Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Pures Appl. 83 (2004) 955–1018
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Canonical Brownian motion on the space of univalent functions and resolution of Beltrami equations by a continuity method along stochastic flows

Hélène Airault [a], Paul Malliavin [b], Anton Thalmaier [c,*]

[a] *INSSET, université de Picardie, 48 rue Raspail, 02100, Saint Quentin (Aisne), laboratoire CNRS UMR 6140, LAMFA 33, rue Saint-Leu, 80039 Amiens, France*

[b] *10, rue Saint Louis en l'Isle, 75004 Paris, France*

[c] *Département de mathématiques, université de Poitiers, téléport 2 – BP 30179, 86962 Futuroscope Chasseneuil, France*

Received 15 May 2004

Dedicated to Lennart Carleson on the occasion of his seventy sixth birthday

Abstract

A.A. Kirillov has given a parametrization of the space $\mathcal{U}^\infty$ of univalent functions on the closed unit disk, which are C^∞ up to the boundary, by $\mathrm{Diff}(S^1)/S^1$ where $\mathrm{Diff}(S^1)$ denotes the group of orientation preserving diffeomorphisms of the circle S^1. In the same spirit, the space $\mathcal{J}^\infty$ of C^∞ Jordan curves in the complex plane can be parametrized by the double quotient $\mathrm{SU}(1,1)\backslash\mathrm{Diff}(S^1)/\mathrm{SU}(1,1)$. As a consequence, $\mathcal{J}^\infty$ carries a canonical Riemannian metric. We construct a canonical Brownian motion on $\mathcal{U}^\infty$. Classical technologies of the theory of univalent functions, like Beurling–Ahlfors extension, Loewner equation, Beltrami equation, developed in the context of Kunita's stochastic flows, are the tools for obtaining this result which should be seen as a first step to the construction of a canonical Brownian motion on $\mathcal{J}^\infty$.

* Corresponding author.
E-mail addresses: helene.airault@insset.u-picardie.fr (H. Airault), sli@ccr.jussieu.fr (P. Malliavin), thalmaier@math.univ-poitiers.fr (A. Thalmaier).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2004.06.001

Résumé

A.A. Kirillov a paramétré l'espace $\mathcal{U}^\infty$ des fonctions C^∞, bijectives dans le disque unité fermé et holomorphes dans son intérieur, par l'espace $\mathrm{Diff}(S^1)/S^1$, où $\mathrm{Diff}(S^1)$ désigne le groupe des difféomorphismes préservant l'orientation du cercle unité. Dans le même esprit, l'espace $\mathcal{J}^\infty$ des courbes de Jordan C^∞ du plan complexe peut être paramétré par $\mathrm{SU}(1,1)\backslash\mathrm{Diff}(S^1)/\mathrm{SU}(1,1)$. Par voie de conséquence, l'espace $\mathcal{J}^\infty$ possède une métrique riemannienne canonique. Nous construisons dans cet article le mouvement brownien canonique sur $\mathcal{U}^\infty$ à l'aide de technologies classiques de la théorie des fonctions univalentes, comme l'extension de Beurling–Ahlfors, l'équation de Loewner, l'équation de Beltrami, revisitées dans le contexte des flots stochastiques à la Kunita. Cet article nous semble être un premier pas vers la construction du mouvement brownien canonique sur $\mathcal{J}^\infty$.

Keywords: Conformal geometry; Jordan curves; Group of diffeomorphisms of the circle; Beurling–Ahlfors extension; Beltrami equation; Stochastic conformal welding; Stochastic flow; Canonical Brownian motion

1. Introduction: A canonical Riemannian metric on the space of Jordan curves

A *Jordan curve* in the complex plane $\mathbb{C}$ is a closed subset $\Gamma \subset \mathbb{C}$ such that there exists a continuous injective map of the circle $\phi : S^1 \mapsto \mathbb{C}$ satisfying $\phi(S^1) = \Gamma$. The parametrization ϕ is not unique: given two parametrizations ϕ_1, ϕ_2 of the same Jordan curve, there exists a homeomorphism h of S^1 such that $\phi_2 = \phi_1 \circ h$. Two parametrizations define the same orientation of Γ if h is an orientation preserving homeomorphism of the circle.

Denote by $\mathcal{J}$ the set of Jordan curves in $\mathbb{C}$, which form a non closed part of the set of compact subsets of the plane. In fact, a continuous family of Jordan curves may converge to a curve having double points.

When Γ is a rectifiable Jordan curve, the arc length provides a canonical parametrization. More generally, we may use *holomorphic parametrizations*, constructed in the following way: the complement of Γ in $\mathbb{C}$ is the union of two connected open subsets D_Γ^+ and D_Γ^- where D_Γ^+ is bounded and D_Γ^- is unbounded. The Riemann mapping theorem gives the existence of a conformal map ψ^+ of D_Γ^+ onto the open unit disk in $\mathbb{C}$, defined up to composition with an element in $\mathrm{SU}(1,1)$, the group of automorphisms of the Poincaré disk of the form $z \mapsto (az+b)/(\bar{b}z+\bar{a})$. By a theorem of Caratheodory [8,25], ψ^+ has a continuous injective extension F^+ to the closure $\bar{D}_\Gamma^+$ of D_Γ^+, and $(F^+)^{-1}|S^1$ gives a parametrization of Γ canonically defined up to an element of $\mathrm{SU}(1,1)$. The advantage of holomorphic parametrizations lies in the fact that its indeterminacy corresponds to the finite dimensional group $\mathrm{SU}(1,1)$. Note that $g_\Gamma := F^- \circ (F^+)^{-1}|S^1$ is an orientation preserving homeomorphism of the circle S^1, where F^- is defined analogously to F^+ with respect to D_Γ^-.

Denote by $\mathcal{J}^\infty$ the space of C^∞ Jordan curves. For $\Gamma = \phi(S^1) \in \mathcal{J}^\infty$ with smooth parametrization $\phi \in C^\infty(S^1;\mathbb{C})$, it can be proved that $g_\Gamma \in \mathrm{Diff}(S^1)$ where $\mathrm{Diff}(S^1)$ denotes the group of C^∞ orientation preserving diffeomorphisms of the circle. Furthermore, the classical theory of C^∞ *conformal welding* shows that the map $\Gamma \mapsto g_\Gamma$ is a surjective map of $\mathcal{J}^\infty$ onto $\mathrm{Diff}(S^1)$; in addition, see [16], $\mathcal{J}^\infty$ is isomorphic to the following double quotient:

$$\mathcal{J}^\infty \simeq \mathrm{SU}(1,1)\backslash \mathrm{Diff}(S^1)/\,\mathrm{SU}(1,1). \tag{1.1}$$

Denote by $\mathrm{diff}(S^1)$ the Lie algebra of $\mathrm{Diff}(S^1)$ of right invariant vector fields on $\mathrm{Diff}(S^1)$. We identify $\mathrm{diff}(S^1)$ with the C^∞ vector fields on S^1: every such vector field is of the form $u\frac{\mathrm{d}}{\mathrm{d}\theta}$ where u is a C^∞ function on S^1; the Lie algebra bracket is given by $[u,v]=u\dot v - v\dot u$. The complexified Lie algebra $\mathrm{diff}(S^1)\otimes\mathbb{C}$ has the natural basis $\tilde e_n := \exp(\mathrm{i}n\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}$ in which the Lie bracket takes the form $[\tilde e_n, \tilde e_m] = \mathrm{i}(m-n)\tilde e_{m+n}$ for $m,n\in\mathbb{Z}$.

The Lie algebra of the Poincaré automorphism group $\mathrm{SU}(1,1)$ is the Lie subalgebra of $\mathrm{diff}(S^1)$ generated by 1, $\cos\theta$, $\sin\theta$: the vector field $1\times\frac{\mathrm{d}}{\mathrm{d}\theta}$ corresponds to a rotation of the disk; the vector field $\sin\theta\times\frac{\mathrm{d}}{\mathrm{d}\theta}$ corresponds to the one parameter family of Poincaré automorphisms h_λ where the relation $z'=h_\lambda(z)$ is $\frac{z'-1}{z'+1}=\exp(\mathrm{i}\lambda)\frac{z-1}{z+1}$.

Theorem 1.1. *There exists, up to multiplication by a constant, a unique Riemannian metric on $\mathcal{J}^\infty$ which is invariant under the left and right action of* $\mathrm{SU}(1,1)$. *This metric is defined by an orthonormal basis as given in* (1.2) *below.*

Proof. We follow [4] and consider the quotient $\mathrm{diff}(S^1)/\,\mathrm{sl}(2,\mathbb{R})$. Since $\mathrm{SL}(2,\mathbb{R})$ is a subgroup, the adjoint action of $\tilde e_j$, $j=0,\pm1$ preserves $\mathrm{sl}(2,\mathbb{R})$; therefore the adjoint action passes to the quotient and acts infinitesimally on $\mathcal{J}^\infty$. Let $\mathrm{ad}^r(\tilde e_j)$ be this action which by hypothesis is unitary, as well as the corresponding action $\mathrm{ad}^l(\tilde e_j)$ on the left. As $\mathrm{ad}^r(\tilde e_0)$ is unitary, the metric is invariant by rotation. Extending the metric as a Hermitian metric on the complex valued functions, invariance by rotation implies that the metric diagonalizes in the basis $\{\tilde e_n\}$: we denote $\{e_n\}$ the orthonormal basis obtained by multiplying each $\tilde e_n$ by a normalizing constant.

