Sensitivity Analysis and Shape Optimisation through a T-spline Isogeometric Boundary Element Method

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Abstract

This paper applies an isogeometric boundary element method (IGABEM) to sensitivity analysis and shape optimisation for three-dimensional linear elastic analysis. In the present work, analysis-suitable T-splines are adopted for the geometry representation and the basis for analysis (utilising the isogeometric concept) thus providing a direct link between Computer Aided Design (CAD) and analysis. The use of a boundary element method is particularly advantageous in this context, since surface modelling CAD software provides a complete surface discretisation allowing analysis to be performed directly. This is contrast to domain based numerical methods that still require certain pre-processing steps.

In the present paper we focus on the application of T-splines through the boundary element method for sensitivity analysis and shape optimisation for elastostatic problems. This represents an attractive application of the method where tight integration between CAD and analysis is paramount. The formulation of the integral equation for shape sensitivity analysis is presented, and validation of the technique against a closed-form solution is outlined. As expected from an integral equation approach which utilises the exact geometry, high accuracies are demonstrated.

Keywords: isogeometric analysis, IGABEM, shape optimisation, T-splines

1. INTRODUCTION

Isogeometric analysis (IGA) was first proposed by Hughes et al. [1] to merge the fields of analysis and computational geometry. The concept relies on the use of basis functions defined by Computer Aided Design (CAD) to define not only the geometry of the problem, but also to approximate the unknown fields during analysis. The use of such a strategy is compelling, since the task of mesh generation is greatly minimised (or completely circumvented) and the exact geometry is used at all stages of analysis. Much of the recent literature has focused on the use of Non-Uniform Rational B-Splines (NURBS) [2] which are ubiquitous throughout the CAD community, but they exhibit limitations such as the inability to produce water-tight geometries and global refinement algorithms that are not amenable for analysis.

The present work makes use of T-splines [3] which produce water-tight geometries and, perhaps more importantly, the ability to locally refine the discretisation. The boundary element method (BEM) has been shown to be a particularly suitable choice for IGA since all quantities defined by CAD pertain to the boundary of the problem, which is all that is required for BEM analysis. The method has been applied using NURBS for 2D linear elasticity [4] and for shape optimisation in 3D [5]. More recently, a BEM T-spline discretisation strategy has been applied [6]. The present paper builds on this work by applying BEM for shape optimisation using T-splines thus allowing extremely complex geometries to be analysed directly from CAD.

In the present paper we first give a brief overview of T-spline technology and then outline the T-spline boundary element formulation for shape sensitivity analysis. This formulation makes use of implicit differentiation and a regularised form of the integral equation to reduce the strength of the singularity for amenable integration. A collocation based BEM is used throughout. To validate the implementation, a simple cube problem with a closed-form solution is demonstrated. Finally, a brief conclusion is made with insights into the future of isogeometric boundary element methods.

2. ISOGEOMETRIC BOUNDARY ELEMENT METHOD (IGABEM)

2.1. T-splines

T-splines were first proposed by Sederberg et al. [3] to overcome the drawbacks of NURBS - primarily the lack of water-tightness between patches and the use of redundant control points. T-splines inherit the attractive properties of

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NURBS (they are in fact a superset of NURBS) but possess the distinctive advantage that water-tightness can be guaranteed. In addition, T-splines allow for local refinement, which is important not only for accurate geometrical modelling but also paramount for efficient and accurate numerical analysis. Conceptually, the power of T-splines stems from the ability to create T-junctions, similar to the concept of the "hanging nodes" and oct/quad-tree meshes popular in finite element methods. In the case that no T-junctions exist, then the T-spline simply reduces to a NURBS discretisation.

Much of the IGA literature is centred on the use of NURBS as a discretisation tool, but the inability to apply local refinement and problems created by a lack of inter-patch continuity has pushed the research field into alternative technologies. T-splines have been adopted in many IGA works such as [7] where T-splines are used to solve two-dimensional and three-dimensional fluid and structural analysis problems. In addition, *a posteriori* error estimation techniques for local *h*-refinement with T-splines have been very recently introduced by Dorfel et al. [8]. But a straightforward application of T-splines for analysis has shown that convergence issues arise, created by a lack of linear independence at extraordinary points. To combat this, "analysis-suitable" T-splines [9] have been formulated to overcome this issue. These are found to be a subset of T-splines and inherit all their attractive properties. Recently, the technique of Bézier extraction has been applied to T-splines in which basis functions can be reduced to a simple product of Bézier "extraction coefficients" and Bernstein polynomials [10]. In this way, the structure of conventional analysis codes can be maintained whereby integration over elements uses a common set of local basis functions.

