J. Math. Pures Appl.,
77, 1998, *p.* 283-315

COMPLETE LIFTS OF CONNECTIONS AND STOCHASTIC JACOBI FIELDS

By M. ARNAUDON and A. THALMAIER

ABSTRACT. – Differentiable families of ∇-martingales on manifolds are investigated: their infinitesimal variation provides a notion of stochastic Jacobi fields. Such objects are known [2] to be martingales taking values in the tangent bundle when the latter is equipped with the complete lift of the connection ∇. We discuss various characterizations of TM-valued martingales. When applied to specific families of ∇-martingales which appear in connection with the heat flow for maps between Riemannian manifolds, our results allow to establish formulas giving a stochastic representation for the differential of solutions to the nonlinear heat equation. As an application, we prove local and global gradient estimates for harmonic maps of bounded dilatation. © Elsevier, Paris

1. Introduction

Let M be a smooth manifold endowed with a connection ∇. The tangent bundle TM inherits a connection ∇', the complete lift of ∇ (*see* [31]), which appears naturally when dealing with variations of ∇-martingales: The derivative of an M-valued martingale depending differentiably on a parameter (with respect to uniform convergence in probability on compact sets) is a TM-valued ∇'-martingale (*see* [2]). This connection on TM has the property that its geodesics are the Jacobi fields on M with respect to ∇.

In a certain sense, the stochastic analogue to geodesics on a manifold M with a connection ∇ is constituted by the class of M-valued ∇-martingales. Their infinitesimal variations define a notion of stochastic Jacobi fields in the same way as variations of geodesics in classical differential geometry lead to the class of (deterministic) Jacobi fields.

Let Mart(M,∇) denote the set of M-valued ∇-martingales defined on a given filtered probability space $\big(\Omega,(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P}\big)$. It is shown in [2] that on Mart(M,∇) the so-called topology of semimartingales coincides with the topology of uniform convergence in probability on compact intervals. Furthermore [2], if an M-valued continuous semimartingale depends on a parameter a in a differentiable manner (with respect to the topology of semimartingales) then also its anti-development depends differentiably on a; the same is true for horizontal lifts to initial conditions varying C^1 in probability in the parameter a. Moreover, the operations of taking anti-developments and horizontal lifts commute with differentiation if the tangent bundle TM is endowed with the complete

---

1991 *Mathematics Subject Classification.* 58G32, 60H30.

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/1998/03

lift ∇' of the connection ∇ on M. See Bismut [3], Meyer [24] for background on semimartingales taking values in vector bundles.

Vector fields on the path space of M-valued Brownian motion or continuous semimartingales have attracted the attention of many authors, e.g. [7], [14], [17], [13], [5], [20]. Recently, T. Lyons and Z.-M. Qian [21] studied vector fields along semimartingales obtained by varying a metric connection. In this paper we characterize variational fields of M-valued semimartingales which give rise to martingales in TM with respect to canonically lifted connections. As observed in [2], smooth variations of M-valued ∇-martingales lead to ∇'-martingales in TM, and under mild regularity assumptions, all ∇'-martingales on TM are obtained in this way.

Canonical examples for differentiable families of martingales on Riemannian manifolds are of the form $u(X_{\bullet}(a))_{a\in M}$ where $u : M \to N$ is a harmonic map, or more generally, of the form $u(t-\bullet, X_{\bullet}(a))_{a\in M}$ where $u : [0,t] \times M \to N$ is a smooth solution to the nonlinear heat equation; in both cases $X_{\bullet}(a)$ is a $\mathrm{BM}(M,g)$ starting from a at time $t=0$. When specialized to such situations, our results lead to stochastic representations for the differential of u, extending well-known derivative formulae from the linear case $N = \mathbb{R}^n$ to general targets.

When applied to the nonlinear heat equation, the corresponding Jacobi fields allow to establish differentiation formulas which are appropriate tools for a priori estimates of the harmonic heat flow. We exemplify our methods by proving explicit gradient estimates for harmonic maps of bounded dilatation. Well-known Liouville type theorems follow directly as corollaries from these estimates.

2. Complete lifts of connections on the frame bundle

We shall exploit the following fact (*see* [25]): If $\pi : P \to M$ is a principal fibre bundle with structure group G, then, in a natural way, $T\pi : TP \to TM$ is a principal fibre bundle with structure group TG. Note that the tangent bundle TG of a Lie group G is a Lie group by the tangent operation on TG as group multiplication. Moreover, each G-connection on $\pi : P \to M$ lifts in a canonical way to a TG-connection on $T\pi : TP \to TM$. The complete lift of a linear connection on M to its tangent bundle TM, as described in [31], derives naturally from this construction in the case of the frame bundle $\pi : L(M) \to M$ with $G = \mathrm{GL}(m, \mathbb{R})$. We start with a brief sketch of these concepts to fix the notation and to clarify the setting.

A G-connection on a principal fibre bundle $\pi : P \to M$ with structure group G is most concisely described as a G-invariant splitting h of the following exact sequence of vector bundles over P:

(2.1)
$$\begin{tikzcd} 0 \arrow[r] & \ker T\pi \arrow[r, "\iota"] & TP \arrow[r, "T\pi"] & \pi^*TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0, \end{tikzcd}$$

i.e., $T\pi \circ h = \mathrm{id}$ and $(TR_g)h = h$ for all $g \in G$, where R_g is the right action of g on P. Such a splitting induces a G-invariant decomposition

$$TP = \ker T\pi \oplus h(\pi^*TM) \equiv V \oplus H$$

of TP in a vertical and a horizontal subbundle. From this point of view, a G-connection is a selection of a horizontal space H_u at each $u \in P$ in a G-invariant way, *i.e.*, $H_{u \cdot g} = (T_u R_g) H_u$ for all $g \in G$. According to $TP = V \oplus H$, vector fields $X \in \Gamma(TP)$ are decomposed as $X = X^{\mathrm{vert}} + X^{\mathrm{hor}}$. The bundle isomorphism

$$h\colon \pi^*TM \xrightarrow{\sim} H, \quad h_u\colon T_{\pi(u)}M \xrightarrow{\sim} H_u, \; u \in P, \tag{2.2}$$

is the horizontal lift of the G-connection (*see* [19]). Note that each $u \in P$ defines an embedding $I_u\colon G \hookrightarrow P$, $g \mapsto u \cdot g$, and thus, via the differential of I_u at $e \in G$,

$$T_e I_u : T_e G \to T_u P, \quad A \longmapsto \hat{A}(u), \tag{2.3}$$

a canonical identification $\kappa_u\colon \mathfrak{g} \xrightarrow{\sim} V_u$ of the Lie algebra $\mathfrak{g} = T_e G$ with the vertical fibre $V_u = \{v \in T_u P\colon (T\pi)v = 0\}$ at u; in particular, $\ker T\pi = P \times \mathfrak{g}$.

A G-invariant splitting of (2.1) may equivalently be expressed in terms of a bundle homomorphism $\bar{\omega} : TP \to \ker T\pi$ such that $\bar{\omega} \circ \iota = \mathrm{id}$, and by this way, $\bar{\omega} = (\mathrm{pr}_P, \omega)$ with $\mathrm{pr}_P : TP \to P$ canonically, it defines an equivariant $\mathfrak{g}$-valued 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ on P, the connection 1-form on P:

$$\omega_u(X_u) = \kappa_u^{-1}(X^{\mathrm{vert}})_u, \quad X \in \Gamma(TP). \tag{2.4}$$

In these terms, G-invariance of the splitting translates to equivariance of ω, *i.e.* $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for all $g \in G$.

If P is a subbundle of the frame bundle $\mathrm{L}(M)$, then a canonical trivialization of TP over P is given as follows: The horizontal subbundle H is trivialized by the standard-horizontal vector fields $L_1, \ldots, L_m \in \Gamma(TP)$ with $L_i(u) = h_u(ue_i)$, the vertical subbundle V by the standard-vertical vector fields $\hat{A} \in \Gamma(TP)$ defined in (2.3), where A runs through a basis of $\mathfrak{g}$, see [19]. In this situation, we have canonically $TP = P \times \mathbb{R}^m \times \mathfrak{g}$.

We now specialize to the case of the frame bundle $\pi : \mathrm{L}(M) \to M$ over a manifold M with structure group $\mathrm{GL}(m, \mathbb{R})$. Linear connections on M uniquely correspond to $\mathrm{GL}(m, \mathbb{R})$-connections on $\mathrm{L}(M)$. By definition, see [19], a G-structure on M where G is a Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$ is a reduction of the structure group $\mathrm{GL}(m, \mathbb{R})$ of $\mathrm{L}(M)$ to G. This paper relies on the following construction: To each G-structure on M there is an associated G'-structure on TM where G' is a certain subgroup of $\mathrm{GL}(2m, \mathbb{R})$. Moreover, if $\pi : P \to M$ is a G-structure on M and $\pi' : P' \to TM$ the induced G'-structure on TM, then G-connections on P naturally lift to G'-connections on P'. Roughly speaking, these lifts are obtained by differentiating all maps in (2.1). To give a more precise formulation, the following Lemma is needed.

LEMMA 2.1. – *There is a natural injection* $\iota_M\colon T(\mathrm{L}(M)) \hookrightarrow \mathrm{L}(TM)$ *as follows: Given the* $\mathrm{GL}(m,\mathbb{R})$*-principal bundle* $\pi : \mathrm{L}(M) \to M$ *and the* $\mathrm{GL}(2m,\mathbb{R})$*-principal bundle* $\tilde{\pi} : \mathrm{L}(TM) \to TM$*, then there exists an embedding* $\iota_m\colon T\mathrm{GL}(m,\mathbb{R}) \hookrightarrow \mathrm{GL}(2m,\mathbb{R})$ *and a bundle homomorphism* ι_M *of* $T(\mathrm{L}(M))$ *into* $\mathrm{L}(TM)$ *such that both*

(2.5)
$$\begin{tikzcd} T\mathrm{L}(M) \arrow[r, "\iota_M"] \arrow[d, "T\pi"'] & \mathrm{L}(TM) \arrow[d, "\tilde{\pi}"] \\ TM \arrow[r, "\mathrm{id}"] & TM \end{tikzcd}$$

and

(2.6)
$$\begin{tikzcd} T\mathrm{L}(M) \times T\mathrm{GL}(m,\mathbb{R}) \arrow[r] \arrow[d, "(\iota_M{,}\iota_m)"'] & T\mathrm{L}(M) \arrow[d, "\iota_M"] \\ \mathrm{L}(TM) \times \mathrm{GL}(2m,\mathbb{R}) \arrow[r] & \mathrm{L}(TM) \end{tikzcd}$$

are commutative diagrams.

Proof. – We sketch the construction for further reference, see [25] for details. First, $T\mathrm{GL}(m,\mathbb{R})$ is considered as a Lie subgroup of $\mathrm{GL}(2m,\mathbb{R})$ in the usual way: If $\rho : \mathrm{GL}(m,\mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m$ denotes the natural operation of $\mathrm{GL}(m,\mathbb{R})$ on $\mathbb{R}^m$, then $T\mathrm{GL}(m,\mathbb{R})$ operates on $T\mathbb{R}^m = \mathbb{R}^m \oplus \mathbb{R}^m$ by the tangent operation

$$T\rho : T\mathrm{GL}(m,\mathbb{R}) \times T\mathbb{R}^m \to T\mathbb{R}^m.$$

Elements $Y \in T\mathrm{GL}(m,\mathbb{R})$ may be represented as $Y \equiv (g,A)$ with $g \in \mathrm{GL}(m,\mathbb{R})$ and $A \in \mathfrak{gl}(m,\mathbb{R})$ such that $Y = (R_g)_*A \in T_g\mathrm{GL}(m,\mathbb{R})$ where R_g denotes right translation by g in $\mathrm{GL}(m,\mathbb{R})$. Then, the tangent operation of $T\mathrm{GL}(m,\mathbb{R})$ on $\mathbb{R}^m \oplus \mathbb{R}^m$ is given by $Y \cdot (x,v) = (gx, Agx + gv)$ for $(x,v) \in \mathbb{R}^m \oplus \mathbb{R}^m$. In these terms:

$$\iota_m(Y) := \begin{pmatrix} g & 0 \\ Ag & g \end{pmatrix} \in \mathrm{GL}(2m,\mathbb{R}).$$

The injection $\iota_M\colon T\mathrm{L}(M) \hookrightarrow \mathrm{L}(TM)$ is most easily described relying on local trivializations. For a fixed coordinate neighbourhood V in M, with the notation $\mathrm{L}(M)/V = \mathrm{L}(V)$ and $T\mathrm{L}(M)/TV = T\mathrm{L}(V)$, we have

$$\varphi_V\colon \mathrm{L}(V) \xrightarrow{\sim} V \times \mathrm{GL}(m,\mathbb{R})$$

and $T\varphi_V\colon T\mathrm{L}(V) \xrightarrow{\sim} TV \times T\mathrm{GL}(m,\mathbb{R})$; in the same way,

$$\phi_V\colon \mathrm{L}(TV) \xrightarrow{\sim} TV \times \mathrm{GL}(2m,\mathbb{R}).$$

The embedding ι_M restricted to $T\mathrm{L}(V)$ is given as composition of the following maps:

$$T\mathrm{L}(V) \xrightarrow[T\varphi_V]{\sim} TV \times T\mathrm{GL}(m,\mathbb{R}) \xrightarrow[(\mathrm{id},\iota_m)]{} TV \times \mathrm{GL}(2m,\mathbb{R}) \xrightarrow[\phi_V^{-1}]{\sim} \mathrm{L}(TV).$$

The fact that φ_V and ϕ_V are constructed with the same trivialization provides the intrinsic nature of ι_M. The claimed commutativity is an immediate consequence of the described construction. □

Notation 2.2. – For a differentiable manifold F let $s_F : TTF \to TTF$ be the canonical isomorphism which makes the following diagram commutative:

$$\begin{tikzcd}
TTF \arrow[r, "s_F"] \arrow[d, "T_{\mathrm{pr}_F}"'] & TTF \arrow[d, "\mathrm{pr}^F_T"] \\
TF \arrow[r, "\mathrm{id}"] & TF
\end{tikzcd}, \tag{2.7}$$

where pr_F and pr_{TF} are the projections $TF \to F$, resp. $T(TF) \to TF$. In explicit terms, the isomorphism s_F is described as follows: if $v = \partial_a \partial_t x(t,a)$ for some smooth map $(t,a) \mapsto x(t,a) \in F$, then $s_F(v) = \partial_t \partial_a x(t,a)$.

Remark 2.3. – The canonical embedding $\iota_M : T\mathrm{L}(M) \to \mathrm{L}(TM)$ in diagram (2.5) may equivalently be described as follows: Let $W = \dot{u}(0) \in T\mathrm{L}(M)$ for some smooth curve $a \mapsto u(a) \in \mathrm{L}(M)$, and $v = \dot{b}(0) \in T\mathbb{R}^m = \mathbb{R}^m \oplus \mathbb{R}^m$ for some smooth curve $a \mapsto b(a) \in \mathbb{R}^m$, then

$$\iota_M(W)\, v = s_M\big((u\,b)^{\cdot}(0)\big),$$

where $s_M : TTM \to TTM$ is given by (2.7).

DEFINITION 2.4. – Let G be a Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$ and $G' = \iota_m(TG) \equiv TG$ the corresponding Lie subgroup of $\mathrm{GL}(2m,\mathbb{R})$. Let $\pi : P \to M$ be a G-structure on M. We denote by π' the restriction of the projection $\tilde{\pi} : \mathrm{L}(TM) \to TM$ to the subbundle $P' := \iota_M(TP)$ of $\mathrm{L}(TM)$. Then $\pi' : P' \to TM$ provides a G'-structure on TM, the *canonical lift* of the G-structure P on M to the tangent bundle TM.

