

An entropy formula for the heat equation on manifolds with time-dependent metric, application to ancient solutions

Hongxin Guo · Robert Philipowski ·
Anton Thalmaier

Received: date / Accepted: date

Abstract We introduce a new entropy functional for nonnegative solutions of the heat equation on a manifold with time-dependent Riemannian metric. Under certain integral assumptions, we show that this entropy is non-decreasing, and moreover convex if the metric evolves under super Ricci flow (which includes Ricci flow and fixed metrics with nonnegative Ricci curvature). As applications, we classify nonnegative ancient solutions to the heat equation according to their entropies. In particular, we show that a nonnegative ancient solution whose entropy grows sublinearly on a manifold evolving under super Ricci flow must be constant. The assumption is sharp in the sense that there do exist nonconstant positive eternal solutions whose entropies grow exactly linearly in time. Some other results are also obtained.

Keywords Ricci flow · Brownian motion · Entropy

Mathematics Subject Classification (2010) 53C44 · 58J65

1 Introduction

Let M be a smooth manifold equipped with a family $(g(t))_{t \geq 0}$ of Riemannian metrics depending smoothly on t , and let u be a nonnegative solution of the backward heat equation

$$\frac{\partial u}{\partial t} + \Delta_{g(t)} u = 0. \quad (1.1)$$

Hongxin Guo
School of Mathematics and Information Science, Wenzhou University Wenzhou, Zhejiang 325035, China
E-mail: guo@wzu.edu.cn

Robert Philipowski · Anton Thalmaier
Unité de Recherche en Mathématiques, FSTC, Université du Luxembourg, 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, Grand Duchy of Luxembourg

Robert Philipowski
E-mail: robert.philipowski@uni.lu

Anton Thalmaier
E-mail: anton.thalmaier@uni.lu

The classical Boltzmann-Shannon entropy functional is defined by

$$\text{Ent}(t) = - \int_M u(t, y) \log u(t, y) \text{vol}_{g(t)}(dy)$$

(provided that the integral exists). If the metric does not depend on t , then, under certain reasonable assumptions on u (for instance if u grows slowly enough so that integration by parts can be justified) the Boltzmann-Shannon entropy is non-increasing, and moreover concave if $\text{Ric} \geq 0$. In the case of a compact manifold Lim and Luo [21] recently studied asymptotic estimates on the time derivative of Ent . However, the classical Boltzmann-Shannon entropy has two important drawbacks:

1. It need not be monotone if the metric depends on t .
2. On noncompact manifolds it is finite only for a relatively narrow class of functions. Even if u is a positive constant $\neq 1$ on \mathbb{R}^n (equipped with the standard metric), its Boltzmann-Shannon entropy equals $\pm\infty$.

In this paper we introduce a new entropy functional of Boltzmann-Shannon type which has much better chances to be finite and which is monotone even if the metric depends on t . We fix a point $x \in M$ and let $p(t, x, y)$ be the heat kernel of the adjoint heat equation

$$\frac{\partial p}{\partial t} = \Delta_{g(t)} p - \frac{1}{2} \text{tr} \frac{\partial g}{\partial t} p.$$

In other words, $p(t, x, \cdot)$ is the density of X_t with respect to $\text{vol}_{g(t)}$,

$$p(t, x, y) \text{vol}_{g(t)}(dy) = \mathbb{P}\{X_t \in dy\},$$

where $(X_t)_{t \geq 0}$ is a $(g(t))_{t \geq 0}$ -Brownian motion started at x and speeded up by the factor $\sqrt{2}$, see [1, 11, 19]. We assume that $\int_M p(t, x, y) \text{vol}_{g(t)}(dy) = 1$ for all $t > 0$, in other words that Brownian motion on M does not explode. By a result of Kuwada and the second author [19] this condition is satisfied, in particular, if $(M, g(t))$ is complete for all $t \geq 0$ and the metric evolves under backward super Ricci flow, i.e.

$$\frac{\partial g}{\partial t} \leq 2 \text{Ric}. \quad (1.2)$$

We define the entropy of u with respect to the heat kernel measure

$$p(t, x, y) \text{vol}_{g(t)}(dy)$$

by

$$\begin{aligned} \mathcal{E}(t) &:= \mathbb{E}[(u \log u)(t, X_t)] \\ &= \int_M (u \log u)(t, y) p(t, x, y) \text{vol}_{g(t)}(dy). \end{aligned} \quad (1.3)$$

(From a physical point of view it would be more natural to call $-\mathcal{E}$ entropy. Our sign convention has the advantage of avoiding unnecessary minus signs.) Note that in contrast to the classical Boltzmann-Shannon entropy this entropy is well-defined for all non-negative solutions u (because the heat kernel has total mass 1 and the function $u \mapsto u \log u$ is bounded from below). Moreover, thanks to the fast decay of the heat kernel our entropy is finite in most cases of interest. In the next section we will show that under certain integral assumptions $\mathcal{E}(t)$ is non-decreasing, and moreover convex in the case of super Ricci flow.

Remark 1 Monotonicity formulas, or the search for quantities which are monotonic along a geometric flow, play a key role in analysis and geometry, e.g. [9, 12]. They are a powerful tools to control geometric evolutions. In 2002, Perelman [26] introduced two geometric quantities, now called Perelman's entropy and Perelman's reduced volume, which are scale-invariant, non-decreasing under Ricci flow, and allow to control the geometry enough to prevent collapsing. Our results on the evolution of $\mathcal{E}(t)$ are clearly inspired by Perelman's \mathcal{F} -functional.

