MODULARITY OF CERTAIN MOD p^n GALOIS REPRESENTATIONS

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Abstract. For a rational prime $p \geq 3$ and an integer $n \geq 2$, we study the modularity of continuous 2-dimensional mod p^n Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ whose residual representations are odd and absolutely irreducible. Under suitable hypotheses on the local structure of these representations and the size of their images we use deformation theory to construct characteristic 0 lifts. We then invoke modularity lifting results to prove that these lifts are modular. As an application, we show that certain unramified mod p^n Galois representations arise from modular forms of weight $p^{n-1}(p-1)+1$.

1. Introduction

Let p be an odd rational prime and $\boldsymbol{k}$ be a finite field of characteristic p. Let W denote the ring of Witt vectors of $\boldsymbol{k}$. For an integer $n \geq 2$ suppose we are given a continuous Galois representation $\rho_n : G_{\mathbb{Q}} \longrightarrow GL_2(W/p^n)$. The goal of this paper is to investigate local and global conditions under which ρ_n lifts to a representation ρ which is modular. The question of modularity of ρ_n naturally leads one to wonder about statements analogous to Serre's modularity conjecture in the mod p^n situation as well. However, näive attempts at such generalizations are bound to fail because, *a fortiori*, a given mod p^n representation might not even lift to characteristic 0. Nevertheless, we show that when the local structure of ρ_n "mimics" the (mod p^n reduction of) the local structure of the p-adic Galois representation attached to a p-ordinary modular eigenform then we can not only construct p-adic lifts but, indeed, we can also prove their modularity. Our main result is the following.

Theorem A. *Let ρ_n: $G_{\mathbb{Q}} \longrightarrow GL_2(W/p^n)$ be a continuous Galois representation whose residual representation $\overline{\rho} := \rho_n$ mod p is odd and has squarefree, prime-to-p Artin conductor $N := N(\overline{\rho})$. Assume the following hypotheses:*

- (C1) *The representation ρ_n has fixed determinant ϵ_n which lifts to $\epsilon = \psi\chi^{k-1}$ where ψ is a finite order character unramified at p, χ is the p-adic cyclotomic character, and $k \geq 2$ is an integer.*
- (C2) *The image of $\overline{\rho}$ contains $SL_2(\boldsymbol{k})$. In addition, if $p = 3$ then the image ρ_n contains a transvection $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.*
- (C3) *At p the restriction of ρ_n to the decomposition group G_p is*

$$\rho_n|_{G_p} \simeq \begin{pmatrix} \psi_{1p}\chi^{k-1} & * \\ 0 & \psi_{2p} \end{pmatrix}$$

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where $\psi_1, \psi_2 \colon G_{\mathbb{Q}} \longrightarrow W^\times$ are some unramified characters lifting ψ_{1p}, ψ_{2p} and $\psi_1\psi_2 = \psi$. Furthermore $\psi_{1p}\chi^{k-1} \not\equiv \psi_{2p} \mod p$ and $\psi_{1p}\chi^{k-2} \not\equiv \psi_{2p} \mod p$.

(C4) *Suppose that ρ_n is ramified at a place $\mathfrak{q} \nmid p$. If $p | \#\overline{\rho}(I_{\mathfrak{q}})$ then*

$$\rho_n|_{G_{\mathfrak{q}}} \simeq \begin{pmatrix} \chi^{k-1} & * \\ 0 & 1 \end{pmatrix} \delta$$

and if $p \nmid \#\overline{\rho}(I_{\mathfrak{q}})$ then $\rho_n|_{G_{\mathfrak{q}}} \simeq \delta^{-1}\chi^{k-1}\psi|_{G_{\mathfrak{q}}} \oplus \delta$ for some unramified δ. Additionally, if $\mathfrak{q} \equiv 1 \mod p$ and $\overline{\rho}$ is unramified at $\mathfrak{q}$ then $p | \#\overline{\rho}(Frob_{\mathfrak{q}})$.

There is then a p-ordinary modular form f of weight k and level prime to p such that its associated p-adic representation ρ_f lifts ρ_n, has determinant $\psi\chi^{k-1}$, and

$$\rho_f|_{G_p} \sim \begin{pmatrix} \psi_1\chi^{k-1} & * \\ 0 & \psi_2 \end{pmatrix}$$

with ψ_2 being an unramified lift of ψ_{2p}.

We emphasize that some of the work that goes into proving Theorem A was done in a previous paper [1] of the author. The focus of that work however was to prove higher companion form theorems. This paper shifts emphasis to a more general setting of mod p^n Galois representations and takes a closer look at the local deformation theory that is used.

The proof relies on being able to prescribe local deformation conditions for ρ_n and using the methods of Ramakrishna [8], generalized by Taylor in [10], to piece this local information together to produce a lift in characteristic 0. *A priori*, the requirement that ρ_n have fixed determinant is forced upon us since we wish to reconcile ρ_n with a Galois representation attached to a modular form. Hypotheses C2 and the last part of C4 are technical assumptions that make the deformation theory work. This is done in Section 2. In Section 3, the lift that we construct is shown to be modular as a consequence of a modularity lifting theorem due to Skinner and Wiles [9]. We note that hypothesis C2 also ensures that $\overline{\rho}$ is absolutely irreducible and, since it is assumed to be continuous and odd, Serre's Modularity Conjecture guarantees the modularity of $\overline{\rho}$. This, along with hypothesis C3, is crucial for the successful application of Skinner-Wiles. We conclude Section 3 by applying the methods used to prove Theorem A to also prove what can be considered a mod p^n analog of [4, Theorem 2.1.1].

