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Derivative Estimates of Semigroups and Riesz Transforms on Vector Bundles

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Abstract. We use versions of Bismut type derivative formulas obtained by Driver and Thalmaier [9], to prove derivative estimates for various heat semigroups on Riemannian vector bundles. As an application, the weak (1, 1) property for a class of Riesz transforms on a vector bundle is established. Some concrete examples of vector bundles (e.g., differential forms) are considered to illustrate the results.

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1. Introduction

The Riesz transform $Hf = \nabla(-\Delta)^{-1/2} f$ on a Riemannian manifold, considered by Strichartz [19], has been investigated in many subsequent papers, see, e.g., [2–4, 6, 7, 16, 17] and the references therein. Since the Riesz transform is bounded in L^2, by the interpolation theorem the weak (1, 1) property implies L^p-boundedness for $p \in (1, 2]$.

In this paper we aim to study the weak (1, 1) property for Riesz transforms on Riemannian vector bundles. We shall follow the lines of recent work by Coulhon and Duong [7], who proved this property for the Riesz transform on a Riemannian manifold. The authors used the doubling volume property and Li–Yau type heat kernel upper bounds, where the former can be taken into account in our case and the latter implies heat kernel bounds of the same type on vector bundles according to Donnelly–Li's semigroup domination. The difficult point for us to follow is that in [7] also derivative estimates of the heat kernel on Riemannian manifolds are used: in the case of a vector bundle E, for given t, x and y, the heat kernel $p_t(x, y)$ is a linear operator from E_y to E_x, so that it seems not easy to follow the

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corresponding argument concerning derivatives of the heat kernel. We are able to overcome this difficulty by using a derivative formula for semigroups on vector bundles derived recently by Driver and the first named author [9]. It turns out that derivative estimates of the semigroups $P_t\alpha$, α being a section of E, rather than of their heat kernels $p_t(x, y)$ are sufficient for our purpose. As in the scalar case, the first ones are easier to establish since, adopting a stochastic approach, they only require estimates of certain functionals of Brownian motion with respect to the Wiener measure, while the second ones depend on estimates with respect to the pinned Wiener measure (Brownian bridge). Nevertheless we like to stress that derivative estimates of the heat kernels $p_t(x, y)$ itself could be derived from the general formulas in [9] as well.

The paper is organized as follows. In Section 2 we present derivative estimates for semigroups on vector bundles (see Theorem 2.1). They are derived from the more general Theorem 3.2 which in turn follows from a derivative formula proved in [9] (see Theorem 3.1). Our derivative estimates are applied in Section 4 to study the L^p-boundedness ($1 < p \leqslant 2$) of Riesz transforms on Riemannian vector bundles with a metric connection (Theorem 4.1). Moreover, a typical example is presented to illustrate our results (Corollary 4.6). Lemma 4.3 gives a local version of the well-known Calderón–Zygmund decomposition which is a key tool in our study of Riesz transforms. A similar decomposition has been used in [7].

2. A Derivative Estimate for Semigroups on Vector Bundles

Let M be a (not necessarily complete) Riemannian manifold of dimension d, and let E, $\tilde{E}$ be Riemannian vector bundles over M, endowed with a metric connection. We denote by Γ, Γ_b and Γ_0 the smooth, the bounded smooth, and the compactly supported smooth sections of a vector bundle, respectively.

For a given "multiplication map" $m \in \Gamma(\mathrm{Hom}(T^*M \otimes E, \tilde{E})) \equiv \Gamma(TM \otimes E^* \otimes \tilde{E})$ consider the Dirac type operator

$$D_m := m\nabla : \Gamma(E) \to \Gamma(\tilde{E})$$

defined as the following composition

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{m} \Gamma(\tilde{E}).$$

Further let $\mathcal{R} \in \Gamma(\mathrm{End}(E))$, $\tilde{\mathcal{R}} \in \Gamma(\mathrm{End}(\tilde{E}))$ and $V \in C^2(M)$. Let $\mathrm{d}\mu = \mathrm{e}^V\,\mathrm{d}x$ where $\mathrm{d}x$ is the Riemannian volume measure on M. Consider

$$L = \Box + \nabla_{\nabla V} - \mathcal{R} \quad \text{on } \Gamma(E)$$

and

$$\tilde{L} = \Box + \nabla_{\nabla V} - \tilde{\mathcal{R}} \quad \text{on } \Gamma(\tilde{E}),$$

where $\Box = -\nabla^*\nabla$ denotes the horizontal Laplacian on a Riemannian vector bundle with a metric connection.

We assume that

$$\varrho := \tilde{L} D_m - D_m L$$

is zero order, i.e. $\varrho \in \Gamma(\mathrm{Hom}(E, \tilde{E}))$, and that m is compatible with the Levi-Civita connection, i.e. for any $X \in \Gamma(TM)$, $\alpha \in \Gamma(E)$, $v \in TM$, one has $\nabla_v m_X \alpha = m_{\nabla_v X}\alpha + m_X \nabla_v \alpha$, where $m_X\alpha := m(\phi \otimes \alpha)$ with $\phi := \langle X, \cdot \rangle$ the one form associated to X.

Assume further that $\mathcal{R} \in \Gamma(\mathrm{End}(E))$ is symmetric (i.e. $\mathcal{R}_x : E_x \to E_x$ is a symmetric linear transformation for each $x \in M$) and bounded below (i.e. there exists $c \in \mathbb{R}$ such that $\langle \mathcal{R}\alpha, \alpha \rangle \geqslant c$ for all $\alpha \in E$).

Since $(L, \Gamma_0(E))$ is then bounded in $L^2(E, \mu)$ from above, it has a canonical self-adjoint extension (i.e. the Friedrichs extension, cf. [18]). Let P_t be the symmetric semigroup corresponding to $\frac{1}{2}L$.

Our goal is to establish estimates on $D_m P_T \alpha$. To this end we introduce the following conditions.

ASSUMPTION A. There exist constants $c(\varrho)$, $c(m) \geqslant 0$ and $a_1, a_2, a_3 \in \mathbb{R}$ such that

A1. $a_1|\alpha|^2 \leqslant \langle \mathcal{R}\alpha, \alpha \rangle \leqslant a_2|\alpha|^2,\ \alpha \in E,$
A2. $a_3|\eta|^2 \leqslant \langle \tilde{\mathcal{R}}\eta, \eta \rangle,\ \eta \in \tilde{E},$
A3. $\|\varrho\| \leqslant c(\varrho),\ \|m\| \leqslant c(m).$

Let P_t^0 denote the (Dirichlet) semigroup generated by the Friedrichs extension of $(\Delta + \nabla V, C_0^\infty(M))$ on $L^2(M, \mu)$.

Denote $B(x, \delta) := \{\rho_x \leqslant \delta\}$ where $\rho_x \equiv \rho(x, \cdot)$ denotes the Riemannian distance function to the point x. We can now state our main result in this section.

THEOREM 2.1. *Let M be complete and assume that for each $x \in M$ there exist $c > 0$ and $\varepsilon \in (0, 1)$ such that*

$$(\Delta + \nabla V)\rho_x \leqslant c\,(\rho_x^{-1} + \rho_x^{\varepsilon}) \tag{2.1}$$

outside $\{x\} \cup \mathrm{cut}(x)$. Assume further that A1, A2, A3 *of Assumption* A *hold on M. Then*

$$|D_m P_T \alpha(x)|^2 \leqslant \mathrm{e}^{-a_1 T}\, \|\alpha\|_\infty \frac{(a_2 - a_3)^+[c(m) + c(\varrho)\sqrt{T}/2]^2}{1 - \exp(-(a_2 - a_3)^+ T)}\, P_T^0|\alpha|(x), \tag{2.2}$$

where again $(a_2 - a_3)^+/[1 - \exp(-(a_2 - a_3)^+ T)] := 1/T$ if $(a_2 - a_3)^+ = 0$.