Existence of the metric. Take

$$e_n := \frac{1}{\sqrt{|n|(n^2-1)}}\exp(\mathrm{i}n\theta), \qquad |n|>1. \tag{1.2}$$

Denote $\rho_j = \mathrm{ad}^r(\tilde e_j)$, $j=0,\pm1$; we have:

$$\rho_j(e_n) = \mathrm{i}\sum_k \beta_n^k(j)e_k, \qquad \beta_n^k(j) := \sqrt{\frac{(n-j)^2|n+j|((n+j)^2-1)}{|n|(n^2-1)}}\,\delta_{n+j}^k,$$

where δ_{n+j}^k is the Kronecker symbol. The unitarity condition $\rho_j^*+\rho_{-j}=0$ is equivalent to

$$\beta_n^k(j) = \beta_k^n(-j), \qquad j=0,1.$$

First remark that $\delta_{n+j}^k = \delta_{k-j}^n$. The verification in case $j=0$ is tautological; in the case $j=1$

$$\frac{(n-1)^2|n+1|((n+1)^2-1)}{|n|(n^2-1)} = \frac{(n+1+1)^2|n|(n^2-1)}{|n+1|((n+1)^2-1)}.$$

The computation above gives unitarity with the exception of the cases where the denominators involved vanish

$$n(n^2-1)=0, \quad (n+1)\big((n+1)^2-1\big)=0, \quad n=-2,-1,0,1.$$

For $n=2$, $\rho_1(e_{-2})=\rho_{-1}(e_{-1})=0$; by passing to the quotient we get unitarity for $\mathrm{ad}^r(e_j)$, $j=-1,0,1$.

Uniqueness of the metric. Any invariant metric diagonalizes in the basis $\{\tilde{e}_n\}$; another invariant metric will thus correspond to an orthonormal basis $\{e_n^\sharp\}$ where $e_n^\sharp = a_n e_n$ and where the a_n are positive constants. Such a transformation gives rise to the multiplication

$$\beta_n^k(\cdot) \mapsto \frac{a_n}{a_k}\beta_n^k(\cdot) = [\beta^\sharp]_n^k(\cdot).$$

The unitarity conditions

$$[\beta^\sharp]_n^k(1) = [\beta^\sharp]_k^n(-1)$$

imply that

$$\frac{a_n}{a_k} = \frac{a_k}{a_n} \quad \forall n,k, \quad \beta_n^k(1) \neq 0,$$

where the last condition corresponds to $n=k-1$; as the a_k are positive we get $a_{k-1}=a_k$ for all k. □

By invariance of the metric, passing from the left to the right action changes the sign of the operator ad. The constructed metric is the Kähler metric associated to a cocycle defining the Virasoro algebra [17,3].

We recall the construction of the *canonical Brownian motion* on $\mathrm{Diff}(S^1)$. The *regularized* canonical Brownian motion on the group of diffeomorphisms of the circle is the stochastic flow on the circle S^1 associated to the Itô SDE,

$$\begin{aligned} &\mathrm{d}g_{x,t}^r(\theta) = \mathrm{d}z_{x,t}^r\big(g_{x,t}^r(\theta)\big), \\ &z_{x,t}^r(\psi) := \sum_{n>1} \frac{r^n}{\sqrt{n^3-n}}\big(x_{2n}(t)\cos n\psi + x_{2n+1}(t)\sin n\psi\big), \end{aligned} \tag{1.3}$$

where $\{x_\bullet\}$ is a sequence of independent scalar-valued Brownian motions and $r \in]0,1[$. It results from Kunita's theory of stochastic flows that $\theta \mapsto g_{x,t}^r(\theta)$ is a C^∞ diffeomorphism. The limit $\lim_{r\to 1} g_{x,t}^r = g_{x,t}$ exists uniformly in θ and defines a random homeomorphism $g_{x,t}$, called *canonical Brownian motion* on $\mathrm{Diff}(S^1)$, see [23,5,9,26]. This random homeomorphism is furthermore Hölder continuous, see [5,9].

Our work is related to the search of unitarizing measures for representations of the Virasoro algebra [3,4,2]. It can also be considered as a contribution to *stochastic conformal geometry of the complex plane*, a new field which has been opened by the SLE theory (see [19]).

2. Beurling–Ahlfors extension of vector fields from the circle to the disk

The circle S^1, considered as Lie group, has the real line $\mathbb{R}$ as Lie algebra. To a C^1 diffeomorphism ϕ of the circle we associate the family of diffeomorphisms $\{\tilde{\phi}\}$ of the line satisfying

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\phi}\right)(x) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\phi\right)(\mathrm{e}^{\mathrm{i}x}).$$

For a diffeomorphism ϕ close to the identity, we select a representative $\tilde{\phi}$ in this family by the choice $\tilde{\phi}(0) = \arg\phi(0) \in \,]-\pi/2, \pi/2[$.

Let $\mathcal{P} = \{\zeta \in \mathbb{C} \mid \zeta = \theta + \mathrm{i}y \text{ where } y > 0\}$. We define the Beurling–Ahlfors extension [6] Φ of $\tilde{\phi}$ to $\mathcal{P}$ by

$$\Phi(\zeta) := \int_{-1}^{1} \tilde{\phi}(\theta - y\psi)\big(1 - |\psi|\big)\,\mathrm{d}\psi - 6\mathrm{i}\int_{-1}^{1} \tilde{\phi}(\theta - y\psi)\big(1 - |\psi|\big)\psi\,\mathrm{d}\psi. \tag{2.1}$$

Since $\tilde{\phi}$ is increasing, we see that $y > 0$ implies $\operatorname{Im}\Phi(\zeta) > 0$. If $\tilde{\phi}(\theta) = \theta$, we get $\Phi(\theta + \mathrm{i}y) = \theta + \mathrm{i}y$. Considering now the case of a flow ϕ_t of diffeomorphisms of S^1 and applying formula (2.1), we obtain a family Φ_t of diffeomorphisms of the plane. Recall that differentiating ϕ_t with respect to t at $t = 0$ gives a vector field on the circle $u(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}$, where u is a 2π-periodic function. Hence differentiating Φ_t with respect to t, we get a vector field U on the half plane $\mathcal{P}$:

$$U(\zeta) = \int_{-1}^{1} u(\theta - y\psi)\big(1 - |\psi|\big)\,\mathrm{d}\psi - 6\mathrm{i}\int_{-1}^{1} u(\theta - y\psi)\big(1 - |\psi|\big)\psi\,\mathrm{d}\psi, \tag{2.2}$$

where we identify vector fields and complex valued functions on the half-plane $\mathcal{P}$.

In terms of the following functions on the real line

$$K(s) := \big(1 - |s|\big)1_{[-1,1]}(s), \qquad S(s) := 6sK(s),$$
$$K_y(s) := \frac{1}{y}K\left(\frac{s}{y}\right), \qquad S_y(s) = \frac{1}{y}S\left(\frac{s}{y}\right),$$

Eq. (2.2) writes as a convolution product:

$$U(\zeta) = \big[(K_y - \mathrm{i}S_y) * u\big](\theta), \qquad \zeta = \theta + \mathrm{i}y. \tag{2.3}$$

Denoting by $\widehat{K}$ the Fourier transform of K, i.e.,

$$\widehat{K}(\xi) = \int_{-\infty}^{+\infty} K(s)\exp(-\mathrm{i}\xi s)\,\mathrm{d}s,$$

we have $\widehat{K}_y(\xi) = \widehat{K}(y\xi)$. Now expanding the periodic function u into a complex Fourier series, i.e., $u(\theta) = \sum_{n=-\infty}^{+\infty} c_n \exp(\mathrm{i}n\theta)$, gives

$$U(\zeta) = \sum_{n=-\infty}^{+\infty} c_n\big(\widehat{K}(yn) + 6\widehat{K}'(yn)\big)\exp(\mathrm{i}n\theta) \quad \text{where } \widehat{K}(\xi) = \frac{\sin^2(\xi/2)}{(\xi/2)^2}. \tag{2.4}$$

In real coordinates, expanding $u(\theta) = a_0 + \sum_{n\geqslant 1} a_n \cos n\theta + b_n \sin n\theta$, we get:

$$U(\zeta) = a_0 + \sum_{n\geqslant 1} \widehat{K}(yn)(a_n \cos n\theta + b_n \sin n\theta) - 6\mathrm{i}\sum_{n\geqslant 1} \widehat{K}'(yn)(b_n \cos n\theta - a_n \sin n\theta).$$

Note that $\widehat{K}(0) = 1$, $\widehat{K}'(0) = 0$ imply that $U(\cdot, y) \to u(\cdot)$ as $y \to 0$. In the sequel the following identities will be used

$$\widehat{K}(\xi) = 2\int_0^1 (1-s)\cos(\xi s)\,\mathrm{d}s = \frac{2(1-\cos\xi)}{\xi^2} = 1 - \frac{\xi^2}{12} + \frac{\xi^4}{360} - \cdots,$$

$$6\widehat{K}''(\xi) + \widehat{K}(\xi) = 2\int_0^1 (1-s)\big(1-6s^2\big)\cos(s\xi)\,\mathrm{d}s,$$

$$\mathcal{Q}(\xi) := \big\{\widehat{K} + 7\widehat{K}' + 6\widehat{K}''\big\}(\xi) = \int_{-1}^1 \big(1-|s|\big)(1-6s^2+7\mathrm{i}s)\mathrm{e}^{\mathrm{i}\xi s}\,\mathrm{d}s. \tag{2.5}$$

It will be proved in Appendix B that

$$\begin{aligned}
\big[\widehat{K}'(\xi)\big]^2 &= \frac{16}{15}\int_0^1 (1-t)^3\big(1-2t-4t^2\big)\big(1-\cos(2\xi t)\big)\,\mathrm{d}t \\
&\quad - \frac{2}{15}\int_0^1 (1-t)^4(3+2t)\big(1-\cos(\xi t)\big)\,\mathrm{d}t, \qquad (2.6)\\
\big[\widehat{K}''(\xi) + \widehat{K}(\xi)\big]^2 &= \int_0^1 \phi_1'(t)\cos(\xi t)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \phi_2'(t)\cos(2\xi t)\,\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(t) &= \frac{9}{35}(1-t)^8 - \frac{48}{35}(1-t)^7 + \frac{8}{5}(1-t)^6 + \frac{8}{5}(1-t)^5 - \frac{5}{3}(1-t)^4,\\
\phi_2(t) &= -\frac{2^7\times 9}{35}(1-t)^8 + \frac{2^{11}\times 3}{35}(1-t)^7 - \frac{2^7\times 13}{5}(1-t)^6 \\
&\quad + 2^8(1-t)^5 - \frac{200}{3}(1-t)^4.
\end{aligned} \tag{2.7}$$

We remark that $\phi_1(0) + \frac{1}{2}\phi_2(0) = 0$.