Remark 2 If we apply the substitution $\tau := -t$, then (1.1) and (1.2) become

$$\frac{\partial u}{\partial \tau} = \Delta_{g(-\tau)} u$$

and

$$\frac{\partial g}{\partial \tau} \geq -2\text{Ric}. \quad (1.4)$$

In other words, with respect to the new time variable τ , the function u is a solution to the forward heat equation and g evolves according to super Ricci flow. In particular, solutions of (1.1) that are defined for all $t \geq 0$ are the same as ancient solutions of the heat equation.

The most important examples of super Ricci flow are of course the Ricci flow itself [6–8, 23, 29], where (1.4) holds with equality, and fixed metrics with nonnegative Ricci curvature. Other interesting examples are the extended Ricci flow introduced by List [22] and Ricci flow coupled with harmonic map flow, as studied by Müller [24].

Ancient solutions to the heat equation are generalizations of harmonic functions. Yau's Liouville theorem for positive harmonic functions states that any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is constant [31]. However, as we can see from the example $u(\tau, y) = e^{\tau+y}$, Yau's Liouville theorem cannot be generalized to positive ancient solutions without any further assumptions.

Starting from this observation, Souplet and Zhang [28, Theorem 1.2] proved the following Liouville type result for positive ancient solutions: Let M be a complete, noncompact manifold with a fixed metric of nonnegative Ricci curvature. If u is a positive ancient solution to the heat equation such that $\log u(\tau, y) = o(d(y) + \sqrt{|\tau|})$ near infinity, then u must be constant.

This result is a direct consequence of a local Li-Yau gradient estimate of Hamilton type for positive solutions of the heat equation, see [28, Theorem 1.1]. It is meanwhile well understood how to establish such estimates by means of Stochastic Analysis using submartingale arguments, see Arnaudon and Thalmaier [2, Section 6]. These arguments can be generalized to positive solutions of the heat equation under Ricci flow, see L.-J. Cheng [4, Theorem 2.4], and allow to generalize Souplet and Zheng's Liouville result to manifolds whose metric evolves under super Ricci flow.

Souplet and Zhang's result for a static metric has been extended to ancient solutions of the nonlinear heat equation taking values in non-positively curved manifolds [30], as well as to certain types of fast diffusion equations on manifold of non-negative Ricci curvature [33]. All these results depend on sharp gradient estimates for the solutions, along with pointwise growth conditions.

Besides the pointwise assumption of Souplet and Zhang, it seems to be desirable to investigate integral conditions. As an application of our entropy formula, among other results we prove the following: Assume that $\frac{\partial g}{\partial t} \leq 2\text{Ric}$, and let u be a nonnegative solution of (1.1). If its entropy $\mathcal{E}(t)$ grows sublinearly, i.e. $\lim_{t \rightarrow \infty} \mathcal{E}(t)/t = 0$, then u is constant.

Our assumption that $\mathcal{E}(t)$ grows sublinearly is an integral one. The result is sharp in the sense that there do exist nonconstant ancient solutions whose entropies grow linearly, for instance $u(t, y) = e^{y-t}$ whose entropy with parameter $x = 0$ satisfies $\mathcal{E}(t) = t$. We also discuss the special case when $\mathcal{E}(t)$ is a linear function of t . In this case under the assumption that $\frac{\partial g}{\partial t} \leq 2\text{Ric}$, we show that u is the product of a function depending only on t and a function depending only on y .

Our methods come from stochastic analysis. Instead of applying classical integration by parts formulas (which are usually hard to justify on non-compact manifolds), all our calculations rely on Itô's formula which does not require extra conditions. We need conditions only to assure that our local submartingales are true submartingales. The wanted inequalities are then obtained by taking expectations of the submartingales.

Probabilistic approaches to flows of time changing metrics have been used recently by several authors. For instance, Kuwada [18] studied couplings of $g(t)$ -Brownian motions, see also [20]. Coulibaly-Pasquier [10] applied the techniques to mean curvature flows. Philipowski [27] constructed stochastic particle approximations for the normalized Ricci flow on surfaces and for the non-normalized Yamabe flow on manifolds of arbitrary dimension. Neel and Popescu [25] developed a stochastic target approach to Ricci flow on surfaces. The authors [13, 14] developed a theory of martingales on manifolds with time-dependent connection and applied this theory to the study of the harmonic map heat flow on manifolds with time-dependent Riemannian metric. Chen, Cheng and Mao [3] recently derived a stochastic representation formula for solutions of heat-type equations on tensor bundles over manifolds with time-dependent Riemannian metric.

2 Monotonicity and convexity of the entropy

In this section we derive formulas for the first two variations of the entropy. We shall see that the entropy $\mathcal{E}(t)$, under certain assumptions, is non-decreasing. Moreover $\mathcal{E}(t)$ is convex if $\frac{\partial g}{\partial t} \leq 2\text{Ric}$.