Theorem 2.1 will be proved in Section 3. Below (see Examples 2.4, 2.5, 2.6) we give some typical examples which are covered by our framework. Note that the right-hand side of (2.2) does not involve derivatives of the section α and can be estimated in terms of $\|\alpha\|_\infty$. Thus estimate (2.2) should be seen as a vector bundle version of the classical Li–Yau type gradient estimates for solutions to the heat equation (or for harmonic functions) on a Riemannian manifold.

REMARK 2.2. In our setting L and $\tilde{L}$ may be any second-order operators on Riemannian vector bundles with a Weitzenböck decomposition, this means they differ (in case $V = 0$) from the corresponding horizontal (or rough) Laplacian $\square$ by a zero-order term (i.e. a homomorphism of the bundle). Note that L and $\tilde{L}$ are linked only by the condition that $\varrho = \tilde{L}D_m - D_m L$ is of zero order. In applications one typically starts with a "Weitzenböck type" operator of the form $L = \square - \mathcal{R}$ on a Riemannian vector bundle E where $\mathcal{R}$ may be any section of End E, together with a canonical first-order "Dirac type" differential operator D_m, for instance $D_m = \nabla$, $\tilde{E} = T^*M \otimes E$ (see Example 2.6 below). The appropriate $\tilde{\mathcal{R}}$, if not yet canonically given, is then constructed from these data in such a way that the "commutator" ϱ meets the zero order condition (see Example 2.6).

REMARK 2.3. Both $\tilde{\mathcal{R}}$ and ϱ appear in the estimate (2.2) only via the bounds in A2 and A3 of Assumption A. In various applications (see below) ϱ actually vanishes.

EXAMPLE 2.4 (cf. Example 2.11 in [9]). Let

$$\Omega^p = \Gamma(\Lambda^p T^*M) \quad (0 \leqslant p \leqslant d)$$

denote the space of p-forms. Let $\mathrm{d} : \Omega^p \to \Omega^{p+1}$ be the exterior differential on Ω^p, and d^*_μ the $L^2(\mu)$-adjoint of d, i.e., $\mu(\langle \mathrm{d}^*_\mu \alpha, \beta\rangle) = \mu(\langle \alpha, \mathrm{d}\beta\rangle)$ for all $\alpha \in \Omega^{p+1}$, $\beta \in \Omega^p$. The weighted Hodge Laplacian is defined by $\Delta^p_\mu := (\mathrm{d}^*_\mu \mathrm{d} + \mathrm{d}\mathrm{d}^*_\mu)|\Omega^p$. Let $E := \Lambda^p T^*M$, $\tilde{E} := \Lambda^{p+1}T^*M$. Then Theorem 2.1 applies to

$$D_m = \mathrm{d}|\Omega^p, \qquad L = -\Delta^p_\mu, \qquad \tilde{L} = -\Delta^{p+1}_\mu$$

with $\varrho = 0$, $\mathcal{R} = \mathcal{R}^p - \mathrm{Hess}_V$ and $\tilde{\mathcal{R}} = \mathcal{R}^{p+1} - \mathrm{Hess}_V$, where $\mathcal{R}^p$ stands for the Weitzenböck curvature operator on Ω^p. Here the multiplication $m \in \Gamma \times (\mathrm{Hom}(T^*M \otimes E, \tilde{E}))$ is given by $m(a \otimes \alpha) = a \wedge \alpha$.

Similarly, we may take $\tilde{E} = \Lambda^{p-1}T^*M$, $\tilde{L} = -\Delta^{p-1}_\mu$, $D_m = \mathrm{d}^*_\mu$ and $\tilde{\mathcal{R}} = \mathcal{R}^{p-1} - \mathrm{Hess}_V$. Now $m \in \Gamma(\mathrm{Hom}(T^*M \otimes E, \tilde{E}))$ is given by $m(a \otimes \alpha) = -(a \llcorner \alpha)$.

EXAMPLE 2.5 (cf. Example 2.13 and 2.14 in [9]). Let M be a spin manifold and $E \to M$ a spinor bundle over M, endowed with the spin connection. Let $\tilde{E} = E$ and $m = \gamma \in \Gamma(\mathrm{Hom}(T^*M \otimes E, E))$ be given by Clifford multiplication. Then $D = D_m$ is the Dirac operator on $\Gamma(E)$ and Theorem 2.1 applies to $L = \tilde{L} := -D^2$ with $\varrho = 0$ and $\mathcal{R} = \tilde{\mathcal{R}} = \frac{1}{4}\,\mathrm{scal}$ where scal is the scalar curvature of M (again with $V = 0$).

The spinor bundle E may be tensored with an auxiliary Riemannian vector bundle F over M (endowed with a connection) to give a twisted Dirac operator of the form

$$D : \Gamma(E \otimes F) \xrightarrow{\nabla^{E\otimes F}} \Gamma(T^*M \otimes E \otimes F)\, l \xrightarrow{\gamma\otimes 1} \Gamma(E \otimes F),$$

where as above γ denotes Clifford multiplication. In this case Theorem 2.1 applies to $L = \tilde{L} = -D^2$ with $\varrho = 0$ and $\mathcal{R} = \tilde{\mathcal{R}} = \frac{1}{4}\,\mathrm{scal} + \mathcal{R}^F$ where $\mathcal{R}^F$ is the Weitzenböck curvature term on F, see [5] for details.

EXAMPLE 2.6 (cf. Proposition 2.15 in [9]). Let $\tilde{E} = T^*M \otimes E$ and $m = \mathrm{id}_{\tilde{E}}$. We take $D_m = \nabla$. Given $\mathcal{R} \in \Gamma(\mathrm{End}(E))$, let

$$\tilde{\mathcal{R}} = \mathrm{Ric}^{\mathrm{tr}} \otimes 1_E - 2R^E \cdot + 1_{T^*M} \otimes \mathcal{R} \in \Gamma(\mathrm{End}(\tilde{E})),$$
$$\varrho = \nabla \cdot R^E + \nabla\mathcal{R} \in \Gamma(\mathrm{Hom}(E, \tilde{E})),$$

where R^E denotes the Riemann curvature tensor to ∇ on E,

$$\mathrm{Ric}^{\mathrm{tr}} \in \Gamma(\mathrm{End}(T^*M))$$

denotes the transpose of the Ricci curvature tensor $\mathrm{Ric} \in \Gamma(\mathrm{End}(TM))$ on M, and for any $\eta \in \tilde{E}_x$, $v \in T_xM$, $a \in E_x$, $\{e_i\} \in \mathrm{O}_x(M)$,

$$(R^E \cdot \eta)(v) := \sum_{i=1}^{d} R^E(v, e_i)\eta(e_i),$$
$$(\nabla \cdot R^E a)(v) := \sum_{i=1}^{d} (\nabla_{e_i} R^E)(e_i, v)a,$$
$$(\nabla\mathcal{R}a)(v) := (\nabla_v\mathcal{R})a.$$

Then Theorem 2.1 applies to $L = \Box - \mathcal{R}$, $\tilde{L} = \tilde{\Box} - \tilde{\mathcal{R}}$ with $m = \mathrm{id}_{\tilde{E}}$ and ϱ as defined above (with $V = 0$).