The present study makes use of analysis-suitable T-splines and Bézier extraction within a boundary element context. In this way, the structure of a conventional BEM code is maintained allowing for T-splines to discretise both the geometry and unknown fields. Therefore, standard techniques required for integration and assembly can be used directly with T-splines. In the present work, extraction coefficients are determined using a Rhinoceros® IGA plugin [11] which also automatically generates collocation point positions.

2.2. *Isogeometric discretisation*

For three dimensional linear elastostatic analysis, the displacement boundary integral equation (DBIE) is given by

$$\mathbf{C}(\mathbf{s})\mathbf{u}(\mathbf{s}) + \fint_S \mathbf{T}(\mathbf{s},\mathbf{x})\mathbf{u}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = \int_S \mathbf{U}(\mathbf{s},\mathbf{x})\mathbf{t}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \tag{1}$$

where $\mathbf{x}$ and $\mathbf{s}$ represent the field point and source point respectively, $\mathbf{u}$ and $\mathbf{t}$ denote displacement and traction around the boundary, and $\mathbf{U}$ and $\mathbf{T}$ are fundamental solutions given by

$$U_{ij}(\mathbf{s},\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}\right] \tag{2}$$

$$T_{ij}(\mathbf{s},\mathbf{x}) = -\frac{1}{8\pi(1-\nu)r^2}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}] + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\right\}. \tag{3}$$

Here, ν is Poisson's ratio, E is Young's modulus, μ is the shear modulus defined as $\mu = \frac{E}{2(1+\nu)}$, r is the distance between the field point x_i and source point s_i, and n_i denotes the outward-pointing normal at the field point. $\fint$ denotes that the integral is evaluated in the Cauchy Principal Value limiting sense. The integrals on the left-hand-side and right-hand-side are found to be strongly singular and weakly singular respectively. In [4], Equation (1) is used directly, but in the the present paper, we will use the following regularised form of the DBIE [12]

$$\int_S \mathbf{T}(\mathbf{s},\mathbf{x})[\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{s})]\,\mathrm{d}S(\mathbf{x}) = \int_S \mathbf{U}(\mathbf{s},\mathbf{x})\mathbf{t}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \tag{4}$$

The above equation contains at most weakly singular integrals which can be solved readily through polar integration. This be can be rearranged such that all unknown terms are placed on the left-hand-side and all terms relating to the prescribed boundary conditions are taken to the right-hand-side as

$$\begin{aligned} &-\int_{S_t} \mathbf{T}(\mathbf{s},\mathbf{x})\mathbf{u}(\mathbf{s})\mathrm{d}S(\mathbf{x}) + \int_{S_t} \mathbf{T}(\mathbf{s},\mathbf{x})\mathbf{u}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) - \int_{S_u} \mathbf{U}(\mathbf{s},\mathbf{x})\mathbf{t}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \\ &= \int_{S_u} \mathbf{T}(\mathbf{s},\mathbf{x})\bar{\mathbf{u}}(\mathbf{s})\mathrm{d}S(\mathbf{x}) - \int_{S_u} \mathbf{T}(\mathbf{s},\mathbf{x})\bar{\mathbf{u}}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) + \int_{S_t} \mathbf{U}(\mathbf{s},\mathbf{x})\bar{\mathbf{t}}(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) \end{aligned} \tag{5}$$

where S_u and S_t are Dirichlet boundary and Neumann boundary respectively. $\bar{\mathbf{u}}(\mathbf{x})$ and $\bar{\mathbf{t}}(\mathbf{x})$ are the prescribed displacement and traction on S_u and S_t respectively. The geometry surface is discretised by T-spline basis functions as:

$$\mathbf{x}_e(\boldsymbol{\xi}) = \sum_a R_{ea}(\boldsymbol{\xi})\mathbf{B}_{ea} \tag{6}$$

where $\boldsymbol{\xi}$ denotes the parametric coordinates of field points in the parent element. R_{ea} are T-spline basis functions and $\mathbf{B}_{ea}$ represent control point coordinates with an element index e and local node index a respectively.

