *F_k converges in distribution to a standard Gaussian random variable $N \sim \mathscr{N}(0,1)$;* **(ii)** *$E[F_k^4] \to 3 = E[N^4]$.*

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(B) *Let* $d \geq 2$ *and* $n_1, ..., n_d$ *be integers, and let* $(F_k^{(1)}, ..., F_k^{(d)})$, $k \geq 1$, *be a sequence of random vectors such that, for every* $i = 1, ..., d$, *the random variable* $F_k^{(i)}$ *lives in the* n_i*th Wiener chaos of* W. *Assume that, as* $k \to \infty$ *and for every* $i, j = 1, ..., d$, $E[F_k^{(i)} F_k^{(j)}] \to c(i,j)$, *where* $c = \{c(i,j) : i, j = 1, ..., d\}$ *is a positive definite symmetric matrix. Then, the following two assertions are equivalent, as* $k \to \infty$: **(i)** $(F_k^{(1)}, ..., F_k^{(d)})$ *converges in distribution to a centered* d*-dimensional Gaussian vector* $(N_1, ..., N_d)$ *with covariance* c; **(ii)** *for every* $i = 1, ..., d$, $F_k^{(i)}$ *converges in distribution to a centered Gaussian random variable with variance* $c(i,i)$.

Roughly speaking, Part (B) of the previous statement means that, for vectors of random variables living inside some fixed Wiener chaoses, *componentwise convergence to Gaussian always implies joint convergence.* The combination of Part (A) and Part (B) of Theorem 1.1 represents a powerful simplification of the so-called 'method of moments and cumulants' (see e.g. [15, Chapter 11] for a discussion of this point), and has triggered a considerable number of applications, refinements and generalizations, ranging from Stein's method to analysis on homogenous spaces, random matrices and fractional processes – see the survey [9] as well as the forthcoming monograph [10] for details and references.

Now, let $(\mathscr{A}, \varphi)$ be a non-commutative tracial W^*-probability space (in particular, $\mathscr{A}$ is a von Neumann algebra and φ is a trace – se Section 2.1 for details), and let $S = \{S_t : t \geq 0\}$ be a free Brownian motion defined on it. It is well-known (see e.g. [2]) that, for every $n \geq 1$ and every $f \in L^2(\mathbb{R}_+^n)$, one can define a free multiple stochastic integral with respect to f. Such an object is usually denoted by $I^S(f)$. Multiple integrals of order n with respect to S compose the so-called nth *Wigner chaos* associated with S. Wigner chaoses play a fundamental role in free stochastic analysis – see again [2].

The following theorem, which is the main result of [4], is the exact free analogous of Part (A) of Theorem 1.1. Note that the value 2 coincides with the fourth moment of the standard semicircular distribution $S(0,1)$.

Theorem 1.2 (See [4]). *Let* $n \geq 2$ *be an integer, and let* $(f_k)_{k \in \mathbb{N}}$ *be a sequence of mirror symmetric (see Section 2.2 for definitions) functions in* $L^2(\mathbb{R}_+^n)$, *each with* $\|f_k\|_{L^2(\mathbb{R}_+^n)} = 1$. *The following statements are equivalent.*

(1) *The fourth moments of the stochastic integrals* $I(f_k)$ *converge to* 2, *that is,*
$$\lim_{k \to \infty} \varphi(I^S(f_k)^4) = 2.$$

(2) *The random variables* $I^S(f_k)$ *converge in law to the standard semicircular distribution* $S(0,1)$ *as* $k \to \infty$.

The aim of this paper is to provide a complete proof of the following Theorem 1.3, which represents a free analogous of Part (B) of Theorem 1.1.