2.1. Passing to the unit disk

Consider the holomorphic chart $\zeta \mapsto z = \exp(\mathrm{i}\zeta)$ and denote $\log^- a = \sup(0, -\log a)$. Letting U^1 be the image of the vector field U in the z coordinate, we have:

$$\exp\bigl(\mathrm{i}\zeta + \varepsilon \mathrm{i} U(\zeta)\bigr) - \exp(\mathrm{i}\zeta) = \varepsilon U^1(z) + \mathrm{o}(\varepsilon)$$

which, by differentiation with respect to ε at $\varepsilon = 0$, gives $\mathrm{i}zU(\zeta) = U^1(z)$ or

$$U^1(z) = \mathrm{i}z\left[\sum_{n=-\infty}^{+\infty} c_n\bigl(\widehat{K}\bigl(n\log^-|z|\bigr) + 6\widehat{K}'\bigl(n\log^-|z|\bigr)\bigr)\exp(\mathrm{i}n\theta)\right]. \tag{2.8}$$

Proposition 2.1. *Given $u \in L^2(S^1)$, the vector field U^1 vanishes at $z = 0$ and is C^1 in $D = \{z\colon |z| < 1\}$.*

Proof. The proof is based on the convergence of the following series:

$$\begin{aligned}
|\nabla U|(\zeta) &\leqslant \sum_n |c_n n|\bigl(\bigl|\widehat{K}(yn)\bigr| + 7\bigl|\widehat{K}'(yn)\bigr| + 6\bigl|\widehat{K}''(yn)\bigr|\bigr)\\
&\leqslant \|u\|_{L^2}\sqrt{\sum_n \bigl(n^2\bigl[\bigl|\widehat{K}(yn)\bigr| + 7\bigl|\widehat{K}'(yn)\bigr| + 6\bigl|\widehat{K}''(yn)\bigr|\bigr]\bigr)^2};\\
\bigl|\nabla U^1\bigr|(z) &\leqslant \bigl|U(\zeta)\bigr| + |z||\nabla U|(\zeta). \qquad \square
\end{aligned}$$

3. Covariances of the extension to the disk of the canonical random field on the circle

In the complex exponential version of Fourier series, expansion (1.3) for $r = 1$ takes the form

$$z_{x,t}(\theta) = \sum_{\substack{n=-\infty\\ n\neq -1,0,1}}^{+\infty} \frac{1}{\sqrt{|n|(n^2-1)}} X_n(t)\exp(\mathrm{i}n\theta), \tag{3.1}$$

where $X_n(t) := \frac{1}{2}(x_{2|n|}(t) - \mathrm{i}\,\mathrm{sign}(n) x_{2|n|+1}(t))$ are complex Brownian motions, independent for $n > 1$, and normalized by the condition $\mathbb{E}[X_n(t)X_{-n}(t)] = t/2$ where $\overline{X}_n(t) = X_{-n}(t)$. We extend (3.1) to the unit disk by means of formula (2.8).

Definition 3.1. Let

$$\widetilde{Z}_{x,t}(z) = \mathrm{i}z \sum_{\substack{n=-\infty \\ n\neq -1,0,1}}^{+\infty} \left(\widehat{K}\left(n \log^- |z|\right) + 6\widehat{K}'\left(n \log^- |z|\right)\right) \times \frac{1}{\sqrt{|n|(n^2-1)}} X_n(t) \exp(\mathrm{i}n\theta). \tag{3.2}$$

It follows from (2.8) that $\widetilde{Z}_{x,t}$ is a C^1 vector field in D.

In a neighbourhood of ∂D the logarithmic chart ζ where $z = \exp(\mathrm{i}\zeta)$, is appropriate for our purpose.

Definition 3.2. Denote by $Z_{x,t}(\zeta)$ the vector field $\widetilde{Z}_{x,t}$ read in the chart $\zeta = \theta + \mathrm{i}y$,

$$Z_{x,t}(\zeta) := \sum_{\substack{n=-\infty \\ n\neq -1,0,1}}^{+\infty} \frac{1}{\sqrt{|n|(n^2-1)}} X_n(t) \left\{\widehat{K}(yn) + 6\widehat{K}'(yn)\right\} \exp(\mathrm{i}n\theta). \tag{3.3}$$

The success of the Beurling–Ahlfors strategy is demonstrated by the following result.

Theorem 3.3. *There exist a constant c, independent of ζ and $y > 0$, such that in terms of $\bar{\partial} = \partial_\theta + \mathrm{i}\partial_y$ the following estimate holds*:

$$\mathbb{E}\left(\left|\bar{\partial} Z_{x,t}(\zeta)\right|^2\right) \leqslant ct. \tag{3.4}$$

Proof. We have:

$$\bar{\partial} Z_{x,t} = \mathrm{i} \sum_{\substack{n=-\infty \\ n\neq -1,0,1}}^{+\infty} \mathrm{sign}(n) \sqrt{\frac{|n|}{n^2-1}} X_n(t) \left\{\widehat{K} + 7\widehat{K}' + 6\widehat{K}''\right\}(yn) \exp(\mathrm{i}n\theta). \tag{3.5}$$

Using the fact that $(\widehat{K} + 7\widehat{K}' + 6\widehat{K}'')(0) = 0$, we get for $\mathcal{Q}(\xi) := \{\widehat{K} + 7\widehat{K}' + 6\widehat{K}''\}(\xi)$,

$$\mathcal{Q}(\xi) = \mathrm{O}(\xi), \quad \text{as } \xi \to 0, \quad \text{and}$$
$$\mathcal{Q}(\xi) = \mathrm{O}(\xi^{-2}), \quad \text{as } \xi \to \infty \quad \text{(see Appendix B).}$$

Since

$$\mathbb{E}\big[\big|\bar\partial Z_{x,t}(\zeta)\big|^2\big] = \frac{t}{2}\sum_{\substack{n=-\infty\\ n\neq 0,\pm 1}}^{+\infty} \frac{|n|}{n^2-1}\big(\mathcal{Q}(yn)\big)^2,$$

estimate (3.4) results from the existence of an absolute constant c_1 such that for all $y > 0$:

$$c_1 \geqslant \sum_n \frac{1}{n}\big[\mathcal{Q}(yn)\big]^2 \simeq y^2 \sum_{n\leqslant y^{-1}} n + y^{-4}\sum_{n>y^{-1}} n^{-5} = \mathrm{O}(1). \qquad \square$$

Non-diagonal estimates of the covariance of $Z_{x,t}$ will be needed later; for this purpose we separate (3.3) in its real and imaginary part:

$$Z_{x,t}(\zeta) = V_{x,t}(\zeta) + \mathrm{i}W_{x,t}(\zeta), \quad \zeta = \theta + \mathrm{i}y, \tag{3.6}$$

where

$$\begin{aligned} V_{x,t} &= \sum_{n>1} \frac{1}{\sqrt{n^3-n}}\widehat{K}(yn)\big[x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\big],\\ W_{x,t} &= -6\sum_{n>1} \frac{1}{\sqrt{n^3-n}}\widehat{K}'(yn)\big[x_{2n+1}(t)\cos n\theta - x_{2n}(t)\sin n\theta\big]. \end{aligned} \tag{3.7}$$

Note that $\widehat{K}$ is an even function, whereas $\widehat{K}'$ is odd.

The covariance is a function of (ζ,ζ') taking its values in the 2×2 matrices: $\left(\begin{smallmatrix} a & c_1\\ c_2 & b\end{smallmatrix}\right)(\zeta,\zeta')$. We denote by $\mathrm{d}_t A * \mathrm{d}_t B$ the Itô contraction of stochastic differentials. Then we have

$$\mathrm{d}_t V_{x,t}(\zeta) * \mathrm{d}_t V_{x,t}(\zeta') = a(\zeta,\zeta')\,\mathrm{d}t, \quad \text{where}$$

$$a(\zeta,\zeta') = \sum_{n>1}\widehat{K}(yn)\widehat{K}(y'n)\frac{1}{n^3-n}\cos n(\theta-\theta'), \tag{3.8}$$

$$\mathrm{d}_t W_{x,t}(\zeta) * \mathrm{d}_t W_{x,t}(\zeta') = b(\zeta,\zeta')\,\mathrm{d}t, \quad \text{where}$$

$$b(\zeta,\zeta') = 36\sum_{n>1}\widehat{K}'(yn)\widehat{K}'(y'n)\frac{1}{n^3-n}\cos n(\theta-\theta'), \tag{3.9}$$

$$\mathrm{d}_t V_{x,t}(\zeta) * \mathrm{d}_t W_{x,t}(\zeta') = c(\zeta,\zeta')\,\mathrm{d}t, \quad \text{where}$$

$$c(\zeta,\zeta') = 6\sum_{n>1}\frac{1}{n^3-n}\widehat{K}'(yn)\widehat{K}(y'n)\sin n(\theta'-\theta). \tag{3.10}$$

In the following, we denote $\mathcal{P}_\delta = \{\theta+\mathrm{i}y\colon\ y>\delta\}$ for $\delta>0$ and $D_\rho = \{z\colon\ |z|<\rho\}$ for $\rho\leqslant 1$.