Theorem B. *Set $k = p^n(p-1)+1$. Let $\rho_n : G_{\mathbb{Q}} \longrightarrow GL_2(W/p^n)$ be a continuous, odd Galois representation with fixed determinant as in Theorem A. Suppose that $\rho_n|_{G_p}$ is unramified and in fact suppose that*

$$\rho_n|_{G_p} \sim \begin{pmatrix} \psi_{1p,x} & 0 \\ 0 & \psi_{2p,x} \end{pmatrix}$$

where ψ_x is the unramified character sending an arithmetic Frobenius element to x. Additionally, suppose that ρ_n satisfies conditions (C2)-(C4) of Theorem A. Then there is a modular eigenform f

of weight $p^n(p-1)+1$ and level prime to p such that $\rho_{f,n} \simeq \rho_n$, where $\rho_{f,n}$ is the mod p^n reduction of the p-adic Galois representation ρ_f attached to f. Moreover,

$$\rho_f|_{G_p} \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

with $T_pf = \psi_2(Frob_p)f$ for some unramified lift ψ_2 of ψ_{2p}.

2. Deformation Theory

In this section, we study the deformation theory that is needed for realizing locally any global obstructions to lifting the given mod p^n representation. The method of Ramakrishna [8] (generalized by Taylor [10]) that we wish to adapt relies on producing sufficiently well-behaved local deformations of a given residual representation and, at the cost of additional ramification, trivializing the dual Selmer group in order to remove the local obstructions and construct the desired lift in characteristic 0.

2.1. **Smooth deformation conditions.** Let $\boldsymbol{k}$ be a finite field of characteristic $p \geq 3$ and W be the ring of Witt vectors of $\boldsymbol{k}$. We follow Mazur's seminal work in [6] to briefly explain deformation conditions. Let $\mathfrak{C}_k$ be the category of complete, local, Noetherian rings over W with fixed residue field $\boldsymbol{k}$. Given a residual representation $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\boldsymbol{k})$, a deformation condition $\mathcal{D}$ on a deformation of $\overline{\rho}$ is a collection of lifts of $G_{\mathbb{Q}}$ defined over W-algebras in $\mathfrak{C}_k$ which is closed under projections (morphisms of objects in $\mathfrak{C}_k$) and satisfies a certain Mayer-Vietoris property. The former property ensures that $\mathcal{D}$ defines a subfunctor while the latter implies that this subfunctor is relatively representable.

If $\mathcal{D}$ is a deformation condition for $\overline{\rho}$ then there is a complete local Noetherian W-algebra R with residue field $\boldsymbol{k}$ and a lift $\rho : \Gamma \longrightarrow GL_2(R)$ in $\mathcal{D}$ with the property that if $\rho' : \Gamma \longrightarrow GL_2(A)$ is a lift of $\overline{\rho}$ in $\mathcal{D}$ then there is a morphism $R \longrightarrow A$ which, when composed with ρ, gives a representation strictly equivalent to ρ'. We also insist that the morphism $R \longrightarrow A$ is unique when A is the ring of dual numbers $\boldsymbol{k}[\epsilon]/(\epsilon^2)$. Letting $\mathrm{ad}^0\overline{\rho}$ be the vector space of traceless 2×2-matrice) over $\boldsymbol{k}$ with $GL_2(\boldsymbol{k})$ acting by conjugation, we identify the tangent space $\mathbf{t}_{\mathcal{D}}$ with a subspace of of $H^1(\Gamma, \mathrm{ad}^0\overline{\rho})$. The fundamental consequence then is the existence of a (uni)versal deformation and the (uni)versal deformation ring R has a presentation $W[[T_1, \ldots, T_n]]/J$ where $n = \dim_{\boldsymbol{k}}\mathbf{t}_{\mathcal{D}}$. A deformation ring is smooth when the ideal of relations J is (0). We construct the smooth global deformation conditions that we are interested in by finding local (uni)versal deformation rings smooth in a number of variables. We now describe local deformation conditions which are relevant to the proof of Theorems A and B.

2.2. **Deformation conditions at** p**.** Let G_p denote the decomposition group at p. Suppose we are given an integer $k \geq 2$ and a representation $\overline{\rho} : G_p \longrightarrow GL_2(\boldsymbol{k})$ such that

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}^{k-1}\overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$$

where $\overline{\psi}_1, \overline{\psi}_2$ are unramified characters. Let ψ be the Teichmüller lift of $\overline{\psi}_1\overline{\psi}_2$. If A is a coefficient ring let $\rho_A : G_{\mathfrak{p}} \longrightarrow GL_2(A)$ be a lift of $\overline{\rho}$ strictly equivalent to a representation of the form

$$\begin{pmatrix} \psi_1\chi^{k-1} & * \\ 0 & \psi_2 \end{pmatrix}$$

for some unramified characters $\psi_1, \psi_2 : G_p \longrightarrow A^\times$ lifting $\overline{\psi}_1, \overline{\psi}_2$ and $\psi_1\psi_2 = \psi$. We then have the following proposition.