3. Heat Equation Derivative Formulas and Estimates

The goal of this section is to explain how Theorem 2.1 follows from the work in [9]. We keep the notations of Section 2. In particular, E and $\tilde{E}$ denote Riemannian vector bundles, endowed with a metric connection, over a (not necessarily complete) Riemannian manifold M. For a multiplication map $m \in \Gamma(TM \otimes E^* \otimes \tilde{E})$, compatible with the Levi-Civita connection, consider the Dirac type operator

$$D_m = m\nabla : \Gamma(E) \to \Gamma(\tilde{E}).$$

Given two sections $\mathcal{R} \in \Gamma(\mathrm{End}(E))$, $\tilde{\mathcal{R}} \in \Gamma(\mathrm{End}(\tilde{E}))$ and a function $V \in C^2(M)$, let

$$L = \Box + \nabla_{\nabla V} - \mathcal{R} \quad \text{on } \Gamma(E), \quad \text{and} \quad \tilde{L} = \Box + \nabla_{\nabla V} - \tilde{\mathcal{R}} \quad \text{on } \Gamma(\tilde{E}),$$

where we assume that $\varrho = \tilde{L}D_m - D_m L$ is zero order, i.e., fibrewise a linear transformation $\varrho(x): E_x \to \tilde{E}_x, x \in M$.

Assuming that $\mathcal{R} \in \Gamma(\mathrm{End}(E))$ is symmetric and bounded below, we consider the semigroup P_t with generator $\frac{1}{2}L$ where L denotes the Friedrichs extension to $(L, \Gamma_0(E))$ on $L^2(E, \mu)$. For fixed $x \in M$, let $(x_t)_{t\geqslant 0}$ be a diffusion process starting from x generated by $\frac{1}{2}(\Delta + \nabla V)$, and $//_t^M$ be the stochastic parallel transport along $(x_t)_{t\geqslant 0}$. Then x_t solves the Stratonovich equation

$$\mathrm{d}x_t = //_t^M \circ \mathrm{d}B_t + \frac{1}{2}\nabla V(x_t)\,\mathrm{d}t, \quad x_0 = x,$$

where B_t is a Brownian motion in $T_x M$. Define Q_t, $\tilde{Q}_t$ as the pathwise solutions of the following linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t = -\frac{1}{2}Q_t\mathcal{R}_{//_t}, \quad Q_0 = \mathrm{id}_{E_x},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Q}_t = -\frac{1}{2}\tilde{Q}_t\tilde{\mathcal{R}}_{//_t}, \quad \tilde{Q}_0 = \mathrm{id}_{\tilde{E}_x},$$

where

$$\mathcal{R}_{//_t} := //_t^{-1}\mathcal{R}\,//_t, \qquad \tilde{R}_{//_t} := //_t^{-1}\tilde{R}\,//_t$$

are linear operators on E_x and $\tilde{E}_x$ respectively. Here $//_t$ denotes stochastic parallel transport in E, resp. in $\tilde{E}$, along the paths of x_t. By definition, the processes Q and $\tilde{Q}$ take values in $\mathrm{End}(E_x)$ and $\mathrm{End}(\tilde{E}_x)$ respectively.

For $\alpha \in \Gamma_b(E)$ and $T > 0$, define $\alpha_t = P_{T-t}\alpha$, $t \in [0, T]$. Let $N_t = Q_t\,//_t^{-1}\alpha_t(x_t)$ and $\tilde{N}_t = \tilde{Q}_t\,//_t^{-1}D_m\alpha_t(x_t)$. Finally let $(\ell_t)_{t\in[0,T]}$ be a finite energy process on $\tilde{E}_x$, i.e., $\mathbb{E}[\int_0^T |\ell_t'|^2\,\mathrm{d}t] < \infty$. Given these data, we define

$$U_t^\ell := \int_0^t (Q_s^{-1})^* m_{\mathrm{d}B_s}^* \tilde{Q}_s^* \ell_s' + \frac{1}{2}\int_0^t (Q_s^{-1})^* \varrho_{//_s}^* \tilde{Q}_s^* \ell_s'\,\mathrm{d}s,$$
$$Z_t^\ell := \langle \tilde{N}_t, \ell_t\rangle - \langle N_t, U_t^\ell\rangle,$$

where $\varrho_{//_s}^* := //_s^{-1}\varrho^*(x_s)\,//_s$ and $\varrho^*(x): \tilde{E}_x \to E_x$, resp. $m_v^*: \tilde{E}_x \to E_x$, denote the adjoints to $\varrho(x): E_x \to \tilde{E}_x$, resp. $m_v: E_x \to \tilde{E}_x$, $v \in T_x M$. Then Z_t^ℓ is a local martingale with (cf. [9] for details)

$$\mathrm{d}Z_t^\ell = \langle \tilde{Q}_t\,//_t^{-1}\nabla_{//_t^M \mathrm{d}B_t} D_m\alpha_t(x_t), \ell_t\rangle - \langle Q_t\,//_t^{-1}\nabla_{//_t^M \mathrm{d}B_t}\alpha_t(x_t), U_t^\ell\rangle -$$
$$- \langle \tilde{Q}_t\, m_{\mathrm{d}B_t}\, Q_t^{-1} N_t, \ell_t'\rangle.$$

Let $\delta_x := \sup\{r > 0 : \{\rho_x \leqslant r\}$ is compact$\}$ where ρ_x is the Riemannian distance function from x. Obviously, one has $\delta_x = \infty$ whenever M is complete. For any $\delta \in (0, \delta_x)$ let $\tau_\delta := \inf\{t \geqslant 0 : \rho_x(x_t) \geqslant \delta\}$. Then $(Z_{t\wedge\tau_\delta}^\ell)_{t\geqslant 0}$ is a martingale, from where the following result is derived by taking expectations.

THEOREM 3.1 (Driver and Thalmaier [9]). *Let $x \in M$, $\delta \in (0, \delta_x)$, $T > 0$, $\xi \in \tilde{E}_x$ and $\alpha \in \Gamma_b(E)$ be fixed. For any finite energy process $(\ell_t)_{t\in[0,T]}$ in $\tilde{E}$ with $\ell_0 = \xi$ and $\ell_t = 0$ for $t \geqslant T^* := \tau_\delta \wedge T$, one has*

$$\langle D_m P_T\alpha(x), \xi\rangle = -\mathbb{E}\big[\langle Q_{T^*}\, //_{T^*}^{-1} P_{T-T^*}\alpha(x_{T^*}), U_{T^*}^{\ell}\rangle\big].$$

We are going to use Theorem 3.1 to establish estimates on $D_m P_T\alpha$. To this end we assume Assumption A to be satisfied.

Let P_t^0 denote the (Dirichlet) semigroup generated by the Friedrichs extension of $(\Delta + \nabla V, C_0^\infty(M))$ on $L^2(M, \mu)$.

Denote $B(x, \delta) := \{\rho_x \leqslant \delta\}$.

THEOREM 3.2. *Let $x \in M$, $\delta \in (0, \delta_x)$, and $T > 0$. Furthermore let $c(\varrho)$, $c(m)$ and a_1, a_2, a_3 be such that* A1, A2, A3 *of Assumption* A *hold on $B(x, \delta)$. Let $f \in C^2(B(x, \delta))$ such that $0 < f \leqslant 1$ in $\{\rho_x < \delta\}$ and $f|\partial B(x, \delta) = 0$. Define*

$$c(f) := \sup_{B(x,\delta)} \{f^2(a_2 - a_3)^+ + 3|\nabla f|^2 - f(\Delta + \nabla V)f\}.$$

Then for $\alpha \in \Gamma_b(E)$,

$$|D_m P_T\alpha(x)|^2 \leqslant f(x)^{-2}\, \mathrm{e}^{-a_1 T}\, \|\alpha\|_\infty\, C(T, f)\, P_T^0|\alpha|(x),$$

where

$$C(T, f) := \frac{c(f)}{1 - \exp(-c(f)T)} \left(c(m) + c(\varrho)\frac{\sqrt{T}}{2}\right)^2$$

with the convention that $c(f)/[1 - \exp(-c(f)T)] := 1/T$ if $c(f) = 0$. In particular, if $\partial B(x, \delta)$ is empty, we may take $f \equiv 1$ and then $c(f) = (a_2 - a_3)^+$.