Remark 2.5. – Let $\pi : P \to M$ be a G-structure on M where G is a Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$, and let $\pi' : P' \to TM$ be the induced G'-structure on TM. Assume there is a G-connection on P splitting the following sequence of vector bundles over P:

$$\begin{tikzcd}
0 \arrow[r] & P \times \mathfrak{g} \arrow[r] & TP \arrow[r, "T\pi"] \arrow[l, dashed, bend left, "\bar{\omega}=(\mathrm{pr}_P{,}\,\omega)"] & \pi^* TM \arrow[r] & 0
\end{tikzcd}, \tag{2.8}$$

defining a connection 1-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ on P. Differentiation of (2.8) gives a G'-connection on P' via:

$$\begin{tikzcd}
0 \arrow[r] & TP \times T\mathfrak{g} \arrow[r, "s_P \circ T\iota"] \arrow[d, "(\iota_M{,}\, T\iota_m \circ s_G)"', "\wr"] & TTP \arrow[r, "TT\pi"] \arrow[d, "T\iota_M"', "\wr"] & (T\pi)^* TTM \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & P' \times \mathfrak{g}' \arrow[r] & TP' \arrow[r, "T\pi'"] & (\pi')^* TTM \arrow[r] & 0
\end{tikzcd}, \tag{2.9}$$

where the isomorphisms $s_P : TTP \to TTP$ and $s_G : TTG \to TTG$ are defined by (2.7).

Note that the elements of $T\mathfrak{g}$ are vertical in TTG, hence s_G maps $T\mathfrak{g}$ to T_0TG where $0 \in T_eG \equiv \mathfrak{g}$, and obviously T_0TG is mapped to $\mathfrak{g}' = T_1G'$ under $T\iota_m$. The first line in (2.9) is a sequence of vector bundles over TP, the second over P'.

From (2.9) we can see that each G-invariant splitting of (2.8) gives rise to a natural G'-connection on P' which is usually called *complete lift* of the G-connection on P to P', e.g. [4]. The following lemma summarizes its main properties.

LEMMA 2.6 (Characterizations of the complete lift). – *Let $\pi : P \to M$ be a G-structure on M where G is a Lie subgroup of* $\mathrm{GL}(m,\mathbb{R})$ *and let $\pi' : P' \to TM$ be the induced G'-structure on TM. The complete lift to P' of a G-connection on P is the G'-connection on P' which is characterized as follows:*

(i) *Its connection* 1*-form* $\omega' \in \Gamma(T^*P' \otimes \mathfrak{g}')$ *is determined by*

$$\iota_M^*\omega' = T\iota_m \circ s_G \circ T\omega \circ s_P, \tag{2.10}$$

where $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ is the connection 1*-form on P and $T\omega$ the differential of ω when considered as a mapping $\omega : TP \to \mathfrak{g}$. In other words, $\omega' \circ T\iota_M$ and $T(\iota_m \circ \omega)$ agree up to canonical identification.*

(ii) *Its horizontal lift $h' : (\pi')^*TTM \to TP'$ is determined by the horizontal lift $h : \pi^*TM \to TP$ of the connection on P via*

$$\iota_M^{-1}h' = T\iota_M \circ s_P \circ Th \circ s_M, \tag{2.11}$$

where $(\iota_M^{-1}h')_u = h'_{\iota_M(u)}$ for $u \in TP$.

(iii) *The splitting of TP' in a vertical and a horizontal subbundle over P' is given by $TP' = (T\iota_M \circ s_P)(TV) \oplus (T\iota_M \circ s_P)(TH) = V' \oplus H'$ if $TP = V \oplus H$ is the corresponding splitting of TP over P.*

In addition, if $\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^m)$ denotes the canonical 1*-form of the connection on P,*

$$\vartheta_u(X_u) = u^{-1}(T_u\pi)X_u, \quad u \in P, \ X \in \Gamma(TP), \tag{2.12}$$

then the canonical 1*-form $\vartheta' \in \Gamma(T^*P' \otimes T\mathbb{R}^m)$ of the induced connection on P' is determined by*

$$\iota_M^*\vartheta' = T\vartheta \circ s_P, \tag{2.13}$$

where $T\vartheta$ is the differential of ϑ when considered as mapping $\vartheta : TP \to \mathbb{R}^n$.

In this paper we are mainly interested in the following two cases:

(i) The frame bundle $\pi : \mathrm{L}(M) \to M$ over a manifold M with structure group $\mathrm{GL}(m,\mathbb{R})$, and its canonical lift to a $T\mathrm{GL}(m,\mathbb{R})$-structure $\pi' : \mathrm{L}'(M) \to TM$ on TM.

(ii) The orthonormal frame bundle $\pi : \mathrm{O}(M) \to M$ over a Riemannian manifold (M,g) with structure group $\mathrm{O}(m)$, and its canonical lift to a $T\mathrm{O}(m)$-structure $\pi' : \mathrm{O}'(M) \to TM$ on TM.

Let again $\pi : P \to M$ be a G-structure on M equipped with a G-connection, where G is a certain Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$, e.g. $G = \mathrm{GL}(m,\mathbb{R})$ itself or $G = \mathrm{O}(m)$ corresponding to $P = \mathrm{L}(M)$, resp. $P = \mathrm{O}(M)$. Each G-connection on P induces a linear connection on M, *i.e.* a connection in the vector bundle $E = TM$, splitting the exact sequence of vector bundles over TM,

$$0 \longrightarrow p^*E \longrightarrow TE \xrightarrow{Tp} p^*TM \longrightarrow 0, \tag{2.14}$$

with $p : E \longrightarrow M$ denoting the canonical projection. In explicit terms, the decomposition

$$TE = p^*E \oplus H(E) \equiv V(E) \oplus H(E) \tag{2.15}$$

in the vertical and horizontal subbundle is given as follows: Let $e \in E$ and choose $u \in P_{p(e)}$. Note that $u\xi = e$ for precisely one $\xi \in \mathbb{R}^m$. Reading ξ as a map $\xi : P \longrightarrow E$, we consider its differential $T_u\xi$ at u. Then, induced from the splitting $TP = V \oplus H$ of TP, we have

$$T_eE = E_{p(e)} \oplus H_e(E), \tag{2.16}$$

where $H_e(E)$ is the image of H_u under $T_u\xi$. The corresponding covariant derivative ∇ is given by

$$\nabla_X Y = \mathrm{pr}_{V(E)} TYX$$

with $\mathrm{pr}_{V(E)}$ denoting the projection of TE onto $V(E) \equiv p^*E$.

In the same way, each G'-connection on P' induces a linear connection on TM, *i.e.* a connection in the vector bundle $E' = TE \equiv TTM$ over $E = TM$. The splitting $TE' = V'(E) \oplus H'(E)$ over E' again is essentially given by taking tangent spaces in (2.15), *i.e.* $TE' = s_E(TV(E)) \oplus s_E(TH(E))$ with $s_E : TTE \longrightarrow TTE$ defined by (2.7). The corresponding covariant derivative ∇' is characterized by the following property: For a vector field $X \in \Gamma(TM)$ on M denote by $X' \in \Gamma(TTM)$ its complete lift to TM, *i.e.*, $X' = s_M TX$. Then

$$\nabla'_{X'} Y' = \mathrm{pr}_{s_E(TV(E))} TY'X' = (\mathrm{pr}_{V(E)} TYX)' = (\nabla_{X'} Y)',$$

in accordance with the definition in [31]. The covariant derivative ∇' is called the *complete lift* of ∇ to TM.

3. Differentiable families of semimartingales

In this section we consider again $P = \mathrm{L}(M)$, resp. $P = \mathrm{O}(M)$, with structure group $G = \mathrm{GL}(m, \mathbb{R})$, resp. $G = \mathrm{O}(m)$, more generally, P may be a G-structure on M where G is a certain Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$. In either case, we suppose that $\pi : P \longrightarrow M$ is endowed with a G-connection. As explained in the previous section, we have a covariant derivative ∇ on M and its complete lift ∇' on TM.

Let X be a continuous M-valued semimartingale. Recall that, by definition, a P-valued semimartingale U is a horizontal lift of X if $\pi \circ U = X$ and $\int_U \omega = 0$, almost surely; here $\int_U \omega = \int \omega(\delta U)$ denotes the (Stratonovich) integral of the connection 1-form ω along U. Furthermore, the anti-development $\mathcal{A}(X)$ of X takes values in $T_{X_0}M$ and is given by $\mathcal{A}(X) = U_0 \int_U \vartheta$ where U is a horizontal lift of X. Each horizontal lift U of X is uniquely determined by its initial value U_0, the parallel transport $//_{0,\bullet} = U_\bullet \circ U_0^{-1} : T_{X_0}M \rightarrow T_{X_\bullet}M$ along $X_\bullet$ is consequently independent of the choice of U_0, and so is the anti-development

$\mathcal{A}(X)$ of X (*see* [8], [12] for details); $\int_U \vartheta = \int \vartheta(\delta U)$ itself will be called anti-development of X into $\mathbb{R}^m$ with initial frame U_0.

Now consider families $(X(a))_{a\in I}$ of continuous semimartingales on M, differentiable (*i.e.* C^1) in the topology of semimartingales. The index set I is allowed to be an arbitrary differentiable manifold. As shown in [2], on the set of continuous martingales this topology coincides with the topology of compact convergence in probability. We denote by $X' \equiv TX$ the family $(X'(v))_{v\in TI}$ of TM-valued semimartingales where $X'(v) := (T_aX)v$ if $v \in T_aI$. The following theorem adapts a similar result obtained in [2].

THEOREM 3.1. – *Let $(X(a))_{a\in I}$ be a C^1-family of continuous semimartingales on M (for simplicity, each with infinite lifetime), and let $U(a)$ be a horizontal lift of $X(a)$ to P such that $a \mapsto U_0(a)$ is C^1 in probability.*

(i) *Then $a \mapsto U(a)$ is C^1 in the topology of semimartingales, and $U' = \iota_M TU$ is the horizontal lift of TX to P with $U_0' = \iota_M TU_0$.*

(ii) *If $Z(a) = \mathcal{A}(X(a))$ is the anti-development of $X(a)$, then $a \mapsto Z(a)$ is C^1 in the topology of semimartingales. Moreover, for $v \in TI$, denoting by $Z'(v) = \mathcal{A}'(TX(v))$ the anti-development of $TX(v)$ (with respect to complete lift of the connection to P'), we get $Z'(v) = s_M(TZ)v$, in other words,*

$$\mathcal{A}'(TX) = s_M T\mathcal{A}(X),$$

where $s_M : TTM \to TTM$ is the isomorphism defined by diagram (2.7).

Proof. – The fact that $a \mapsto U(a)$ and $a \mapsto Z(a)$ are C^1 in the topology of semimartingales follows from [2]. We check the claimed identities.

i) Here it is enough to show that U' is above TX, *i.e.*, $\pi' \circ U' = TX$, and horizontal, *i.e.*, $\omega'(\delta U') = 0$. But $\pi' \circ U' = \tilde{\pi} \circ U' = \tilde{\pi} \circ \iota_M \circ TU = T\pi \circ TU = T(\pi \circ U) = TX$. On the other hand, since $\omega(\delta U) = 0$, we get by differentiation, $0 = (T\omega \circ s_P)(\delta TU)$, and thus

$$\omega'(\delta U') = \omega'(T\iota_M\, \delta TU) = (T\iota_m \circ s_G \circ T\omega \circ s_P)(\delta TU) = 0.$$

ii) Let $\bar{Z}(a) = \int \vartheta(\delta U(a))$ be the anti-development of $X(a)$ into $\mathbb{R}^m$ with initial frame $U_0(a)$. Since by definition of ι_M (*see* Remark 2.3)

$$U_0' T\bar{Z} \equiv \iota_M(TU_0)(T\bar{Z}) = s_M T(U_0\bar{Z}),$$

it suffices to verify that

$$(T\bar{Z})v = \bar{Z}'(v), \tag{3.1}$$

where $\bar{Z}'(v) = \int \vartheta'(\delta U'(a))$ is the anti-development of $TX(v)$ into $\mathbb{R}^m \oplus \mathbb{R}^m$ with initial frame $U_0'(a)$. But, using the explicit form of ϑ given in (2.13), we immediately get $T\big(\vartheta(\delta U)\big) = (T\vartheta \circ s_P)\big(\delta(TU)\big) = (\vartheta' \circ T\iota_M)\big(\delta(TU)\big) = \vartheta'\big(\delta(\iota_M TU)\big) = \vartheta'(\delta U')$ which concludes the proof. □

We give P a linear connection $\tilde{\nabla}$ which has the following property: If $\widetilde{/\!/}_{0,\bullet}$ is the parallel transport with respect to $\tilde{\nabla}$ along a curve $t \mapsto u(t)$ in P, and $/\!/_{0,\bullet}$ the parallel transport with respect to ∇ along $t \mapsto x(t) = \pi\big(u(t)\big)$ in M, then the following diagram is supposed to commute:

(3.2)
$$\begin{tikzcd}
H_{u(0)} \arrow[r, "\widetilde{//}_{0,t}"] & H_{u(t)} \\
T_{x(0)}M \arrow[u, "h_{u(0)}", "\cong"'] \arrow[r, "//_{0,t}"'] & T_{x(t)}M \arrow[u, "\cong", "h_{u(t)}"']
\end{tikzcd},$$

where $H \subset TP$ denotes the horizontal subbundle. Note that, for instance, the canonical flat connection ∇^{flat} on P, induced by the trivialization of TP over P:

$$TP = H \oplus V = \operatorname{span}\{L_1, ..., L_m\} \oplus \operatorname{span}\{\hat{A} : A \in \text{basis of } \mathfrak{g}\},$$

i.e., $\nabla^{\text{flat}}_X Y = \sum_i X(a^i)L_i + \sum_{i,j} X(b^{ij})\hat{A}_{ij}$ if $Y = \sum_i a^i L_i + \sum_{i,j} b^{ij}\hat{A}_{ij} \in \Gamma(TP)$, has property (3.2). Also the horizontal lift ∇^h of ∇ to P (*see* [4]) satisfies (3.2). It is easily checked that (3.2) is equivalent to

$$\widetilde{\nabla}_{X^h} Y^h = (\nabla_X Y)^h \quad \text{for } X, Y \in \Gamma(TM),$$

where $X^h \in \Gamma(TP)$, $X^h_u = h_u(X_{\pi(u)})$, $u \in P$, is the horizontal lift of X to P. Given a linear connection $\widetilde{\nabla}$ on P with property (3.2), it is immediately seen that the horizontal lift U to P of a ∇-martingale X on M is a $\widetilde{\nabla}$-martingale on P: Indeed, for the corresponding anti-developments we have

$$\widetilde{\mathcal{A}}(X)(U) = h_{U_0}(\mathcal{A}(X)).$$

In the same way, replacing the roles of M, ∇, H by TM, ∇', H', we give P' a connection $\widetilde{\nabla}'$ which satisfies the analogous property to (3.2).

COROLLARY 3.2. – *Let M be a manifold with a linear connection ∇ and $(X(a))_{a \in I}$ a family of M-valued continuous semimartingales, C^1 in the topology of semimartingales. Let $a \mapsto U_0(a) \in P$ be C^1 in probability such that $\pi\big(U_0(a)\big) = X(a)$. Let $(U(a))_{a \in I}$ be the horizontal lifts of $(X(a))_{a \in I}$ to P, and $(Z(a))_{a \in I}$ the antidevelopments. As in Theorem* (3.1), *consider the families $X'(v)_{v \in TI}$ on TM, $U'(v)_{v \in TI}$ on P' and $Z'(v)_{v \in TI}$ on TTM. In this situation the following constructions commute:*

(3.3)
$$\begin{tikzcd}
U \arrow[r, mapsto, "\iota_M \circ T"] & U' \\
X \arrow[u, mapsto, "\text{h.l.}"] \arrow[r, mapsto, "T"] \arrow[d, mapsto, "\mathcal{A}"'] & X' \arrow[u, mapsto, "\text{h.l.}"'] \arrow[d, mapsto, "\mathcal{A}'"] \\
Z \arrow[r, mapsto, "s_M \circ T"] & Z'
\end{tikzcd},$$

where h.l. stands for the horizontal lift to P induced by ∇, resp. to P' induced by ∇'. In addition, if one of the families in (3.3) *consists of martingales* (*with respect to the specified connection*) *then already all others are martingales.*

Note that each family in (3.3) allows to construct the whole diagram.