Proposition 2.1. *Let $\overline{\rho} : G_p \longrightarrow GL_2(\boldsymbol{k})$ be as above and further assume that $\overline{\chi}^{k-1}\psi_1 \neq \overline{\chi}\psi_2$. Then the deformation condition consisting of lifts ρ_A of $\overline{\rho}$ is a smooth deformation condition. The dimension of its tangent space is equal to $1 + dim_{\boldsymbol{k}} H^0(G_p, ad^0\overline{\rho})$.*

Proof. See Example 3.4 in [5]. □

2.3. **Deformation conditions at $\mathfrak{q} \nmid p$.** For the rest of this subsection let $G_{\mathfrak{q}}$ denote the decomposition group at $\mathfrak{q}$ for some place $\mathfrak{q} \nmid p$. Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\boldsymbol{k})$ be a residual representation with fixed determinant d. We describe deformation conditions for $\overline{\rho}|_{G_{\mathfrak{q}}}$ in two cases according as $\overline{\rho}$ is ramified at $\mathfrak{q}$ or not. In the former case we have the following proposition.

Proposition 2.2. *Suppose that $\mathfrak{q}|N(\overline{\rho})$ where $N(\overline{\rho})$ is the Artin conductor of $\overline{\rho}$. If $p \nmid \#\overline{\rho}(I_{\mathfrak{q}})$ then the collection of lifts of $\overline{\rho}$ which factor through $G_{\mathfrak{q}}/(I_{\mathfrak{q}} \cap \ker\overline{\rho})$ and have fixed determinant lifting d is a smooth deformation condition. The tangent space has dimension $dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, ad^0\overline{\rho})$. If $p|\#\overline{\rho}(I_{\mathfrak{q}})|$ then suppose that*

$$\overline{\rho} \sim \begin{pmatrix} \overline{\chi} & * \\ 0 & 1 \end{pmatrix} \overline{\varepsilon}$$

for some character $\overline{\varepsilon} : G_{\mathfrak{q}} \longrightarrow \boldsymbol{k}^\times$. Moreover, assume that if $\overline{\rho}$ is semi-simple then $\overline{\chi}$ is non-trivial. Fix a character $\varepsilon : G_{\mathfrak{q}} \longrightarrow W^\times$ lifting $\overline{\varepsilon}$. Then the collection of lifts strictly equivalent to

$$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix} \varepsilon$$

is a smooth deformation condition. The tangent space has dimension $dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, ad^0\overline{\rho})$.

Proof. For the case $p \nmid \#\overline{\rho}(I_{\mathfrak{q}})|$ see Example E1 in [3] and for case $p|\#\overline{\rho}(I_{\mathfrak{q}})|$ see Example 3.3 in [5]. □

Now, for the remaining case, set $F = \mathbb{Q}_{\mathfrak{q}}$ and let $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ be unramified at $\mathfrak{q}$ with $\overline{\rho}(\text{Frob}) = \begin{pmatrix} \mathfrak{q}\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Of course, any lift of $\overline{\rho}$ necessarily factors through the maximal tamely ramified extension F^{tr} of F. The group $Gal(F^{\text{tr}}/F)$ is topologically generated by two elements τ and σ which satisfy the relation $\sigma\tau\sigma^{-1} = \tau^{\mathfrak{q}}$ and such that τ is a (topological) generator of the tame inertia subgroup and σ is a lift of Frobenius to $Gal(F^{\text{tr}}/F)$.

First define polynomials $h_n(T) \in \mathbb{Z}[T], n \geq 1$, by the recursion $h_{n+2} = Th_{n+1} - h_n$ and initial values $h_1 := 1, h_2 := T$. The following properties of h_n are easily verified by induction:

- $h_n(2) = n$
- If M is a 2×2 matrix over any commutative ring with trace t and determinant 1 then, $M^n = h_n(t)M - h_{n-1}(t)I$
- $h_n^2 - Th_nh_{n-1} + h_{n-1}^2 = 1$

We then have the following proposition.

Proposition 2.3. *Let $\overline{\rho}$ be as above with $\mathfrak{q}\alpha \neq \alpha^{-1}$. Denote by $\hat{\alpha}$ the Teichmuller lift of α. Let R, resp. $\rho : Gal(F^{tr}/F) \to GL_2(R)$, be the versal deformation ring, resp. the versal representation, for lifts of $\overline{\rho}$ with determinant χ.*

(i) *Suppose $\alpha^2 \neq 1$ and $\mathfrak{q}^2\alpha^2 \neq 1$. If $\mathfrak{q} \equiv 1 \mod p$ then $R \cong W(k)[[S,T]]/\langle (1+T)^{\mathfrak{q}} - (1+T)\rangle$ and*

$$\rho(\sigma) = \begin{pmatrix} \mathfrak{q}\hat{\alpha}(1+S) & 0 \\ 0 & (\hat{\alpha}(1+S))^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 1+T & 0 \\ 0 & (1+T)^{-1} \end{pmatrix}.$$

If $\mathfrak{q} \not\equiv 1 \mod p$ then $R \cong W[[S]]$ and $\rho(\sigma) = \left(\begin{smallmatrix} \mathfrak{q}\hat{\alpha}(1+S) & 0 \\ 0 & (\hat{\alpha}(1+S))^{-1} \end{smallmatrix}\right)$. In any case, a deformation condition is smooth with tangential dimension $dim_{\mathbf{k}} H^0(G_{\mathfrak{q}}, ad^0\overline{\rho})$ if and only if it is unramified.