Theorem 1 *Let u be a solution of the backward heat equation (1.1). Suppose that for $t > 0$,*

$$\int_M |\nabla(u \log u)|^2(t, y) p(t, x, y) \text{vol}_{g(t)}(dy) < \infty \quad (2.5)$$

and

$$\int_M \left| \nabla \left(\frac{|\nabla u|^2}{u} \right) \right|^2(t, y) p(t, x, y) \text{vol}_{g(t)}(dy) < \infty. \quad (2.6)$$

Then as long as $\mathcal{E}(t)$ is finite its first derivative is given by

$$\mathcal{E}'(t) = \int_M \frac{|\nabla u|^2}{u}(t, y) p(t, x, y) \text{vol}_{g(t)}(dy), \quad (2.7)$$

and its second derivative by

$$\begin{aligned} \mathcal{E}''(t) = \int_M \left(2u \left(|\nabla \nabla \log u|^2 + \left(\text{Ric} - \frac{1}{2} \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \right) \right)(t, y) \\ p(t, x, y) \text{vol}_{g(t)}(dy). \end{aligned} \quad (2.8)$$

For the proof we need the following lemma:

Lemma 1 *If u solves the backward heat equation (1.1), we have*

$$\left(\frac{\partial}{\partial t} + \Delta_{g(t)}\right)(u \log u) = \frac{|\nabla u|^2}{u} \quad (2.9)$$

and

$$\left(\frac{\partial}{\partial t} + \Delta_{g(t)}\right)\left(\frac{|\nabla u|^2}{u}\right) = u \left(2|\nabla \nabla \log u|^2 + \left(2\text{Ric} - \frac{\partial g}{\partial t}\right)(\nabla \log u, \nabla \log u)\right). \quad (2.10)$$

Proof The first equality is straight-forward. The second one is well-known in the case of a fixed Riemannian metric (e.g. [15]; for a proof see [21, Proposition 2.1]). The additional term $-\frac{\partial g}{\partial t}(\nabla \log u, \nabla \log u)$ appearing here comes from the time-derivative of $|\nabla u|^2$ via the formula

$$\frac{\partial}{\partial t}(|\nabla f|^2) = -\frac{\partial g}{\partial t}(\nabla f, \nabla f), \quad f \in C^\infty(M). \quad (2.11)$$

Note that not only $|\cdot|$, but also ∇ depends on t , which is the reason for the minus sign in formula (2.11). \square

Using Lemma 1 it is easy to give a formal proof of Theorem 1 via integration by parts. However, since M is not assumed to be compact, the feasibility of integration by parts is difficult to justify, and therefore we present a proof based on stochastic analysis. In this proof the assumptions (2.5) and (2.6) are used to show that certain local martingales are indeed true martingales. One should note that thanks to the exponential decay of the heat kernel (see e.g. [5] for the case of a fixed metric and [32, Section 6.5] for the case of Ricci flow) the conditions (2.5) and (2.6) are satisfied in most cases of interest.

Remark 3 In terms of a $(g(t))_{t \geq 0}$ -Brownian motion $(X_t)_{t \geq 0}$ started at x , conditions (2.5) and (2.6) read as

$$\mathbb{E}[|\nabla(u \log u)|^2(t, X_t)] < \infty, \quad (2.12)$$

$$\mathbb{E}\left[\left|\nabla\left(\frac{|\nabla u|^2}{u}\right)\right|^2(t, X_t)\right] < \infty, \quad (2.13)$$

and imply that

$$\int_0^t \mathbb{E}[|\nabla(u \log u)|^2(s, X_s)] ds < \infty \quad \text{and} \quad \int_0^t \mathbb{E}\left[\left|\nabla\left(\frac{|\nabla u|^2}{u}\right)\right|^2(s, X_s)\right] ds < \infty \quad (2.14)$$

where we simply used that the left-hand sides of (2.12) and (2.13) are locally bounded in the time variable. The last two conditions are standard conditions to assure that the martingale parts of the processes

$$(u \log u)(s, X_s), \quad \frac{|\nabla u|^2}{u}(s, X_s), \quad 0 \leq s \leq t, \quad (2.15)$$

are true martingales (even L^2 -martingales). The condition, analogous to (2.12) resp. (2.13), guaranteeing that the local martingale

$$u(s, X_s), \quad 0 \leq s \leq t,$$

is a true L^2 -martingale reads as

$$\mathbb{E}[|\nabla u|^2(t, X_t)] < \infty. \quad (2.16)$$

In many cases it may be straight-forward to verify directly that the martingale parts of (2.15) are true martingales (for instance, if estimates for $|\nabla u|^2/u$ are available [4, Theorem 2.4], or if $\sup_{[0,T] \times M} u \leq C_T$). Then the technical conditions (2.5) and (2.6), resp. (2.12) and (2.13), are dispensable.

Proof (of Theorem 1) Denote $\dim M = n$ and let U be a horizontal lift of the $(g(t))_{t \geq 0}$ -Brownian motion X and Z the corresponding anti-development of X . (Note that Z is an \mathbb{R}^n -valued Brownian motion speeded up by the factor $\sqrt{2}$). By Itô's formula (see [19, Lemma 1])

$$d(u \log u)(t, X_t) = \left(\frac{\partial}{\partial t} + \Delta_{g(t)} \right) (u \log u)(t, X_t) dt + dM_t, \quad (2.17)$$

where thanks to (2.5) the local martingale

$$M_t := \sum_{i=1}^n \int_0^t \langle \nabla(u \log u)(s, X_s), (U_s e_i) dZ_s^i \rangle$$

is a true martingale (as stochastic integral of a square-integrable process, see e.g. [17, Chapt. 3, Def. 2.9]). Combining (2.17) and (2.9) we obtain

$$\mathbb{E}[(u \log u)(t, X_t)] = (u \log u)(0, x) + \mathbb{E} \left[\int_0^t \frac{|\nabla u|^2}{u}(s, X_s) ds \right],$$

or in other words,

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \mathbb{E} \left[\frac{|\nabla u|^2}{u}(s, X_s) \right] ds,$$

and hence

$$\mathcal{E}'(t) = \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right] = \int_M \frac{|\nabla u(t, y)|^2}{u(t, y)} p(t, x, y) \text{vol}_{g(t)}(dy),$$

as claimed.