4. Stochastic Jacobi fields

We begin by studying TM-valued semimartingales which are martingales, either with respect to the horizontal lift ∇^h or with respect to the complete lift ∇' of a connection ∇ on M. Recall that the horizontal lift ∇^h of ∇ is the linear connection on TM determined by the following condition [31]: If $t \mapsto J(t)$ is a curve in TM, and $t \mapsto X(t) = \pi(J(t))$ the projection to M, then the parallel transport $//^h_{0,t}W$ along J (with respect to ∇^h) of a vector $W \in T_{J(0)}TM$ (with decomposition $W = W^{\text{vert}} \oplus W^{\text{hor}}$ in $T_{J(0)}TM = V_{J(0)} \oplus H_{J(0)}$) is given by

$$(4.1) \qquad //^h_{0,t}W = v_{J(t)} \circ //_{0,t} \circ (v_{J(0)})^{-1}(W^{\text{vert}}) \oplus h^{\nabla}_{J(t)} \circ //_{0,t} \circ (h^{\nabla}_{J(0)})^{-1}(W^{\text{hor}})$$

where $v\colon \pi^*TM \xrightarrow{\sim} V$ and $h^{\nabla}\colon \pi^*TM \xrightarrow{\sim} H$ denote vertical, resp. horizontal lift, and $//_{0,\bullet}$ parallel transport on TM along $t \to X(t)$.

For any vector field $A \in \Gamma(X^*TM)$ on M along $t \mapsto X(t)$ with covariant derivative

$$(4.2) \qquad \nabla_D A \equiv //_{0,\bullet}\tfrac{d}{dt}\,//^{-1}_{0,\bullet} A \equiv v_A^{-1}(\dot{A}^{\text{vert}}),$$

let $A^h, A^v \in \Gamma(J^*TTM)$ denote the corresponding horizontally, resp. vertically lifted vector field on TM along $t \mapsto J(t)$. The following formulae for the covariant derivatives with respect to ∇^h are immediate from (4.1)

$$(4.3) \qquad \nabla^h_D A^h = (\nabla_D A)^h, \quad \nabla^h_D A^v = (\nabla_D A)^v.$$

For vector fields $A, B \in \Gamma(TM)$, we get the usual formulae, appearing as definition in [31] II (7.3),

$$(4.4) \qquad \nabla^h_{A^v} B^v = 0, \quad \nabla^h_{A^v} B^h = 0, \quad \nabla^h_{A^h} B^v = (\nabla_A B)^v, \quad \nabla^h_{A^h} B^h = (\nabla_A B)^h,$$

where $A^h, A^v, B^h, B^v \in \Gamma(TTM)$ denote the horizontal, resp. vertical lifts of A, B. Note however that in [31] the splitting

$$TTM = V(TM) \oplus H(TM)$$

in the vertical and horizontal subbundle, which appears in (4.1), is taken with respect to the adjoint connection

$$(4.5) \qquad \hat{\nabla}_A B := \nabla_B A + [A, B] = \nabla_A B - \mathrm{T}^M(A, B), \quad A, B \in \Gamma(TM),$$

where T^M is the torsion tensor on M. For $C \in \Gamma(T^*M \otimes TM)$, let $\gamma C \in \Gamma(TTM)$ be defined via $(\gamma C)_{(x,u)} = v_u\big(C_x(u)\big)$ for all $u \in T_xM$. Then, the horizontal lift of a vector field A as defined in [31] p. 87, corresponds to $A^h - \gamma\mathrm{T}^M(\,\cdot\,, A)$ in our notation. This leads to slightly different formulae in case of connections with torsion. The following formulae, relating complete and horizontal lift of a connection to each other, will be crucial for our purpose, see [31] II (7.1) and (7.2):

$$(4.6) \qquad \begin{aligned} &\nabla'_{A^v} B^v = 0, \quad \nabla'_{A^v} B^h = \mathrm{T}^M(A, B)^v, \quad \nabla'_{A^h} B^v = (\nabla_A B)^v, \\ &\nabla'_{A^h} B^h = (\nabla_A B)^h + \gamma\big(\mathrm{R}^M(\,\cdot\,, A)B + (\nabla_A \mathrm{T}^M)(\,\cdot\,, B)\big). \end{aligned}$$

for vector fields $A, B \in \Gamma(TM)$. Here T^M and R^M denote the torsion, resp. curvature tensor of the connection ∇. In particular, one has

$$\begin{aligned} &\nabla'_{A^v} B^v - \nabla^h_{A^v} B^v = \nabla'_{A^h} B^v - \nabla^h_{A^h} B^v = 0, \\ &\nabla'_{A^v} B^h - \nabla^h_{A^v} B^h = \mathrm{T}^M(A, B)^v, \\ &\nabla'_{A^h} B^h - \nabla^h_{A^h} B^h = \gamma\big(\mathrm{R}^M(\cdot, A)B + (\nabla_A \mathrm{T}^M)(\cdot, B)\big). \end{aligned} \tag{4.7}$$

Similarly, for vector fields $A \in \Gamma(X^*TM)$ on M along a curve $t \mapsto X(t)$, we get for the lifted fields A^v, A^h along J with $\pi \circ J = X$ the formulae:

$$\begin{aligned} &\nabla'_D A^v = \big(\nabla_D A\big)^v, \\ &\nabla'_D A^h = \big(\nabla_D A\big)^h + v_J\Big(\mathrm{R}^M(J, \dot{X})A + \nabla_D\big(\mathrm{T}^M(J, A)\big) - \mathrm{T}^M(J, \nabla_D A)\Big), \end{aligned} \tag{4.8}$$

hence, by means of $\nabla_D\big(\mathrm{T}^M(J, A)\big) = \big(\nabla_D \mathrm{T}^M\big)(J, A) + \mathrm{T}^M(\nabla_D J, A) + \mathrm{T}^M(J, \nabla_D A)$,

$$\begin{aligned} &\nabla'_D A^v - \nabla^h_D A^v = 0, \\ &\nabla'_D A^h - \nabla^h_D A^h = v_J\Big(\mathrm{R}^M(J, \dot{X})A + \big(\nabla_D \mathrm{T}^M\big)(J, A) + \mathrm{T}^M(\nabla_D J, A)\Big). \end{aligned} \tag{4.9}$$

Following Emery [12], if ∇ is a connection on M, the Itô differential $d^\nabla X$ of a continuous semimartingale X on M can be written as the projection $F^\nabla(\mathcal{D}X)$ of the second order differential $\mathcal{D}X$ under the canonical projection $F^\nabla : \tau M \to TM$,

$$FL = \begin{cases} L & \text{for } L = A \in \Gamma(TM), \\ \frac{1}{2}(\nabla_A B + \nabla_B A + [A, B]) & \text{for } L = AB \text{ with } A, B \in \Gamma(TM). \end{cases}$$

In this way, each connection ∇ on M gives rise to a splitting of the following exact sequence of vector bundles over M:

$$\begin{tikzcd} 0 \arrow[r] & TM \arrow[r, "\iota"] & \tau M \arrow[r] \arrow[l, bend left, dashed, "F"] & TM \odot TM \arrow[r] & 0, \end{tikzcd} \tag{4.10}$$

here ι is the canonical inclusion of TM in τM, and $TM \odot TM$ the symmetrical tensor product.

Lemma 4.1. – *Let J be a semimartingale with values in TM and $X = \pi \circ J$ its projection to M. Then*

$$d^{\nabla'} J = d^{\nabla^h} J + \tfrac{1}{2} v_J\Big(\mathrm{R}^M(J, dX)dX + \nabla \mathrm{T}^M(dX, J, dX) + \mathrm{T}^M\big(//_{0,\bullet} d //_{0,\bullet}^{-1} J, dX\big)\Big),$$

where $v_{J_t} : T_{X_t}M \to T_{J_t}TM$ denotes the vertical lift. The covariant derivative $\nabla \mathrm{T}^M \in \Gamma(T^*M^{\otimes 3} \otimes TM)$ of the torsion T^M is given by $\nabla \mathrm{T}^M(A, B, C) = (\nabla_A \mathrm{T}^M)(B, C)$ for $A, B, C \in \Gamma(TM)$.

Proof. – To calculate the difference $F^{\nabla'} - F^{\nabla^h}$ it is sufficient to consider

$$\big(F^{\nabla'} - F^{\nabla^h}\big)\, \dot{J}(0) \quad \text{and} \quad \big(F^{\nabla'} - F^{\nabla^h}\big)\, \ddot{J}(0)$$

where J is a smooth curve in TM. Clearly $(F^{\nabla'} - F^{\nabla^h})\,\dot{J}(0) = 0$. For the second term, we have by definition

$$(F^{\nabla'} - F^{\nabla^h})\,\ddot{J}(0) = \big((\nabla'_D - \nabla^h_D)\dot{J}\big)(0).$$

By decomposing $\dot{J}$ according to $\dot{J} = h^{\nabla}_J(\dot{X}) + v_J(\nabla_D J)$, we conclude from (4.9)

$$(\nabla'_D \dot{J} - \nabla^h_D \dot{J})(0) = v_J\Big(\mathrm{R}^M(J,\dot{X})\dot{X} + \nabla\mathrm{T}^M(\dot{X},J,\dot{X}) + \mathrm{T}^M(\nabla_B J,\dot{X})\Big)$$

which implies the claimed formula. □

THEOREM 4.2. – *Let J be a TM-valued semimartingale and $X = \pi \circ J$. The antidevelopment $\mathcal{A}^h(J)$ of J with respect to ∇^h is given by*

$$\mathcal{A}^h(J) = h^{\nabla}_{J_0}\big(\mathcal{A}(X)\big) + v_{J_0}\big(//_{0,\bullet}^{-1} J - J_0\big), \tag{4.11}$$

using the canonical identification $T_{J_0}TM \equiv H_{J_0} \oplus V_{J_0} \cong T_{X_0}M \oplus T_{X_0}M$ via horizontal lift $h^{\nabla}_{J_0} : T_{X_0}M \to H_{J_0}$ and vertical lift $v_{J_0} : T_{X_0}M \to V_{J_0}$.

Proof. – Let $w \in T_{J_0}TM$. By (4.1), $//^h_{0,t}w = h^{\nabla}_{J_t} \circ //_{0,t} \circ (h^{\nabla}_{J_0})^{-1}w$ if $w \in H_{J_0}$, and $//^h_{0,t}w = v_{J_t} \circ //_{0,t} \circ (v_{J_0})^{-1}w$ if $w \in V_{J_0}$ (indeed, for this, we may assume that J is just a smooth path, the claim is then equivalent to (4.1)). Further note that $Z := \mathcal{A}^h(J)$ is determined by the following Stratonovich equation

$$\delta Z = (//^h_{0,\bullet})^{-1}\delta J, \quad Z_0 = 0 \in T_{J_0}TM \tag{4.12}$$

(with the identification of $T_{J_0}TM$ and its tangent space). This gives in particular

$$\delta\,(v_{J_0}^{-1} Z^{\mathrm{vert}}) = v_{J_0}^{-1}\,(//^h_{0,\bullet})^{-1}(\delta J)^{\mathrm{vert}} = //_{0,\bullet}^{-1}\, v_J^{-1}(\delta J)^{\mathrm{vert}}. \tag{4.13}$$

On the other hand, (4.2) extends to continuous semimartingales J as

$$//_{0,\bullet}\delta\,(//_{0,\bullet}^{-1} J) = v_J^{-1}(\delta J)^{\mathrm{vert}}.$$

Thus $//_{0,\bullet}^{-1} J - J_0$ and $v_{J_0}^{-1}(Z^{\mathrm{vert}})$ satisfy the same stochastic differential equation with identical starting points, so that they are equal. In the same way, we conclude from (4.12)

$$\delta\,\big((h^{\nabla}_{J_0})^{-1} Z^{\mathrm{hor}}\big) = (h^{\nabla}_{J_0})^{-1}(//^h_{0,\bullet})^{-1}(\delta J)^{\mathrm{hor}} = //_{0,\bullet}^{-1}(h^{\nabla}_J)^{-1}(\delta J)^{\mathrm{hor}}.$$

But $(h^{\nabla}_J)^{-1}(\delta J)^{\mathrm{hor}} = \delta X$, since for a vector $u \in T_jTM$, $j \in TM$,

$$\big(h^{\nabla}_j\big)^{-1}(u^{\mathrm{hor}}) = \pi_*(u)$$

with $\pi : TM \to M$ the canonical projection. Hence, also $\mathcal{A}(X)$ and $\big(h^{\nabla}_{J_0}\big)^{-1}(Z^{\mathrm{hor}})$ satisfy the same stochastic differential equation with identical starting points, they are consequently equal. □

Remark 4.3. – Let J be a semimartingale taking values in TM. We denote by DJ the covariant differential of J, *i.e.*, $DJ = //_{0,\bullet} d(//_{0,\bullet}^{-1} J)$ where $//_{0,\bullet}$ is parallel translation along $\pi \circ J$. (Note that this notion is consistent with the notation $\nabla_D \equiv \frac{D}{dt}$ commonly used in classical differential geometry.) Instead of Stratonovich equation (4.12), the anti-development $Z := \mathcal{A}^h(J)$ of J can also be characterized by means of an Itô equation:

$$dZ = (//_{0,\bullet}^{h})^{-1}\, d^{\nabla^h} J, \quad Z_0 = 0 \in T_{J_0}TM. \tag{4.14}$$

Analogous to (4.13), this equation gives

$$d(v_{J_0}^{-1} Z^{\mathrm{vert}}) = //_{0,\bullet}^{-1}\, v_J^{-1} \big(d^{\nabla^h} J\big)^{\mathrm{vert}}. \tag{4.15}$$

Combining (4.15) with $d(//_{0,\bullet}^{-1} J) = d(v_{J_0}^{-1} Z^{\mathrm{vert}})$, we see that DJ is the projection of the vertical part of $d^{\nabla^h} J$, in other words, $DJ = v_J^{-1}(d^{\nabla^h} J)^{\mathrm{vert}}$.

COROLLARY 4.4. – *Let M be a manifold endowed with a connection ∇. Let J be a continuous semimartingale with values in TM. Then J is a ∇^h-martingale on TM if and only if*

(i) *$X = \pi \circ J$ is a ∇-martingale on M and*

(ii) $d(//_{0,\bullet}^{-1} J) \overset{\mathrm{m}}{=} 0$

where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales and $//_{0,t}$ parallel translation on M along X.

Proof. – The claim follows immediately from Theorem 4.2. ☐

THEOREM 4.5. – *Let J be a TM-valued semimartingale and $X = \pi \circ J$. The antidevelopment $\mathcal{A}'(J)$ of J with respect to ∇' is given by*

$$\begin{aligned} \mathcal{A}'(J) = h_{J_0}^{\nabla}\big(\mathcal{A}(X)\big) + v_{J_0}\bigg(//_{0,\bullet}^{-1} J - J_0 + \int_0^{\bullet} //_{0,s}^{-1} \mathrm{T}^M(J_s, \delta X_s) - \mathrm{T}^M\big(J_0, \mathcal{A}(X)\big) \\ + \int_0^{\bullet} \bigg(\int_0^s //_{0,r}^{-1} \mathrm{R}^M(J_r, \delta X_r)\, //_{0,r} \bigg) \delta\mathcal{A}(X)_s \bigg), \end{aligned}$$

where T^M, R^M is the torsion, resp. curvature tensor of the connection ∇ on M. Thus, the difference between $\mathcal{A}'(J)$ and $\mathcal{A}^h(J)$ is vertical and coincides with the vertical lift of

$$\int_0^{\bullet} //_{0,s}^{-1} \mathrm{T}^M(J_s, \delta X_s) - \mathrm{T}^M\big(J_0, \mathcal{A}(X)\big) + \int_0^{\bullet} \int_0^s //_{0,r}^{-1} (\mathrm{R}^M(J_r, \delta X_r)(//_{0,r} \delta\mathcal{A}(X)_s)).$$

Note that $\delta X_s = //_{0,s}\, \delta\mathcal{A}(X)_s$. A similar formula for the derivative of the anti-development of a semimartingale with respect to a parameter appears in T. Lyons and Z.-M. Qian [21], Theorem 1; see also [17], Theorem 2.1, for a related result.