Theorem 1.3. *Let $d \geq 2$ and $n_1, \ldots, n_d$ be some fixed integers, and consider a positive definite symmetric matrix $c = \{c(i,j) : i,j = 1,...,d\}$. Let $(s_1, \ldots, s_d)$ be a semicircular family with covariance c (see Definition 2.10). For each $i = 1, \ldots, d$, we consider a sequence $(f_k^{(i)})_{k\in\mathbb{N}}$ of mirror-symmetric functions in $L^2(\mathbb{R}_+^{n_i})$ such that, for all $i,j = 1, \ldots, d$,*

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})] = c(i,j). \tag{1.1}$$

The following three statements are equivalent as $k \to \infty$.
(1) The vector $((I^S(f_k^{(1)}), \ldots, I^S(f_k^{(d)}))$ converges in distribution to $(s_1, \ldots, s_d)$.
(2) For each $i = 1, \ldots, d$, the random variable $I^S(f_k^{(i)})$ converges in distribution to s_i.
(3) For each $i = 1, \ldots, d$,

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)})^4] = 2\,c(i,i)^2.$$

Remark 1.4. In the previous statement, the quantity $\varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})]$ equals $\langle f_k^{(i)}, f_k^{(j)}\rangle_{L^2(\mathbb{R}_+^{n_i})}$ if $n_i = n_j$, and equals 0 if $n_i \neq n_j$. In particular, the limit covariance matrix c is necessarily such that $c(i,j) = 0$ whenever $n_i \neq n_j$.

Remark 1.5. Two additional references deal with non-semicircular limit theorems inside the free Wigner chaos. In [11], one can find necessary and sufficient conditions for the convergence towards the so-called Marčenko-Pastur distribution (mirroring analogous findings in the classical setting – see [8]). In [3], conditions are established for the convergence towards the so-called 'tetilla law' (or 'symmetric Poisson distribution' – see also [6]).

Combining the content of Theorem 1.3 with those in [4, 16], we can finally state the following Wiener-Wigner transfer principle, establishing an equivalence between multidimensional limit theorems on the classical and free chaoses.

Theorem 1.6. *Let $d \geq 1$ and $n_1, \ldots, n_d$ be some fixed integers, and consider a positive definite symmetric matrix $c = \{c(i,j) : i,j = 1,...,d\}$. Let $(N_1, \ldots, N_d)$ be a d-dimensional Gaussian vector and $(s_1, \ldots, s_d)$ be a semicircular family, both with covariance c. For each $i = 1, \ldots, d$, we consider a sequence $(f_k^{(i)})_{k\in\mathbb{N}}$ of fully-symmetric functions (cf. Definition 2.2) in $L^2(\mathbb{R}_+^{n_i})$. Then:*

1. *For all $i,j = 1, \ldots, d$ and as $k \to \infty$, $\varphi[I^S(f_k^{(i)})I^S(f_k^{(j)})] \to c(i,j)$ if and only if $E[I^W(f_k^{(i)})I^W(f_k^{(j)})] \to \sqrt{(n_i)!(n_j)!}\,c(i,j)$.*
2. *If the asymptotic relations in (1) are verified then, as $k \to \infty$,*

$$\big(I^S(f_k^{(1)}), \ldots, I^S(f_k^{(d)})\big) \overset{\text{law}}{\to} (s_1, \ldots, s_d)$$

if and only if

$$\big(I^W(f_k^{(1)}),\dots,I^W(f_k^{(d)})\big) \stackrel{\text{law}}{\to} \big(\sqrt{(n_1)!}N_1,\dots,\sqrt{(n_d)!}N_d\big).$$

The remainder of this paper is organized as follows. Section 2 gives concise background and notation for the free probability setting. Theorems 1.3 and 1.6 are then proved in Section 3.

2. Relevant definitions and notations

We recall some relevant notions and definitions from free stochastic analysis. For more details, we refer the reader to [2, 4, 7].