(ii) *If $\alpha^2 = 1$ and $\mathfrak{q}^2 \not\equiv 1 \mod p$ then $R \cong W(k)[[S,T]]/(ST)$ and*

$$\rho(\sigma) = \hat{\alpha} \begin{pmatrix} l(1+S) & 0 \\ 0 & (1+S)^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.$$

A deformation condition for $\overline{\rho}$ with determinant χ is smooth if and only if it is either unramified or of the type considered in Proposition 2.2.

(iii) *If $\alpha^2 = 1$ and $\mathfrak{q} \equiv -1 \mod p$, then $R := W(k)[[S, T_1, T_2]]/J$ where*

$$J := \left\langle T_1\big(\mathfrak{q}(1+S)^2 - h_q(2\sqrt{1+T_1T_2})\big), T_2\big(1 - \mathfrak{q}(1+S)^2 h_q(2\sqrt{1+T_1T_2})\big)\right\rangle, \quad \text{and}$$

$$\rho(\sigma) = \hat{\alpha} \begin{pmatrix} \mathfrak{q}(1+S) & 0 \\ 0 & (1+S)^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} \sqrt{1+T_1T_2} & T_1 \\ T_2 & \sqrt{1+T_1T_2} \end{pmatrix}.$$

The only ramified smooth deformation condition for $\overline{\rho}$ is the of the type considered in Proposition 2.2 and it corresponds to the quotient $W(k)[[S, T_1, T_2]]/(S, T_2)$.

Proof. Let A be a coefficient ring with maximal ideal $\mathfrak{m}_A$, and let be ρ_A a lifting of $\overline{\rho}$ with determinant χ. By Hensel's Lemma, we can assume that $\rho_A(\sigma)$ is diagonal. Let

$$\rho_A(\sigma) = \begin{pmatrix} \mathfrak{q}\hat{\alpha}(1+s) & 0 \\ 0 & (\hat{\alpha}(1+s))^{-1} \end{pmatrix}, \quad \rho_A(\tau) = \begin{pmatrix} a & t_1 \\ t_2 & d \end{pmatrix} \tag{2.1}$$

with $s, t_1, t_2, a-1, d-1 \in \mathfrak{m}_A$ and $ad - t_1t_2 = 1$. Since $\sigma\tau\sigma^{-1} = \tau^{\mathfrak{q}}$, we have

$$\begin{pmatrix} a & t_1\mathfrak{q}(\hat{\alpha}(1+s))^2 \\ t_2\mathfrak{q}^{-1}(\hat{\alpha}(1+s))^{-2} & d \end{pmatrix} = \begin{pmatrix} ah_{\mathfrak{q}}(t) - h_{\mathfrak{q}-1}(t) & t_1h_{\mathfrak{q}}(t) \\ t_2h_{\mathfrak{q}}(t) & dh_{\mathfrak{q}}(t) - h_{\mathfrak{q}-1}(t) \end{pmatrix} \tag{2.2}$$

where $t = a + d$ is the trace. Note that $t \equiv 2 \mod \mathfrak{m}_A$ and so $h_{\mathfrak{q}}(t) \equiv \mathfrak{q} \mod \mathfrak{m}_A$.

If $\alpha^2 \neq 1$ then $\mathfrak{q}(\hat{\alpha}(1+s))^2 - h_{\mathfrak{q}}(t)$ is a unit and we get $t_1 = 0$. Similarly, if $\mathfrak{q}^2\alpha^2 \neq 1$ then $t_2 = 0$. The claims made in part (i) of the proposition are now immediate.

We now continue our analysis of ρ_A under the assumption that $\alpha^2 = 1$ and $\mathfrak{q} \not\equiv \pm 1 \mod p$. For ease of notation, we shall in fact assume that $\hat{\alpha} = 1$. Since $1 - h_{\mathfrak{q}}(t)$ is a unit, taking the difference of the diagonal entries on both sides of (2.2) gives $a - d = 0$ and so $a = d = \sqrt{1 + t_1t_2}$. Comparison of the off-diagonal entries of (2.2) (followed by multiplication) produces $t_1t_2(1 - h_{\mathfrak{q}}(t)^2) = 0$.

Suppose now that $\mathfrak{q} \not\equiv -1 \mod p$. Then $t_1t_2 = 0$ and so $t = 2, h_{\mathfrak{q}}(t) = l$. We can now simplify the two relations from the off-diagonal entries to get $t_1 = t_1(1+s)^2$ and $t_2 = t_2\mathfrak{q}^2(1+s)^2$, and finally deduce that $st_1 = 0, t_2 = 0$. Part (ii) of the proposition now follows easily.