The formula for the second derivative of \mathcal{E} can be proved in the same way, using (2.6) and (2.10). \square

3 Gradient-entropy estimates

In this section we give gradient estimates for positive solutions of the backward heat equation (1.1) in terms of their entropy.

Proposition 1 *Let $u : [0, T] \times M \rightarrow \mathbb{R}_+$ be a positive solution of the backward heat equation (1.1) satisfying the conditions (2.12), (2.13) and (2.16) for $t = T$. Assume that $\frac{\partial g}{\partial t} \leq 2 \text{Ric}$. Then, for each $t \in]0, T]$ and $x \in M$,*

$$t \left| \frac{\nabla u}{u} \right|^2(0, x) \leq \mathbb{E} \left[\frac{u(t, X_t)}{u(0, x)} \log \frac{u(t, X_t)}{u(0, x)} \right] \quad (3.18)$$

where (X_s) is a $(g(s))_{s \geq 0}$ -Brownian motion starting at x . In other words, if u is normalized such that $u(0, x) = 1$, then

$$|\nabla u|^2(0, x) \leq \frac{\mathcal{E}(t)}{t}. \quad (3.19)$$

Proof Consider the process

$$N_s := (t-s) \frac{|\nabla u|^2}{u}(s, X_s) + (u \log u)(s, X_s), \quad 0 \leq s \leq t, \quad (3.20)$$

which is easily seen to be a submartingale under the given conditions. This enables us to exploit the inequality $\mathbb{E}[N_0] \leq \mathbb{E}[N_t]$ which gives

$$t \frac{|\nabla u|^2}{u}(0, x) + (u \log u)(0, x) \leq \mathbb{E}[(u \log u)(t, X_t)]. \quad (3.21)$$

Combining this with the fact that $u(0, x) = \mathbb{E}[u(t, X_t)]$ which follows from the martingale property of $(u(s, X_s))_{0 \leq s \leq t}$, the claimed inequality is obtained. \square

Corollary 1 *Let u be a positive solution of the backward heat equation (1.1) on $[0, T] \times M$. We keep the assumptions of Theorem 1. Let $x \in M$ and $0 < t \leq T$.*

1. *Then, for any $\delta > 0$,*

$$\left| \frac{\nabla u}{u} \right|(0, x) \leq \frac{\delta}{2t} + \frac{1}{2\delta} \mathbb{E} \left[\frac{u(t, X_t)}{u(0, x)} \log \frac{u(t, X_t)}{u(0, x)} \right] \quad (3.22)$$

2. *If $m_t := \sup_{[0, t] \times M} u$, then*

$$\left| \frac{\nabla u}{u} \right|(0, x) \leq \frac{1}{t^{1/2}} \sqrt{\log \frac{m_t}{u(0, x)}}. \quad (3.23)$$

Note that estimate (3.23) is Hamilton's gradient estimate in global form. Estimate (3.18) shows that the entropy

$$\mathbb{E}[(v \log v)(t, X_t)] = \mathbb{E} \left[\frac{u(t, X_t)}{u(0, x)} \log \frac{u(t, X_t)}{u(0, x)} \right]$$

of a (normalized) positive solution $v(t, \cdot) := \frac{u(t, \cdot)}{u(0, x)}$ to the backward heat equation grows at least linearly in t , and trivially at most as $\log(m_t/u(0, x))$ with $m_t = \sup u|_{[0, t] \times M}$.

4 Entropy and linear growth

We now investigate positive solutions of the backward heat equation (1.1) according to their entropy. Recall that by Remark 2 any global solution to the backward equation (1.1) gives rise to an ancient solution of the forward heat equation.

Theorem 2 *Let $u : \mathbb{R}_+ \times M \rightarrow \mathbb{R}_+$ be a positive solution of the backward heat equation (1.1) satisfying (2.5) and (2.6) for all $t > 0$. If $\frac{\partial g}{\partial t} \leq 2 \text{Ric}$ and if the entropy of u grows sublinearly, i.e. $\lim_{t \rightarrow \infty} \mathcal{E}(t)/t = 0$, then u is constant.*

Proof Since \mathcal{E} is convex, the condition $\lim_{t \rightarrow \infty} \frac{\mathcal{E}(t)}{t} = 0$ implies that \mathcal{E} is constant. Therefore

$$\mathcal{E}'(t) = \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right] \equiv 0,$$

so that u is constant. \square

Remark 4 Note that Theorem 2 also immediately follows from the results of the last section. Indeed, if the conditions (2.5) and (2.6), or equivalently, the conditions (2.12), (2.13) hold for all $t > 0$, we have that (3.20) is a true submartingale. Rewriting $\mathbb{E}[N_0] \leq \mathbb{E}[N_t]$, resp. $\mathbb{E}[N_{t/2}] \leq \mathbb{E}[N_t]$, we have

$$\begin{aligned} \frac{|\nabla u|^2}{u}(0, x) + \frac{1}{t} (u \log u)(0, x) &\leq \frac{1}{t} \mathbb{E}[(u \log u)(t, X_t)], \quad \text{resp.,} \\ \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t/2, X_{t/2}) \right] + \frac{2}{t} \mathbb{E}[(u \log u)(t/2, X_{t/2})] &\leq \frac{2}{t} \mathbb{E}[(u \log u)(t, X_t)], \end{aligned}$$

and it suffices to take the limit as $t \rightarrow \infty$.