Lemma 3.4. *The covariance of $Z_{x,t}$ is a C^5 function on $\mathcal{P}\times\mathcal{P}$. Denoting*

$$\Delta(\zeta,\zeta') := a(\zeta,\zeta)+b(\zeta,\zeta)+a(\zeta',\zeta')+b(\zeta',\zeta')-2a(\zeta,\zeta')-2b(\zeta,\zeta'),$$

then

$$\Delta(\zeta,\zeta')\leqslant c|\zeta-\zeta'|^2,\quad (\zeta,\zeta')\in\mathcal{P}_\delta\times\mathcal{P}_\delta; \tag{3.11}$$

$$\Delta(\zeta,\zeta')\leqslant c\|\zeta-\zeta'\|^2\big(\log^-\big(\|\zeta-\zeta'\|\big)+1\big),\quad (\zeta,\zeta')\in\mathcal{P}\times\mathcal{P}. \tag{3.12}$$

Proof. The covariance of $Z_{x,t}$ is given by the series (3.8)–(3.10). We have:

$$\Delta(\zeta,\zeta')=\sum_{n>1}\frac{1}{n^3-n}\Delta_n(\zeta,\zeta')$$

with $\Delta_n(\zeta,\zeta')=A_n(\widehat{K})+36A_n(\widehat{K}')$, where

$$A_n(F):=\big[F(yn)-F(y'n)\big]^2+4F(yn)F(y'n)\sin^2\left(\frac{n(\theta-\theta')}{2}\right).$$

Differentiation with respect to θ or y has essentially the effect of multiplying the nth term of the series by n; for $\zeta\in\mathcal{P}_\delta$ the nth term is of order $\mathrm{O}(n^{-7})$. Passing to the disk introduces for $\widetilde{Z}_{x,t}$ a singularity by the differentiation of $\log^-|z|$, which for the first derivative in the variables (x,y) where $z=x+\mathrm{i}y$ gives a singularity of order $\mathrm{O}(|z|^{-1})$; this singularity is balanced by the factor z appearing in (3.2). We conclude with the estimate $|A_n(F)|\leqslant c\delta^{-2}(|y-y'|^2+|\theta-\theta'|^2)$ where $F=\widehat{K}$ or $\widehat{K}'$. This establishes (3.11).

As $\widehat{K}$ and $\widehat{K}'$ are bounded, the second term in $A_n(F)$ is bounded by $\sin^2(n(\theta-\theta')/2)$ which leads for Δ to a contribution bounded by

$$\sum_{n>1}\frac{1}{n^3-n}\sin^2\left(\frac{n(\theta-\theta')}{2}\right)\simeq|\theta-\theta'|^2\big(\log^-|\theta-\theta'|+1\big).$$

The elementary inequality $2(\alpha+\beta)(\log^-(\alpha+\beta)+1)\geqslant\alpha\log^-\alpha$ for $\alpha,\beta>0$, implies that

$$\sum_{n>1}\frac{1}{n^3-n}\sin^2\left(\frac{n(\theta-\theta')}{2}\right)\leqslant c\|\zeta-\zeta'\|^2\big(\log^-\|\zeta-\zeta'\|+1\big).$$

Assume that $y\leqslant y'$ and split the contribution of the first term of $A_n(F)$ to Δ in two partial sums: $B_1=\sum_{n^{-1}<y'}$ and $B_2=\sum_{n^{-1}\geqslant y'}$. The estimates of B_i, $i=1,2$, are achieved by the mean value theorem: along with the asymptotics

$$\widehat{K}'(\xi)=\mathrm{O}\left(\frac{1}{|\xi|^2+1}\right),\quad \widehat{K}''(\xi)=\mathrm{O}\left(\frac{1}{|\xi|^2+1}\right),\quad \xi\to\infty,$$

we obtain the estimate:

$$B_1 \leqslant (y-y')^2 \sum_{n>(y')^{-1}} \frac{1}{(y'n)^2+1}\frac{n^2}{n^3-1} \simeq (y-y')^2 \int_{(y')^{-1}}^{\infty} \frac{1}{(y't)^2+1}\frac{\mathrm{d}t}{t}$$
$$= (y-y')^2 \int_1^{\infty} \frac{1}{t^2+1}\frac{\mathrm{d}t}{t}$$

which gives (3.12).

Remark 1. We can as well establish the existence of a constant C such that for all $y>0$, $y'>0$, $y \neq y'$,

$$\sum_{n>1} \frac{1}{n^3-n}\big(\widehat{K}(yn)-\widehat{K}(y'n)\big)^2 \leqslant C\left[(y-y')^2 \log\left|\frac{1}{y-y'}\right| + (y-y')^2\right]. \quad (3.13)$$

Namely, from the first part of Eq. (2.5) we get $\widehat{K}(yn)-\widehat{K}(y'n) = 2\int_0^1 (1-s)(\cos(yns) - \cos(y'ns))\,\mathrm{d}s$; thus

$$\sum_{n>1} \frac{1}{n^3-n}\big(\widehat{K}(yn)-\widehat{K}(y'n)\big)^2 \leqslant c \sum_{n\geqslant 1} \frac{1}{n^3} \int_0^1 (1-s)^2 \sin^2\left(\frac{y-y'}{2}ns\right)\mathrm{d}s.$$

By [5, (3.2.1), p. 402], the last term is dominated by

$$\mathrm{const}(y-y')^2 \int_0^1 (1-s)^2\left[s^2 + s^2 \log\left(\frac{1}{(y-y')s}\right)\right]\mathrm{d}s;$$

integration then gives (3.13).

3.1. Covariance of $\bar{\partial} Z_{x,t}$

Formula (3.5) represents $\bar{\partial} Z_{x,t}$ as a complex Fourier series. Passing again to real coordinates $Z_{x,t} = V_{x,t} + \mathrm{i}W_{x,t}$, we have:

$$\bar{\partial} Z_{x,t} = (\partial_\theta V_{x,t} - \partial_y W_{x,t}) + \mathrm{i}(\partial_\theta W_{x,t} + \partial_y V_{x,t}).$$

Differentiation of (3.7) gives $6\partial_y V_{x,t} = \partial_\theta W_{x,t}$ and therefore

$$\bar{\partial} Z_{x,t} = A + \mathrm{i}B,$$

where

$$\begin{aligned} A &= \sum_{n>1} \frac{n}{\sqrt{n^3-n}} \big(6\widehat{K}''(yn) + \widehat{K}(yn)\big)\big(x_{2n+1}(t)\cos n\theta - x_{2n}(t)\sin n\theta\big), \\ B &= 7\partial_y V_{x,t}. \end{aligned} \tag{3.14}$$

Denoting

$$\begin{aligned} \mathbb{E}\big[\mathrm{d}_t A(\zeta_1) * \mathrm{d}_t A(\zeta_2)\big] &= \alpha(\zeta_1,\zeta_2)\,\mathrm{d}t, \\ \mathbb{E}\big[\mathrm{d}_t B(\zeta_1) * \mathrm{d}_t B(\zeta_2)\big] &= \beta(\zeta_1,\zeta_2)\,\mathrm{d}t, \\ \mathbb{E}\big[\mathrm{d}_t A(\zeta_1) * \mathrm{d}_t B(\zeta_2)\big] &= \gamma(\zeta_1,\zeta_2)\,\mathrm{d}t, \end{aligned} \tag{3.15}$$

we have:

$$\alpha(\zeta_1,\zeta_2) = \sum_{n>1} \frac{n^2}{n^3-n} \big(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\big)\big(6\widehat{K}''(y_2 n) + \widehat{K}(y_2 n)\big)\cos n(\theta_1-\theta_2),$$

$$\beta(\zeta_1,\zeta_2) = \sum_{n>1} \frac{49n^2}{n^3-n} \widehat{K}'(y_1 n)\widehat{K}'(y_2 n)\cos n(\theta_1-\theta_2),$$

$$\gamma(\zeta_1,\zeta_2) = \sum_{n>1} \frac{7n^2}{n^3-n} \big(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\big)\widehat{K}'(y_2 n)\sin n(\theta_2-\theta_1).$$

Lemma 3.5. *There exist strictly positive numerical constants c_1, c_2 such that*

$$\begin{aligned} \big|\alpha(\zeta_1,\zeta_2)\big| &\leqslant c_1 \exp\big(-c_2 d(\zeta_1,\zeta_2)\big), \\ \big|\beta(\zeta_1,\zeta_2)\big| &\leqslant c_1 \exp\big(-c_2 d(\zeta_1,\zeta_2)\big), \\ \big|\gamma(\zeta_1,\zeta_2)\big| &\leqslant c_1 \exp\big(-c_2 d(\zeta_1,\zeta_2)\big), \end{aligned} \tag{3.16}$$

where $d(\zeta_1,\zeta_2)$ denotes the distance between ζ_1 and ζ_2 in the Poincaré metric of the half plane $\{y>0\}$.