2.1. **Free probability, free Brownian motion and stochastic integrals.** In this note, we consider as given a so-called *(tracial)* W^* *probability space* $(\mathscr{A},\varphi)$, where $\mathscr{A}$ is a von Neumann algebra (with involution $X\mapsto X^*$), and $\varphi :\to \mathbb{C}$ is a *tracial state* (or *trace*). In particular, φ is weakly continuous, positive (that is, $\varphi(Y)\geq 0$ whenever Y is a nonnegative element of $\mathscr{A}$), faithful (that is, $\varphi(YY^*)=0$ implies $Y=0$, for every $Y\in\mathscr{A}$) and tracial (that is, $\varphi(XY)=\varphi(YX)$, for every $X,Y\in\mathscr{A}$). The self-adjoint elements of $\mathscr{A}$ are referred to as *random variables*. The *law* of a random variable X is the unique Borel measure on $\mathbb{R}$ having the same moments as X (see [7, Proposition 3.13]). For $1\leq p\leq\infty$, one writes $L^p(\mathscr{A},\varphi)$ to indicate the L^p space obtained as the completion of $\mathscr{A}$ with respect to the norm $\|a\|_p=\tau(|a|^p)^{1/p}$, where $|a|=\sqrt{a^*a}$, and $\|\cdot\|_\infty$ stands for the operator norm.

Definition 2.1. Let $\mathscr{A}_1,\dots,\mathscr{A}_n$ be unital subalgebras of $\mathscr{A}$. Let $X_1,\dots,X_m$ be elements chosen from among the $\mathscr{A}_i$'s such that, for $1\leq j<m$, X_j and X_{j+1} do not come from the same $\mathscr{A}_i$, and such that $\varphi(X_j)=0$ for each j. The subalgebras $\mathscr{A}_1,\dots,\mathscr{A}_n$ are said to be *free* or *freely independent* if, in this circumstance, $\varphi(X_1X_2\cdots X_n)=0$. Random variables are called freely independent if the unital algebras they generate are freely independent.

Definition 2.2. The (centered) *semicircular distribution* (or Wigner law) $S(0,t)$ is the probability distribution

$$S(0,t)(dx)=\frac{1}{2\pi t}\sqrt{4t-x^2}\,dx,\quad |x|\leq 2\sqrt{t}. \tag{2.1}$$

Being symmetric around 0, the odd moments of this distribution are all 0. Simple calculations (see e.g. [7, Lecture 2]) show that the even moments can be expressed in therms of the so-called *Catalan numbers*: for non-negative integers m,

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m}S(0,t)(dx)=C_m t^m,$$

where $C_m=\frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number. In particular, the second moment (and variance) is t while the fourth moment is $2t^2$.

Definition 2.3. A *free Brownian motion* S consists of: (i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann sub-algebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$, for $0 \leq s < t$), (ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}$ such that: (a) $S_0 = 0$ and $S_t \in \mathscr{A}_t$ for every t, (b) for every t, S_t has a semicircular distribution with mean zero and variance t, and (c) for every $0 \leq u < t$, the increment $S_t - S_u$ is free with respect to $\mathscr{A}_u$, and has a semicircular distribution with mean zero and variance $t - u$.

For the rest of the paper, we consider that the W^*-probability space $(\mathscr{A}, \varphi)$ is endowed with a free Brownian motion S. For every integer $n \geq 1$, the collection of all operators having the form of a multiple integral $I^S(f)$, $f \in L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n)$, is defined according to [2, Section 5.3], namely: (a) first define $I^S(f) = (S_{b_1} - S_{a_1}) \cdots (S_{b_n} - S_{a_n})$ for every function f having the form

$$f(t_1, ..., t_n) = \mathbf{1}_{(a_1,b_1)}(t_1) \times \ldots \times \mathbf{1}_{(a_n,b_n)}(t_n), \tag{2.2}$$

where the intervals (a_i, b_i), $i = 1, ..., n$, are pairwise disjoint; (b) extend linearly the definition of $I^S(f)$ to 'simple functions vanishing on diagonals', that is, to functions f that are finite linear combinations of indicators of the type (2.2); (c) exploit the isometric relation

$$\langle I^S(f), I^S(g) \rangle_{L^2(\mathscr{A},\varphi)} = \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n) \overline{g(t_n, \ldots, t_1)} dt_1 \ldots dt_n, \tag{2.3}$$

where f, g are simple functions vanishing on diagonals, and use a density argument to define $I(f)$ for a general $f \in L^2(\mathbb{R}_+^n)$.