Proof. Without loss of generality we may assume $c(f) > 0$. Indeed if $c(f) = 0$ we first replace $c(f)$ in the proof below by $c_\varepsilon(f) := c(f) + \varepsilon$ where $\varepsilon > 0$, and take the limit as $\varepsilon \searrow 0$ at the end.

Note that $c(f) \geqslant 0$. If $\partial B(x, \delta)$ is non-empty, this is trivial since $f|\partial B(x, \delta) = 0$; otherwise $M = B(x, \delta)$ is compact and we see from $\int_M [(\Delta + \nabla V)f]\, \mathrm{e}^V\, \mathrm{d}x = 0$ that $(\Delta + \nabla V)f$ is zero somewhere on M, which again implies $c(f) \geqslant 0$.

We shall apply Theorem 3.1 by constructing a proper finite energy process ℓ_t as in [20]. Letting $T(t) := \int_0^t f^{-2}(x_s)\, \mathrm{d}s$ for $t \leqslant \tau_\delta$, we have

$$\tau(t) := \inf\{s \geqslant 0 : T(s) \geqslant t\} \leqslant t \wedge \tau_\delta.$$

Then $T(\tau(t)) = t$ for all $t \geqslant 0$ and $\tau(T(t)) = t$ for $t \leqslant \tau_\delta$. Let

$$h_0(t) = \int_0^t f^{-2}(x_s) 1_{\{s<\tau(T)\}}\, \mathrm{d}s, \quad t \geqslant 0.$$

We have $h_0(t) = T$ for $t \geqslant \tau(T)$. Following the argument in Section 4 of [20], we obtain

$$\mathbb{E}[f^{-2}(x_{\tau(t)}) \exp((a_2 - a_3)^+\tau(t))] \leqslant f(x)^{-2} \exp(c(f)t), \quad t > 0. \tag{3.1}$$

For $\xi \in \tilde{E}_x$ with $|\xi| = 1$, take

$$h_1(t) = \frac{c(f)}{1 - \exp(-c(f)T)} \int_0^t \exp(-c(f)s)\,\mathrm{d}s, \quad \ell_t = (1 - h_1 \circ h_0(t))\xi.$$

Then $\ell_0 = \xi$ and $\ell_t = 0$ for $t \geqslant \tau(T)$ ($\leqslant T \wedge \tau_\delta$), $|\ell_t| \leqslant 1$, and

$$|\ell_t'| = \frac{c(f)}{1 - \exp(-c(f)T)} \exp(-c(f)h_0(t))\, f^{-2}(x_t)\, 1_{\{t<\tau(T)\}}.$$

Therefore, by Theorem 3.1 and by semigroup domination $|P_t\alpha| \leqslant \exp(-a_1 t/2) \times P_t^0|\alpha|$,

$$\begin{aligned}|D_m P_T\alpha(x)|^2 &\leqslant \mathbb{E}[\exp(-a_1(\tau_\delta \wedge T))]\, |P_{T-\tau_\delta\wedge T}\alpha(x_{\tau_\delta\wedge T})|^2\, \mathbb{E}|U^\ell_{\tau_\delta\wedge T}|^2 \\ &\leqslant \exp(-a_1 T)\, \|\alpha\|_\infty\, P_T^0|\alpha|(x)\, \mathbb{E}|U^\ell_{\tau_\delta\wedge T}|^2.\end{aligned}$$

It remains to estimate $\mathbb{E}|U^\ell_{T\wedge\tau_\delta}|^2$. For any $r > 0$ one has

$$\begin{aligned}\mathbb{E}|U^\ell_{T\wedge\tau_\delta}|^2 &\leqslant (1+r)c(m)^2\, \mathbb{E}\int_0^{T\wedge\tau_\delta} \|Q_s^{-1}\|^2\, \|\tilde{Q}_s\|^2\, |\ell_s'|^2\,\mathrm{d}s + \\ &\quad + \frac{1+r^{-1}}{4} c(\varrho)^2 T\, \mathbb{E}\int_0^{T\wedge\tau_\delta} \|Q_s^{-1}\|^2\, \|\tilde{Q}_s\|^2\, |\ell_s'|^2\,\mathrm{d}s \\ &\leqslant \frac{c(f)^2[(1+r)c(m)^2 + (1+r^{-1})c(\varrho)^2 T/4]}{[1-\exp(-c(f)T)]^2} \times \\ &\quad \times \int_0^{\tau(T)} \mathrm{e}^{(a_2-a_3)s - 2c(f)h_0(s)}\, f^{-2}(x_s)\,\mathrm{d}T(s).\end{aligned}$$

Noting that $h_0(\tau(t)) = t$, by (3.1) we obtain

$$\begin{aligned}\mathbb{E}|U^\ell_{T\wedge\tau_\delta}|^2 &\leqslant \frac{c(f)^2[c(m) + c(\varrho)\sqrt{T}/2]^2}{[1-\exp(-c(f)T)]^2} \times \\ &\quad \times \int_0^T \mathrm{e}^{-2c(f)s}\, \mathbb{E}[f^{-2}(x_{\tau(s)})\, \mathrm{e}^{(a_2-a_3)\tau(s)}]\mathrm{d}s \\ &\leqslant \frac{c(f)^2[c(m) + c(\varrho)\sqrt{T}/2]^2}{[1-\exp(-c(f)T)]^2 f(x)^2} \int_0^T \exp(-c(f)s)\,\mathrm{d}s \\ &= \frac{c(f)}{1-\exp(-c(f)T)} \left(\frac{c(m) + c(\varrho)\sqrt{T}/2}{f(x)}\right)^2.\end{aligned}$$

This completes the proof. □

We are now able to prove Theorem 2.1 of Section 2, which is an immediate consequence of Theorem 3.2.

Proof of Theorem 2.1. If M is compact we may take $B(x,\delta) = M$ and $f \equiv 1$, then the desired result follows from Theorem 3.2. When M is noncompact, we have $\delta_x = \infty$. Let $f_n(y) = \cos \frac{\pi\rho(x,y)}{2n}$ on $B(x, n)$. By (2.1) there exists $c_1(x) > 0$ such that

$$c(f_n) \leqslant (a_2 - a_3)^+ + c_1(x)(n^{\varepsilon-1} + n^{-2})$$

provided $\mathrm{cut}(x) = \varnothing$. We obtain the desired result by applying Theorem 3.2 to f_n and letting $n \to \infty$. In the case where $\mathrm{cut}(x) \neq \varnothing$, we prove the same result by a trick used in part (2) of the proof to Corollary 5.1 in [20]. □

4. Riesz Transforms on Vector Bundles

In this section we assume that M is complete. Moreover A1, A2, A3 of Assumption A are assumed to hold on M throughout the section. Consider the operator on $L^2(E, \mu)$

$$T_\sigma := D_m(-L + \sigma)^{-1/2} = \frac{1}{\sqrt{\pi}} D_m \int_0^\infty \mathrm{e}^{-\sigma s}\, P_{2s} \frac{\mathrm{d}s}{\sqrt{s}} \tag{4.1}$$

for $\sigma \geqslant 0$ suitable. (In the sequel we ignore the normalization constant $1/\sqrt{\pi}$ in (4.1) which is irrelevant for our purpose.) We aim to study the weak $(1, 1)$ property of T_σ: there exists $c > 0$ such that

$$\sup_{\lambda>0} \lambda\, \mu\{|T_\sigma \alpha| > \lambda\} \leqslant c\, \mu(|\alpha|), \quad \alpha \in \Gamma_0(E). \tag{4.2}$$

Let p_t^0 be the heat kernel of P_t^0 with respect to μ, and let $B(x, r)$ be the (closed) geodesic ball with center x and radius r. Recall that P_t^0 is generated by $\frac{1}{2}(\Delta + \nabla V)$ on $L^2(M, \mu)$ with $\mu = \mathrm{e}^V\, \mathrm{d}x$. We need the following assumptions.