Proof. First, we use expression (3.5) for $\bar\partial Z$. Define

$$\mathcal{D}(z_1,z_2)\,\mathrm{d}t := \mathbb{E}\big[\mathrm{d}_t\bar\partial Z_{x,t}(y_1,\theta_1) * \overline{\mathrm{d}_t\bar\partial Z_{x,t}(y_2,\theta_2)}\big], \quad z_k = \exp(-y_k + \mathrm{i}\theta_k),\ k=1,2.$$

Since $\mathbb{E}[\mathrm{d}_t X_n * \mathrm{d}_t \overline{X}_m] = \delta_m^n\,\mathrm{d}t$, by taking into account that $\mathcal{Q}$ is a real valued function, we get:

$$\mathcal{D}(z_1,z_2) = \sum_{\substack{n=-\infty \\ n\neq -1,0,1}}^{+\infty} \frac{|n|}{n^2-1} \mathcal{Q}(y_1 n)\mathcal{Q}(y_2 n)\exp\big(\mathrm{i}n(\theta_1-\theta_2)\big).$$

Denote $u_y(\theta) := \sum_{n=-\infty}^{+\infty} \mathcal{Q}(yn)\exp(-\mathrm{i}n\theta)$ and $u(s) = (1-|s|)1_{[-1,1]}(s)(1+7\mathrm{i}s-6s^2)$. Because of (2.5), we have

$$u_y(s) = \frac{2\pi}{y} u\left(\frac{s}{y}\right).$$

Let

$$\psi(\theta) = 2\sum_{n>1} \frac{n}{n^2-1}\cos n\theta \equiv 2\log\frac{1}{\theta} + \text{a bounded function},$$

then $\mathcal{D}(z_1, z_2) = (u_{y_1} * u_{y_2} * \psi)(\theta_1 - \theta_2)$ where $*$ denotes convolution on the circle. Define

$$\mathcal{D}_{y_1,y_2}(\theta)\,\mathrm{d}t := \mathcal{D}(z_1, z_2)\,\mathrm{d}t = \mathbb{E}\big[\mathrm{d}_t\bar{\partial} Z_{x,t}(y_1,\theta) * \overline{\mathrm{d}_t\bar{\partial} Z_{x,t}(y_2, 0)}\big]$$

and consider the norm in the Wiener algebra $\mathcal{A}$ of absolutely convergent Fourier series:

$$\|\mathcal{D}_{y_1,y_2}\|_{\mathcal{A}} := \sum_{\substack{n=-\infty\\ n\neq -1,0,1}}^{+\infty} \frac{|n|}{n^2-1}\big|\mathcal{Q}(y_1n)\mathcal{Q}(y_2n)\big| \simeq \int_{-\infty}^{+\infty}\big|\mathcal{Q}(y_1t)\mathcal{Q}(y_2t)\big|\frac{\mathrm{d}t}{|t|};$$

$$\|\mathcal{D}_{y_1,y_2}\|_{L^2}^2 := \sum_{\substack{n=-\infty\\ n\neq -1,0,1}}^{+\infty} \frac{n^2}{(n^2-1)^2}\big|\mathcal{Q}(y_1n)\mathcal{Q}(y_2n)\big|^2 \simeq \int_{-\infty}^{+\infty}\big|\mathcal{Q}(y_1t)\mathcal{Q}(y_2t)\big|^2\frac{\mathrm{d}t}{|t|^2}.$$

In the last two identities, the comparison of the series with the integral can be justified by writing $\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = \sum_{k=-\infty}^{+\infty}\int_k^{k+1} f(t)\,\mathrm{d}t$ and computing $\int_k^{k+1} f(t)\,\mathrm{d}t$ via applying Taylor's formula with remainder term to the function f. $\square$

Theorem 3.6. *We have*

$$\begin{aligned}&\|\mathcal{D}_{y_1,y_2}\|_{L^\infty} \leqslant \|\mathcal{D}_{y_1,y_2}\|_{\mathcal{A}} \simeq \rho(y_1,y_2) := \frac{\inf(y_1,y_2)}{\sup(y_1,y_2)};\\ &\|\mathcal{D}_{y_1,y_2}\|_{L^2}^2 \simeq \inf(y_1,y_2)\times\rho(y_1,y_2).\end{aligned} \tag{3.17}$$

Proof. Assume that $y_1 \leqslant y_2 \leqslant 1$. Then we have

$$\|\mathcal{D}_{y_1,y_2}\|_{\mathcal{A}} \simeq \int_{-\infty}^{+\infty}\big|\mathcal{Q}(ty_1)\mathcal{Q}(ty_2)\big|\frac{\mathrm{d}t}{|t|};$$

$$\int_0^{+\infty} \simeq \int_0^{1/y_2} y_1y_2t^2\frac{\mathrm{d}t}{t} + \int_{1/y_2}^{1/y_1}\frac{y_1t}{y_2^2t^2}\frac{\mathrm{d}t}{t} + \int_{1/y_1}^{+\infty}\frac{1}{(y_1y_2t^2)^2}\frac{\mathrm{d}t}{t} \simeq \frac{y_1}{y_2},$$

and

$$\|\mathcal{D}_{y_1,y_2}\|_{L^2}^2 \simeq \int_{-\infty}^{+\infty} \left|\mathcal{Q}(ty_1)\mathcal{Q}(ty_2)\right|^2 \frac{\mathrm{d}t}{t^2};$$

$$\int_0^{+\infty} \simeq \int_0^{1/y_2} (y_1y_2)^2 t^4 \frac{\mathrm{d}t}{t^2} + \int_{1/y_2}^{1/y_1} \frac{(y_1t)^2}{(y_2t)^4}\frac{\mathrm{d}t}{t^2} + \int_{1/y_1}^{+\infty} \frac{1}{(y_1y_2t^2)^4}\frac{\mathrm{d}t}{t^2} \simeq y_1 \times \rho(y_1, y_2). \qquad \square$$

Remark 2. For $y_i \to 0$, the L^∞ norm of the function $\mathcal{D}_{y_1,y_2}$ is much larger than its L^2 norm: this means that its support is localized on a set of small measure. We shall see that the support of the covariance is concentrated nearby zero which means that $\bar\partial Z_{x,t}$ has a quite unexpected behaviour of white noise when $y_i \to 0$.

This fact is made explicit by the following theorem:

Theorem 3.7. *We have*

$$\left|\mathcal{D}_{y_1,y_2}(\theta)\right| \leqslant c \inf\left\{\rho, \frac{y_1y_2}{\theta^2}\right\} \quad \textit{and} \quad \int_{S^1} \left|\mathcal{D}_{y_1,y_2}(\theta)\right| \mathrm{d}\theta \leqslant 2\sqrt{y_1y_2}. \tag{3.18}$$

Proof. In the sense of distribution we have

$$u_y = cy\delta_0' + \mathrm{O}\big(y^2\big), \quad y \to 0, \ c := \frac{7\mathrm{i}}{6},$$

where δ_0' denotes the derivative of the Dirac mass at the origin. Given a C^∞ function f of compact support, as the support of u_y converges to 0, by using a Taylor expansion of f, we are reduced to calculate the integral $\int s^a u_y(s)\,\mathrm{d}s$ for $a \in \{0, 1\}$. Then we use that ψ is C^2 in the complement of 0 and that

$$\ddot\psi(\theta) = \frac{2}{\theta^2} + \mathrm{O}(1).$$

The second inequality in (3.18) follows from the first one as follows: if $y_1 \leqslant y_2$, then $\rho = y_1/y_2$. Observe that $y_1y_2/\theta^2 > \rho$ is equivalent to $\theta^2 < y_2^2$, i.e., $\theta < y_2$. We decompose

$$\int_{S^1} \left|\mathcal{D}_{y_1,y_2}(\theta)\right| \mathrm{d}\theta = \int_0^{y_2} \left|\mathcal{D}_{y_1,y_2}(\theta)\right| \mathrm{d}\theta + \int_{y_2}^{2\pi} \left|\mathcal{D}_{y_1,y_2}(\theta)\right| \mathrm{d}\theta.$$

Then $\int_0^{y_2} |\mathcal{D}_{y_1,y_2}(\theta)|\,\mathrm{d}\theta \leqslant \rho y_2$, since $\rho y_2 = y_1 \leqslant \sqrt{y_1y_2}$, we obtain the domination for $\int_0^{y_2} |\mathcal{D}_{y_1,y_2}(\theta)|\,\mathrm{d}\theta$. For the second integral, we have $\int_{y_2}^{2\pi} y_1y_2\theta^{-2}\,\mathrm{d}\theta \leqslant y_1$. $\square$

It is useful to rewrite the estimates in (3.16)–(3.17) in term of conformal invariants. Let μ_ζ be the harmonic measure of the point ζ given by the Poisson kernel. Then, for two points ζ_1, ζ_2 the measures μ_{ζ_1} and μ_{ζ_2} are mutually absolutely continuous. Let $q(\zeta_1, \zeta_2)$ be the Radon–Nikodym derivative defined by the relation $\mathrm{d}\mu_{\zeta_2} = q(\zeta_1, \zeta_2)\,\mathrm{d}\mu_{\zeta_1}$. Defining the Harnack deviation as

$$e_H(\zeta_1, \zeta_2) = \|q\|_{L^\infty} + \|q^{-1}\|_{L^\infty},$$

we get $e_H(\zeta_1, \zeta_2) \simeq \sup_{h\in\mathcal{H}^+} h(\zeta_1)/h(\zeta_2)$ where $\mathcal{H}^+$ denotes the cone of positive harmonic functions. The Poincaré distance $d_{\mathcal{P}}$ is the Riemannian distance for the Poincaré metric $|\mathrm{d}\zeta|^2/y^2$. Thus, using the invariance of the Poincaré metric and of the Harnack deviation under Möbius transformations, we can compute these expressions on the unit disk in the special case where $z_1 = 0$; then

$$e_H(0, z_2) = \frac{1+|z_2|}{1-|z_2|} = \exp\bigl(d_{\mathcal{P}}(0, z_2)\bigr). \tag{3.19}$$

Theorem 3.8. *There exists a numerical constant c such that*

$$\bigl|\mathcal{D}(\zeta_1, \zeta_2)\bigr| \leqslant c \exp\bigl(-d_{\mathcal{P}}(\zeta_1, \zeta_2)\bigr). \tag{3.20}$$