As recalled in the Introduction, for $n \geq 1$, the collection of all random variables of the type $I^S(f)$, $f \in L^2(\mathbb{R}_+^n)$, is called the nth *Wigner chaos* associated with S. One customarily writes $I^S(a) = a$ for every complex number a, that is, the Wigner chaos of order 0 coincides with $\mathbb{C}$. Observe that (2.3) together with the above sketched construction imply that, for every $n, m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$,

$$\varphi[I^S(f) I^S(g)] = \mathbf{1}_{n=m} \times \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n) \overline{g(t_n, \ldots, t_1)} dt_1 \ldots dt_n, \tag{2.4}$$

where the right hand side of the previous expression coincides by convention with the inner product in $L^2(\mathbb{R}_+^0) = \mathbb{C}$ whenever $m = n = 0$.

2.2. Mirror Symmetric Functions and Contractions.

Definition 2.4. Let n be a natural number, and let f be a function in $L^2(\mathbb{R}_+^n)$.

(1) The *adjoint* of f is the function $f^*(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$.
(2) f is called *mirror symmetric* if $f = f^*$, i.e. if

$$f(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$$

for almost all $t_1, \dots, t_n \geq 0$ with respect to the product Lebesgue measure

(3) f is called *fully symmetric* if it is real-valued and, for any permutation σ in the symmetric group Σ_n, $f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$ for almost every $t_1, \dots, t_n \geq 0$ with respect to the product Lebesgue measure.

An operator of the type $I^S(f)$ is self-adjoint if and only if f is mirror symmetric.

Definition 2.5. Let n, m be natural numbers, and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $p \leq \min\{n, m\}$ be a natural number. The *pth contraction* $f \overset{p}{\frown} g$ of f and g is the $L^2(\mathbb{R}_+^{n+m-2p})$ function defined by nested integration of the middle p variables in $f \otimes g$:

$$f \overset{p}{\frown} g\,(t_1, \dots, t_{n+m-2p})$$
$$= \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.$$

Notice that when $p = 0$, there is no integration, just the products of f and g with disjoint arguments; in other words, $f \overset{0}{\frown} g = f \otimes g$.

2.3. **Non-crossing Partitions.** A *partition* of $[n] = \{1, 2, \dots, n\}$ is (as the name suggests) a collection of mutually disjoint nonempty subsets B_1, $\dots, B_r$ of $[n]$ such that $B_1 \sqcup \cdots \sqcup B_r = [n]$. The subsets are called the *blocks* of the partition. By convention we order the blocks by their least elements; i.e. $\min B_i < \min B_j$ iff $i < j$. If each block consists of two elements, then we call the partition a *pairing*. The set of all partitions on $[n]$ is denoted $\mathscr{P}(n)$, and the subset of all pairings is $\mathscr{P}_2(n)$.

Definition 2.6. Let $\pi \in \mathscr{P}(n)$ be a partition of $[n]$. We say π has a *crossing* if there are two distinct blocks B_1, B_2 in π with elements $x_1, y_1 \in B_1$ and $x_2, y_2 \in B_2$ such that $x_1 < x_2 < y_1 < y_2$.

If $\pi \in \mathscr{P}(n)$ has no crossings, it is said to be a *non-crossing partition.* The set of non-crossing partitions of $[n]$ is denoted $NC(n)$. The subset of non-crossing pairings is denoted $NC_2(n)$.

Definition 2.7. Let $n_1, \dots, n_r$ be positive integers with $n = n_1 + \cdots + n_r$. The set $[n]$ is then partitioned accordingly as $[n] = B_1 \sqcup \cdots \sqcup B_r$ where $B_1 = \{1, \dots, n_1\}$, $B_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, and so forth through $B_r = \{n_1 + \cdots + n_{r-1} + 1, \dots, n_1 + \cdots + n_r\}$. Denote this partition as $n_1 \otimes \cdots \otimes n_r$.