The unknown displacement and traction are also discretised with T-spline basis functions as

$$\mathbf{u}_e(\boldsymbol{\xi}) \approx \tilde{\mathbf{u}}_e(\boldsymbol{\xi}) = \sum_a R_{ea}(\boldsymbol{\xi})\tilde{\mathbf{u}}_{ea} \tag{7}$$

$$\mathbf{t}_e(\boldsymbol{\xi}) \approx \tilde{\mathbf{t}}_e(\boldsymbol{\xi}) = \sum_a R_{ea}(\boldsymbol{\xi})\tilde{\mathbf{t}}_{ea} \tag{8}$$

where $\tilde{\mathbf{u}}_{ea}$ and $\tilde{\mathbf{t}}_{ea}$ are the nodal parameters associated with displacement and traction respectively. After inserting Equation (7) and Equation (8) into Equation (5), and choosing an appropriate set of collocation points, the DBIE can be written as

$$\begin{aligned} &-\sum_a \int_{S_t} \mathbf{T}^c R_{e_c a} J_{e_c} \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}_{e_c a} + \sum_e \sum_a \int_{S_t} \mathbf{T}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}_{ea} - \sum_e \sum_a \int_{S_u} \mathbf{U}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{t}}_{ea} \\ &= \int_{S_u} \mathbf{T}^c \bar{\mathbf{u}}_{e_c} J_{e_c} \mathrm{d}S(\boldsymbol{\xi}) - \sum_e \int_{S_u} \mathbf{T}^c \bar{\mathbf{u}}_e J_e \mathrm{d}S(\boldsymbol{\xi}) + \sum_e \int_{S_t} \mathbf{U}^c \bar{\mathbf{t}}_e J_e \mathrm{d}S(\boldsymbol{\xi}) \end{aligned} \tag{9}$$

where c is the collocation point index, e_c the element containing the collocation point and J_e the Jacobian matrix for the transition from the parent element to the physical element. For simplicity, we ignore independent variables in Equation (9) and simplify all terms to the following system of equations:

$$\mathbf{A}\mathbf{x} = \mathbf{z} \tag{10}$$

where $\mathbf{A}$ is a matrix containing all integrals relating to unknown parameters, $\mathbf{z}$ is a vector of integral terms multiplied by the associated boundary condition values and $\mathbf{x}$ is the vector of unknown nodal parameters associated with displacement and traction.

3. SHAPE SENSITIVITY ANALYSIS WITH IGABEM

Sensitivity analysis plays a key role in gradient-based shape optimisation problems. We adopt an implicit differentiation method, i.e. take shape derivatives with respect to the design parameter for both side of Equation (9)

$$\begin{aligned} &-\sum_a \left[\int_{S_t} \mathbf{T}^c R_{e_c a} J_{e_c} \mathrm{d}S(\boldsymbol{\xi})\right]' \tilde{\mathbf{u}}_{e_c a} - \sum_a \int_{S_t} \mathbf{T}^c R_{e_c a} J_{e_c} \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}'_{e_c a} + \sum_e \sum_a \left[\int_{S_t} \mathbf{T}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\right]' \tilde{\mathbf{u}}_{ea} \\ &+ \sum_e \sum_a \int_{S_t} \mathbf{T}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}'_{ea} - \sum_e \sum_a \left[\int_{S_u} \mathbf{U}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\right]' \tilde{\mathbf{t}}_{ea} - \sum_e \sum_a \int_{S_u} \mathbf{U}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{t}}'_{ea} \\ &= \left[\int_{S_u} \mathbf{T}^c \bar{\mathbf{u}}_{e_c} J_{e_c} \mathrm{d}S(\boldsymbol{\xi})\right]' - \sum_e \left[\int_{S_u} \mathbf{T}^c \bar{\mathbf{u}}_e J_e \mathrm{d}S(\boldsymbol{\xi})\right]' + \sum_e \left[\int_{S_t} \mathbf{U}^c \bar{\mathbf{t}}_e J_e \mathrm{d}S(\boldsymbol{\xi})\right]' \end{aligned} \tag{11}$$