Remark 5 Let $\frac{\partial g}{\partial t} \leq 2\text{Ric}$. For a positive solution $u: \mathbb{R}_+ \times M \rightarrow \mathbb{R}_+$ of the backward heat equation (1.1) we may consider the constant

$$\theta := \lim_{t \rightarrow \infty} \mathcal{E}'(t)$$

which is well-defined by the monotonicity resulting from formula (2.8). The value of θ may be zero, a positive constant or $+\infty$. Theorem 2 can then be rephrased to the statement that a positive solution $u: \mathbb{R}_+ \times M \rightarrow \mathbb{R}_+$ of the backward heat equation (1.1), satisfying (2.5) and (2.6) for all $t \geq 0$, is trivial if and only if $\theta = 0$.

Theorem 3 *Let $u: [0, T] \times M \rightarrow \mathbb{R}_+$ be a positive solution of the backward heat equation (1.1) satisfying (2.5) and (2.6) for $t = T$. If $\frac{\partial g}{\partial t} \leq 2\text{Ric}$ and $\mathcal{E}(t)$ is an exactly linear function of t , then u has the form*

$$u(t, y) = \psi(y)\phi(t) \quad (4.24)$$

for some functions ψ and ϕ . Moreover, ψ and ϕ satisfy the differential equation

$$\frac{\partial \phi / \partial t}{\phi} = -\frac{\Delta \psi}{\psi}. \quad (4.25)$$

Proof Since $\mathcal{E}(t)$ is exactly linear, we have

$$\begin{aligned} \mathcal{E}''(t) &= \mathbb{E} \left[2u \left(|\nabla \nabla \log u|^2 + \left(\text{Ric} - \frac{1}{2} \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \right) (t, X_t) \right] \\ &= \int_M 2u \left(|\nabla \nabla \log u|^2 + \left(\text{Ric} - \frac{1}{2} \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \right) p dy \equiv 0. \end{aligned}$$

In particular, $\nabla \nabla \log u \equiv 0$ implies that

$$\nabla \left(|\nabla \log u|^2 \right) = 2 \nabla \nabla \log u (\nabla \log u, \cdot) = 0$$

so that $|\nabla \log u|^2$ is a function of t only. Since

$$0 = \text{tr}(\nabla \nabla \log u) = \Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -\frac{\partial \log u}{\partial t} - |\nabla \log u|^2,$$

we have

$$\frac{\partial \log u}{\partial t} = -|\nabla \log u|^2$$

and hence

$$\log u(t, y) - \log u(0, y) = - \int_0^t |\nabla \log u|^2(s) ds.$$

It follows that

$$u(t, y) = u(0, y) \exp \left(- \int_0^t |\nabla \log u|^2(s) ds \right),$$

and this proves Eq. (4.24). Then Eq. (4.25) follows immediately from (4.24) and the backward heat equation for u . \square

We also have the following simple observation:

Proposition 2 *If $2\text{Ric} - \frac{\partial g}{\partial t}$ is positive definite everywhere, then no nonconstant positive solution to the backward heat equation satisfying (2.5) and (2.6) can have linear entropy.*

Proof If $\mathcal{E}(t)$ is linear, we have $\mathcal{E}''(t) \equiv 0$ which implies that

$$\left(2\text{Ric} - \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \equiv 0,$$

as the heat kernel $p(t, x, y)$ is strictly positive everywhere. Since $2\text{Ric} - \frac{\partial g}{\partial t}$ is strictly positive everywhere, we get $\nabla \log u \equiv 0$. \square

Example 1 Let $u(t, y) = e^{y-t}$ on \mathbb{R} equipped with the standard metric. Choose $x = 0$ in the heat kernel, so that $p(t, x, y) = e^{-y^2/4t} / \sqrt{4\pi t}$. We first check that the assumptions (2.5) and (2.6) are satisfied: By elementary and straightforward calculations, for every $t > 0$ we have

$$\int_{\mathbb{R}} |\nabla(u \log u)|^2(t, y) p(t, x, y) dy = (9t^2 + 8t + 1)e^{2t} < \infty$$

and

$$\int_{\mathbb{R}} \left| \nabla \left(\frac{|\nabla u|^2}{u} \right) \right|^2(t, y) p(t, x, y) dy = e^{2t} < \infty.$$

An easy calculation shows that

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} (-t + y) e^{-\frac{y^2}{4t} + y - t} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} ((y - 2t) + t) e^{-\frac{1}{4t}(y - 2t)^2} dy = t, \end{aligned}$$

so that \mathcal{E} grows exactly linearly.