We say that a pairing $\pi \in \mathscr{P}_2(n)$ *respects* $n_1 \otimes \cdots \otimes n_r$ if no block of π contains more than one element from any given block of $n_1 \otimes \cdots \otimes n_r$. The set of such respectful pairings is denoted $\mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. The set of non-crossing pairings that respect $n_1 \otimes \cdots \otimes n_r$ is denoted $NC_2(n_1 \otimes \cdots \otimes n_r)$.

Definition 2.8. Let $n_1, \dots, n_r$ be positive integers, and let $\pi \in \mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. Let B_1, B_2 be two blocks in $n_1 \otimes \cdots \otimes n_r$. Say that π *links* B_1 *and* B_2 if there is a block $\{i, j\} \in \pi$ such that $i \in B_1$ and $j \in B_2$.

Define a graph C_π whose vertices are the blocks of $n_1 \otimes \cdots \otimes n_r$; C_π has an edge between B_1 and B_2 iff π links B_1 and B_2. Say that π *is connected* with respect to $n_1 \otimes \cdots \otimes n_r$ (or that π *connects the blocks of* $n_1 \otimes \cdots \otimes n_r$) if the graph C_π is connected. We shall denote by $NC_2^c(n_1 \otimes \cdots \otimes n_r)$ the set of all non-crossing pairings that both respect and connect $n_1 \otimes \cdots \otimes n_r$.

Definition 2.9. Let n be an even integer, and let $\pi \in \mathscr{P}_2(n)$. Let $f \colon \mathbb{R}_+^n \to \mathbb{C}$ be measurable. The *pairing integral* of f with respect to π, denoted $\int_\pi f$, is defined (when it exists) to be the constant

$$\int_\pi f = \int f(t_1, \dots, t_n) \prod_{\{i,j\} \in \pi} \delta(t_i - t_j)\, dt_1 \cdots dt_n.$$

We finally introduce the notion of a semicircular family (see e.g. [7, Definition 8.15]).

Definition 2.10. Let $d \geq 2$ be an integer, and let $c = \{c(i,j) : i,j = 1, ..., d\}$ be a positive definite symmetric matrix. A d-dimensional vector $(s_1, ..., s_d)$ of random variables in $\mathscr{A}$ is said to be a *semicircular family with covariance* c if for every $n \geq 1$ and every $(i_1, ..., i_n) \in [d]^n$

$$\varphi(s_{i_1} s_{i_2} \cdots s_{i_n}) = \sum_{\pi \in NC_2(n)} \prod_{\{a,b\} \in \pi} c(i_a, i_b).$$

The previous relation implies in particular that, for every $i = 1, ..., d$, the random variable s_i has the $S(0, c(i,i))$ distribution – see Definition 2.2.

For instance, one can rephrase the defining property of the free Brownian motion $S = \{S_t : t \geq 0\}$ by saying that, for every $t_1 < t_2 < \cdots < t_d$, the vector $(S_{t_1}, S_{t_2} - S_{t_1}, ..., S_{t_d} - S_{t_{d-1}})$ is a semicircular family with a diagonal covariance matrix such that $c(i,i) = t_i - t_{i-1}$ (with $t_0 = 0$), $i = 1, ..., d$.

3. Proof of the main results

A crucial ingredient in the proof of Theorem 1.3 is the following statement, showing that contractions control all important pairing integrals. This is the generalization of Proposition 2.2. in [4] to our situation.