Proof. We have

$$\frac{y_1}{y_2} \sup_x \frac{(x^2+y_2^2)}{(x-\theta)^2+y_1^2} = \frac{y_1}{y_2} \sup_x \frac{u(x)}{v(x)}, \quad \theta > 0.$$

At the extremum, we get $u/v = u'/v' = x/(x-\theta)$, or $x^2\theta - x(\theta^2 + y_1^2 - y_2^2) - \theta y_2^2 = 0$. This equation is satisfied for

$$x_M = \frac{\theta^2 + y_1^2 - y_2^2 + \sqrt{(\theta^2 + y_1^2 - y_2^2)^2 + 4\theta^2 y_2^2}}{2\theta},$$

where the value of the maximum is

$$\frac{y_1}{y_2} \frac{\theta^2 + y_1^2 - y_2^2 + \sqrt{(\theta^2 + y_1^2 - y_2^2)^2 + 4y_2^2\theta^2}}{\theta^2 + y_1^2 - y_2^2 + \sqrt{(\theta^2 + y_1^2 - y_2^2)^2 + 4y_2^2\theta^2} - 2\theta^2}.$$

For y_1/y_2 small, the maximum is essentially reached at $x = \theta$ and takes the value

$$\frac{y_1}{y_2}\frac{\theta^2 + y_2^2}{y_1^2} = \frac{\theta^2}{y_1 y_2} + \frac{y_2}{y_1} < 2\max\left\{\frac{\theta^2}{y_1 y_2}, \frac{y_2}{y_1}\right\},$$
$$\frac{y_1 y_2}{\theta^2 + y_2^2} = \left\{\frac{\theta^2}{y_1 y_2} + \frac{y_2}{y_1}\right\}^{-1} > \frac{1}{2}\left\{\frac{y_1 y_2}{\theta^2}, \frac{y_1}{y_2}\right\} > \frac{1}{2}\bigl|\mathcal{D}(y_2, y_1 + \mathrm{i}\theta)\bigr|. \qquad \square$$

We shall need an estimate of the covariance $\mathbb{E}[\bar\partial Z_{x,t}(\zeta_1)\bar\partial Z_{x,t}(\zeta_2)]$. For this purpose it is convenient to use real coordinates. Exploiting (3.14)–(3.15), we have

$$\mathbb{E}\big[\bar\partial Z_{x,t}(\zeta_1)\bar\partial Z_{x,t}(\zeta_2)\big] = \big(\alpha(\zeta_1,\zeta_2)-\beta(\zeta_1,\zeta_2)\big)+\mathrm{i}\big(\gamma(\zeta_2,\zeta_1)+\gamma(\zeta_1,\zeta_2)\big),$$
$$\mathbb{E}\big[\bar\partial Z_{x,t}(\zeta_1)\overline{\bar\partial Z_{x,t}(\zeta_2)}\big] = \big(\alpha(\zeta_1,\zeta_2)+\beta(\zeta_1,\zeta_2)\big)+\mathrm{i}\big(\gamma(\zeta_2,\zeta_1)-\gamma(\zeta_1,\zeta_2)\big).$$

It remains to calculate $\beta(\zeta_1,\zeta_2)$ and $\gamma(\zeta_2,\zeta_1)+\gamma(\zeta_1,\zeta_2)$. Expressions of the covariance functions are given in Appendix B. In particular, it is proved that

$$\begin{aligned}\beta\big((y_1,\theta),(y_2,\theta)\big)/36 &= \sum_{n>1}\frac{n}{n^2-1}\widehat{K}'(ny_1)\widehat{K}'(ny_2)\\ &= \frac{2}{3}\frac{y_2}{y_1}\int_0^1(1-s)h(y_1s)\,\mathrm{d}s + g\times\left[\frac{y_2}{y_1}\right]^2\end{aligned}\tag{3.21}$$

where $y_2\leqslant y_1\leqslant\pi/4$ and where g is a function bounded by a numerical constant; the function h is explicitly given.

4. Canonical Brownian motion on Diff(S^1) extended to the disk

The purpose of this paragraph is to integrate the random vector $\widetilde{Z}_{x,t}$ defined in (3.2), that is to construct the stochastic flow $\widetilde{\Psi}_{x,t}$ defined *formally* by the Stratonovich SDE,

$$\delta_t\widetilde{\Psi}_{x,t}(z) = (\delta_t\widetilde{Z}_{x,t})\big(\widetilde{\Psi}_{x,t}(z)\big),\quad \widetilde{\Psi}_{x,0}(z)=z,\tag{4.1a}$$

where δ_t denotes the Stratonovich differential. The first step is to rewrite (4.1a) in the framework of Itô calculus. To this end we have to calculate the underlying stochastic contractions.

In the chart $z=\exp(\mathrm{i}\zeta)$, we get $\widetilde{\Psi}_{x,t}(z)=\exp[\mathrm{i}\Psi_{x,t}(-\mathrm{i}\log z)]$. The logarithm is defined up to a multiple of 2π; as the vector $Z_{x,t}$ is periodic, the flow $\Psi_{x,t}$ commutes with the group operation generated by $\zeta\mapsto\zeta+2\pi$; therefore the indeterminacy of the logarithm does not matter for the construction of $\widetilde{\Psi}_{x,t}$. Let $\mathscr{P}$ denote again the half plane $\{y>0\}$. Fixing $\zeta_0\in\mathscr{P}$ the trajectory $t\mapsto\Psi_{x,t}(\zeta_0)=\theta_{x,t}+\mathrm{i}y_{x,t}$ is driven by the Stratonovich SDE,

$$\delta_t\psi_{x,t}(\zeta) = (\delta_t Z_{x,t})\big(\psi_{x,t}(\zeta)\big),\quad \psi_{x,0}(\zeta)=\zeta,\tag{4.1b}$$

where $Z_{x,t}$ is the vector field defined by (3.3). The following system is equivalent to (4.1b),

$$\begin{aligned}\delta\theta_{x,t} &= \sum_{n>1}\frac{1}{\sqrt{n^3-n}}\widehat{K}(ny_{x,t})\big(\cos(n\theta_{x,t})\delta x_{2n}(t)+\sin(n\theta_{x,t})\delta x_{2n+1}(t)\big),\\ \delta y_{x,t} &= -6\sum_{n>1}\frac{1}{\sqrt{n^3-n}}\widehat{K}'(ny_{x,t})\big(\cos(n\theta_{x,t})\delta x_{2n+1}(t)-\sin(n\theta_{x,t})\delta x_{2n}(t)\big).\end{aligned}\tag{4.1c}$$

4.1. The stochastic contractions

The regularized Brownian motion on $\mathrm{Diff}(S^1)$ has been defined through the Itô SDE (1.3). The drift induced by writing (1.3) in Stratonovich form vanishes:

$$\sum_{k>1} \frac{r^k}{(k^2-1)}(\mathrm{d}_t x_{2k}\cos k\theta' + \mathrm{d}_t x_{2k+1}\sin k\theta') * (-\mathrm{d}_t x_{2k}\sin k\theta' + \mathrm{d}_t x_{2k+1}\cos k\theta') = 0.$$

Therefore $g^r_{x,t}$ satisfies the Stratonovich SDE

$$\delta g^r_{x,t}(\theta) = \sum_{k>1} \frac{r^k}{\sqrt{k(k^2-1)}}\big[\cos\big(k g^r_{x,t}(\theta)\big)\delta x_{2k}(t) + \sin\big(k g^r_{x,t}(\theta)\big)\delta x_{2k+1}(t)\big].$$

Proposition 4.1. *Define the vector field* $A(\zeta) = v(\zeta) + \mathrm{i}w(\zeta)$, $\zeta = \theta + \mathrm{i}y$, *where*

$$v = 0, \quad w(y,\theta) = 6\sum_{n>1}\frac{1}{n^2-1}\big(6\widehat{K}''(yn) + \widehat{K}(yn)\big)\widehat{K}'(yn). \tag{4.2}$$

The following Stratonovich and Itô equations are equivalent:

$$\begin{aligned}
&\delta_t \widetilde{\Psi}_{x,t}(z) = \delta_t \widetilde{Z}_{x,t}\big(\widetilde{\Psi}_{x,t}(z)\big), \quad \widetilde{\Psi}_{x,0}(z) = z,\\
&\mathrm{d}_t \widetilde{\Psi}_{x,t}(z) = \mathrm{d}_t \widetilde{Z}_{x,t}\big(\widetilde{\Psi}_{x,t}(z)\big) + \frac{1}{2}A\big(\widetilde{\Psi}_{x,t}(z)\big)\,\mathrm{d}t, \quad \widetilde{\Psi}_{x,0}(z) = z.
\end{aligned} \tag{4.3}$$

Proof. This can be seen from system (4.1c) or directly from (4.1b). According to Itô's calculus, the stochastic contraction is

$$\frac{1}{2}\left(\mathrm{i}\,\mathrm{d}_t\frac{\partial W_{x,t}}{\partial y} * \mathrm{d}_t W_{x,t} + \mathrm{i}\,\mathrm{d}_t\frac{\partial W_{x,t}}{\partial \theta} * \mathrm{d}_t V_{x,t} + \mathrm{d}_t\frac{\partial V_{x,t}}{\partial y} * \mathrm{d}_t W_{x,t} + \mathrm{d}_t\frac{\partial V_{x,t}}{\partial \theta} * \mathrm{d}_t V_{x,t}\right).$$

The last term vanishes since

$$\mathrm{d}_t\frac{\partial}{\partial\theta}\big(x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\big) * \mathrm{d}_t\big(x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\big) = 0;$$

the third term vanishes since

$$\mathrm{d}_t\big(x_{2n+1}(t)\cos n\theta - x_{2n}(t)\sin n\theta\big) * \mathrm{d}_t\big(x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\big) = 0.$$