More generally, for any constants $a > 0$ and b , the function $u(t, y) = ae^{by - b^2 t}$ is a positive solution of the backward heat equation, and its entropy $\mathcal{E}(t) = a(\log a + b^2 t)$ grows exactly linearly. Similar examples can be constructed on \mathbb{R}^n .

Example 2 Let $M = \mathbb{R}^3 \setminus \{0\}$ be equipped with the standard metric. The function

$$u(x) = \|x\|^{-1}$$

is harmonic on M , and thus $u(t, x) \equiv u(x)$ provides trivially a stationary solution of the backward heat equation (1.1) with respect to the static Euclidean metric. Let X be a Brownian motion on \mathbb{R}^3 (with generator Δ) starting at $e_1 = (1, 0, 0)$. Since

$$\nabla u(x) = -\frac{x}{\|x\|^3} \quad \text{and} \quad \frac{|\nabla u|^2}{u}(x) = \frac{1}{\|x\|^3},$$

it is easy to check that

$$\mathcal{E}(t) = \mathbb{E}[(u \log u)(X_t)] = -\frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3 \setminus \{0\}} \frac{\log \|x\|}{\|x\|} \exp\left(-\frac{\|x - e_1\|^2}{4t}\right) dx,$$

which is clearly bounded as function of t , whereas

$$\mathbb{E}\left[\frac{|\nabla u|^2}{u}(X_t)\right] = \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3 \setminus \{0\}} \frac{1}{\|x\|^3} \exp\left(-\frac{\|x - e_1\|^2}{4t}\right) dx = +\infty.$$

This shows that Theorem 1 fails in general without assumptions on u . Since the entropy $\mathcal{E}(t)$ of u grows sublinearly, it also shows that Theorem 2 fails without extra assumptions, like the conditions (2.5) and (2.6).

5 Monotonicity and convexity of local entropy

The results presented in the previous sections depend on the technical conditions (2.5) and (2.6) which have been used to assure that certain local martingales are true martingales. As indicated, due to the fast decay of the heat kernel on complete non-compact manifolds, the required conditions are rather weak. In this section we describe some ideas how stochastic analysis can be used to localize the entropy.

Definition 1 For a relatively compact domain $D \subset M$ we define local entropies as follows:

$$\mathcal{E}_D(t) := \mathbb{E}[(u \log u)(t \wedge \tau_D, X_{t \wedge \tau_D})] \quad \text{and} \quad \mathcal{E}_D := \mathbb{E}[(u \log u)(\tau_D, X_{\tau_D})],$$

where τ_D is the first exit time of X from D (with the convention $\tau_D := 0$ if X does not start in D).

Note that since $(X_t)_{t \geq 0}$ is an elliptic diffusion and since D is relatively compact, the stopping time τ_D is finite almost surely. By Fatou's lemma we trivially have

$$\mathcal{E}_D \leq \lim_{t \rightarrow \infty} \mathcal{E}_D(t).$$

Remark 6 As before let X be a $(g(t))_{t \geq 0}$ -Brownian motion X . Itô's formula, along with Eq. (2.9), implies

$$\mathbb{E}[(u \log u)(t \wedge \tau_D, X_{t \wedge \tau_D})] = (u \log u)(0, x) + \mathbb{E}\left[\int_0^{t \wedge \tau_D} \frac{|\nabla u|^2}{u}(s, X_s) ds\right],$$

in other words,

$$\mathcal{E}_D(t) = \mathcal{E}_D(0) + \int_0^t \mathbb{E}\left[\frac{|\nabla u|^2}{u}(s, X_s) \cdot 1_{\{s \leq \tau_D\}}\right] ds, \quad (5.26)$$

and in particular,

$$\mathcal{E}_D^t(t) = \mathbb{E}\left[\frac{|\nabla u|^2}{u}(t, X_t) \cdot 1_{\{t \leq \tau_D\}}\right] \geq 0.$$

Notation Equation (5.26) shows that $\mathcal{E}_D(t)$ is monotone both as a function of t and D . We define

$$\mathcal{E}_M(t) := \lim_{D \uparrow M} \mathcal{E}_D(t) \equiv \mathcal{E}_D(0) + \int_0^t \mathbb{E} \left[\frac{|\nabla u|^2}{u}(s, X_s) \right] ds. \quad (5.27)$$

By Fatou's lemma we observe that

$$\mathcal{E}_M(t) \geq \mathcal{E}(t), \quad t \geq 0.$$

Note that, as long as $\mathcal{E}_M(t)$ is finite, we always have

$$\mathcal{E}'_M(t) := \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right] \equiv \lim_{D \uparrow M} \mathcal{E}'_D(t).$$

Remark 7 With a similar argument we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t \wedge \tau_{D_n}, X_{t \wedge \tau_{D_n}}) \right] &= \frac{|\nabla u|^2}{u}(0, x) \\ &+ \int_0^t \mathbb{E} \left[\left(2u |\nabla \nabla \log u|^2 + 2u \left(\text{Ric} - \frac{1}{2} \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \right) (s, X_s) \right] ds. \end{aligned}$$

Theorem 4 Suppose that $\frac{\partial g}{\partial t} \leq 2\text{Ric}$, and let $(D_n)_{n \in \mathbb{N}}$ be an increasing sequence of relatively compact domains in M satisfying $\cup_{n \in \mathbb{N}} D_n = M$. Let $u : \mathbb{R}_+ \times M \rightarrow \mathbb{R}_+$ be a positive solution of the backward heat equation (1.1) such that

$$\sup_{[0, t] \times M} \frac{|\nabla u|^2}{u} \leq C_t \quad (5.28)$$

for each t with a constant C_t depending on t . Then if the entropy of u is of sublinear growth, i.e.

$$\frac{\mathcal{E}_M(t)}{t} \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad (5.29)$$

then u is constant.