Proposition 3.1. *Let $d \geq 2$ and $n_1, \dots, n_d$ be some fixed positive integers. Consider, for each $i = 1, \dots, d$, sequences of mirror-symmetric functions $(f_k^{(i)})_{k \in \mathbb{N}}$ with $f_k^{(i)} \in L^2(\mathbb{R}_+^{n_i})$, satisfying:*

- *There is a constant $M > 0$ such that $\|f_k^{(i)}\|_{L^2(\mathbb{R}_+^{n_i})} \leq M$ for all $k \in \mathbb{N}$ and all $i = 1, \dots, d$.*
- *For all $i = 1, \dots, d$ and all $p = 1, \dots, n_i - 1$,*

$$\lim_{k \to \infty} f_k^{(i)} \overset{p}{\frown} f_k^{(i)} = 0 \qquad in \quad L^2(\mathbb{R}_+^{2n_i - 2p}).$$

Let $r \geq 3$, and let π be a connected non-crossing pairing that respects $n_{i_1} \otimes \cdots \otimes n_{i_r}$: $\pi \in NC_2^c(n_{i_1} \otimes \cdots \otimes n_{i_r})$. Then

$$\lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = 0.$$

Proof. In the same way as in [4] one sees that without restriction (i.e., up to a cyclic rotation and relabeling of the indices) one can assume that

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \int_{\pi'} (f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}) \otimes (f_k^{(i_3)} \otimes \cdots \otimes f_k^{(i_r)}),$$

where $0 < 2p < n_{i_1} + n_{i_2}$ and

$$\pi' \in NC_2^c\big((n_{i_1} + n_{i_2} - 2p) \otimes n_{i_3} \otimes \cdots \otimes n_{i_r}\big).$$

Note that $0 < 2p < n_{i_1}+n_{i_2}$ says that $f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}$ is not a trivial contraction (trivial means that either nothing or all arguments are contracted); of course, in the case $n_{i_1} \neq n_{i_2}$ it is allowed that $p = \min(n_{i_1}, n_{i_2})$.

By Lemma 2.1. of [4] we have then

$$\begin{aligned} &|\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)}| \\ &\qquad \leq \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot \|f_k^{(i_3)}\|_{L^2(\mathbb{R}_+^{n_{i_3}})} \cdots \|f_k^{(i_r)}\|_{L^2(\mathbb{R}_+^{n_{i_r}})} \\ &\qquad \leq \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot M^{r-2}. \end{aligned}$$

Now we only have to observe that, by also using the mirror symmetry of $f_k^{(i_1)}$ and $f_k^{(i_2)}$, we have

$$\begin{aligned} &\|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|^2_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} = \Big\langle f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)}, f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)} \Big\rangle_{L^2(\mathbb{R}_+^{2p})} \\ &\leq \|f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)}\|_{L^2(\mathbb{R}_+^{2p})} \cdot \|f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{2p})}. \end{aligned}$$

According to our assumption we have, for each $i = 1, \dots, d$ and each $q = 1, \dots, n_i - 1$, that

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}).$$

Since now at least one of the two contractions $\overset{n_{i_1}-p}{\frown}$ and $\overset{n_{i_2}-p}{\frown}$ is non-trivial, we can choose either $q = n_{i_1} - p$, $i = i_1$ or $q = n_{i_2} - p$, $i = i_2$ in the above, and this implies that

$$\lim_{k\to\infty} \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} = 0,$$

which gives our claim. □

We can now provide a complete proof of Theorem 1.3.

Proof of Theorem 1.3. The equivalence between (2) and (3) follows from [4]. Clearly, (1) implies (3), so we only have the prove the reverse implication. So let us assume (3). Note that, by Theorem 1.6 of [4], this is equivalent to the fact that all non-trivial contractions of $f_k^{(i)}$ converge to 0; i.e., for each $i = 1, \dots, d$ and each $q = 1, \dots, n_i - 1$ we have

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}). \tag{3.1}$$

We will use statement (3) in this form. In order to show (1), we have to show that any moment in the variables $I(f_k^{(1)}), \dots, I(f_k^{(d)})$ converges, as $k \to \infty$, to the corresponding moment in the semicircular variables $s_1, \dots, s_d$. So, for $r \in \mathbb{N}$ and positive integers $i_1, \dots, i_r$, we consider the moments