ASSUMPTION B. M is complete and the following conditions hold:

B1. For any $x \in M$ there exist $c > 0$ and $\varepsilon \in (0, 1)$ such that

$$(\Delta + \nabla V)\rho_x \leqslant c\, (\rho_x^{-1} + \rho_x^{\varepsilon}) \tag{4.3}$$

outside of $\{x\} \cup \mathrm{cut}(x)$.

B2. There exist two constants $c \geqslant 1$ and $\gamma \in [0, 2)$ such that

$$\mu(B(x, tr)) \leqslant c\, t^c \mu(B(x, r)) \exp(t^\gamma + r^\gamma), \quad t \geqslant 1,\ x \in M,\ r > 0. \tag{4.4}$$

B3. There exists a constant $c > 0$ such that

$$p_t^0(x, x) \leqslant \frac{c}{\mu(B(x, \sqrt{t}))}, \quad t \in (0, 1],\ x \in M. \tag{4.5}$$

We note that condition B1 holds, in particular, if there exist $o \in M$ and $c > 0$ such that Ric is bounded below by $-c\,(1+\rho_o^{2\varepsilon})$ and $|\nabla V| \leqslant c\,(1+\rho_o^{\varepsilon})$. Condition B2 is a generalization to the doubling volume property, and is not directly comparable with the local condition used in Theorem 1.2 of [7], since γ here is allowed to be larger than 1. Finally, condition B3 holds if $\mathrm{Ric} - \mathrm{Hess}_V$ is bounded from below, see, e.g., Lemma 2.5 in [10]. When $V = 0$, see [11] for a necessary and sufficient geometric condition of (4.5).

THEOREM 4.1. *Let Assumptions* A, B *be satisfied and assume that* $\|T_\sigma\|_{2\to 2} < \infty$. *Let* a_1 *be the lower bound of* $\mathcal{R}$ *specified in* A1 *of Assumption* A, *and* γ *be the constant specified in* B2 *of Assumption* B.

(i) *If* $\sigma > -a_1$, *then* T_σ *is weak* $(1, 1)$ *and hence bounded in* $L^p(E, \mu)$ *for* $p \in (1, 2]$.
(ii) *If* $\varrho = 0$, *and either* $a_1 > 0$, *or* $a_1 = \gamma = 0$ *and* (4.5) *holds for all* $t > 0$, *then the conclusion holds also for* T_0.

To prove Theorem 4.1, we need the following lemmas.

LEMMA 4.2. *Let* B2 *and* B3 *be satisfied with* $\gamma \in (0, 2)$. *There exist two constants* $c, \delta_1 > 0$ *such that*

$$p_t^0(x, y) \leqslant \frac{c}{\mu(B(y, \sqrt{t}))} \exp\biggl(-\frac{\delta_1\,\rho^2(x, y)}{t} + 4\,t^{\gamma/2}\biggr), \quad t > 0,\ x, y \in M. \tag{4.6}$$

If B2 *is satisfied for* $\gamma = 0$ *and* B3 *holds for all* $t > 0$, *then* (4.6) *holds for* $\gamma = 0$ *as well.*

Proof. When $\gamma = 0$, then condition (4.4) reduces to the doubling volume property. In this case, the validity of (4.5) for all $t > 0$ implies (4.6) for $\gamma = 0$, see p. 1155 in [7].

We now consider the case $\gamma \in (0, 2)$. By (4.5) and Theorem 1.1 in [14] (with a proof valid also for $V \neq 0$), one has

$$p_t^0(x, y) \leqslant \frac{c_1}{\sqrt{\mu(B(x, \sqrt{t}))\,\mu(B(y, \sqrt{t}))}} \exp\biggl(-\frac{2\alpha\,\rho^2(x, y)}{t}\biggr) \tag{4.7}$$

for some constants $c_1, \alpha > 0$, all $t \in (0, 1]$ and all $x, y \in M$. On the other hand, it follows from (4.4) that

$$\begin{aligned} \mu(B(y, \sqrt{t})) &\leqslant \mu(B(x, \sqrt{t} + \rho(x, y))) \\ &\leqslant \mu(B(x, \sqrt{t}))\, c\,(1 + \rho(x, y)t^{-1/2})^c \times \\ &\quad \times \exp[(1 + \rho(x, y)t^{-1/2})^\gamma + t^{\gamma/2}] \\ &\leqslant c_2\, \mu(B(x, \sqrt{t})) \exp[\alpha\, \rho^2(x, y)/t] \end{aligned} \tag{4.8}$$

for some constant $c_2 > 0$ and all $t \in (0, 1]$. Thus we obtain the existence of a constant $c_3 > 0$ such that

$$p_t^0(x, y) \leqslant \frac{c_3}{\mu(B(y, \sqrt{t}))} \exp\left(-\frac{\alpha\,\rho^2(x, y)}{t}\right)$$

for all $t \in (0, 1]$. Since the integral maximum principle (see [12, 13]) implies that

$$\int_M p_t^0(x, y)^2 \exp\left(\frac{\alpha\,\rho^2(x, y)}{t}\right) \mu(\mathrm{d}x)$$

is non-increasing in t, it follows from (4.4) that

$$\begin{aligned}
&\int_M p_t^0(x, y)^2 \exp\left(\frac{\alpha\,\rho^2(x, y)}{t}\right) \mu(\mathrm{d}x) \\
&\quad\leqslant \int_M p_1^0(x, y)^2 \exp(\alpha\rho^2(x, y))\, \mu(\mathrm{d}x) \\
&\quad\leqslant \frac{c_3}{\mu(B(y, 1))} \\
&\quad\leqslant \frac{c_4}{\mu(B(y, \sqrt{t}))} \exp(2t^{\gamma/2})
\end{aligned}$$

for some $c_4 > 0$ and all $t \geqslant 1$. Therefore, according to [11] (see also (3.4) in [14]) we obtain

$$p_t^0(x, y) \leqslant \frac{c}{\sqrt{\mu(B(y, \sqrt{t}))\mu(B(x, \sqrt{t}))}} \exp\left(-\frac{\alpha\,\rho^2(x, y)}{2t} + 2t^{\gamma/2}\right),$$
$$t > 0,\ x, y \in M,$$

for some constant $c > 0$. The proof is then completed by applying one more time the first two lines of (4.8). □

We shall use the following local version of Calderón–Zygmund decomposition. We omit the proof since it is similar to that of the classical one (cf. [1]).

LEMMA 4.3. *Let (M, ρ) be a metric space such that any bounded closed subset is compact. Let $X \subset M$ be compact, and μ a Borel measure on M such that $\mu(B) > 0$ for any ball B of positive radius and $\mu(A) < \infty$ for any compact set A. Let $D := \sup\{\rho(x, y) : x, y \in X\} < \infty$ and assume that*

$$C := \sup\left\{\frac{\mu(2B(x, r))}{\mu(B(x, r))} : x \in X,\ r \in (0, D]\right\} < \infty,$$

where $B(x, r) := \{y : \rho(x, y) \leqslant r\}$ and $2B(x, r) := B(x, 2r)$.