where the superscript " $'$ " denotes differentiation with repsect to the design parameters. By noting that the parent element coordinates are independent of design parameters, (12) can be rewritten as

$$\begin{aligned} &-\sum_a \int_{S_t} (\mathbf{T}^c R_{e_c a} J_{e_c})' \,\mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}_{e_c a} - \sum_a \int_{S_t} \mathbf{T}^c R_{e_c a} J_{e_c} \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}'_{e_c a} + \sum_e \sum_a \int_{S_t} (\mathbf{T}^c R_{ea} J_e)' \,\mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}_{ea} \\ &+ \sum_e \sum_a \int_{S_t} \mathbf{T}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{u}}'_{ea} - \sum_e \sum_a \int_{S_u} (\mathbf{U}^c R_{ea} J_e)' \,\mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{t}}_{ea} + \sum_e \sum_a \int_{S_u} \mathbf{U}^c R_{ea} J_e \mathrm{d}S(\boldsymbol{\xi})\tilde{\mathbf{t}}'_{ea} \\ &= \int_{S_u} (\mathbf{T}^c \bar{\mathbf{u}}_{e_c} J_{e_c})' \,\mathrm{d}S(\boldsymbol{\xi}) - \sum_e \int_{S_u} (\mathbf{T}^c \bar{\mathbf{u}}_e J_e)' \,\mathrm{d}S(\boldsymbol{\xi}) + \sum_e \int_{S_t} (\mathbf{U}^c \bar{\mathbf{t}}_e J_e)' \,\mathrm{d}S(\boldsymbol{\xi}) \end{aligned} \tag{12}$$

The above equation can be written in a compact form

$$\mathbf{A}'\mathbf{x} + \mathbf{A}\mathbf{x}' = \mathbf{z}' \tag{13}$$

i.e.

$$\mathbf{A}\mathbf{x}' = \mathbf{z}' - \mathbf{A}'\mathbf{x} \tag{14}$$

where $\mathbf{A}'$ and $\mathbf{z}'$ are the shape derivatives of $\mathbf{A}$ and $\mathbf{z}$ respectively, and $\mathbf{x}'$ contains unknown displacement and traction shape derivatives.

4. NUMERICAL EXAMPLE

A simple problem of a cube under tension is investigated in the present paper. The geometry, T-spline mesh and control points are shown in Figure (1). The cube's length $l = 1$, Young's modulus $E = 1 \times 10^5$ and poisson ration $\nu = 0.3$. The boundary conditions are: $u_x = 0$ at $x = 0$, $u_y = 0$ at $y = 0$, $u_z = 0$ at $z = 0$. The specified traction is $t_x = 100$ at $x = 1$. The length l along the x direction of the cube is taken as the design parameter.

The analytical solution for this problem is found to be:

$$\varepsilon_{xx} = \frac{t_x}{E}, \quad \varepsilon_{yy} = -\nu\varepsilon_{xx}, \quad \varepsilon_{zz} = -\nu\varepsilon_{xx}. \tag{15}$$

$$u_x = \varepsilon_{xx} x = \tfrac{t_x}{E} x \tag{16}$$
$$u_y = \varepsilon_{yy} y = -\nu \tfrac{t_x}{E} y \tag{17}$$
$$u_z = \varepsilon_{zz} z = -\nu \tfrac{t_x}{E} z \tag{18}$$

Hence, the analytical displacement sensitivity is

$$u'_x = \varepsilon_{xx} x' = \tfrac{t_x}{E} x' \tag{19}$$
$$u'_y = \varepsilon_{xx} y' = -\nu \tfrac{t_x}{E} y' \tag{20}$$
$$u'_z = \varepsilon_{xx} z' = -\nu \tfrac{t_x}{E} z' \tag{21}$$

To investigate the accuracy of IGABEM sensitivity analysis, we illustrate the analytical and numerical displacement shape derivatives and the pointwise relative error in Figures (2), (3) and (4) respectively. As can be seen, the numerical results agree with the analytical shape derivatives very well, even with a very coarse mesh. It should be noted that the present geometry is produced using Rhinoceros® with an Autodesk® T-spline plugin [11] without any subsequent mesh generation. The advantages of this feature may not be obvious for the present simple geometry, but represents a significant advantage for three dimensional industrial geometries.