Finally, we consider the case $\mathfrak{q} \equiv -1 \mod p$. The presentation for R and ρ follows from the presentation of an arbitrary lift along with the fact that $\dim_{\boldsymbol{k}} H^1(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}) = 3$. We now indicate how to determine the smooth deformation conditions. Take A to be characteristic 0 (and $\hat{\alpha} = 1$). In the presentation (2.1) the trace of $\rho_A(\tau)$ is $2\sqrt{1 + t_1t_2}$. If $t_1t_2 \neq 0$ then $\rho_A(\tau)$ has distinct eigenvalues - contradicting the fact that ρ_A is twist equivalent to $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)$ over its field of fractions. Hence we must have $t_2 = 0$ and $s = 0$. □

Let ρ_n be as in Theorem A. We summarize the various local deformation conditions described in this section to show that there exists a global deformation condition for ρ_n *i.e.* a global deformation condition $\mathcal{D}$ for $\overline{\rho} := \rho_n \mod p$ with determinant ϵ such that ρ_n is a deformation of type $\mathcal{D}$.

Proposition 2.4. *There exists a global deformation condition $\mathcal{D}$ for ρ_n such that each local component $\mathcal{D}_q$ is a smooth deformation condition whose tangent space has dimension $\dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, ad^0\overline{\rho}) + \delta$ where δ is 1 when $p = q$ and 0 otherwise.*

Proof. Let S denote the finite set of places of $\mathbb{Q}$ at which ρ_n is ramified along with places dividing $p\infty$. We define a global deformation condition $\mathcal{D}$ for $\overline{\rho}$ by the following requirements:

- (a) Deformations are unramified outside S and have determinant $\psi\chi^{k-1}$. By Proposition 2.3, ρ_n is a smooth deformation condition at these primes with tangent space dimension $\dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, \mathrm{ad}^0\overline{\rho})$.
- (b) At p, the local condition $\mathcal{D}_p$ consists of deformations as in Proposition 2.1. Then this proposition, along with hypothesis C3 in Theorem A ensures that ρ_n is a smooth deformation condition with tangent space dimension $\dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, \mathrm{ad}^0\overline{\rho}) + 1$.
- (c) Let $\mathfrak{q} \neq p$ be a place in S. We need to distinguish two cases:
  - (i) If $p | \#\overline{\rho}(I_{\mathfrak{q}})$ then $\overline{\rho}|_{G_{\mathfrak{q}}} \sim \left(\begin{smallmatrix} \overline{\chi} & * \\ 0 & 1 \end{smallmatrix}\right) \overline{\epsilon}$ for some character $\overline{\epsilon}$. We then take $\mathcal{D}_{\mathfrak{q}}$ to be local lifts with determinant $\psi\chi^{k-1}$ of the type considered in the first part of Proposition 2.2.
  - (ii) If $p \nmid \#\overline{\rho}(I_{\mathfrak{q}})$ we take $\mathcal{D}_{\mathfrak{q}}$ as in the second part of Proposition 2.2 *i.e.* lifts with determinant $\psi\chi^{k'-1}$ which factor through $G_{\mathfrak{q}}/(I_{\mathfrak{q}} \cap \ker \overline{\rho})$.

  In either case, hypothesis C4 of Theorem A and Proposition 2.2 imply that ρ_n is a smooth deformation condition at these places with tangent space dimension being $\dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, \mathrm{ad}^0\overline{\rho})$.

It then follows that ρ_n is a deformation of type $\mathcal{D}$. □

2.4. **Deformations of mod p^n representations to W.** We now show that any obstructions to lifting ρ_n to characteristic 0 can be realized locally and these, in turn, can be overcome by trivializing certain dual Selmer groups. We denote by ad^0 the the vector space of 2×2-matrices over $\boldsymbol{k}$ with $GL_2(W/p^n)$ acting by conjugation, and by $\mathrm{ad}^0(i)$ its twist by the i-th power of the determinant. Given an global deformation condition $\mathcal{D}$ we denote, by $\mathcal{D}_{\mathfrak{q}}$, the local component at a prime q, its tangent space by $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}$, and by $\mathbf{t}^{\perp}_{\mathcal{D}_{\mathfrak{q}}} \subseteq H^1(G_{\mathfrak{q}}, \mathrm{ad}^0\overline{\rho}(1))$ the orthogonal complement of $\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}$ under the pairing induced by $\mathrm{ad}^0\overline{\rho}\times\mathrm{ad}^0\overline{\rho}(1) \xrightarrow{\text{trace}} \boldsymbol{k}(1)$. The tangent space for $\mathcal{D}$ is the Selmer group $H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho})$ which we define as the preimage of $\bigoplus_{\mathfrak{q}\in S}\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}$ under the restriction map

$$H^1(G_{\mathbb{Q}}, ad^0\overline{\rho}) \to \bigoplus_{\mathfrak{q}\in S} H^1(G_{\mathfrak{q}}, ad^0\overline{\rho}).$$

Similarly we will let the dual Selmer group $H^1_{\{\mathbf{t}^{\perp}_{\mathcal{D}_{\mathfrak{q}}}\}}(G_{\mathbb{Q}}, ad^0\overline{\rho}(1))$ denote the preimage of $\bigoplus_{\mathfrak{q}\in S}\mathbf{t}^{\perp}_{\mathcal{D}_{\mathfrak{q}}}$ under the restriction map

$$H^1(G_{\mathbb{Q}}, ad^0\overline{\rho}(1)) \to \bigoplus_{\mathfrak{q}\in S} H^1(G_{\mathfrak{q}}, ad^0\overline{\rho}(1)).$$

We also set

$$\delta(\mathcal{D}) := \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{D}_{\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}) - \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}^{\perp}_{\mathcal{D}_{\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)).$$

Before we state and prove the lifting result for ρ_n we record the following lemma about some properties of certain subgroups of $GL_2(W/p^n)$.