Proof. – We start by proving that the parallel transport $//'_{0,\bullet}$ on TM (with respect to ∇') along J of a vector $w \in T_{J_0}TM$ with $u = \pi_*(w)$ satisfies

$$\begin{aligned} (//'_{0,t})\, w = (//^{h}_{0,t})\, w - v_{J_t}\Big(\mathrm{T}^M(J_t, //_{0,t}\, u) - //_{0,t} \mathrm{T}^M(J_0, u) \Big) \\ - v_{J_t}\bigg(//_{0,t} \bigg(\int_0^t //_{0,s}^{-1} \Big(\mathrm{R}^M(J_s, \delta X_s)(//_{0,s} u) \Big) \bigg) \bigg). \end{aligned} \tag{4.16}$$

Let w_t denote the right hand side of (4.16), $w_t^h = //_{0,t}^h w$ and $u_t = //_{0,t}\, u$. To prove (4.16), we may assume that J is a smooth path. First, by exploiting (4.8), we have

$$\nabla'_D(//_{0,t}^h\, w^{\mathrm{vert}}) = \nabla'_D\big(v_{J_t} \circ //_{0,t} \circ v_{J_0}^{-1}(w^{\mathrm{vert}})\big) = v_{J_t}\Big(\nabla_D\big(//_{0,t}\, v_{J_0}^{-1}(w^{\mathrm{vert}})\big)\Big) = 0$$

and in the same way

$$\nabla'_D\Big(v_{J_t}\big(//_{0,t}\mathrm{T}^M(J_0, u)\big)\Big) = 0,$$

hence

$$\nabla'_D\, w_t = \nabla'_D(//_{0,t}^h\, w^{\mathrm{hor}}) - \nabla'_D\, v_{J_t}\bigg(\mathrm{T}^M(J_t, u_t) + //_{0,t} \int_0^t //_{0,s}^{-1}(\mathrm{R}^M(J_s, \dot X_s)\, u_s)ds\bigg).$$

Again by (4.9), and the fact that $\nabla_D u_t = 0$, we find

$$\begin{aligned}\nabla'_D\, w_t &= v_{J_t}\bigg(\mathrm{R}^M(J_t, \dot X_t)u_t - \nabla_D //_{0,t} \int_0^t //_{0,s}^{-1}(\mathrm{R}^M(J_s, \dot X_s)\, u_s)ds\bigg)\\ &= v_{J_t}\bigg(\mathrm{R}^M(J_t, \dot X_t)u_t - //_{0,t}\frac{d}{dt} \int_0^t //_{0,s}^{-1}(\mathrm{R}^M(J_s, \dot X_s)\, u_s)ds\bigg) = 0\end{aligned}$$

which establishes formula (4.16) for the parallel transport. From here we find a formula for the inverse parallel transport (using the expression of $(//_{0,t}^h)^{-1}$ on vertical vectors):

$$\begin{aligned}(//'_{0,t})^{-1}(w') = (//_{0,t}^h)^{-1}(w') &+ v_{J_0}\Big(//_{0,t}^{-1}\mathrm{T}^M\big(J_t, \pi_* w'\big) - \mathrm{T}^M\big(J_0, //_{0,t}^{-1}\pi_* w'\big)\Big)\\ &+ v_{J_0}\bigg(\int_0^t //_{0,s}^{-1}\Big(\mathrm{R}^M(J_s, \delta X_s)\big(//_{0,s}\, //_{0,t}^{-1}\pi_* w'\big)\Big)\bigg).\end{aligned}$$

Now, to calculate $\mathcal{A}'(J) - \mathcal{A}^h(J)$, we work with Stratonovich equations. By means of

$$\delta\mathcal{A}'(J) = (//'_{0,\bullet})^{-1}(\delta J)$$

and

$$\delta\mathcal{A}^h(J) = (//_{0,\bullet}^h)^{-1}(\delta J),$$

we get

$$\begin{aligned}&\delta\big(\mathcal{A}'(J) - \mathcal{A}^h(J)\big) = \big((//'_{0,\bullet})^{-1} - (//_{0,\bullet}^h)^{-1}\big)(\delta J)\\ &= v_{J_0}\bigg(//_{0,\bullet}^{-1}\mathrm{T}^M\big(J, \pi_*(\delta J)\big) - \mathrm{T}^M\big(J_0, //_{0,\bullet}^{-1}\pi_*(\delta J)\big)\\ &\qquad\qquad + \int_0^\bullet //_{0,s}^{-1}\mathrm{R}^M(J_s, \delta X_s)\big(//_{0,s}//_{0,\bullet}^{-1}\pi_*(\delta J_\bullet)\big)\bigg)\\ &= v_{J_0}\bigg(//_{0,\bullet}^{-1}\mathrm{T}^M\big(J, \delta X\big) - \mathrm{T}^M\big(J_0, \delta\mathcal{A}(X)\big) + \int_0^\bullet //_{0,s}^{-1}\mathrm{R}^M(J_s, \delta X_s)//_{0,s}\delta\mathcal{A}(X)_\bullet\bigg).\end{aligned}$$

Integration with respect to time gives

$$\begin{aligned}\mathcal{A}'(J) = \mathcal{A}^h(J) + v_{J_0}\Big(\int_0^\bullet //_{0,s}^{-1}\mathrm{T}^M\big(J_s, \delta X_s\big) - \mathrm{T}^M\big(J_0, \mathcal{A}(X)\big)\Big) \\ + v_{J_0}\Big(\int_0^\bullet \Big(\int_0^s //_{0,r}^{-1}\mathrm{R}^M(J_r, \delta X_r)\, //_{0,r}\Big)\delta\mathcal{A}(X)_s\Big)\end{aligned}$$

which is using (4.11) seen to be the claimed formula. □

COROLLARY 4.6. – *Let M be a manifold endowed with a connection ∇. Let J be a continuous semimartingale with values in TM. Then J is a ∇'-martingale on TM if and only if*

(i) *$X = \pi \circ J$ is a ∇-martingale on M and*

(ii) $d\,(//_{0,\bullet}^{-1}J) + \frac{1}{2}\,//_{0,\bullet}^{-1}\Big(\nabla\mathrm{T}^M(dX, J, dX) + \mathrm{T}^M(DJ, dX) + \mathrm{R}^M(J, dX)\, dX\Big) \overset{\mathrm{m}}{=} 0$. *Here $DJ = //_{0,\bullet}d\,(//_{0,\bullet}^{-1}J)$ denotes again the covariant differential of J.*

Proof. – The claim could be derived from Lemma 4.1. We conclude directly with Theorem 4.1. Let X be a ∇-martingale (equivalently, $\mathcal{A}(X)$ a local martingale). Since obviously $\mathrm{T}^M\big(J_0, \mathcal{A}(X)\big)$ is then a local martingale, it remains to show that

$$\begin{aligned}//_{0,\bullet}^{-1}\mathrm{T}^M(J, \delta X) + \int_0^\bullet //_{0,r}^{-1}\big(\mathrm{R}^M(J_r, \delta X_r)(//_{0,r}\delta\mathcal{A}(X)_\bullet)\big) \\ \overset{\mathrm{m}}{=} \tfrac{1}{2}\,//_{0,\bullet}^{-1}\nabla\mathrm{T}^M(dX, J, dX) + \tfrac{1}{2}\,//_{0,\bullet}^{-1}\mathrm{T}^M(DJ, dX) + \tfrac{1}{2}\,//_{0,\bullet}^{-1}\mathrm{R}^M(J, dX)dX.\end{aligned}$$

Indeed, first note that

$$//_{0,\bullet}^{-1}\mathrm{T}^M(J, \delta X) \overset{\mathrm{m}}{=} \tfrac{1}{2}\,//_{0,\bullet}^{-1}\nabla\mathrm{T}^M(dX, J, dX) + \tfrac{1}{2}\,//_{0,\bullet}^{-1}\mathrm{T}^M(DJ, dX).$$

On the other hand, denoting

$$A := \mathcal{A}(X) = \int_0^\bullet //_{0,r}^{-1}\delta X_r \quad \text{and} \quad C := \int_0^\bullet //_{0,r}^{-1}\mathrm{R}^M(J_r, \delta X_r)//_{0,r},$$

we have

$$\int_0^\bullet //_{0,r}^{-1}\big(\mathrm{R}^M(J_r, \delta X_r)(//_{0,r}\delta\mathcal{A}(X)_\bullet)\big) = C\,\delta A = C\,dA + \tfrac{1}{2}\,d[C, A],$$

where the bracket $[C, A] \equiv \sum_{k,i}[C_{ik}, A_k]\, e_i$ (in terms of a basis (e_i) of $T_{X_0}M$) is given by

$$[C, A] = \int_0^\bullet //_{0,\bullet}^{-1}\mathrm{R}^M(J, dX)dX.$$

Thus, the claim is a consequence of Theorem 4.5. □

There are characterizations of geodesics in TM analogous to the characterizations of martingales given in Corollary 4.4 and 4.6:

Remark 4.7 [31]. – Let J be a curve in TM and $\gamma = \pi \circ J$ its projection to M.

(1) The curve J is a ∇^h-geodesic in TM if and only if γ is a ∇-geodesic on M and J has vanishing second covariant derivative along γ, *i.e.*,

$$\nabla_D \dot\gamma = 0 \quad \text{and} \quad \nabla_D^2 J \equiv 0.$$

(2) The curve J is a ∇'-geodesic in TM if and only if

$$\nabla_D \dot\gamma = 0 \quad \text{and} \quad \nabla_D^2 J + \nabla_D\big(\mathrm{T}^M(J,\dot\gamma)\big) + \mathrm{R}^M(J,\dot\gamma)\dot\gamma \equiv 0.$$

In other words, ∇'-geodesics in TM are just Jacobi fields along geodesics on M. Note that $\nabla_D\big(\mathrm{T}^M(J,\dot\gamma)\big) = (\nabla_D \mathrm{T}^M)(J,\dot\gamma) + \mathrm{T}^M(\nabla_D J,\dot\gamma)$ since $\nabla_D\dot\gamma = 0$.

Let ∇ be a linear connection on M and $\bar\nabla$ its symmetrization. By definition, $\bar\nabla = \frac{1}{2}(\nabla + \hat\nabla)$ where $\hat\nabla$ is the adjoint connection (4.5), *i.e.*

$$\bar\nabla AB = \tfrac{1}{2}(\nabla_A B + \hat\nabla_A B) \equiv \nabla_A B - \tfrac{1}{2}\mathrm{T}^M(A,B), \quad A, B \in \Gamma(TM). \tag{4.17}$$

In terms of $\bar\nabla$, Corollary 4.6 may be formulated as follows.

COROLLARY 4.8. – *Let M be a manifold endowed with a connection ∇. Let J be a continuous semimartingale with values in TM. Then J is a ∇'-martingale on TM if and only if*

(i) *$X = \pi \circ J$ is a ∇-martingale on M and*

(ii) $d(\bar{/\!/}_{0,\bullet}^{-1} J) + \frac{1}{2}\, \bar{/\!/}_{0,\bullet}^{-1} \bar{\mathrm{R}}^M(J, dX)\, dX \overset{\mathrm{m}}{=} 0$

where $\bar{/\!/}_{0,\bullet} : T_{X_0}M \to T_{X_t}M$ denotes parallel transport on M along X with respect to the symmetrized connection $\bar\nabla$, and $\bar{\mathrm{R}}$ the curvature tensor to $\bar\nabla$.

Proof. – Indeed, since the torsion of the complete lift ∇' on TM is just the complete lift of the torsion of ∇ on M (*see* [31] p. 41), we see that symmetrization and complete lift commute: $(\bar\nabla)' = \bar\nabla'$. On the other hand, symmetrization of a connection does not change the class of martingales, hence the ∇'-martingales in TM are exactly the $(\bar\nabla)'$-martingales in TM. By these observations the claim is reduced to Corollary 4.6. □

A special type of ∇'-martingales in TM are those arising as infinitesimal variations of ∇-martingales on M, *i.e.* which are of the form

$$J_t = (T_a X_t)v, \quad v \in T_a I, \tag{4.18}$$

where $I \ni a \mapsto X(a)$ is a C^1 family of ∇-martingales on M. Let $a \mapsto U_t(a)$ be a horizontal lift of $X_t(a)$ to the frame bundle $\mathrm{L}(M)$ over M such that $a \mapsto U_0(a)$ is C^1, then

$$/\!/_{0,t}^{-1} J_t = U_0(a)\, \vartheta\big((T_a U_t)v\big) \tag{4.19}$$

with ϑ the canonical 1-form of the connection defined by (2.12). In this case, it is possible to recover condition (ii) of Corollary 4.6 from the structural equations of the connection.

Following the ideas of P. Malliavin [22], [23], a TM-valued ∇'-martingale J of the form (4.18) will be called a *stochastic Jacobi field* on M along $X = \pi \circ J$. As shown in [2] each TM-valued ∇'-martingale J is a stochastic Jacobi field in the following sense: Let J be defined up to lifetime ζ. Then there is a C^1-family $(X(a))_{a\in\mathbb{R}}$ of ∇-martingales on M such that, if $\zeta(a)$ is the lifetime of $X(a)$, then $\zeta(a)\to\zeta$ in probability as $a\to 0$, and $J=\partial_a|_{a=0}X(a)$ holds. On a complete Riemannian manifold M one can choose $(X(a))_{a\in\mathbb{R}}$ such that, in addition, each $X(a)$ is defined on $[0,\zeta[$.

Example 4.9 (Gradient Brownian motions). – For a Riemannian manifold (M,g) let $J_t=(T_aX_t)v$ where $v\in T_aM$ and $X(a)$ is $\mathrm{BM}(M,g)$ such that $X_0(a)=a$. Then (*see* [8])

$$d\,(//_{0,t}^{-1}J_t)\overset{\mathrm{m}}{=}-\frac{1}{2}//_{0,t}^{-1}\mathrm{Ric}^{\#}(J_t,\cdot\,)\,dt. \tag{4.20}$$

For $u\in T_xM$, by definition, $\mathrm{Ric}^{\#}(u,\,\cdot\,)\in T_xM$ is given through $\langle\mathrm{Ric}^{\#}(u,\,\cdot\,),w\rangle=\mathrm{Ric}(u,w)$ for all $w\in T_xM$.

Example 4.10 (Stratonovich SDE). – Let M be an m-dimensional smooth manifold endowed with a torsion-free connection ∇. Given $r\in\mathbb{N}$, let $A\in\Gamma(\mathbb{R}^r\otimes TM)$ and $A_0\in\Gamma(TM)$. Consider the Stratonovich SDE on M

$$\delta X=A(X)\,\delta B+A_0(X)\,dt \tag{4.21}$$

where B is an $\mathbb{R}^r$-valued Brownian motion on some filtered probability space satisfying the usual completeness conditions. Solutions X to (4.21) are ∇-martingales on M if

$$A_0+\frac{1}{2}\sum_{i=1}^{r}\nabla_{A_i}A_i=0$$

where $A_i\equiv A(\cdot)e_i\in\Gamma(TM)$ for $i=1,\dots,r$. For some $v\in T_aM$ let $J_t=(T_aX_t)v$. Then [3] we have:

$$\begin{aligned} d\,(//_{0,\bullet}^{-1}J)&=//_{0,\bullet}^{-1}\nabla_J\Big(A_0+\frac{1}{2}\sum_{i=1}^{r}\nabla_{A_i}A_i\Big)\,dt\\ &\quad-\frac{1}{2}\sum_{i=1}^{r}//_{0,\bullet}^{-1}\mathrm{R}^M\big(J,A_i(X)\big)A_i(X)\,dt+\sum_{i=1}^{r}//_{0,\bullet}^{-1}(\nabla_JA_i)\,dB^i.\end{aligned} \tag{4.22}$$

Equation (4.22) for the covariant differential $DJ=//_{0,\bullet}d\,//_{0,\bullet}^{-1}J$ of J is a special case of more general results in [2], dealing with variations of solutions of arbitrary Itô SDE's, see [2] Corollary 3.17.