For any $\lambda > 0$ and any nonnegative $f \in L^1(\mu)$ such that $\|f\|_1 > 0$, $\{f \neq 0\} \subset X$ and $\mu(f 1_{B(x,r)})$ is continuous in $r \in (0, D]$ for any $x \in X$, there exist $g \in L^\infty(\mu)$ and $h_i \in L^1(\mu)$, $B(x_i, r_i)$ with $x_i \in X$ and $r_i \in (0, D]$, $i = 1, \dots, k$ for some $k \geqslant 1$, such that

(1) $f = g + \sum_{i=1}^{k} h_i$.
(2) $0 \leqslant g \leqslant C\lambda$, *$\mu$-a.e. and* $\|h_i\|_\infty < \infty$ *for each* i *if* $\|f\|_\infty < \infty$.
(3) $\{h_i \neq 0\}_{i=1}^{k}$ *are pairwise disjoint and*

$$\{h_i \neq 0\} \subset \{f \neq 0\} \cap B(x_i, 2r_i) \quad \text{for all } i.$$

(4) $\sum_{i=1}^{k} \mu(2B(x_i, r_i)) \leqslant C\lambda^{-1}\|f\|_1$ *and* $\|h_i\|_1 \leqslant 2\lambda\, \mu(2B(x_i, r_i))$ *for all* i.

Finally, we need the following lemma analogous to Lemma 2.1 in [7].

LEMMA 4.4. *Let* B2 *and* B3 *be satisfied. For* $a > 0$ *there exists* $C > 0$ *such that*

$$\int_{M\setminus B(y,\sqrt{t})} e^{-2a\rho^2(x,y)/s}\, \mu(dx) \leqslant C\mu(B(y,\sqrt{t}))\, e^{-at/s}, \quad y \in M,\ s, t > 0.$$

Proof. For $s, t > 0$ one has

$$\int_{M\setminus B(y,\sqrt{t})} e^{-2a\rho^2(x,y)/s}\, \mu(dx) \leqslant e^{-at/s} \int_M e^{-a\rho^2(x,y)/s}\, \mu(dx).$$

If $s \in (0, 1]$, then it follows from (4.4) that

$$\begin{aligned}\int_M e^{-a\rho^2(x,y)/s}\, \mu(dx) &\leqslant \mu(B(y,\sqrt{s})) + \sum_{n=1}^{\infty} e^{-a(n-1)}\, \mu(B(y,\sqrt{ns}))\\ &\leqslant \mu(B(y,\sqrt{s}))\left(1 + \sum_{n=1}^{\infty} cn^c\, e^{-a(n-1)+n^{\gamma/2}+1}\right)\\ &\leqslant c_1\, \mu(B(y,\sqrt{s}))\end{aligned}$$

for some constant $c_1 > 0$. If $s \geqslant 1$, then (4.4) yields on the other hand

$$\begin{aligned}&\int_M e^{-a\rho^2(x,y)/s}\, \mu(dx)\\ &\quad\leqslant \mu(B(y,\sqrt{s})) + \sum_{n=1}^{\infty} e^{-a\,[(n-1)s^2+s]/s}\, \mu(B(y,\sqrt{ns^2+s}))\\ &\quad\leqslant \mu(B(y,\sqrt{s}))\times\\ &\qquad\times\left(1 + \sum_{n=1}^{\infty} c\, n^c \exp\bigl(-a(n-1)s^2 - a + (1+ns)^{\gamma/2} + s^{\gamma/2}\bigr)\right)\\ &\quad\leqslant c_2\, \mu(B(y,\sqrt{s}))\end{aligned}$$

for some constant $c_2 > 0$ since $\gamma < 2$. □

Proof of Theorem 4.1. For a given $\alpha \in \Gamma_0(E)$, let $\{X_n\}_{n=1}^{N}$ be a partition of $\operatorname{supp}\alpha$ such that each X_n is a bounded domain with diameter less than 1. By

Lemma 4.3 for $X = \bar{X}_n$, there is $c > 0$ determined by the constants in (4.4) such that for each n and any $\lambda > 0$ the following decomposition holds:

$$|\alpha| 1_{X_n} = g_n + h_n := g_n + \sum_{i=1}^{k_n} h_{n_i},$$

where k_n, g_n, h_{n_i} and $|\alpha| 1_{X_n}$ (in place of k, g, h_i and f respectively) satisfy (1), (2), (3) and (4) of Lemma 4.3 with $r_i \leqslant 1$. Letting

$$g = \sum_{n=1}^{N} g_n, \qquad h = \sum_{n=1}^{N} \sum_{i=1}^{k_n} h_{n_i},$$

we therefore obtain

$$|\alpha| = g + h := g + \sum_{i=1}^{k} h_i$$

for some $k \geqslant 1$, where g, h_i satisfy the following conditions (recall that $\{g_n \neq 0\} \cup \{h_{n_i} \neq 0\} \subset X_n$ for all n and i):

(a) $0 \leqslant g \leqslant c\,\lambda$ and h_i is bounded for each i.
(b) There exist k many balls $\{B_i := B(x_i, r_i)\}_{i=1}^{k}$ with $r_i \leqslant 1$ such that $h_i \equiv 0$ outside B_i.
(c) $\sum_i \mu(B_i) \leqslant c\, \|\alpha\|_1 / \lambda$ and $\|h_i\|_1 \leqslant c\lambda\, \mu(B_i)$ for all i.

By (c) we have $\sum_i \|h_i\|_1 \leqslant c\, \|\alpha\|_1$ and $\|g\|_1 \leqslant c\, \|\alpha\|_1$ for some constant $c > 0$.

For a function f on M, let $\tilde{f} := f \frac{\alpha}{|\alpha|} 1_{\{|\alpha| > 0\}}$. Since

$$\mu\{|T_\sigma \alpha| > \lambda\} \leqslant \mu\{|T_\sigma \tilde{g}| > \lambda/2\} + \mu\{|T_\sigma \tilde{h}| > \lambda/2\},$$

and

$$\begin{aligned}\mu\{|T_\sigma \tilde{g}| > \lambda/2\} &\leqslant 4\lambda^{-2}\, \|T_\sigma \tilde{g}\|_2^2 \\ &\leqslant 4\lambda^{-2}\, \|T_\sigma\|_2^2\, \|g^2\|_1 \\ &\leqslant 4\lambda^{-1}\, \|T_\sigma\|_2^2\, C \|g\|_1 \leqslant c\lambda^{-1} \|\alpha\|_1\end{aligned}$$

for some $c > 0$, estimate (4.2) follows from

$$\mu\{|T_\sigma \tilde{h}| > \lambda\} \leqslant \frac{c}{\lambda} \|\alpha\|_1, \quad \lambda > 0. \tag{4.9}$$

Let $t_i = r_i^2$ and

$$T_\sigma \tilde{h} = \sum_i T_\sigma P_{2t_i} \tilde{h}_i + \sum_i T_\sigma (1 - P_{2t_i}) \tilde{h}_i.$$

Noting that $t_i \leqslant 1$ and

$$|P_t \tilde{h}_i| \leqslant \mathrm{e}^{-a_1 t/2}\, P_t^0 |h_i| = \mathrm{e}^{-a_1 t/2} \int p_t^0(\cdot, y)\, |h_i|(y)\, \mu(\mathrm{d}y), \quad t > 0,$$

by the argument in [7], pp. 1158–1160, leading to the corresponding L^2-estimate on p. 1158 therein, we obtain

$$\Big\| \sum_i P_{2t_i}\tilde{h}_i \Big\|_2^2 \leqslant c\lambda \|\alpha\|_1, \quad \lambda > 0 \tag{4.10}$$

for some $c > 0$ independent of α. The only difference between the procedure in [7] and the one leading to (4.10) is that instead of

$$\mu(B(y, r_i)) \leqslant (1 + \rho(x, y) r_i^{-1})^D \mu(B(x, r_i)),$$

used in [7], we now use

$$\begin{aligned}&\mu(B(y, r_i))\\ &\quad\leqslant c\,\mu(B(x, r_i))\,(1 + \rho(x, y) r_i^{-1})^c \exp[\rho(x, y)^\gamma r_i^{-\gamma}], \quad i = 1, \ldots, k,\end{aligned}$$

for some $c > 0$ and all $x, y \in M$, which is an immediate consequence of (4.4).