Proof Under condition (5.28) the local submartingale

$$\frac{|\nabla u|^2}{u}(t, X_t), \quad t \geq 0, \quad (5.30)$$

is bounded on compact time intervals, and hence is a true submartingale. In particular, the expectations

$$t \rightarrow \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right]$$

are non-decreasing. On the other hand, the condition

$$\frac{\mathcal{E}_M(t)}{t} \equiv \frac{\mathcal{E}_M(0)}{t} + \frac{1}{t} \int_0^t \mathbb{E} \left[\frac{|\nabla u|^2}{u}(s, X_s) \right] ds \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

implies that

$$\mathbb{E} \left[\frac{|\nabla u|^2}{u}(t_n, X_{t_n}) \right] \rightarrow 0$$

for a sequence $t_n \uparrow \infty$. Hence,

$$\mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right] \equiv 0$$

and consequently, $\nabla u(t, \cdot) \equiv 0$ for all t , so that u is constant in space. Since u solves the heat equation, this implies $\partial u / \partial t = 0$ and u is constant in space and time. \square

Theorem 5 *Let $(D_n)_{n \in \mathbb{N}}$ be an increasing sequence of relatively compact domains in M satisfying $\cup_{n \in \mathbb{N}} D_n = M$. Let $u : [0, T] \times M \rightarrow \mathbb{R}_+$ be a positive solution of the backward heat equation (1.1) such that*

$$\sup_{[0, T] \times M} \frac{|\nabla u|^2}{u} \leq C_T. \quad (5.31)$$

Suppose that $t \mapsto \mathcal{E}_M(t)$ is a linear function on $[0, T]$.

(1) If $\frac{\partial g}{\partial t} \leq 2\text{Ric}$, then u has the form

$$u(t, y) = \psi(y)\phi(t)$$

for some functions ψ and ϕ . Moreover, ψ and ϕ satisfy the differential equation

$$\frac{\partial \phi / \partial t}{\phi} = -\frac{\Delta \psi}{\psi}.$$

(2) If $2\text{Ric} - \frac{\partial g}{\partial t}$ is positive definite everywhere, then u is constant.

Proof Since \mathcal{E}_M is linear, we get

$$t \mapsto \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right]$$

is a constant function. By Remark 7 we may conclude that

$$\mathbb{E} \left[\left(2u |\nabla \log u|^2 + 2u \left(\text{Ric} - \frac{1}{2} \frac{\partial g}{\partial t} \right) (\nabla \log u, \nabla \log u) \right) (t, X_t) \right] = 0.$$

One can now apply the same arguments as in the proofs of Theorem 3 and Proposition 2. \square

Remark 8 For the results above, condition (5.28) has been only used to assure that (5.30) is a true submartingale. In terms of $\tau_n := \tau_{D_n}$ a necessary and sufficient condition for the true submartingale property is that

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\frac{|\nabla u|^2}{u}(\tau_n, X_{\tau_n}) 1_{\{\tau_n \leq t\}} \right] = 0, \quad (5.32)$$

see for instance [16].

Remark 9 One may always write

$$\mathbb{E} \left[\frac{|\nabla u|^2}{u}(t \wedge \tau_n, X_{t \wedge \tau_n}) \right] = \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) 1_{\{t < \tau_n\}} \right] + \mathbb{E} \left[\frac{|\nabla u|^2}{u}(\tau_n, X_{\tau_n}) 1_{\{\tau_n \leq t\}} \right] \quad (5.33)$$

where the left-hand side of Eq. (5.33) is monotone in t and n . Thus, if u is of sublinear growth, by letting $t \rightarrow \infty$ in (5.33), monotonicity of $n \mapsto \mathbb{E} \left[\frac{|\nabla u|^2}{u}(\tau_n, X_{\tau_n}) \right]$ is obtained (without extra conditions). In the proof to Theorem 4 we used however monotonicity along deterministic times, i.e. monotonicity of $t \mapsto \mathbb{E} \left[\frac{|\nabla u|^2}{u}(t, X_t) \right]$ which follows from Eq. (5.33), as $n \rightarrow \infty$, but under only the additional hypothesis (5.32).