The first term gives rise to

$$36\sum_{n>1}\frac{1}{n^2-1}\widehat{K}''(yn)\widehat{K}'(yn); \tag{4.4}$$

for the second term we use

$$\mathrm{d}_t \frac{\partial}{\partial\theta}\big(x_{2n+1}(t)\cos n\theta - x_{2n}(t)\sin n\theta\big) * \mathrm{d}_t\big(x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\big) = -n\,\mathrm{d}t. \quad \square$$

Remark 3. The Itô system associated to (4.3) is

$$\begin{cases} \mathrm{d}\theta_{x,t} = \displaystyle\sum_{n>1} \frac{1}{\sqrt{n^3-n}} \widehat{K}(ny_{x,t})\big(\cos(n\theta_{x,t})\,\mathrm{d}x_{2n}(t) + \sin(n\theta_{x,t})\,\mathrm{d}x_{2n+1}(t)\big), \\ \mathrm{d}y_{x,t} = -6\displaystyle\sum_{n>1} \frac{1}{\sqrt{n^3-n}} \widehat{K}'(ny_{x,t})\big(\cos(n\theta_{x,t})\,\mathrm{d}x_{2n+1}(t) - \sin(n\theta_{x,t})\,\mathrm{d}x_{2n}(t)\big) \\ \qquad\qquad + \dfrac{1}{2} w(y_{x,t})\,\mathrm{d}t. \end{cases} \tag{4.5}$$

For the drift function w given by (4.2), the following asymptotic expansions hold nearby $y=0$:

$$w \simeq \sum_{ny<1} \frac{y^2}{n^4} + \sum_{yn>1} \frac{1}{y^4 n^6} \simeq y, \quad \frac{\partial w}{\partial y} \simeq -\sum_{ny<1} \frac{y}{n^2} + \sum_{yn>1} \frac{1}{y^4 n^5} \simeq 1.$$

As $y \to \infty$, we have $w(y) \to 0$ and $\partial_y w \to 0$.

Let $\zeta_\omega(t) = \theta_\omega(t) + \mathrm{i}y_\omega(t)$ be the generic trajectory of the diffusion associated to the elliptic operator

$$\mathcal{L}_\zeta = \frac{1}{2} b(\zeta,\zeta) \frac{\partial^2}{\partial y^2} + \frac{1}{2} a(\zeta,\zeta) \frac{\partial^2}{\partial\theta^2} + w(\zeta)\frac{\partial}{\partial y}.$$

There exists a measurable probability preserving map $\chi : X \to \Omega$; let $\omega = \chi(x)$ be such that $\theta_{x,t} + \mathrm{i}y_{x,t} = \theta_{\chi(x)}(t) + \mathrm{i}y_{\chi(x)}(t)$.

Lemma 4.2. *The trajectories of the diffusions associated to $\mathcal{L}$ have infinite lifetime and stay in $\mathcal{P}$ up to their lifetime.*

Proof. Note that $t \mapsto y_\omega(t)$ is the one-dimensional diffusion associated to the ODE

$$\frac{1}{2} q(y) \frac{\mathrm{d}^2}{\mathrm{d}y^2} + w(y)\frac{\mathrm{d}}{\mathrm{d}y}, \quad q(y) = 36 \sum_{n>1} \frac{1}{n^3-n} \big(\widehat{K}'\big)^2(yn).$$

Since $\widehat{K}'(s) = -s/6 + \mathrm{O}(s^3)$ as $s \to 0$, and $\widehat{K}'(s) = \mathrm{O}(s^{-2})$ as $s \to \infty$, we have

$$q(y) = \frac{1}{4} y^2 \sum_{n<y^{-1}} \frac{1}{n} + \mathrm{O}\big(y^2\big) < y^2 \log^- y, \quad y \to 0.$$

From [12,13] and using the fact that $w(y) > 0$, we deduce the existence of a probability preserving map $\omega \mapsto \tilde{\omega}$ with the following property: denoting $y_{\tilde{\omega}}$ the diffusion associated to the ODE

$$\frac{1}{2} y^2 \log^-(y) \frac{\mathrm{d}^2}{\mathrm{d}y^2}, \tag{4.6}$$

satisfying $y_{\tilde{\omega}}(0) = y_\omega(0)$, then

$$\min_{t\in[0,T]} y_{\tilde{\omega}}(t) \leqslant \min_{t\in[0,T]} y_\omega(t); \qquad \max_{t\in[0,T]} y_\omega(t) \leqslant \max_{t\in[0,T]} y_{\tilde{\omega}}(t).$$

We may integrate (4.6) by the change of variable $\lambda(y) := [\log^-(y)]^{1/2}$ and by using Itô calculus.

The other possibility to leave $\mathcal{P}$ is by $y_\omega(t) \to \infty$ as $t \to T < \infty$: this can be excluded as for $y > y_1$; we may use as comparison equation the usual Brownian motion, along with $w(y) \to 0$ as $y \to \infty$. $\square$

Theorem 4.3. *Eqs.* (4.1)–(4.3) *define a unique stochastic flow* $\widetilde{\Psi}_{x,t}$ *of* C^1 *diffeomorphisms of* D_1; *moreover,*

$$\lim_{\rho\to 1} \widetilde{\Psi}_{x,t}\big(\rho \exp(\mathrm{i}\theta)\big) = g_{x,t}(\theta) \quad \textit{uniformly in } \theta \in S^1.$$

Proof. We apply Theorem 8.2 of [20], p. 852, which shows that (4.3) and (4.5) are localized on D_δ. It remains to pass from D_δ to D; this is a consequence of Lemma 3.4. The proof of the convergence for $r \to 1$ results from [18, Theorem 5.4.2, p. 245]. $\square$

5. Canonical Brownian motion on the space of univalent functions and stochastic conformal welding

As shown in Lemma 3.4, the covariance of $Z_{x,t}$ is C^5 on $\mathcal{P}$; the covariance of $\widetilde{Z}_{x,t}$ is therefore C^5 on $D^\delta = \{z\colon\ \delta < |z| < 1\}$; near $z = 0$ however, $\widetilde{Z}_{x,t}$ is only C^1. Hence $\widetilde{\Psi}_{x,t}$ defines a C^4 diffeomorphism on D^δ. As at $z = 0$ the vector field $\widetilde{Z}_{x,t}$ vanishes and is of class C^1, we conclude that $\widetilde{\Psi}_{x,t}$ is a C^1 orientation preserving diffeomorphism of the open disk D.

We define the complex modulus of quasi-conformality as

$$\mu_{\widetilde{\Psi}}(z) := \frac{\bar{\partial}\widetilde{\Psi}_{x,t}}{\partial\widetilde{\Psi}_{x,t}}(z), \quad |z| < 1, \tag{5.1}$$

and $\nu_{x,t}^\rho$ by

$$\nu_{x,t}^\rho(z) := \mu_{\widetilde{\Psi}}(z), \quad |z| \leqslant \rho < 1, \qquad \nu_{x,t}^\rho(z) := 0, \quad |z| > \rho. \tag{5.2}$$

By a compactness argument, we get

$$\sup_z \bigl|\nu^\rho_{x,t}(z)\bigr| < 1. \tag{5.3}$$

Now let $F^\rho_{x,t}$ be a solution of the following Beltrami equation, defined on the whole complex plane:

$$\frac{\bar\partial F^\rho_{x,t}}{\partial F^\rho_{x,t}}(z) = \nu^\rho_{x,t}(z). \tag{5.4}$$

Following [1, p. 91], we normalize the solution by the conditions

$$\partial_z F^\rho_{x,t}(z) - 1 \in L^p, \quad F^\rho_{x,t}(0) = 0.$$

The solution is analytically expressible in terms of the Ahlfors–Volterra series (see [1, p. 92]) giving a solution of the Beltrami equation; the solution is unique and provides a functional on the probability space X.

Theorem 5.1 (Smooth welding theorem). *Denote $D_\rho := \{z\colon |z| < \rho\}$ and let*

$$f^\rho_{x,t}(z) = F^\rho_{x,t} \circ \bigl(\widetilde\Psi_{x,t}\bigr)^{-1}(z), \quad z \in \widetilde\Psi_{x,t}(D_\rho);$$
$$g^\rho_{x,t}(z) = F^\rho_{x,t}(z), \quad z \notin \widetilde\Psi_{x,t}(D_\rho).$$

Then

$$\begin{aligned} &f^\rho_{x,t} \text{ is holomorphic and univalent on } \widetilde\Psi_{x,t}(D_\rho),\\ &g^\rho_{x,t} \text{ is holomorphic and univalent on } \bigl(\widetilde\Psi_{x,t}(D_\rho)\bigr)^c, \end{aligned} \tag{5.5}$$

and

$$(f^\rho_{x,t})^{-1} \circ g^\rho_{x,t}(z) = \widetilde\Psi_{x,t}(z), \quad z \in \partial D_\rho. \tag{5.6}$$

Proof. The composition formula for quasi-conformal moduli (see [21, p. 24]),

$$\mu_{u\circ v^{-1}}(z') = \frac{\mu_u - \mu_v}{1 - \mu_u\bar\mu_v}(z)\left(\frac{\partial v}{|\partial v|}\right)^2(z), \quad z' = v(z), \tag{5.7}$$

implies that $f^\rho_{x,t}$ is holomorphic and univalent on $\widetilde\Psi_{x,t}(D_\rho)$. As $\nu^\rho_{x,t} = 0$ on $(D_\rho)^c$, we get holomorphicity of $g^\rho_{x,t}$. □

The following compositions are well defined for $z \in \partial D_\rho$:

$$\bigl((f^\rho_{x,t})^{-1} \circ g^\rho_{x,t}\bigr)(z) = \bigl(\widetilde\Psi_{x,t} \circ (F^\rho_{x,t})^{-1} \circ F^\rho_{x,t}\bigr)(z) = \widetilde\Psi_{x,t}(z).$$

The map $\rho \mapsto \widetilde\Psi_{x,t}(D_\rho)$ is increasing and $\bigcup_{\rho<1} \widetilde\Psi_{x,t}(D_\rho) = D_1$.