Lemma 2.5. *Let G be a subgroup of $GL_2(W/p^n)$. Suppose the mod p reduction of G contains $SL_2(\boldsymbol{k})$. Furthermore, assume that if $p=3$ then G contains the transvection $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Then the following statements hold.*

(a) *G contains $SL_2(W/p^n)$.*
(b) *Suppose that $p\geq 5$. If $\boldsymbol{k}=\mathbb{F}_5$ assume that $G \mod 5 = GL_2(\mathbb{F}_5)$. Then $H^1(G, ad^0(i))=0$ for $i=0,1$.*
(c) *The restriction map $H^1\left(G, ad^0(i)\right) \longrightarrow H^1\left(<\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)>, ad^0(i)\right)$ is an injection (for all $p\geq 3$).*

Proof. This is Proposition 2.7 in [1]. We sketch a slightly simpler argument for parts (b) and (c). We first make the following observation:

For $i=0,1$ and $p\geq 3$, $H^1(GL_2(\boldsymbol{k}), \mathrm{ad}^0(i))=0$ if $k=\mathbb{F}_5$ and $H^1(SL_2(\boldsymbol{k}), \mathrm{ad}^0(i))=0$ if $k\neq\mathbb{F}_5$.

It is well known—see Lemma 2.48 of [2], for instance—that $H^1(SL_2(\boldsymbol{k}), \mathrm{ad}^0)=0$ except when $\boldsymbol{k}=\mathbb{F}_5$. We prove the exceptional case. Let $B\supset U$ be the subgroups of $GL_2(\mathbb{F}_5)$ consisting of matrices of the form $\left(\begin{smallmatrix}*&*\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$ respectively. We then need to verify that $H^1(B, \mathrm{ad}^0(i)) \cong H^1(U, \mathrm{ad}^0(i))^{B/U} = (0)$. Let $\sigma := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \tau := \left(\begin{smallmatrix}3&0\\0&1\end{smallmatrix}\right)$. It follows that $(\sigma-1)\mathrm{ad}^0$ is the subspace of upper triangular matrices in ad^0. Thus if $0\neq\xi\in H^1(U, \mathrm{ad}^0(i))$ then we can assume that $\xi(\sigma)=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, and ξ is fixed by B/U if and only if $(\tau*\xi)(\sigma)-\xi(\sigma)$ is upper triangular. Now $(\tau*\xi)(\sigma)=\tau\xi(\sigma^2)\tau^{-1}=\left(\begin{smallmatrix}1&2\\-1&1\end{smallmatrix}\right)$ and so $\xi\in H^1(U, \mathrm{ad}^0(i))^{B/U}$ if and only if $1+3^i=0$.

Now, to prove part (b), we use the Inflation-Restriction exact sequence

$$0 \longrightarrow H^1(G/H, \mathrm{ad}^0(i)^H) \longrightarrow H^1(G, \mathrm{ad}^0(i)) \longrightarrow H^1(H, \mathrm{ad}^0(i))^{G/H}$$

where $H := ker\{G \longrightarrow G \mod p^n\}$. Then, the first term vanishes by the hypothesis and the observation made above while the last term vanishes because $\mathrm{ad}^0(i)$ has trivial fixed points under the action of $SL_2(\boldsymbol{k})$ which G/H contains.

The sequence above also shows that all non-zero classes in $H^1(G, \mathrm{ad}^0(i))$ must come from $H^1(G \bmod p, \mathrm{ad}^0(i))$ via inflation. Since $H^1(SL_2(\boldsymbol{k}), \mathrm{ad}^0(i))$ is trivial, (c) follows unless $k = \mathbb{F}_5$ in which case the assertion is the general fact from group cohomology that restriction to a p-Sylow subgroup is injective if the coefficients are p-primary.

□

Another crucial input that we will use is a well-known formula due to Wiles which we state for the sake of convenience.

Wiles' formula. The following formula is a specialization of [7, Theorem 8.7.9].

$$\delta(\mathcal{D}) = \dim_{\boldsymbol{k}} H^0(G_{\mathbb{Q}}, \mathrm{ad}^0(\overline{\rho})) - \dim_{\boldsymbol{k}} H^0(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)) + \sum_{q \le \infty} (\dim_{\boldsymbol{k}} \mathbf{t}_{\mathcal{D}_q} - \dim_{\boldsymbol{k}} H^0(G_q, \mathrm{ad}^0\overline{\rho})).$$

In the context of our work $H^0(G_{\mathbb{Q}}, \mathrm{ad}^0(\overline{\rho}) = H^0(G_{\mathbb{Q}}, \mathrm{ad}^0(\overline{\rho}(1)) = 0$ since $\overline{\rho}$ is absolutely irreducible. Moreover it follows from easy group cohomological considerations that $\dim_{\boldsymbol{k}} \mathbf{t}_{\mathcal{D}_\infty} = 0$ and $\dim_{\boldsymbol{k}} H^0(G_\infty, \mathrm{ad}^0\overline{\rho})) = 1$. Therefore we will only need the simpler formula:

$$\delta(\mathcal{D}) = \sum_{q < \infty} (\dim_{\boldsymbol{k}} \mathbf{t}_{\mathcal{D}_q} - \dim_{\boldsymbol{k}} H^0(G_q, \mathrm{ad}^0\overline{\rho})) - 1.$$

We are now ready to prove the following lifting result. (*cf.* [1, Proposition 3.1 and Theorem 3.2].)