Since $\|T_\sigma\|_{2\to 2} < \infty$, estimate (4.10) implies that

$$\begin{aligned}\mu\Big\{\Big|T_\sigma \sum_i P_{2t_i}\tilde{h}_i\Big| > \lambda\Big\} &\leqslant \frac{\|T_\sigma\|_{2\to 2}^2}{\lambda^2} \Big\| \sum_i P_{2t_i}\tilde{h}_i \Big\|_2^2\\ &\leqslant \frac{c}{\lambda}\|\alpha\|_1\end{aligned}$$

for some $c > 0$. Therefore, to establish (4.9) it remains to prove

$$\mu\Big\{\Big|T_\sigma \sum_i (1 - P_{2t_i})\tilde{h}_i\Big| > \lambda\Big\} \leqslant c\|\alpha\|_1/\lambda, \quad \lambda > 0, \tag{4.11}$$

for some $c > 0$. Since

$$\begin{aligned}&\mu\Big\{\Big|T_\sigma \sum_i (1 - P_{2t_i})\tilde{h}_i\Big| > \lambda\Big\}\\ &\quad\leqslant \sum_i \mu(B(x_i, 2r_i)) + \mu\Big\{1_{M\setminus \cup_i B(x_i, 2r_i)} \Big|T_\sigma \sum_i (1 - P_{2t_i})\tilde{h}_i\Big| > \lambda\Big\},\end{aligned}$$

by (4.4) and (c), estimate (4.11) follows if

$$\int_{M\setminus B(x_i, 2r_i)} |T_\sigma (1 - P_{2t_i})\tilde{h}_i|\, \mathrm{d}\mu \leqslant c\, \|h_i\|_1 \tag{4.12}$$

for some constant $c > 0$. Observing that (up to a multiplication constant which plays no role)

$$(-\Box - \nabla_{\nabla V} + \mathcal{R} + \sigma)^{-1/2} = \int_0^\infty \frac{\exp(-\sigma s) P_{2s}}{\sqrt{s}}\, \mathrm{d}s,$$

we have

$$\begin{aligned}
&(-\Box - \nabla_{\nabla V} + \mathcal{R} + \sigma)^{-1/2}(1 - P_{2t_i}) \\
&\quad = \int_0^\infty \frac{\exp(-\sigma s) P_{2s}}{\sqrt{s}}\,\mathrm{d}s - \int_0^\infty \frac{\exp(-\sigma s) P_{2(t_i+s)}}{\sqrt{s}}\,\mathrm{d}s \\
&\quad = \int_0^\infty \left[\frac{\exp(-\sigma s)}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\exp(-\sigma(s-t_i))}{\sqrt{s-t_i}}\right] P_{2s}\,\mathrm{d}s.
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}
&T_\sigma(1 - P_{2t_i})\tilde{h}_i \\
&\quad = \int_0^\infty \left[\frac{\exp(-\sigma s)}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\exp(-\sigma(s-t_i))}{\sqrt{s-t_i}}\right] D_m P_{2s}\tilde{h}_i\,\mathrm{d}s
\end{aligned} \tag{4.13}$$

provided

$$\int_0^\infty \left|\frac{\exp(-\sigma s)}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\exp(-\sigma(s-t_i))}{\sqrt{s-t_i}}\right| \|D_m P_{2s}\tilde{h}_i\|_\infty\,\mathrm{d}s < \infty. \tag{4.14}$$

By Theorem 2.1 we have

$$\|D_m P_{2s}\tilde{h}_i\|_\infty \leqslant \mathrm{e}^{-a_1 s}\,\|h_i\|_\infty\sqrt{C(2s)},$$

where

$$C(s) := \frac{(a_2 - a_3)^+[c(m) + c(\varrho)\sqrt{s}/2]^2}{1 - \exp(-(a_2 - a_3)^+ s)}.$$

Then it is easy to deduce (4.14) and hence (4.13) under the conditions given in either (i) or (ii) of Theorem 4.1. Combining (4.13) with Theorem 2.1, we obtain

$$\begin{aligned}
&|T_\sigma(1 - P_{2t_i})\tilde{h}_i| \\
&\quad \leqslant \int_0^\infty \left|\frac{\mathrm{e}^{-\sigma s}}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\,\mathrm{e}^{-\sigma(s-t_i)}}{\sqrt{s-t_i}}\right| \times \\
&\qquad \times \left\{\|P_s\tilde{h}_i\|_\infty\, P_s^0|P_s\tilde{h}_i|\ \mathrm{e}^{-a_1 s}\,C(s)\right\}^{1/2}\mathrm{d}s.
\end{aligned} \tag{4.15}$$

Next, by Lemma 4.2, inequality (4.6) holds under our conditions. For any $x \in M$, by (4.6) one has

$$\begin{aligned}
|P_s\tilde{h}_i(x)| &\leqslant \mathrm{e}^{-a_1 s/2}\int_{B(x_i,r_i)} p_s^0(x,y)\,|h_i|(y)\,\mu(\mathrm{d}y) \\
&\leqslant c\,\|h_i\|_1\,\mathrm{e}^{-a_1 s/2}\sup_{y\in B(x_i,r_i)}\frac{\exp(4s^{\gamma/2})}{\mu(B(y,\sqrt{s}))}.
\end{aligned} \tag{4.16}$$

Let $\varepsilon = \delta_1/8$. By (4.6), (b), and Lemma 4.4 we obtain

$$\begin{aligned}
\mathrm{I} &:= \int_{M\setminus B(x_i,2r_i)} \sqrt{P_s^0|P_s\tilde h_i|}\,\mathrm{d}\mu \leqslant \int_{M\setminus B(x_i,2r_i)} \mathrm{e}^{-a_1 s/4}\sqrt{P_{2s}^0|\tilde h_i|}\,\mathrm{d}\mu \\
&\leqslant \mathrm{e}^{-a_1 s/4}\left[\sup_{y\in B(x_i,r_i)}\int_{M\setminus B(x_i,2r_i)} \mathrm{e}^{-2\varepsilon\rho(y,\cdot)^2/s}\,\mathrm{d}\mu\right]^{1/2}\times \\
&\quad\times\left[\int_{[M\setminus B(x_i,2r_i)]\times B(x_i,r_i)} \mathrm{e}^{2\varepsilon\rho(y,x)^2/s}\, p_{2s}^0(x,y)\,|h_i|(y)\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)\right]^{1/2} \\
&\leqslant c_1\,\mathrm{e}^{4s^{\gamma/2}-a_1 s/4}\left[\sup_{y\in B(x_i,r_i)}\mu(B(y,\sqrt{s}))\,\mathrm{e}^{-\varepsilon t_i/s}\right]^{1/2}\times \\
&\quad\times\left[\int_{B(x_i,r_i)}|h_i|(y)\mu(\mathrm{d}y)\int_{\rho(x,y)>r_i}\frac{\mathrm{e}^{-(\delta_1-4\varepsilon)\rho(x,y)^2/2s}}{\mu(B(y,\sqrt{2s}))}\,\mu(\mathrm{d}x)\right]^{1/2} \\
&\leqslant c_2\sqrt{\|h_i\|_1}\,\mathrm{e}^{4s^{\gamma/2}-\delta_1 t_i/8s-a_1 s/4}\sup_{y\in B(x_i,r_i)}\sqrt{\mu(B(y,\sqrt{s}))},
\end{aligned}$$