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right].$$

We have to show that they converge, for $k \to \infty$, to the corresponding moment $\varphi(s_{i_1} \cdots s_{i_r})$. Note that our assumption (1.1) says that

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})] = c(i,j) = \varphi(s_i s_j).$$

By Proposition 1.38 in [4] we have

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)}.$$

By Remark 1.33 in [4], any $\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ can be uniquely decomposed into a disjoint union of connected pairings $\pi = \pi_1 \sqcup \cdots \sqcup \pi_m$ with $\pi_q \in NC_2^c(\bigotimes_{j\in I_q} n_{i_j})$, where $\{1, \dots, r\} = I_1 \sqcup \cdots \sqcup I_m$ is a partition of the index set $\{1, \dots, r\}$. The above integral with respect to π factors then accordingly into

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \prod_{q=1}^m \int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}.$$

Consider now one of those factors, corresponding to π_q. Since π_q must respect $\bigotimes_{j\in I_q} n_{i_j}$, the number $r_q := \# I_q$ must be strictly greater than 1. On the other hand, if $r_q \geq 3$, then, from (3.1) and Proposition 3.1, it follows that the corresponding pairing integral $\int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}$ converges to 0 in L^2. Thus, in the limit, only those π make a contribution, for which all r_q are equal to 2, i.e., where each of the π_q in the decomposition of π corresponds to a complete contraction between two of the appearing functions. Let $NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ denote the set of those pairings π. So we get

$$\lim_{k\to\infty} \varphi\left[I(f_k^{(i_1)}) \cdots I(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)},$$

We continue as in [4]: each $\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ is in bijection with a non-crossing pairing $\sigma \in NC_2(r)$. The contribution of such a π is the product of the complete contractions for each pair of the corresponding $\sigma \in NC_2(r)$; but the complete contraction is just the L^2 inner product between the paired functions, i.e.,

$$\lim_{k\to\infty} \varphi\left[I^S(f_k^{(i_1)})\cdots I^S(f_k^{(i_r)})\right] = \sum_{\sigma\in NC_2(r)} \prod_{\{s,t\}\in\sigma} c(i_s,i_t).$$

This is exactly the moment $\varphi(s_{i_1}\cdots s_{i_r})$ of a semicircular family $(s_1,\ldots,s_d)$ with covariance matrix c, and the proof is concluded. □

We conclude this paper with the proof of Theorem 1.6.

Proof of Theorem 1.6. Point (1) is a simple consequence of the Wigner isometry (3.2) (since each $f_k^{(i)}$ is fully symmetric, $f_k^{(i)}$ is in particular mirror-symmetric), together with the classical Wiener isometry which states that

$$E[I^W(f)I^W(g)] = \mathbf{1}_{n=m} \times n!\langle f,g\rangle_{L^2(\mathbb{R}_+^n)} \tag{3.2}$$

for every $n,m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$. For point (2), we observe first that the case $d=1$ is already known, as it corresponds to [4, Theorem 1.8]. Consider now the case $d \geq 2$. Let us suppose that $\big(I^S(f_k^{(1)}),\ldots,I^S(f_k^{(d)})\big) \overset{\text{law}}{\to} (s_1,\ldots,s_d)$. In particular, $I^S(f_k^{(i)}) \overset{\text{law}}{\to} s_i$ for all $i=1,\ldots,d$. By [4, Theorem 1.8] (case $d=1$), this implies that $I^W(f_k^{(i)}) \overset{\text{law}}{\to} \sqrt{(n_i)!}N_i$. Since the asymptotic relations in (1) are verified, Theorem 1.1**(B)** leads then to $\big(I^W(f_k^{(1)}),\ldots,I^W(f_k^{(d)})\big) \overset{\text{law}}{\to} \big(\sqrt{(n_i)!}N_1,\ldots,\sqrt{(n_d)!}N_d\big)$, which is the desired conclusion. The converse implication follows exactly the same lines,and the proof is concluded. □

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