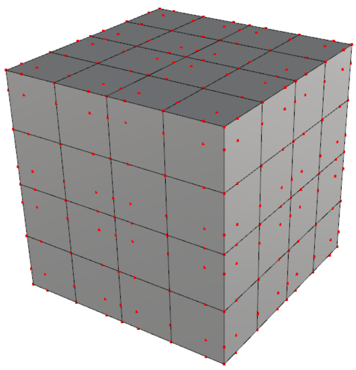

Figure 1: Cube geometry, T-spline mesh and control points

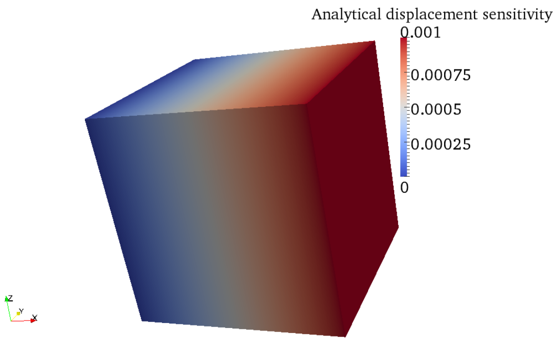

Figure 2: Analytical displacement sensitivity

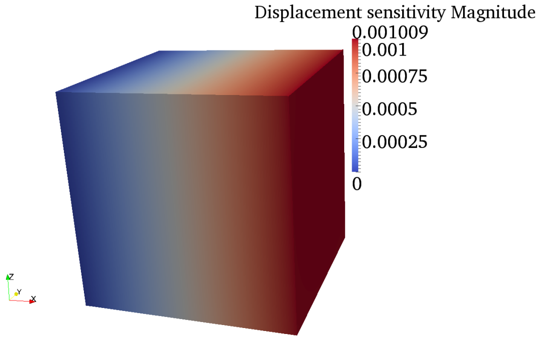

Figure 3: IGABEM displacement sensitivity

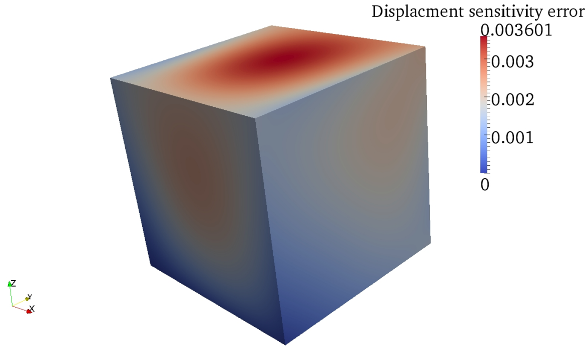

Figure 4: Displacement sensitivity relative error $\left(\frac{|\mathbf{u}' - \mathbf{u}'^h|}{||\mathbf{u}'||_\infty} \right)$

5. Conclusions

A T-spline isogeometric boundary element method for shape sensitivity analysis applied to elastostatic problems has been briefly outlined. T-splines have been adopted due to their attractive properties both from a computational geometry and analysis standpoint which allow, through a boundary integral approach, a direct link to be made between CAD and analysis. This is particularly relevant for the application of shape sensitivity where both computational geometry and analysis must communicate seamlessly with one another. The boundary element formulation for shape sensitivity analysis has been outlined using analysis-suitable T-splines to discretise the geometry and unknown fields. The implementation of the method has been verified against a closed-form solution where high accuracies are demonstrated.

However, much work still needs to be done on the topic of IGABEM. This includes a rigorous study on the effect of integration accuracy when using functions such as NURBS and T-splines for the approximation of fields. In addition, the application of Galerkin BEM in an IGA context still needs to be assessed including a convergence study. Many other applications of the technique have yet to be studied including acoustics, electromagnetics and fluid flow, and it is expected that the topic of IGABEM will flourish into a rich research topic.

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