Example 4.11 (Stochastic Jacobi fields on a Lie group). – Let G be a Lie group, $\mathfrak{g}\equiv T_eG$ its Lie algebra, ∇ the canonical left invariant connection such that $\nabla_AB=0$ if $B\in\mathfrak{g}$ (we identify $\mathfrak{g}$ and the left invariant vector fields on G in the usual way). The torsion is given by $\mathrm{T}(A_g,B_g)=-[A,B]_g$ for $A,B\in\mathfrak{g}$, the curvature tensor by $\mathrm{R}\equiv 0$. Let M be a semimartingale with values in $T_eG=\mathfrak{g}$ such that $M_0=0$, and $\mathcal{E}(M)$ its stochastic exponential, *i.e.* the solution X of

$$\delta X=X\,\delta M,\quad X_0=e$$

(writing $X\,\delta M$ for $(L_X)_*(\delta M)$). Then, to a family $X(a)$ of semimartingales with values in G, there exist T_eG-valued semimartingales $M(a)$ such that $X(a) = X_0(a)\,\mathcal{E}(M(a))$, and we have $Z(a) \equiv \mathcal{A}(X(a)) = X_0(a)\,M(a)$. But, see e.g. [1],

$$\begin{aligned}\mathcal{E}(M(a)) &= \mathcal{E}\Big(\big(M(a) - M(0)\big) + M(0)\Big)\\ &= \mathcal{E}\bigg(\int_0^\bullet \mathrm{Ad}\,\mathcal{E}(M(0))\,\delta\big(M(a) - M(0)\big)\bigg)\mathcal{E}(M(0))\end{aligned}$$

which shows

$$X(a) = X_0(a)\,\mathcal{E}\bigg(\int_0^\bullet \mathrm{Ad}\,\mathcal{E}\big(M(0)\big)\,\delta\big(M(a) - M(0)\big)\bigg)\mathcal{E}\big(M(0)\big). \tag{4.23}$$

Let $J = \partial_a|_{a=0}X$. Differentiating (4.23) at $a = 0$ gives

$$J = X_0\bigg(\int_0^\bullet \mathrm{Ad}\,\mathcal{E}(M)\,\delta M'\bigg)\mathcal{E}(M) + J_0\,\mathcal{E}(M) \tag{4.24}$$

(we omit the parameter a when equal to 0). Taking into account that

$$//_{0,\bullet}^{-1}J = X_0\,X^{-1}J = X_0\,\mathcal{E}(M)^{-1}X_0^{-1}J,$$

we get from (4.24) the equation

$$\mathrm{Ad}\,\mathcal{E}(M)(X_0^{-1}//_{0,\bullet}^{-1}J) = \int_0^\bullet \mathrm{Ad}\,\mathcal{E}(M)\,\delta M' + X_0^{-1}J_0,$$

hence

$$\mathrm{Ad}\,\mathcal{E}(M)(X_0^{-1}\delta(//_{0,\bullet}^{-1}J)) + \delta(\mathrm{Ad}\,\mathcal{E}(M))(X_0^{-1}//_{0,\bullet}^{-1}J) = \mathrm{Ad}\,\mathcal{E}(M)\,\delta M',$$

or since $T\mathrm{Ad}g = \mathrm{Ad}g \circ \mathrm{ad}$ with $\mathrm{ad}(A)B = [A,B]$ for $A, B \in \mathfrak{g}$,

$$\mathrm{Ad}\,\mathcal{E}(M)\Big(X_0^{-1}\delta(//_{0,\bullet}^{-1}J) + \big[\delta M, X_0^{-1}//_{0,\bullet}^{-1}J\big]\Big) = \mathrm{Ad}\,\mathcal{E}(M)\,\delta M'.$$

This shows

$$\delta(//_{0,\bullet}^{-1}J) - \mathrm{T}(\delta Z, //_{0,\bullet}^{-1}J) = X_0\,\delta M'. \tag{4.25}$$

But X_0M' is the vertical part of Z', thus

$$v^{-1}\big((Z')^{\mathrm{vert}}\big) = //_{0,\bullet}^{-1}J - J_0 + \int_0^\bullet //_{0,\bullet}^{-1}\mathrm{T}(J, \delta X)$$

and with $s \equiv s_G$ (as defined by (2.7)) we obtain:

$$v^{-1}\big((s(Z'))^{\mathrm{vert}}\big) = //_{0,\bullet}^{-1}J - J_0 + \int_0^\bullet //_{0,\bullet}^{-1}\mathrm{T}(J, \delta X) - \mathrm{T}(J_0, Z),$$

in accordance with Theorem 4.5. Note that:

$$v^{-1}\big((\mathcal{A}(X)')^{\mathrm{vert}}\big) - v^{-1}\big((\mathcal{A}'(J))^{\mathrm{vert}}\big) = \mathrm{T}\big(J_0, \mathcal{A}(X)\big).$$

THEOREM 4.12. – *Let M be a manifold endowed with a connection ∇. On TM consider the complete lift ∇' of ∇. Let J be a continuous semimartingale with values in TM. Then J is a ∇'-martingale on TM if and only if*

(i) $X = \pi \circ J$ *is a ∇-martingale on M, and*

(ii) $\Theta_{0,\bullet}^{-1} J \equiv (\Theta_{0,t}^{-1} J_t)_{t \geq 0}$ *is a local martingale*

where the linear transports $\Theta_{0,t}$: $T_{X_0}M \rightarrow T_{X_t}M$ are defined by the following $(\bar{\nabla})$-covariant equation along X

$$\begin{cases} d\,(\overline{/\!/}_{0,\bullet}^{-1}\Theta_{0,\bullet}) = -\frac{1}{2}\,\overline{/\!/}_{0,\bullet}^{-1}\bar{\mathrm{R}}^M(\Theta_{0,\bullet}, dX)dX, \\ \qquad\qquad \Theta_{0,0} = \mathrm{id}, \end{cases} \tag{4.26}$$

with $\overline{/\!/}_{0,\bullet} : T_{X_0}M \rightarrow T_{X_t}M$ denoting parallel transport on M along X with respect to the symmetrized connection $\bar{\nabla}$ and $\bar{\mathrm{R}}$ the curvature tensor to $\bar{\nabla}$.

Moreover, if ∇ is the Levi-Civita connection on a Riemannian manifold M, then $\Theta_{0,t} : T_{X_0}M \rightarrow T_{X_t}M$ on M along X is norm-increasing if all sectional curvatures along X are nonpositive, and norm-decreasing if nonnegative.

The deformed (damped) parallel transport $\Theta_{0,\bullet} : T_{X_0}M \rightarrow T_{X_\bullet}M$ on M along X has been studied by several authors; in the physical literature it is sometimes called Dohrn-Guerra parallel translation [6]. P.A. Meyer deals with it in terms of local coordinates under the name "transport géodésique", see [24], formula (27); he also notes the monotonicity of the norm (in the Riemannian case) depending on the sign of the curvature ([24], Proposition 4). Note that (4.26) reduces to the so-called Ricci flow [8] if X is a Brownian motion on a Riemannian manifold.

Proof (of Theorem 4.12). – Without loss of generality, we may assume that ∇ is torsion-free, following the same argumentation as in the proof of Corollary 4.8. It suffices to show that $d(\Theta_{0,\bullet}^{-1} J) \overset{\mathrm{m}}{=} 0$ if and only if

$$d\,/\!/_{0,\bullet}^{-1} J + \tfrac{1}{2}\,/\!/_{0,\bullet}^{-1}\mathrm{R}^M(J, dX)\,dX \overset{\mathrm{m}}{=} 0. \tag{4.27}$$

Let $\theta_{0,t} := /\!/_{0,t}^{-1}\Theta_{0,t} : T_{X_0}M \rightarrow T_{X_0}M$. Then

$$\begin{cases} d\theta_{0,\bullet} = -\frac{1}{2}\,/\!/_{0,\bullet}^{-1}\mathrm{R}^M(/\!/_{0,\bullet}\theta_{0,\bullet}, dX)dX, \\ \theta_{0,0} = \mathrm{id}, \end{cases} \tag{4.28}$$

and

$$\begin{cases} d\theta_{0,\bullet}^{-1} = \frac{1}{2}\,\theta_{0,\bullet}^{-1}\,/\!/_{0,\bullet}^{-1}\mathrm{R}^M(/\!/_{0,\bullet}\,\cdot, dX)dX, \\ \theta_{0,0}^{-1} = \mathrm{id}. \end{cases} \tag{4.29}$$

Indeed, from $\theta_{0,t}^{-1} \circ \theta_{0,t} = \mathrm{id}$ we get $d\theta_{0,t}^{-1}\,\theta_{0,t} + \theta_{0,t}^{-1}\,d\theta_{0,t} = 0$ and hence

$$\begin{aligned} d\theta_{0,t}^{-1} &= -\theta_{0,t}^{-1}\,d\theta_{0,t}\,\theta_{0,t}^{-1} \\ &= \tfrac{1}{2}\,\theta_{0,t}^{-1}\,//_{0,t}^{-1}\mathrm{R}^M(//_{0,t}\,\cdot, dX_t)dX_t. \end{aligned}$$

Thus,

$$\begin{aligned} d(\Theta_{0,\bullet}^{-1}J) &= d(\theta_{0,\bullet}^{-1}\,//_{0,\bullet}^{-1}J) \\ &= d\theta_{0,\bullet}^{-1}\,//_{0,\bullet}^{-1}J + \theta_{0,\bullet}^{-1}\,d(//_{0,\bullet}^{-1}J) \\ &= \tfrac{1}{2}\,\theta_{0,\bullet}^{-1}\,//_{0,\bullet}^{-1}\mathrm{R}^M(J, dX)dX + \theta_{0,\bullet}^{-1}\,d(//_{0,\bullet}^{-1}J), \end{aligned}$$

which shows that $d(\Theta_{0,\bullet}^{-1}J) \overset{\mathrm{m}}{=} 0$ if and only if (4.27) holds.

Now assume that ∇ is the Levi-Civita connection on a Riemannian manifold M, let $w \in T_{X_0}M$. Since $\Theta_{0,t} = //_{0,t}\,\theta_{0,t}$, it remains to show that $\|\theta_{0,t}\,w\| \geq \|\theta_{0,0}\,w\|$ if $\mathrm{R}^M \leq 0$, and $\|\theta_{0,t}\,w\| \leq \|\theta_{0,0}\,w\|$ if $\mathrm{R}^M \geq 0$. But, by means of (4.28), we have,

$$\begin{aligned} \tfrac{1}{2}d\,\|\theta_{0,\bullet}w\|^2 &= \langle \theta_{0,\bullet}w, d\theta_{0,\bullet}w \rangle \\ &= -\tfrac{1}{2}\,\langle //_{0,\bullet}\theta_{0,\bullet}w, \mathrm{R}^M(//_{0,\bullet}\theta_{0,\bullet}w, dX)dX \rangle, \end{aligned} \tag{4.30}$$

which gives the claim. □

For a Levi-Civita connection ∇, the above argument actually shows that the maps

$$\Theta_{s,t} := \Theta_{0,t} \circ \Theta_{0,s}^{-1} : T_{X_s}M \to T_{X_t}M, \quad s \leq t,$$

are norm-increasing if $\mathrm{R}^M \leq 0$, and norm-decreasing if $\mathrm{R}^M \geq 0$.

Remark 4.13. – Let M be a manifold endowed with a torsion-free connection ∇. In terms of the deformed parallel transport (4.26), resp. (4.28), we can rewrite the formula in Theorem 4.5 for the anti-development of J with respect to ∇' as

$$\begin{aligned} \mathcal{A}'(J) = h_{J_0}^{\nabla}\big(\mathcal{A}(X)\big) + v_{J_0}\bigg(&\int_0^\bullet \theta_{0,s}\,d(\Theta_{0,\bullet}^{-1}J)_s \\ &+ \int_0^\bullet \Big(\int_0^s //_{0,r}^{-1}\mathrm{R}^M(J_r, \delta X_r)\,//_{0,r}\Big)d\mathcal{A}(X)_s\bigg) \end{aligned} \tag{4.31}$$

where the last integral is now an Itô integral, and hence a local martingale if X is a ∇-martingale.

5. Families of martingales and the harmonic map heat flow

Let M, N be Riemannian manifolds, each endowed with the Levi-Civita connection, and $u_0 \in C^\infty(M, N)$. We are interested in the deformation of u_0 under the heat flow

$$\tfrac{\partial}{\partial t}u = \tfrac{1}{2}\tau(u), \quad u|_{t=0} = u_0. \tag{5.1}$$

Recall that $\tau(f) = \mathrm{tr}\nabla df \in \Gamma(f^*TN)$ denotes the tension field along a smooth map $f : M \to N$. Let $X(a)$ be a $\mathrm{BM}(M, g)$, started at $a \in M$ at time $t = 0$. Hence, the anti-development $\mathcal{A}\big(X(a)\big)$ of $X(a)$ is a (flat) Brownian motion B in T_aM. Occasionally we will assume (M,g) to be BM-complete, *i.e.*, that Brownian motions on (M,g) have infinite lifetime, but in general M is not required to be (metrically) complete. If $u : [0,t] \times M \to N$ is a smooth solution of (5.1), then

$$\tilde{X}_r(a) = u\big(t-r, X_r(a)\big), \quad 0 \le r \le t, \ a \in M, \tag{5.2}$$

defines a differentiable family of ∇^N-martingales on N, see [28], and consequently

$$T_a\tilde{X}_r\, v = T_{X_r(a)}u(t-r, \,\cdot\,)\, T_aX_r\, v, \quad 0 \le r \le t, \ v \in T_aN, \tag{5.3}$$

a family of martingales in TN with respect to the complete lift of ∇^N to TN. For not necessarily BM-complete M, the martingales (5.2) and (5.3) are only defined up to $t \wedge \zeta(a)$ where $\zeta(a)$ is the lifetime of $X(a)$.

Remark 5.1. – Let $u : [0,t] \times M \to N$ be a smooth solution to (5.1). Fixing $v \in T_aM$ and writing $u_r = u(r, \cdot)$, we get the stochastic Jacobi fields $J_r = (TX_r)v$ and $\tilde{J}_r = Tu_{t-r}J_r \equiv Tu_{t-r}(TX_r)v$ on M, resp. N. By Corollary 4.6, we have

$$d/\!/_{0,r}^{-1}\tilde{J}_r + \tfrac{1}{2}\, /\!/_{0,r}^{-1}\mathrm{tr}\big[\mathrm{R}^N\big(\tilde{J}_r, Tu_{t-r}(\,\cdot\,)\big)\, Tu_{t-r}(\,\cdot\,)\big]\, dr \overset{\mathrm{m}}{=} 0. \tag{5.4}$$

Note that equation (4.20) is a special case of (5.4) for $M = N$ and $u_0 = \mathrm{id}$.