Acknowledgements

The first author was supported by Zhejiang Provincial Natural Science Foundation of China (Project No. LY13A010009)

References

1. Arnaudon, M., Coulibaly, K.A., Thalmaier, A.: Brownian motion with respect to a metric depending on time: definition, existence and applications to Ricci flow. *C. R. Math. Acad. Sci. Paris* **346** (13–14), 773–778 (2008).
2. Arnaudon, M., Thalmaier, A.: Li-Yau type gradient estimates and Harnack inequalities by stochastic analysis. In: Probabilistic approach to geometry, *Adv. Stud. Pure Math.*, vol. 57, pp. 29–48. Math. Soc. Japan, Tokyo (2010).
3. Chen, X., Cheng, L.J., Mao, J.: A probabilistic method for the gradient estimates of some geometric flows. Preprint (2013), arXiv:1312.6484.
4. Cheng, L.J.: A probabilistic approach for gradient estimates on time-inhomogeneous manifolds. *Statist. Probab. Lett.* **88**, 174–183 (2014).
5. Cheng, S.Y., Li, P., Yau, S.T.: On the upper estimate of the heat kernel of a complete Riemannian manifold. *Amer. J. Math.* **103** (5), 1021–1063 (1981).
6. Chow, B., Chu, S.C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci flow: Techniques and applications. Part I: Geometric aspects. Part II: Analytic aspects. Part III: Geometric-analytic aspects. *Mathematical Surveys and Monographs*, vols. 135, 144, 163. American Mathematical Society, Providence, RI (2007, 2008, 2010).
7. Chow, B., Knopf, D.: The Ricci flow: an introduction, *Mathematical Surveys and Monographs*, vol. 110. American Mathematical Society, Providence, RI (2004).
8. Chow, B., Lu, P., Ni, L.: Hamilton’s Ricci flow, *Graduate Studies in Mathematics*, vol. 77. American Mathematical Society, Providence, RI; Science Press, New York (2006).
9. Colding, T.H., Minicozzi II, W.P.: Monotonicity and its analytic and geometric implications. *Proc. Natl. Acad. Sci. USA* **110** (48), 19,233–19,236 (2013).
10. Coulibaly-Pasquier, K.A.: Some stochastic process without birth, linked to the mean curvature flow. *Ann. Probab.* **39** (4), 1305–1331 (2011).
11. Coulibaly-Pasquier, K.A.: Brownian motion with respect to time-changing Riemannian metrics, applications to Ricci flow. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2), 515–538 (2011).
12. Grotowski, J.F.: Heat kernels in geometric evolution equations. In: The ubiquitous heat kernel, *Contemp. Math.*, vol. 398, pp. 193–201. Amer. Math. Soc., Providence, RI (2006).
13. Guo, H., Filipowski, R., Thalmaier, A.: Martingales on manifolds with time-dependent connection. *J. Theor. Probab.*, in press.
14. Guo, H., Filipowski, R., Thalmaier, A.: A stochastic approach to the harmonic map heat flow on manifolds with time-dependent Riemannian metric. *Stochastic Process. Appl.* **124** (11), 3535–3552 (2014).
15. Hamilton, R.S.: A matrix Harnack estimate for the heat equation. *Comm. Anal. Geom.* **1** (1), 113–126 (1993).
16. Kabanov, Y., Stricker, C.: On the true submartingale property, d’après Schachermayer. In: Séminaire de Probabilités, XXXVI, *Lecture Notes in Math.*, vol. 1801, pp. 413–414. Springer, Berlin (2003).
17. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus, *Graduate Texts in Mathematics*, vol. 113. Springer-Verlag, New York (1988).
18. Kuwada, K.: Convergence of time-inhomogeneous geodesic random walks and its application to coupling methods. *Ann. Probab.* **40** (5), 1945–1979 (2012).
19. Kuwada, K., Filipowski, R.: Non-explosion of diffusion processes on manifolds with time-dependent metric. *Math. Z.* **268** (3–4), 979–991 (2011).
20. Kuwada, K., Filipowski, R.: Coupling of Brownian motions and Perelman’s \mathcal{L} -functional. *J. Funct. Anal.* **260** (9), 2742–2766 (2011).
21. Lim, A. P. C., Luo, D.: Asymptotic estimates on the time derivative of entropy on a Riemannian manifold. *Adv. Geom.* **13** (1), 97–115 (2013).
22. List, B.: Evolution of an extended Ricci flow system. *Comm. Anal. Geom.* **16** (5), 1007–1048 (2008).
23. Morgan, J., Tian, G.: Ricci flow and the Poincaré conjecture, *Clay Mathematics Monographs*, vol. 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA (2007).
24. Müller, R.: Ricci flow coupled with harmonic map flow. *Ann. Sci. Éc. Norm. Supér. (4)* **45** (1), 101–142 (2012).

25. Neel, R. W., Popescu, I.: A Stochastic Target Approach to Ricci Flow on Surfaces. Preprint arXiv: 1209.4150v2 (2012).
26. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159v1 (2002).
27. Philipowski, R.: Stochastic particle approximations for the Ricci flow on surfaces and the Yamabe flow. *Potential Anal.* **35** (4), 353–371 (2011).
28. Souplet, P., Zhang, Q.S.: Sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds. *Bull. London Math. Soc.* **38** (6), 1045–1053 (2006).
29. Topping, P.: Lectures on the Ricci flow, *London Mathematical Society Lecture Note Series*, vol. 325. Cambridge University Press, Cambridge (2006).
30. Wang, M.: Liouville theorems for the ancient solution of heat flows. *Proc. Amer. Math. Soc.* **139** (10), 3491–3496 (2011).
31. Yau, S.T.: Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.* **28**, 201–228 (1975).
32. Zhang, Q.S.: Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture. CRC Press, Boca Raton, FL (2011).
33. Zhu, X.: Hamilton’s gradient estimates and Liouville theorems for fast diffusion equations on noncompact Riemannian manifolds. *Proc. Amer. Math. Soc.* **139** (5), 1637–1644 (2011).