for some constants $c_1, c_2 > 0$. Combining this with (4.15) and (4.16) we obtain

$$\begin{aligned}
\mathrm{II} &:= \int_{M\setminus B(x_i,2r_i)} |T_\sigma(1-P_{2t_i})\tilde h_i|\,\mathrm{d}\mu \\
&\leqslant c\,\|h_i\|_1\int_0^\infty \exp\left(-a_1 s + 8s^{\gamma/2} - \frac{\delta_1 t_i}{8s}\right)\sqrt{C(s)}\times \\
&\quad\times\left|\frac{\mathrm{e}^{-\sigma s}}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\,\mathrm{e}^{-\sigma(s-t_i)}}{\sqrt{s-t_i}}\right|\sup_{x,y\in B(x_i,r_i)}\sqrt{\frac{\mu(B(x,\sqrt{s}))}{\mu(B(y,\sqrt{s}))}}\,\mathrm{d}s
\end{aligned}$$

for some $c > 0$. Noting that for $x, y \in B(x_i, r_i)$ by (4.4)

$$\begin{aligned}
\mu(B(x,\sqrt{s})) &\leqslant \mu(B(y,\sqrt{s}+r_i)) \\
&\leqslant c(1+r_i s^{-1/2})^c\mu(B(y,\sqrt{s}))\exp[(\sqrt{s}+r_i)^\gamma s^{-\gamma/2} + s^{\gamma/2}] \\
&\leqslant c(\varepsilon)\,\mu(B(y,\sqrt{s}))\exp(t_i\varepsilon/s + s^{\gamma/2}), \quad \varepsilon > 0,
\end{aligned}$$

we obtain for any $\varepsilon > 0$,

$$\mathrm{II} \leqslant c_3(\varepsilon)\,\|h_i\|_1\int_0^\infty G(t_i,s)\,\mathrm{d}s,$$

where

$$\begin{aligned}
G(t_i,s) &:= \exp\left(-a_1 s + 9s^{\gamma/2} - \frac{(\delta_1-\varepsilon)t_i}{8s}\right)\times \\
&\quad\times\left|\frac{\mathrm{e}^{-\sigma s}}{\sqrt{s}} - \frac{1_{\{s>t_i\}}\,\mathrm{e}^{-\sigma(s-t_i)}}{\sqrt{s-t_i}}\right|\sqrt{C(s)}.
\end{aligned}$$

But we have

$$\int_0^{t_i} G(t_i, s)\,\mathrm{d}s \leqslant c \int_0^1 \frac{1}{s} \exp\left(\frac{\varepsilon - \delta_1}{8s}\right) \mathrm{d}s < \infty, \qquad \varepsilon \in (0, \delta_1).$$

On the other hand, it is easy to see that

$$\sup_{t_i \in (0,1]} \int_{t_i}^{\infty} G(t_i, s)\,\mathrm{d}s < \infty, \quad \text{if } \sigma > -a_1.$$

This proves (4.12), and hence (4.2), provided that $\sigma > -a_1$.

Finally, let $\sigma = 0$ and $\varrho = 0$. If either $a_1 > 0$ or $a_1 = \gamma = 0$, then there is constant $c > 0$ such that

$$\begin{aligned}\int_{t_i}^{\infty} G(t_i, s)\,\mathrm{d}s &\leqslant c \int_{t_i}^{\infty} \left|\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s - t_i}}\right| \left(1 + \frac{1}{\sqrt{s}}\right) \mathrm{d}s \\ &= c \int_1^{\infty} \left|\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-1}}\right| \left(\sqrt{t_i} + \frac{1}{\sqrt{s}}\right) \mathrm{d}s \\ &\leqslant c \int_1^{\infty} \left|\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s-1}}\right| \left(1 + \frac{1}{\sqrt{s}}\right) \mathrm{d}s < \infty.\end{aligned}$$

Thus (4.12) holds. □

REMARK 4.5. In the general situation of Theorem 4.1 the L^2-boundedness of T_σ is still an assumption. In most geometric applications however (see [15]), the operator $-L$ is the square of a Dirac operator D, i.e. $L = -D^2$, and $D_m = D$, like in the case of the Hodge Laplacian on forms with $D = d + d^*$. Then L^2-boundedness of T_σ is naturally satisfied.

The following is a direct consequence of Theorem 4.1 when applied to the case of differential forms as in Example 2.4. Recall that in this case exterior differentiation (resp. co-differentiation) commutes with the Hodge Laplacian, implying that $\varrho = 0$. Obviously, Theorem 4.1 applies to other vector bundles, for instance those given in Examples 2.5 and 2.6.

COROLLARY 4.6. *Suppose Assumption* B *holds. Let* $E = \Lambda^p T^*M$ *and* $\Omega^p = \Gamma(E)$ *be the space of p-forms. As in Example* 2.4*, let* $L = -\Delta_\mu^p$ *where* Δ_μ^p *is the weighted Hodge Laplacian. We have* $\mathcal{R} = \mathcal{R}^p - \mathrm{Hess}_V$ *where* $\mathcal{R}^p$ *is the Weitzenböck curvature operator on* Ω^p*. Let* γ *be the constant specified in Assumption* B1.

(1) *Let* $\tilde{E} = \Lambda^{p+1} T^*M$ *and* $\Omega^{p+1} = \Gamma(\tilde{E})$*,* $D_m = \mathrm{d}$*,* $\tilde{L} = -\Delta_\mu^{p+1}$*,* $\tilde{\mathcal{R}} = \mathcal{R}^{p+1} - \mathrm{Hess}_V$*. Assume that* $\mathcal{R}$ *is bounded and* $\tilde{\mathcal{R}}$ *is bounded from below. Let* a_1 *denote the lower bound of* $\mathcal{R}$*. If* $\sigma > -a_1$ *then* T_σ *is weak* (1, 1)*. If* $a_1 > 0$*, or if* $a_1 = \gamma = 0$ *and* (4.4) *holds globally for all* $t > 0$*, then* T_0 *is weak* (1, 1).

(2) *The same conclusions as in* (1) *hold if we let* $\tilde{E} = \Lambda^{p-1}T^*M$, $\tilde{L} = -\Delta_\mu^{p-1}$, $D_m = \mathrm{d}_\mu^*$ (*the* L_μ^2*-adjoint of* d) *and* $\tilde{\mathcal{R}} = \mathcal{R}^{p-1} - \mathrm{Hess}_V$.

We remark that Riesz transforms on differential forms have also been studied by Bakry [4] under lower bounds on the curvature term in the Weitzenböck decomposition. The author does not investigate the weak (1, 1) property, but he also treats the case $p > 2$. The conditions for his estimates (see [4], Théorème 5.1, Corollaire 5.3) are given in terms of lower bounds on $\mathcal{R}$ and $\tilde{\mathcal{R}}$ as well. In the case of tensor fields over an Einstein manifold ([4], Section VI), Bakry establishes analogous results for the horizontal Laplacian ($L = \square$) and the covariant derivative ($D_m = \nabla$) which are similar to what one gets from our Theorem 4.1, applied to the situation of Example 2.6 (in the special case $\mathcal{R} = 0$). Note that the condition of M being Einstein leads to the commutation rule $\varrho = 0$ corresponding to Equation (6.1) in [4].

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