THEOREM 5.2. – *Let M, N be Riemannian manifolds endowed with the Levi-Civita connection where M is assumed to be* BM-*complete. Let $u : [0,t] \times M \to N$ be a smooth solution to the heat equation* (5.1). *For $v \in T_aM$, consider the stochastic Jacobi field*

$$\tilde{J} = Tu_{t-\bullet}\, TX_\bullet\, v$$

along the N-valued martingale $\tilde{X} = u(t-\cdot, X_\bullet(a))$ and let $\Theta_{0,\bullet}$ be the linear transport along $\tilde{X}$ determined by

$$\begin{cases} d\,(/\!/_{0,\bullet}^N)^{-1}\Theta_{0,\bullet} = -\tfrac{1}{2}\,(/\!/_{0,\bullet}^N)^{-1}\mathrm{R}^N(\Theta_{0,\bullet}, d\tilde{X})d\tilde{X}, \\ \qquad\qquad \Theta_{0,0} = \mathrm{id}, \end{cases} \tag{5.5}$$

where $/\!/_{0,\bullet}^N$ is the parallel transport on N along $\tilde{X}$. Suppose that the local martingale $\Theta_{0,\bullet}^{-1}\tilde{J}$ in $T_{u(t,a)}N$ is already a martingale; then

$$(Tu_t)_a v = \tilde{J}_0 = \mathbb{E}\,[\Theta_{0,r}^{-1}\tilde{J}_r], \quad 0 < r \le t.$$

In other words,

$$\begin{aligned} (Tu_t)_a v &= \mathbb{E}\big[\theta_{0,r}^{-1}\,(/\!/_{0,r}^N)^{-1}T_{X_r(a)}u_{t-r}\,T_aX_r\,v\big] \\ &= \mathbb{E}\big[\theta_{0,t}^{-1}\,(/\!/_{0,t}^N)^{-1}T_{X_t(a)}u_0\,T_aX_t\,v\big], \quad 0 < r \le t, \end{aligned} \tag{5.6}$$

where the linear maps $\theta_{0,r} : T_{u(t,a)}N \to T_{u(t,a)}N$ are determined by the pathwise equation

$$\begin{cases} \frac{d}{dr}\theta_{0,r} = -\tfrac{1}{2}\,(/\!/_{0,r}^N)^{-1}\mathrm{tr}\mathrm{R}^N\big(/\!/_{0,r}^N\theta_{0,r}, T_{X_r(a)}u_{t-r}(\,\cdot\,)\big)T_{X_r(a)}u_{t-r}(\,\cdot\,), \\ \ \theta_{0,0} = \mathrm{id}. \end{cases} \tag{5.7}$$

In addition, we have

$$\|\theta_{0,r}^{-1}\| \le 1, \quad \big[\|\theta_{0,r}^{-1}\| \ge 1\big], \quad 0 < r \le t, \tag{5.8}$$

if N has nonpositive [*nonnegative*] *sectional curvature.*

Note that, equivalently to (5.7), $\theta_{0,\bullet}^{-1}$ is determined by

$$\begin{cases} \frac{d}{dr}\,\theta_{0,r}^{-1} = \frac{1}{2}\,\theta_{0,r}^{-1}\,(//_{0,r}^{N})^{-1}\mathrm{tr}\mathrm{R}^{N}\big(//_{0,r}^{N}, T_{X_r(a)}u_{t-r}(\,\boldsymbol{\cdot}\,)\big)T_{X_r(a)}u_{t-r}(\,\boldsymbol{\cdot}\,), \\ \quad \theta_{0,0}^{-1} = \mathrm{id}. \end{cases} \tag{5.9}$$

In Theorem 5.2 the assumption of $\Theta_{0,\bullet}^{-1}\tilde{J}$ actually being a martingale in $T_{u(t,a)}N$ is obviously satisfied, if M, N are compact manifolds. In case M is compact, for instance, it is well-known that we can choose the family of Brownian motions $X(a)$ in such a way that $\sup_{0 \le r \le t} \|T_aX_r\| \in L^p$ for any $1 \le p < \infty$. Furthermore, by means of (5.8), equation (5.6) allows to give *a priori* estimates for u_t in terms of u_0 if the target N is negatively curved. This provides a stochastic explanation of the famous result of Eells and Sampson that there is no blow-up in the heat flow for harmonic maps in case of nonpositively curved targets.

Now, we want to extend derivative formulae as developed in [29] to the nonlinear case of a curved target N. To this end, we consider the linear transport $W_{0,\bullet} \equiv \Theta_{0,\bullet} : T_aM \to T_{X_\bullet(a)}M$ along the Brownian motion $X_\bullet(a)$ on M, defined by (4.26). For $v \in T_aM$, the TM-valued process $W_{0,\bullet}(v)$ above $X(a)$ satisfies the following covariant equation along $X(a)$:

$$\begin{cases} \frac{D}{dr}\,W_{0,r}(v) = -\frac{1}{2}\mathrm{Ric}^{M}\big(W_{0,r}(v),\,\boldsymbol{\cdot}\,\big)^{\#}, \\ \quad W_{0,0}(v) = v. \end{cases} \tag{5.10}$$

By definition, $\frac{D}{dr}W_{0,r}(v) = //_{0,r}\,\frac{d}{dr}\,//_{0,r}^{-1}\,W_{0,r}(v)$. Further, denote by

$$Z(a) = \mathcal{A}^{N}\big(u(t-\boldsymbol{\cdot}, X_\bullet(a))\big)$$

the anti-development of $\tilde{X}_\bullet(a) = u\big(t-\boldsymbol{\cdot}, X_\bullet(a)\big)$, taking values in $T_{u(t,a)}N$. Note that

$$dZ(a) = (//_{0,s}^{N})^{-1}\,Tu(t-s,\,\boldsymbol{\cdot}\,)\,//_{0,s}^{M}\,dB_s. \tag{5.11}$$

Furthermore, for $a \in M$, let

$$\mathcal{F}_r(a) := \mathcal{F}_r^{X(a)} \equiv \sigma\big\{X_s(a) : 0 \le s \le r\big\} \tag{5.12}$$

be the filtration generated by X, when started at a.

THEOREM 5.3. – *Let M, N be compact Riemannian manifolds, and assume that $u : [0,t] \times M \to N$ is a smooth solution of the nonlinear heat equation* (5.1). *Let $v \in T_aM$. Then the following formula holds:*

$$\begin{aligned}(T_au_t)v = -\,\mathbb{E}\Big[\mathcal{A}^{N}\big(u(t-\boldsymbol{\cdot}, X_\bullet(a))\big)_t \int_0^t \big\langle W_{0,s}(\dot{h}_s), //_{0,s}^{M}\,dB_s\big\rangle\Big] \\ + \tfrac{1}{2}\,\mathbb{E}\Big[\int_0^t (//_{0,s}^{N})^{-1}\mathrm{tr}\big[\mathrm{R}^{N}\big(Tu_{t-s}\,W_{0,s}(h_s), Tu_{t-s}(\,\boldsymbol{\cdot}\,)\big)\,Tu_{t-s}(\,\boldsymbol{\cdot}\,)\big]\,ds\Big]\end{aligned}$$

for any bounded $\mathcal{F}_\bullet(a)$*-adapted process* h *with sample paths in the Cameron-Martin space* $\mathbb{H}([0,t],T_aM)$ *such that* $\big(\int_0^t \|\dot{h}_s\|^2\,ds\big)^{1/2} \in L^{1+\varepsilon}$ *for some* $\varepsilon > 0$*, and the property that* $h_0 = v$*,* $h_t = 0$.

Proof. – We may assume that the Brownian motions $X(\,\cdot\,)$ on M (for varying starting points) are all constructed as solutions of a fixed Stratonovich SDE of the type

$$\delta X = A(X)\,\delta\tilde{B} + A_0(X)\,dt \tag{5.13}$$

with $A_0 \in \Gamma(TM)$, $A \in \Gamma(\mathbb{R}^m \otimes TM)$, and $\tilde{B}$ an $\mathbb{R}^m$-valued Brownian motion, $m \in \mathbb{N}$, defined on some filtered probability space $\big(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P}\big)$ satisfying the usual conditions, such that the Levi-Civita connection ∇^M on M equals the Le Jan-Watanabe connection induced from (5.13), see [10]. Then, in particular,

$$B \equiv \mathcal{A}^M\big(X(a)\big) = \int_0^\bullet (/\!/_{0,s}^M)^{-1} A\big(X_s(a)\big)\,d\tilde{B}_s,$$

or $A\big(X_s(a)\big)d\tilde{B}_s = /\!/_{0,s}^M\,dB_s$. Using the fact [11], [10] that

$$W_{0,r}(v) := \mathbb{E}^{\mathcal{F}_r(a)}\big[(T_aX_r)\,v\big] \equiv /\!/_{0,r}^M\,\mathbb{E}^{\mathcal{F}_r(a)}\big[(/\!/_{0,r}^M)^{-1}(T_aX_r)\,v\big] \tag{5.14}$$

solves (5.10), it is sufficient to verify the formula

$$\begin{aligned}(T_au_t)v = &-\mathbb{E}\Big[\mathcal{A}^N\big(u(t-\cdot, X_\bullet(a))\big)_t \int_0^t \big\langle (T_aX_s)\,\dot{h}_s, A\big(X_s(a)\big)\,d\tilde{B}_s\big\rangle\Big]\\ &+ \tfrac{1}{2}\,\mathbb{E}\Big[\int_0^t (/\!/_{0,s}^N)^{-1}\mathrm{tr}\big[\mathrm{R}^N\big(Tu_{t-s}\,T_aX_s\,h_s, Tu_{t-s}(\,\cdot\,)\big)\,Tu_{t-s}(\,\cdot\,)\big]\,ds\Big].\end{aligned} \tag{5.15}$$

Since h is $\mathcal{F}_\bullet(a)$-adapted, the formula in Theorem 5.3 follows then upon conditioning with respect to $\mathcal{F}_\bullet(a)$.

Now, for $x \in M$, let $A(x)^* : T_xM \to \mathbb{R}^m$ be the adjoint to $A(x) : \mathbb{R}^m \to T_xM$, thus $A(x)A(x)^* = \mathrm{id}_{T_xM}$. As in [29], let $H_r^\varepsilon : M \to M$, parameterized by $0 \le r \le t$, be defined as the pathwise solution to

$$\begin{cases} \frac{\partial}{\partial\varepsilon}H_r^\varepsilon(a) = A\big(H_r^\varepsilon(a)\big)A(a)^*h_r,\\ H_r^0(a) = a. \end{cases} \tag{5.16}$$

Set $X_r^\varepsilon(a) = X_r\big(H_r^\varepsilon(a)\big)$. Then, in particular $X_r^0(a) = X_r(a)$, and the perturbed process X^ε satisfies

$$\delta X^\varepsilon = A(X^\varepsilon)\,\delta\tilde{B} + A_0(X^\varepsilon)\,dr + (TX_r)\,dH_r^\varepsilon$$

with $dH_r^\varepsilon = \big(\frac{\partial}{\partial r}H_r^\varepsilon\big)\,dr = \dot{H}_r^\varepsilon dr$. In other words,

$$\delta X^\varepsilon(x) = A\big(X^\varepsilon(a)\big)\big[\delta\tilde{B} + A\big(X^\varepsilon(a)\big)^*(T_{H_r^\varepsilon(a)}X_r)\,dH_r^\varepsilon(a)\big] + A_0\big(X^\varepsilon(a)\big)\,dr.$$

This is an SDE of the same type as (5.13) but with the perturbed driving process $d\tilde{B}^\varepsilon(x) = d\tilde{B} + A\big(X^\varepsilon(a)\big)^*(T_{H_r^\varepsilon(a)}X_r)\,dH_r^\varepsilon(a)$. Again, as in [29], we compensate this perturbation by changing the measure by means of the Girsanov exponential $G^\varepsilon_r = \exp\bigl(M^\varepsilon_r - \frac{1}{2}[M^\varepsilon]_r\bigr)$ where

$$M^\varepsilon_r = -\int_0^r \bigl\langle A(X^\varepsilon_s)^*(T_{H^\varepsilon_s} X_s)\, \dot H^\varepsilon_s, d\tilde B_s\bigr\rangle. \tag{5.17}$$

Let $\tilde X^\varepsilon_r(a) = u\bigl(t-r, X^\varepsilon_r(a)\bigr)$ and $Z^\varepsilon(a) = \mathcal{A}^N\bigl(u(t-\bullet\,, X^\varepsilon_\bullet(a)\bigr)$. Then $Z^\varepsilon(a)\cdot G^\varepsilon(a)$ is a local martingale in $T_{u(t,H^\varepsilon_0(a))}N$. Consequently,

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} Z^\varepsilon_r(a)\cdot G^\varepsilon_r(a) = -\mathcal{A}^N\bigl(u(t-\bullet\,, X_\bullet(a))\bigr)_r \int_0^r \langle T_aX_s\,\dot h_s, //^M_{0,s}\,dB_s\rangle + \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} Z^\varepsilon_r(a)$$

is also a local martingale, and so is $Q_r = s_M\bigl(\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0} Z^\varepsilon_r(a)\cdot G^\varepsilon_r(a)\bigr)$ with s_M being defined in diagram (2.7). By Theorem 3.1 (ii), we obtain:

$$s_M\Bigl(\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} Z^\varepsilon(a)\Bigr) = (\mathcal{A}^N)'\Bigl(\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\tilde X^\varepsilon\Bigr).$$

For a fixed, let $\tilde X \equiv \tilde X(a)$ and $\tilde J = \frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\tilde X^\varepsilon(a)$, thus $\tilde J_r = Tu_{t-r}\,T_aX_r\,h_r$. According to Theorem 4.5, we have

$$\begin{aligned}(\mathcal{A}^N)'(\tilde J) &= h_{\tilde J_0}(\mathcal{A}^N(\tilde X))\\ &\quad + v_{\tilde J_0}\biggl((//^N_{0,\bullet})^{-1}\tilde J_\bullet - \tilde J_0 + \int_0^\bullet\int_0^s (//^N_{0,r})^{-1}\Bigl(\mathrm{R}^N(\tilde J_r, \delta\tilde X_r)(//^N_{0,r}\,\delta\mathcal{A}^N(\tilde X)_s)\Bigr)\biggr),\end{aligned}$$

where

$$\begin{aligned}&\int_0^\bullet\int_0^s (//^N_{0,r})^{-1}\Bigl(\mathrm{R}^N(\tilde J_r, \delta\tilde X_r)(//^N_{0,r}\,\delta\mathcal{A}^N(\tilde X)_s)\Bigr)\\ &= \int_0^\bullet\int_0^s (//^N_{0,r})^{-1}\Bigl(\mathrm{R}^N(\tilde J_r, \delta\tilde X_r)(//^N_{0,r}\,d\mathcal{A}^N(\tilde X)_s)\Bigr) + \frac{1}{2}(//^N_{0,\bullet})^{-1}\mathrm{R}^N(\tilde J, d\tilde X)\,d\tilde X.\end{aligned}$$

But we have:

$$\begin{aligned}&(//^N_{0,\bullet})^{-1}\mathrm{R}^N(\tilde J, d\tilde X)\,d\tilde X\\ &\qquad = (//^N_{0,s})^{-1}\mathrm{tr}\bigl[\mathrm{R}^N\bigl(T_{X_s(a)}u_{t-s}\,T_aX_s\,h_s, T_{X_s(a)}u_{t-s}(\,\cdot\,)\bigr)\,T_{X_s(a)}u_{t-s}(\,\cdot\,)\bigr]\,ds.\end{aligned}$$

Now, take the vertical part Q^{vert} of the local martingale Q which is easily seen to be already a martingale under our assumptions. The claimed formula follows by evaluating

$$\mathbb{E}\,(Q^{\mathrm{vert}}_0) = \mathbb{E}\,(Q^{\mathrm{vert}}_t),$$

using the fact that $\tilde J_0 = (T_au_t)v$ and $\tilde J_t = 0$, as a consequence of our assumptions on the process h. □

Formula (5.15) has been derived by Elworthy [9] in the special case $h_s = (1-s/t)v$. Note that in the proof of Theorem 5.3 compactness of the manifolds was only needed to assure that the local martingale Q^{vert} is actually a martingale. As in [29], this can also be achieved by putting appropriate conditions on the process h. For instance, we may formulate the following modified version to Theorem 5.3.