Theorem 2.6. *Suppose we are given a deformation condition $\mathcal{D}$ for ρ_n with determinant ϵ. Let S be a fixed finite set of primes of $\mathbb{Q}$ including primes where $\mathcal{D}$ is ramified and all primes dividing $p\infty$. If $\delta(\mathcal{D}) \ge 0$ then we can find a smooth deformation condition $\mathcal{E}$ for ρ_n with determinant ϵ such that the local conditions $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ differ only at primes $\mathfrak{q} \notin S$ and, at these primes, $\mathcal{E}_{\mathfrak{q}}$ is a smooth deformation condition with tangent space having dimension $\dim_{\boldsymbol{k}} H^0(G_{\mathfrak{q}}, ad^0\overline{\rho})$. Moreover, $H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{\mathfrak{q}}}\}}(\mathbb{Q}, ad^0\overline{\rho}(1)) = (0)$ and the universal deformation ring is a power series ring over W in $\delta(\mathcal{D})$ variables. In particular, there is a representation $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(W)$ of type $\mathcal{E}$ lifting ρ_n.*

Proof. We define a deformation condition $\mathcal{E}_0$ for ρ_n with determinant ϵ as follows. If $p \ge 5$ and the projective image of $\overline{\rho}$ strictly contains $PSL_2(\mathbb{F}_5)$ then $\mathcal{E}_0$ is $\mathcal{D}$. Now suppose that either $p = 3$ or the projective image of $\overline{\rho}$ is A_5 (so $\boldsymbol{k}$ is necessarily $\mathbb{F}_5$). Let K be the splitting field of ρ_n adjoined with the p^n-th roots of unity. We can find an element $h \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\rho_n(h) \sim a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\chi(h) = 1 \mod p^n$ as follows: We write $\epsilon = \chi\epsilon_0\epsilon_1^2$ where ϵ_0 is a finite order character of order coprime to p. Lemma 2.5(a) then implies that the image of the twist of $\rho_n \otimes \epsilon_1^{-1}$ contains $SL_2(W/p^n)$. Thus we can find $h_1 \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\rho_n(h_1) = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\epsilon_1(h_1)$ and we get $\epsilon_0(h_1)\chi(h_1) = 1$. We can then take h to be $h_1^{p^k-1}$ where p^k is the cardinality of $\boldsymbol{k}$. By applying Chebotarev to h we can find a prime $\mathfrak{q}_0 \notin S$ with $\mathfrak{q}_0 \equiv 1 \mod p^n$ and $\rho_n(\mathrm{Frob}_{\mathfrak{q}_0}) = a\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Let $\mathcal{E}_0$ be the deformation condition of $\overline{\rho}$ with determinant ϵ such that $\mathcal{E}_{0\mathfrak{q}} = \mathcal{D}_{\mathfrak{q}}$ at primes $\mathfrak{q} \neq \mathfrak{q}_0$, and at $\mathfrak{q}_0$, $\mathcal{E}_{0\mathfrak{q}_0}$ consists of deformations of the form $\begin{pmatrix}\chi & * \\ 0 & 1\end{pmatrix}\epsilon'$ where $\epsilon' : G_{v_0} \longrightarrow W^\times$ is unramified and $\epsilon|_{G_{\mathfrak{q}_0}} = \chi{\epsilon'}^2$. Proposition 2.2 then ensures that $\mathcal{E}_{0\mathfrak{q}_0}$ is a smooth local deformation condition and hence that $\mathcal{E}_0$ is a global deformation condition for ρ_n. Further, Wiles' formula shows that $\delta(\mathcal{E}_0)$ remains invariant after this adjustment so that $\delta(\mathcal{E}_0) = \delta(\mathcal{D})$.

Next, we claim that the restriction maps

$$H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho}) \quad \text{and} \quad H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho}(1))$$

are injective. When $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains A_5 an easy calculation using Lemma 2.5(b) shows that the kernels $H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{ad}^0\overline{\rho})$ and $H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{ad}^0\overline{\rho}(1))$ are trivial, and so the injectivity follows. In the exceptional case *i.e.* when $p = 3$ or the projective image of $\overline{\rho}$ is A_5, we observe that $\xi \in \ker\left(H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho})\right)$ is naturally an element of $H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{ad}^0\overline{\rho})$. Thus ξ is unramified at $\mathfrak{q}_0$ and so the restriction of ξ to the decomposition group at $\mathfrak{q}_0$ must be trivial. Using Lemma 2.5(c) it follows that $\xi \in H^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{ad}^0\overline{\rho})$ is trivial. A similar argument works for $\mathrm{ad}^0\overline{\rho}(1)$.