THEOREM 5.4. – *Let M, N be Riemannian manifolds. Let $u : [0,t] \times M \to N$ be a smooth solution of* (5.1), $v \in T_aM$. *Then*

$$\begin{aligned}(T_a u_t)v = &-\mathbb{E}\Big[\mathcal{A}^N\big(u(t-\bullet\,, X_\bullet(a))\big)_{\tau(a)\wedge t}\int_0^{\tau(a)\wedge t}\langle W_{0,s}(\dot h_s), //^M_{0,s}\,dB_s\rangle\Big]\\ &+\tfrac{1}{2}\mathbb{E}\Big[\int_0^{\tau(a)\wedge t}(//^N_{0,s})^{-1}\mathrm{tr}\big[\mathrm{R}^N\big(Tu_{t-s}\,W_{0,s}(h_s), Tu_{t-s}(\,\cdot\,)\big)\,Tu_{t-s}(\,\cdot\,)\big]\,ds\Big]\end{aligned}$$

for any bounded $\mathcal{F}_\bullet(a)$-adapted process h with sample paths in the Cameron-Martin space $\mathbb{H}([0,t], T_aM)$ such that $\big(\int_0^{\tau(a)\wedge t}\|\dot h_s\|^2\,ds\big)^{1/2} \in L^{1+\varepsilon}$ for some $\varepsilon > 0$, and the property that $h_0 = v$, $h_s = 0$ for all $s \geq \tau(a) \wedge t$; here $\tau(a)$ is the first exit time of $X(a)$ from some relatively compact neighbourhood D of a.

Note that, in contrast to formula (5.6) in Theorem 5.2, the differential Tu_0 of the initial map u_0 does not appear in the formulae of Theorem 5.3 and Theorem 5.4. In terms of $W_{0,\bullet}(v)$, a version of Theorem 5.2 may be formulated as follows:

THEOREM 5.5. – *Let M, N be Riemannian manifolds, where* Ric^M *is bounded below by some constant and the sectional curvatures* R^N *are bounded above by some constant; in addition M is supposed to be* BM-*complete. Let $u : [0,t] \times M \to N$ be a smooth solution to the heat equation such that $\|Tu\|$ is bounded on $[0,t] \times M$; then*

$$(T_a u_t)v = \mathbb{E}\big[\Theta^{-1}_{0,t}\,\big(T_{X_t(a)}u_0\big)\,W_{0,t}(v)\big], \quad v \in T_aM, \tag{5.18}$$

where $W_{0,\bullet} : T_aM \to T_{X_\bullet}M$ is determined by the following covariant equation along the Brownian motion $X = X(a)$:

$$d\,\big[(//^M_{0,\bullet})^{-1}W_{0,\bullet}\big] = -\tfrac{1}{2}\,(//^M_{0,\bullet})^{-1}\mathrm{R}^M(W_{0,\bullet}, dX)dX, \quad W_{0,0} = \mathrm{id},$$

or equivalently

$$\frac{D}{dr}\,W_{0,r} = -\frac{1}{2}\mathrm{Ric}^M\big(W_{0,r}, \,\cdot\,\big)^\#, \quad W_{0,0} = \mathrm{id}, \tag{5.19}$$

and $\Theta_{0,\bullet} : T_{u(t,a)}N \to T_{\tilde X_\bullet}N$ by the covariant equation along $\tilde X = u(t-\bullet\,, X_\bullet(a))$:

$$d\,\big[(//^N_{0,\bullet})^{-1}\Theta_{0,\bullet}\big] = -\tfrac{1}{2}\,(//^N_{0,\bullet})^{-1}\mathrm{R}^N(\Theta_{0,\bullet}, d\tilde X)d\tilde X, \quad \Theta_{0,0} = \mathrm{id}. \tag{5.20}$$

Proof. – We know that

$$\Theta^{-1}_{0,r}\,T_{X_r(a)}u(t-r,\,\cdot\,)\,T_aX_r\,v, \quad 0 \leq r \leq t, \tag{5.21}$$

is a local martingale. Again, we may assume that the Brownian motions $X(\,\cdot\,)$ on M are constructed as solutions of an SDE of the type (5.13). Filtering out extraneous noise, by conditioning with respect to $\mathcal{F}_\bullet(a)$ as in the proof of Theorem 5.3, shows that with (5.21) also

$$\Theta_{0,r}^{-1} T_{X_r(a)} u(t-r, \cdot\,)\, W_{0,r}(v), \quad 0 \le r \le t, \tag{5.22}$$

is a local martingale, where $W_{0,\bullet}$ satisfies (5.19). It remains to show that (5.22) is already a martingale under the given assumptions. First, since Ric^M is bounded below, say $\mathrm{Ric}^M \ge \alpha$, we conclude from (5.19)

$$\|W_{0,r}(v)\| \le \|v\|\, e^{-\alpha r/2}. \tag{5.23}$$

On the other hand, writing $\Theta_{0,t} = //_{0,t}^N\, \theta_{0,t}$, we get for $w \in T_{u(t,a)}N$ from (4.30)

$$\begin{aligned} \tfrac{1}{2}\, d\|\theta_{0,r} w\|^2 &= -\tfrac{1}{2}\, \big\langle //_{0,r}^N \theta_{0,r} w, \mathrm{R}^N (//_{0,r}^N \theta_{0,r} w, d\tilde{X}) d\tilde{X} \big\rangle \\ &= -\tfrac{1}{2}\, \big\langle //_{0,r}^N \theta_{0,r} w, \mathrm{tr}\, \mathrm{R}^N \big(//_{0,r}^N \theta_{0,r} w, T_{X_r(a)} u_{t-r}(\,\cdot\,)\big)\, T_{X_r(a)} u_{t-r}(\,\cdot\,) \big\rangle\, dr. \end{aligned}$$

By assumption, there is an upper bound for R^N, say $\mathrm{R}^N \le \beta$ for some $\beta \ge 0$, and a bound on the operator norm of Tu, say $\|Tu\| \le c$, which together yields (with $m = \dim M$)

$$\tfrac{d}{dr} \|\theta_{0,r} w\|^2 \ge -m\, \beta\, \|\theta_{0,r} w\|^2\, \big\|T_{X_r(a)} u_{t-r}(\,\cdot\,)\big\|^2 \ge -m\, \beta\, c^2\, \|\theta_{0,r} w\|^2.$$

Thus, $\frac{d}{dr}(\log \|\theta_{0,r} w\|^2) \ge -\beta\, c^2$, or $\|\theta_{0,r} w\|^2 \ge \|w\|^2\, e^{-m\beta c^2 r}$, and hence

$$\|\theta_{0,r}^{-1} w\| \le \|w\|\, e^{m\beta c^2 r/2} \tag{5.24}$$

which gives the claim. □

Note that (5.23) and (5.24) show that, in particular, if $\mathrm{Ric}^M \ge \alpha > 0$ and $\mathrm{R}^N \le 0$, then $u_t \to u_\infty$, where u_∞ is a constant mapping; indeed, by means of (5.18), we have $\|T_a u_t\| \le \mathbb{E}\big[\|T_{X_t(a)} u_0\|\, e^{-\alpha t/2}\big]$.

In contrast to (5.18), the right hand side in formula of Theorem 5.4 expresses the differential Tu_t as a sum of two terms. This can be avoided by working with the "deformed parallel transport" $\Theta_{0,\bullet} : T_{\tilde{X}_0} N \to T_{\tilde{X}_\bullet} N$ along $\tilde{X}$ instead of the ordinary parallel transport. Given a solution $u : [0,t] \times M \to N$ to the nonlinear heat equation (5.1) and $a \in M$, we continue using the notations $\tilde{X}_r = u_{t-r}\big(X_r(a)\big)$ and $\tilde{J}_r = (T_{X_r(a)} u_{t-r})(T_a X_r)\, h_r$ where h is some bounded $\mathcal{F}_\bullet(a)$-adapted process taking sample paths in the Cameron-Martin space $\mathbb{H}([0,t], T_a M)$. Note that the proof of Theorem 5.3 relies on the fact that in the given situation

$$(//_{0,\bullet}^N)^{-1} \tilde{J}_\bullet + \frac{1}{2} \int_0^\bullet (//_{0,\bullet}^N)^{-1} \mathrm{R}^N (\tilde{J}, d\tilde{X})\, d\tilde{X} - \mathcal{A}^N(\tilde{X})_\bullet \int_0^\bullet \langle T_a X_s\, \dot{h}_s, //_{0,s}^M\, dB_s \rangle$$

is a local martingale. Using this, we see that also

$$\Theta_{0,\bullet}^{-1} \tilde{J}_\bullet - \int_0^\bullet \theta_{0,r}^{-1}\, d\Big(\mathcal{A}^N(\tilde{X})_r \int_0^r \langle T_a X_s\, \dot{h}_s, //_{0,s}^M\, dB_s \rangle\Big) \tag{5.25}$$

is a local martingale. Taking into account that

$$\mathcal{A}^N\big(u(t-\bullet\,,X_\bullet(a))\big)_r = \int_0^r (//^N_{0,s})^{-1}\,T_{X_s(a)}u(t-s,\bullet)\,//^M_{0,s}\,dB_s, \tag{5.26}$$

we finally get the local martingale property of

$$\Theta_{0,\bullet}^{-1}\tilde{J}_\bullet - \int_0^\bullet \Theta_{0,s}^{-1}\,(T_{X_s(a)}u_{t-s})\,T_aX_s\,\dot{h}_s\,ds. \tag{5.27}$$

THEOREM 5.6. – *Let M, N be Riemannian manifolds. Let $u : [0,t]\times M \to N$ be a smooth solution of* (5.1), $v \in T_aM$. *Then*

$$\Theta_{0,\bullet}^{-1}\,(T_{X_\bullet}u_{t-\bullet})\,W_{0,\bullet}\,H_\bullet - \int_0^\bullet \Theta_{0,s}^{-1}\,(T_{X_s}u_{t-s})\,W_{0,s}\,dH_s \tag{5.28}$$

is a local $\mathcal{F}_\bullet(a)$-martingale for any $\mathcal{F}_\bullet(a)$-adapted process H of locally bounded variation, taking values in T_aM. In particular,

$$\Theta_{0,\bullet}^{-1}\,(T_{X_\bullet}u_{t-\bullet})\,W_{0,\bullet}\,h_\bullet - \int_0^\bullet \Theta_{0,s}^{-1}\,(T_{X_s}u_{t-s})\,W_{0,s}\,\dot{h}_s\,ds \tag{5.29}$$

is a local $\mathcal{F}_\bullet(a)$-martingale for any $\mathcal{F}_\bullet(a)$-adapted process h with sample paths in the Cameron-Martin space $\mathbb{H}([0,t],T_aM)$.

Proof. – Of course, the second part can be reduced to (5.27). But we may also argue more elementary. Indeed, by Itô's formula

$$d\big(\Theta_{0,r}^{-1}\,T_{X_r}u_{t-r}\,W_{0,r}\,H_r\big) = \big(\Theta_{0,r}^{-1}\,T_{X_r}u_{t-r}\,W_{0,r}\big)\,dH_r + d\big(\Theta_{0,r}^{-1}\,T_{X_r}u_{t-r}\,W_{0,r}\big)\,H_r,$$

hence (5.29) coincides with

$$\int_0^\bullet d\big(\Theta_{0,s}^{-1}\,T_{X_s(a)}u_{t-s}\,W_{0,s}\big)\,H_s$$

which is a local martingale by Theorem 4.12. □

COROLLARY 5.7. – *Let M, N be Riemannian manifolds. Let $u : [0,t]\times M \to N$ be a smooth solution of* (5.1), $v \in T_aM$. *Then*

$$(T_au_t)v = -\mathbb{E}\left[\int_0^{\tau(a)\wedge t} \Theta_{0,s}^{-1}\,(T_{X_s(a)}u_{t-s})\,W_{0,s}\,\dot{h}_s\,ds\right] \tag{5.30}$$

for any bounded $\mathcal{F}_\bullet(a)$-adapted process h with sample paths in the Cameron-Martin space $\mathbb{H}([0,t],T_aM)$ such that $\big(\int_0^{\tau(a)\wedge t}\|\dot{h}_s\|^2\,ds\big)^{1/2} \in L^{1+\varepsilon}$ for some $\varepsilon > 0$, and the property that $h_0 = v$, $h_s = 0$ for all $s \geq \tau(a)\wedge t$; here $\tau(a)$ is again the first exit time of $X(a)$ from some relatively compact neighbourhood D of a.

Consider the general situation of a manifold M with a linear connection ∇, and a continuous semimartingale X taking values in M with anti-development

$$\mathcal{A}(X) = \int_0^{\bullet} //_{0,s}^{-1} \delta X_s \equiv \int_0^{\bullet} //_{0,s}^{-1} d^{\nabla} X_s. \tag{5.31}$$

Replacing formally the parallel transport $//_{0,\bullet}$ along X in (5.31) by the deformed parallel transport $\Theta_{0,\bullet}$ along X, as defined by equation (4.26), leads to the notion of a "deformed anti-development"

$$\mathcal{A}_{\mathrm{def}}(X) = \int_0^{\bullet} \Theta_{0,s}^{-1} \delta X_s \equiv \int_0^{\bullet} \Theta_{0,s}^{-1} d^{\nabla} X_s. \tag{5.32}$$

Remark 5.8. – Returning to the situation of Theorem 5.6 and Corollary 5.7, in terms of the "deformed anti-development" to $\tilde{X} = u(t-\cdot\,, X_\bullet(a))$, *i.e.*,

$$\mathcal{A}_{\mathrm{def}}^{N}\big(u(t-\cdot\,, X_\bullet(a))\big)_r = \int_0^r \Theta_{0,s}^{-1}\, T_{X_s(a)} u(t-s,\ \cdot\,)\, //_{0,s}^{M}\, dB_s,$$

formula (5.30) may be rewritten as

$$(T_a u_t)v = -\mathbb{E}\Big[\mathcal{A}_{\mathrm{def}}^{N}\big(u(t-\cdot\,, X_\bullet(a))\big)_{\tau(a)\wedge t} \int_0^{\tau(a)\wedge t} \langle W_{0,s}\, \dot{h}_s, //_{0,s}^{M}\, dB_s\rangle\Big] \tag{5.33}$$

giving an expression completely analogous to the linear case $\mathrm{R}^N \equiv 0$, with the only difference that anti-developments are taken with respect to the deformed parallel transport. Note that the right-hand side of (5.33) does not involve derivatives of u; deformed anti-developments are well-defined for any continuous semimartingale on N.

6. Gradient estimates for harmonic maps of bounded dilatation

Explicit differentiation formulae, as given in Theorem 5.4, Theorem 5.5 and Corollary 5.7 seem to be appropriate tools in deriving *a priori* estimates for the harmonic heat flow or gradient estimates for harmonic maps.

By a well-known theorem [15], originally proved in purely geometric terms, a harmonic map of bounded dilatation, from a complete Riemannian manifold M with $\mathrm{Ric}^M \geq 0$ to a Riemannian manifold N with $\mathrm{R}^N \leq -\beta < 0$, is necessarily constant. Brownian motion techniques were employed by several authors (e.g. [16], [18], [27]) to obtain results in this direction; the usual strategy to prove the nonexistence of particular maps is by showing that they link random processes of incompatible behaviour.

In this section, we demonstrate how the differentiation formulae of section 5, e.g. formula (5.30), can be directly used to achieve local and global gradient estimates, for instance, for harmonic maps of bounded dilatation. Under appropriate curvature assumption these estimates specialize to Liouville type theorems for harmonic maps of bounded dilatation.

Let M, N be Riemannian manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^2 map, and denote by

$$\lambda_1(p) \geq \lambda_2(p) \geq \ldots \geq \lambda_m(p) \geq 0$$

the eigenvalues of $T_p f^* \circ T_p f : T_p M \to T_p M$. If there is a positive number K such that $\lambda_1(p) \leq K^2 \lambda_2(p)$ for every $p \in M$, the f is said to be of K-bounded dilatation, e.g. [15]. Following Shen [26], the map $f : M \to N$ is said to be of generalized K-bounded dilatation if $\lambda_1 \leq K^2(\lambda_2 + \ldots \lambda_m)$ everywhere on M.

Now let $u : [0,t] \times M \to N$ be a smooth solution of the nonlinear heat equation. Assume that $\mathrm{R}^N \leq -\beta$ for some $\beta \geq 0$. Then, using the notions of the last section, we have:

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dr}\|\theta_{0,r}w\|^2 &= -\tfrac{1}{2}\langle //^N_{0,r}\theta_{0,r}w, \operatorname{tr}\mathrm{R}^N\big(//^N_{0,r}\theta_{0,r}w, T_{X_r(a)}u_{t-r}(\,\cdot\,)\big)\,T_{X_r(a)}u_{t-r}(\,\cdot\,)\rangle \\ &= -\tfrac{1}{2}\sum_{i=1}^m \langle //^N_{0,r}\theta_{0,r}w, \mathrm{R}^N\big(//^N_{0,r}\theta_{0,r}w, T_{X_r(a)}u_{t-r}(e_i)\big)\,T_{X_r(a)}u_{t-r}(e_i)\rangle \\ &\geq \tfrac{1}{2}\sum_{i=1}^m \beta\Big\{\|\theta_{0,r}w\|^2\,\|T_{X_r(a)}u_{t-r}(e_i)\|^2 - \big\langle //^N_{0,r}\theta_{0,r}w, T_{X_r(a)}u_{t-r}(e_i)\big\rangle^2\Big\}, \end{aligned}$$

where $(e_i)_{1\leq i\leq m}$ denotes an orthonormal basis for $T_{X_r(a)}M$. Choosing (e_i) such that

$$(Tu_{t-r})^*\,(Tu_{t-r})\,e_i = \lambda_i\,e_i$$

with $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$, and using

$$\sum_{i=1}^m \big\langle //^N_{0,r}\theta_{0,r}w, T_{X_r(a)}u_{t-r}(e_i)\big\rangle^2 = \|(T_{X_r(a)}u_{t-r})^*\,//^N_{0,r}\theta_{0,r}w\|^2 \leq \lambda_1\,\|\theta_{0,r}w\|^2,$$

we get

$$\tfrac{d}{dr}\|\theta_{0,r}w\|^2 \geq \beta\,\|\theta_{0,r}w\|^2\Big\{\Big(\sum_{i=1}^m \lambda_i\Big) - \lambda_1\Big\} = \beta\,\|\theta_{0,r}w\|^2\Big(\sum_{i=2}^m \lambda_i\Big).$$

Now, specializing to the case of a harmonic mapping $f = u_0 : M \to N$, hence $u_r = u_0$, and assuming that u_0 has generalized K-bounded dilatation, we see that

$$\tfrac{d}{dr}\|\theta_{0,r}w\|^2 \geq \beta\,\|\theta_{0,r}w\|^2\,\lambda_1/K^2.$$

Thus, we obtain

$$\|\theta_{0,r}w\|^2 \geq \|w\|^2 \exp\Big\{\beta/K^2 \int_0^r \lambda_1\big(X_s(a)\big)\,ds\Big\}$$

and therefore

$$\|\theta_{0,r}^{-1}w\| \leq \|w\| \exp\Big\{-\beta/(2K^2)\int_0^r \lambda_1\big(X_s(a)\big)\,ds\Big\}. \tag{6.1}$$

THEOREM 6.1. – *Let M be a compact Riemannian manifold of dimension m with nonempty smooth boundary ∂M and $\mathrm{Ric}^M \geq k$ for some constant $k \in \mathbb{R}$, and let N be an arbitrary Riemannian manifold with $\mathrm{R}^N \leq -\beta$ for some constant $\beta > 0$. Let $u : M \to N$ be a harmonic map of generalized K-bounded dilatation, $a \in \overset{\circ}{M} := M \setminus \partial M$. Then, for any $\varphi \in C^2(M)$ with $\varphi | \partial M = 0$ and $\varphi > 0$ on $\overset{\circ}{M}$, we have the estimate*

$$\|T_a u\|^2 \leq \frac{K^2}{\beta}\, \varphi^{-2}(a)\, c(\varphi), \tag{6.2}$$

where $c(\varphi) = \sup_{\overset{\circ}{M}} \{\varphi^2 \alpha + 3\, \|\nabla \varphi\|^2 - \varphi \Delta_M \varphi\}$ and $\alpha = -(k \wedge 0)$. In particular,

$$\|T_a u\|^2 \leq \frac{K^2}{\beta} C\big(\mathrm{dist}(a, \partial M)\big), \tag{6.3}$$

where

$$C(r) = \frac{\pi^2}{4}(m+3)\, r^{-2} + \frac{\pi}{2}\sqrt{\alpha\,(m-1)}\, r^{-1} + \alpha.$$

If M is a regular geodesic ball $B(a_0, R)$ of radius R about a_0, contained in a complete Riemannian manifold, then

$$\|T_a u\|^2 \leq \frac{K^2}{\beta}\left[\sin\left(\frac{\pi \mathrm{dist}(a, \partial M)}{2R}\right)\right]^{-1} C(R). \tag{6.4}$$

Proof. – We work with Corollary 5.7 and the formula

$$(T_a u) v = -\mathbb{E}\left[\int_0^{\tau(a)} \Theta_{0,s}^{-1}\, (T_{X_s(a)} u)\, W_{0,s}\, \dot{h}_s\, ds\right],$$

where $\tau(a)$ now denotes the exit time of $X(a)$ from $\overset{\circ}{M}$. Thus

$$\|T_a u\|^2 \leq \mathbb{E}\left[\int_0^{\tau(a)} \left\|\Theta_{0,s}^{-1}\, (T_{X_s(a)} u\right\|^2 ds\right] \cdot \mathbb{E}\left[\int_0^{\tau(a)} \left\|W_{0,s}\, \dot{h}_s\right\|^2 ds\right] =: A_1 \cdot A_2.$$

Since $\lambda_1(p) = \|T_p u\|^2$, we get from (6.1) for the first term

$$A_1 \leq \mathbb{E}\left[\int_0^{\tau(a)} \lambda_1\big(X_s(a)\big) \exp\Big\{-\beta / K^2 \int_0^s \lambda_1\big(X_r(a)\big)\, dr\Big\}\, ds\right] \leq \frac{K^2}{\beta}$$

where the last inequality comes from the fact that

$$\lambda_1\big(X_s(a)\big) \exp\Big\{-\frac{\beta}{K^2} \int_0^s \lambda_1\big(X_r(a)\big)\, dr\Big\} = -\frac{K^2}{\beta} \frac{d}{ds} \exp\Big\{-\frac{\beta}{K^2} \int_0^s \lambda_1\big(X_r(a)\big)\, dr\Big\}.$$

The second term A_2 can be estimated using the methods of [30]. We confine ourselves to a brief sketch of the main idea. Consider the strictly increasing process

$$T(r) = \int_0^r \varphi^{-2}\big(X_s(a)\big)\, ds, \quad r \leq \tau(a),$$

and define

$$\sigma(r) = \inf\{s \geq 0 : T(s) \geq r\};$$

then $T(\sigma(r)) = r$ since $\tau(a) < \infty$, and $\sigma(T(r)) = r$ for $r \leq \tau(a)$. The time-changed Brownian motion $X'_{\cdot} = X_{\sigma(\cdot)}$ has generator $L' = 1/2\,\varphi^2\Delta_M$ and lifetime $T(\tau(a))$ which is infinite by Proposition 2.3 [30]. Now, for some fixed $t_0 > 0$, let

$$h_s = v\left(1 - \frac{1}{t_0}\rho\left(\int_0^s \varphi^{-2}(X_r(x))\,1_{\{r<\sigma(t_0)\}}\,dr\right)\right), \tag{6.5}$$

where $\rho \in C^1([0,t_0],\mathbb{R})$ such that $\rho(0) = 0$ and $\rho(t_0) = t_0$. Then $h_0 = v$ and $h_s = 0$ for $s \geq \sigma(t_0)$; in particular $h_s = 0$ for $s \geq \tau(a)$. It is elementary to check that $\int_0^{\sigma(t_0)} \|\dot{h}_s\|^2\,ds \in L^1$. By means of estimate (5.23) and taking h as given by (6.5) with a proper choice for t_0 and ρ, it is possible to verify $A_2 \leq \varphi^{-2}\,c(\varphi)$, see [30] for details. This establishes (6.2). Finally, the estimates (6.3) and (6.4) are obtained with a specific choice for the function φ; basically one works with

$$\varphi(\,\cdot\,) = \cos\left(\frac{\pi\,\mathrm{dist}(a,\,\cdot\,)}{2\,\mathrm{dist}(a,\partial M)}\right) \tag{6.6}$$

with the minor technical difficulty that (6.6) is not C^2 on the cut locus of a, see [30]. □

COROLLARY 6.2 [26]. – *Let (M,g), (N,h) be Riemannian manifolds where M is complete with its Ricci curvature bounded below by a nonpositive constant, and the sectional curvatures of N bounded above by a negative constant, say*

$$\mathrm{Ric}^M \geq -\alpha, \quad \mathrm{R}^N \leq -\beta, \quad \alpha \geq 0,\ \beta > 0. \tag{6.7}$$

Let $u : M \to N$ be a harmonic map of generalized K-bounded dilatation; then

$$u^*h \leq \frac{\alpha K^2}{\beta}\,g. \tag{6.8}$$

In particular, if $\mathrm{Ric}^M \geq 0$, $\mathrm{R}^N \leq -\beta$, then harmonic maps $u : M \to N$ of generalized K-bounded dilatation are constant.

Proof. – Inequality (6.8) follows from (6.3) or (6.4) by exhausting the manifold through a sequence of regular domains (relatively compact open subsets with nonempty smooth boundary). □

Note that (6.8) in particular says that

$$\mathrm{dist}^N(u(x),u(y)) \leq \frac{\alpha K^2}{\beta}\,\mathrm{dist}^M(x,y) \quad \text{for all } x,y \in M.$$

Thus u is distance-decreasing if $\alpha K^2 \leq \beta$.

Acknowledgments

The second named author wishes to thank K. D. Elworthy for helpful discussions on the subject and for showing him his notes [9]. Thanks also to T. J. Lyons for bringing [21] to our attention.

REFERENCES

[1] M. Arnaudon, Connexions et martingales dans les groupes de Lie. In: J. Azéma, P. A. Meyer and M. Yor (Eds.) *Séminaire de Probabilités* XXVI. Lect. Notes in Math., *1526*, 1992, pp. 146-156.

[2] M. Arnaudon and A. Thalmaier, Stability of stochastic differential equations in manifolds. *Séminaire de Probabilités*, to appear.

[3] J.-M. Bismut, *Mécanique aléatoire*. Lect. Notes in Math., *866*, 1981.

[4] L. A. Cordero, C. T. J. Dodson and M. De León, *Differential Geometry of Frame Bundles*. Dordrecht: Kluwer Academic Publishers, 1989.

[5] A.-B. Cruzeiro and P. Malliavin, Renormalized differential geometry on path space: structural equation, curvature. *J. Funct. Anal.*, *139*, 1996, pp. 119-181.

[6] D. Dohrn and F. Guerra, Nelson's stochastic mechanics on Riemannian manifolds. *Lettere al Nuovo Cimento 22*, 1978, pp. 121-127.

[7] B. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. *J. Funct. Anal.*, *110*, 1992, pp. 272-376.

[8] K. D. Elworthy, Geometric aspects of diffusions on manifolds. In: P. L. Hennequin (Ed.) *École d'Été de Probabilités de Saint-Flour* XV-XVII. Lect. Notes in Math., *1362*, 1988, pp. 277-425.

[9] K. D. Elworthy, Harmonic maps and the non-linear heat equation. Unpublished notes, 1995.

[10] K. D. Elworthy, Y. Le Jan and X.-M. Li, Concerning the geometry of stochastic differential equations and stochastic flows. In: K. D. Elworthy, S. Kusuoka and I. Shigekawa (Eds.) *New Trends in Stochastic Analysis*. Proc. Taniguchi Symposium 1994. World Scientific Press, 1997.

[11] K. D. Elworthy and M. Yor, Conditional expectations for derivatives of certain stochastic flows. In: J. Azéma et al. (Eds.) *Séminaire de Probabilités* XXVII. Lect. Notes in Math., *1557*, 1993, pp. 159-172.

[12] M. Emery, *Stochastic Calculus in Manifolds* (with an appendix by P. A. Meyer). Berlin: Springer, 1989.

[13] O. Enchev and D. W. Stroock, Towards a Riemannian geometry on the path space over a Riemannian manifold. *J. Funct. Anal.*, *134*, 1995, pp. 392-416.

[14] S. Fang and P. Malliavin, Stochastic analysis on the path space of a Riemannian manifold. I: Markovian stochastic calculus. *J. Funct. Anal.*, *118*, 1993, pp. 249-274.

[15] S. I. Goldberg, T. Ishihara and N. C. Petridis, Mappings of bounded dilatation of Riemannian manifolds. *J. Differ. Geom.*, *10*, 1975, pp. 619-630.

[16] S. I. Goldberg and C. Mueller, Brownian motion, geometry, and generalizations of Picard's little theorem. *Ann. of Prob.*, *11*, 1983, pp. 833-846.

[17] E. P. Hsu, Quasi-invariance of the Wiener measure on the path space over a compact Riemannian manifold. *J. Funct. Anal.*, *134*, 1995, pp. 417-450.

[18] W. S. Kendall, Brownian motion and the generalised little Picard's theorem. *Trans. Am. Math. Soc.*, *275*, 1983, pp. 751-760.

[19] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*. Vol. I. New York: Interscience Publishers, 1963.

[20] T. Lyons and Z.-M. Qian, A class of vector fields on path spaces. *J. Funct. Anal.*, *145*, 1997, pp. 205-223.

[21] T. Lyons and Z.-M. Qian, Stochastic Jacobi fields and vector fields induced by varying area on path spaces. *Probab. Theory Relat. Fields*, to appear.

[22] P. Malliavin, Champs de Jacobi stochastiques. *C. R. Acad. Sci. Paris, Sér. I.*, *285*, 1977, pp. 789-792.

[23] P. Malliavin, Stochastic Jacobi fields. In: C. I. Byrnes (Ed.) *Partial differential equations and geometry* (Proc. Conf., Park City, Utah, 1977). Lect. Notes in Pure and Appl. Math. *48*. New York: Marcel Dekker, Inc., 1979, pp. 158-167.

[24] P. A. MEYER, Géométrie différentielle stochastique (bis). In: J. Azéma and M. Yor (Eds.) *Séminaire de Probabilités* XVI, 1980/81. Lect. Notes in Math., *921*, 1982, pp. 165-207.
[25] A. MORIMOTO, Prolongations of G-structures to tangent bundles. *Nagoya Math. J.*, *32*, 1968, pp. 67-108.
[26] C. L. SHEN, A generalization of the Schwarz-Ahlfors lemma to the theory of harmonic maps. *J. Reine Angew. Math.*, *348*, 1984, pp. 23-33.
[27] S. STAFFORD, A probabilistic proof of S.-Y. Cheng's Liouville theorem. *Ann. Probab.*, *18*, 1990, pp. 1816-1822.
[28] A. THALMAIER, Brownian motion and the formation of singularities in the heat flow for harmonic maps. *Probab. Theory Relat. Fields*, *105*, 1996, pp. 335-367.
[29] A. THALMAIER, On the differentiation of heat semigroups and Poisson integrals. *Stochastics and Stochastics Reports*, *61*, 1997, pp. 297-321.
[30] A. THALMAIER and F.-Y. WANG, Gradient estimates for harmonic functions on regular domains in Riemannian manifolds. *J. Funct. Anal.*, to appear.
[31] K. YANO and S. ISHIHARA, *Tangent and Cotangent Bundles*. New York: Marcel Dekker, Inc., 1973.

(Manuscript received June, 1997;
accepted October, 1997.)

M. ARNAUDON
Institut de Recherche Mathématique avancée
Université Louis Pasteur et CNRS
7, rue René Descartes
F-67084 Strasbourg Cedex, France
arnaudon@math.u-strasbg.fr

A. THALMAIER
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstr. 6
D-53115 Bonn, Germany
anton@wiener.iam.uni-bonn.de