As previously noted $\delta(\mathcal{E}_0) = \delta(\mathcal{D})$ and since $\delta(\mathcal{D}) \geq 0$ (by hypothesis) we conclude that if the dual Selmer group for $\mathcal{E}_0$ is non-trivial then we can find

$$0 \neq \xi \in H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}), \quad 0 \neq \psi \in H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)).$$

Taking $g \in \mathrm{Gal}(K/\mathbb{Q})$ to be complex conjugation we get $\rho_n(g) \sim \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\chi(g) = -1 \mod p^n$ and, by applying Proposition 2.2 of [5] to g, we find a prime $\mathfrak{r} \notin S \cup \{\mathfrak{q}_0\}$ lifting g such that the restrictions of ξ, ψ to $G_{\mathfrak{r}}$ are not in $H^1(G_{\mathfrak{r}}, N_1), H^1(G_{\mathfrak{r}}, N_2)$ for $\{\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\} = N_1 \subset M_1 = \mathrm{ad}^0\overline{\rho}$, $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} = N_2 \subset M_2 = \mathrm{ad}^0\overline{\rho}(1)$.We now adjust $\mathcal{E}_0$ to produce a new deformation condition $\mathcal{E}_1$ with determinant ϵ such that $\mathcal{E}_1$ and $\mathcal{E}_0$ differ only at $\mathfrak{r}$ where the local component consists of deformations of the form $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)(\epsilon/\chi)^{1/2}$ considered in Proposition 2.2. Here, $(\epsilon/\chi)^{1/2}$ is the unramified character determined by taking the square-root of $\epsilon(\mathrm{Frob}_{\mathfrak{r}})\chi^{-1}(\mathrm{Frob}_{\mathfrak{r}})$. Since $\mathrm{Frob}_{\mathfrak{r}}$ lifts g we have $\chi(\mathrm{Frob}_{\mathfrak{r}}) \equiv -1 \mod p^n$, and consequently $\mathcal{E}_1$ is a smooth deformation condition for ρ_n. A dimension calculation identical to [5, Proposition 4.2] shows that $\dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{1\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)) = \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{0\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}(1)) - 1$. By arguing recursively, we find the desired global deformation condition $\mathcal{E}$ with trivial dual Selmer group. The existence of the lift ρ follows by arguing as in the proof of [10, Lemma 1.1]. Finally, the claim that the universal deformation ring is a power series ring over W in $\delta(\mathcal{D})$ variables follows because $\dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{E}_{\mathfrak{q}}}\}}(G_{\mathbb{Q}}, \mathrm{ad}^0\overline{\rho}) = \delta(\mathcal{E}) = \delta(\mathcal{D})$. □

3. Modularity of ρ: Proof of Theorems A and B

We now have all the tools required to prove Theorems A and B. On the one hand we have lifted ρ_n to a representation in characteristic 0 and on the other we already know that its mod p reduction is modular. This presents an ideal scenario for the application of a suitable modularity lifting theorem and, indeed, we invoke a theorem of Skinner and Wiles.

Proof of Theorems A and B: Let ρ_n be as in Theorem A or Theorem B. Proposition 2.4 implies that there exists a global deformation condition $\mathcal{D}$ for ρ_n such that each local component $\mathcal{D}_q$ is a smooth deformation condition with the required tangential dimension. Therefore, if we knew that $\delta(\mathcal{D}) \geq 0$, we may apply Theorem 2.6 to get a $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(W)$ lifting ρ_n. We verify this inequality as follows. By Wiles' formula $\delta(\mathcal{D}) = \sum_{q<\infty} (\mathbf{t}_{\mathcal{D}_q} - \dim_{\boldsymbol{k}} H^0(G_q, \mathrm{ad}^0\overline{\rho})) - 1$. However, $\mathbf{t}_{\mathcal{D}_q} = \dim_{\boldsymbol{k}} H^0(G_q, \mathrm{ad}^0\overline{\rho})$ for $q \nmid p$ (Propositions 2.2 and 2.3) and $\mathbf{t}_{\mathcal{D}_q} = 1 + \dim_{\boldsymbol{k}} H^0(G_q, \mathrm{ad}^0\overline{\rho})$ for $q = p$ (Proposition 2.1.) Consequently $\delta(\mathcal{D}) = 0$. As noted in the Introduction, the irreducibility of $\overline{\rho}$ is ensured by hypothesis C2 and so it is modular by Serre's modularity conjecture. The p-distinguishedness property is part of hypothesis C3 and our choice of local deformations at p

(Proposition 2.1 and the preceding paragraph) ensures that the image of $\rho|_{I_p}$ has 1 as its lower right matrix entry. A direct application of the main theorem of [7] then shows that ρ arises from a p-ordinary modular form of weight k and level prime to N.

We conclude with the following remarks.

Remark 3.1. All the work in Section 2, and consequently Theorems A and B, can be generalized to the setting of 2-dimensional Galois representations of G_F – where F is a totally real field in which p ramifies – and Hilbert modular forms of parallel weight. The caveat for Theorem A is that Serre's Modularity Conjecture is known only for $F = \mathbb{Q}$. Theorem B, however would become readily available because in this case the modularity of $\overline{\rho}$ follows from [4, Theorem 2.1.1].

Remark 3.2. As we mentioned in the Section 1, Theorem B can be thought of as an analog of [4, Theorem 2.1.1] where it is used as a "first step" in proving a much deeper result about the existence of weight 1 companion forms. A mod p^n version of that result however would involve geometric arguments which are beyond the scope and intent of this work.

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