Theorem 5.2. *For each $r < 1$, the limit*

$$\lim_{\rho\to 1} f^{\rho}_{x,t}(z) =: \varphi_{x,t}(z)$$

exists uniformly in $z \in D_r$ and defines a univalent function $\varphi_{x,t}$ on D_1.

Proof. We follow the idea of Loewner. In the case of a conformal map of a slit domain, we consider an infinitesimal deformation written in a "multiplicative way" which is the composition of maps.

Lemma 5.3. *Denote*

$$\delta_\rho(F^{\rho}_{x,t}) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\bigl\{(F^{\rho}_{x,t})^{-1} \circ F^{\rho+\varepsilon}_{x,t} - \mathrm{Id}\bigr\}. \tag{5.8}$$

Then

$$\delta_\rho(F^{\rho}_{x,t})(z) = \frac{1}{\pi}\int_{\partial D_\rho} \sigma(z')\frac{z}{(z-z')z'}\,\mathrm{d}z', \quad z \in D_\rho, \tag{5.9}$$

where $\sigma := \mu_{\tilde{\Psi}_{x,t}}$.

Proof. Denote $v_\varepsilon := (F^{\rho}_{x,t})^{-1} \circ F^{\rho+\varepsilon}_{x,t}$. Then $v_\varepsilon(z) \to z$, as $\varepsilon \to 0$. The complex dilatation is calculated by the composition formula which gives

$$\mu_{v_\varepsilon} = \sigma 1_{\{z:\ \rho<|z|<\rho+\varepsilon\}} + \chi_\varepsilon, \quad \lim_{\varepsilon\to 0} \chi_\varepsilon = 0. \tag{5.10}$$

We use Eq. (3) of [1, p. 91]

$$(\partial v_\varepsilon - 1)(z) = -\frac{1}{\pi}\int_{\{z:\ \rho<|z'|<\rho+\varepsilon\}} \frac{1}{(z-z')^2}\sigma\,\partial v_\varepsilon\,\mathrm{d}z' \wedge \mathrm{d}\bar{z}'.$$

Then $\partial v_\varepsilon \to 1$ as $\varepsilon \to 0$, and the integral with respect to the area measure converges towards the curvilinear integral on ∂D_ρ, and we get

$$\partial\bigl[\delta_\rho(F^{\rho}_{x,t})\bigr](z) = -\frac{1}{\pi}\int_{\partial D_\rho} \frac{\sigma(z')}{(z'-z)^2}\,\mathrm{d}z'. \tag{5.11}$$

As $\delta_\rho(F^{\rho}_{x,t})$ is holomorphic in D_ρ and vanishes at zero, we conclude by integrating with respect to z. □

Corollary 5.4. *As $\rho \to 1$, $F^{\rho}_{x,t}$ converges uniformly with all its derivatives in a neighbourhood of $z = 0$. For the univalent function $f^{\rho}_{x,t}$, all coefficients of its Taylor expansion at the origin converge.*

Proof. The kernel $(z - z')^{-2}$ has no singularity for z close to zero and $z' \in D_\rho$, where $\rho > 1/2$. As a dilatation, σ is bounded by 1. □

Corollary 5.5. *We have*

$$\left|\delta_\rho(F^\rho_{x,t})\right| \leqslant 2\frac{|z|}{\rho(\rho - |z|)} \tag{5.12}$$

and

$$\delta_{\rho+\varepsilon}(F^{\rho+\varepsilon}_{x,t}) = \delta_\rho(F^\rho_{x,t}) + \mathrm{o}(\varepsilon).$$

Proof. Since $|\sigma| < 1$, we deduce (5.12) from (5.9). □

To prove that $\lim_{\rho\to 1} F^\rho_{x,t}(z_0)$ exists, we use the following Lemma.

Lemma 5.6. *For ρ fixed, consider the differential equation*

$$\dot{z}(s) = \delta_{\rho+s}(F^{\rho+s}_{x,t})\big(z(s)\big), \quad z(0) = z_0. \tag{5.13}$$

If

$$z(s) \in D_\rho \quad \forall s \in [0, 1 - \rho_0[, \tag{5.14}$$

then

$$F^{\rho+s}_{x,t}(z_0) = F^\rho_{x,t}\big(z(s)\big). \tag{5.15}$$

Proof. Fixing $\varepsilon > 0$ sufficiently small, from (5.4) we have

$$F^{\rho+\varepsilon} = F^\rho\big[I + \varepsilon\delta_\rho(F^\rho)\big] \tag{5.16}$$

which gives a discrete approximation

$$z(\varepsilon) = z_0 + \varepsilon\delta_\rho(F^\rho). \tag{5.17}$$

We deduce from (5.16) the approximative identities:

$$\begin{aligned} F^{\rho+k\varepsilon} &\simeq F^{\rho+(k-1)\varepsilon}\big[I + \varepsilon\delta_{(k-1)\varepsilon}(F)\big] \simeq F^{\rho+(k-2)\varepsilon}\big[I + \varepsilon\delta_{(k-2)\varepsilon}(F)\big]\big[I + \varepsilon\delta_{(k-1)\varepsilon}(F)\big] \\ &\simeq F^\rho\big[I + \varepsilon\delta_0(F)\big]\big[I + \varepsilon\delta_\varepsilon(F)\big]\cdots\big[I + \varepsilon\delta_{(k-2)\varepsilon}(F)\big]\big[I + \varepsilon\delta_{(k-1)\varepsilon}(F)\big], \end{aligned}$$

and we are left to associate to this identity the recursion formula

$$\begin{aligned} &z(\varepsilon) = z_0 - \varepsilon\delta_0(F), \quad z(2\varepsilon) = z(\varepsilon) - \varepsilon\delta_\varepsilon(F), \quad \ldots, \\ &z\big((j+1)\varepsilon\big) = z(j\varepsilon) - \varepsilon\delta_{j\varepsilon}, \quad j \in \{0, \ldots, k-1\}. \qquad \square \end{aligned}$$

Remark 4. As $F^{\rho}_{x,t}$ is continuous, $\lim_{\rho\to 1} F^{\rho}_{x,t}(z_0)$ exists uniformly on the set of z_0 satisfying (5.14).

This set is described more precisely by the following lemma.

Lemma 5.7. *Given $r < 1$, there exists ρ_0 such that hypothesis* (5.14) *is satisfied for any $z_0 \in D_r$.*

Proof. Denote $u(s) = |z(s)|$. Using (5.13), along with the fact that $|u(s)| < 1$, we get the inequality

$$\frac{\mathrm{d}u(s)}{\mathrm{d}s} \leqslant 2\frac{1}{\rho(\rho - u(s))}, \quad u(0) = |z_0|.$$

Comparing $u(s)$ with the solution of the ODE

$$\rho(\rho - v)\,\mathrm{d}v = 2\,\mathrm{d}s, \quad v(0) = r,$$

we have to show that there exists ρ sufficiently close to 1, such that this ODE has a regular solution for $s \in [0, 1-\rho]$. The solution of the ODE is

$$\rho^2 v - \frac{1}{2}\rho v^2 = 2s + \rho^2 r - \frac{1}{2}\rho r^2.$$

Equivalence of this solution in implicit form with the solution of the original ODE holds on the interval where the discriminant $\Delta(s)$ of the second order equation is non-zero: $\Delta(s) \neq 0$, $s \in [0, 1-\rho]$. We have

$$\Delta(s) = \rho^4 - 2\rho\left(2s + \rho^2 r - \frac{1}{2}\rho r^2\right) \neq 0, \quad s \in [0, 1-\rho]$$

or $(\rho - r)^2\rho \neq 2s$. This relation is assured by choosing ρ such that $\rho(\rho - r)^2 > 2(1-\rho)$. $\square$

End of proof of Theorem 5.2. It is known that if a sequence of univalent functions converges on compact subsets of the open unit disk to a non-constant function $\varphi_{x,t}$, then $\varphi_{x,t}$ is univalent in the open unit disk (the proof is a direct consequence of Rouché's theorem). On the contrary, univalence on the boundary needs an extra assumption:

$$\varphi_{x,t} \text{ is continuous and injective on } \bar{D}_1. \tag{5.18}$$

Theorem 5.8 (Stochastic welding theorem). *Assume that* (5.18) *holds true. There exists a function $\gamma_{x,t}$ univalent outside the disk such that*

$$\big((\varphi_{x,t})^{-1} \circ \gamma_{x,t}\big)(\exp \mathrm{i}\theta) = g_{x,t}(\exp \mathrm{i}\theta) \tag{5.19}$$

where $g_{x,t}$ is defined through (1.3).

Proof. Under assumption (5.18) we have $(\varphi_{x,t})^{-1} = \lim_{\rho\to 1}(f^{\rho}_{x,t})^{-1}$ and the stochastic welding is obtained as the limit of the smooth welding. □

See [11] for generalized conformal welding.

6. Infinitesimal Beltrami equations

We consider again

$$\mathrm{d}g^r_{x,t} = (\mathrm{d}_t z^r_{x,t}) \circ g^r_{x,t} \quad \text{where } z^r_{x,t}(\theta) = \sum_{n>1} \frac{r^n}{\sqrt{n^3-n}} \bigl(x_{2n}(t)\cos n\theta + x_{2n+1}(t)\sin n\theta\bigr).$$

As before, let $\widetilde{Z}^r_{x,t}$ be the extension of the random field $z^r_{x,t}$ from the circle to the unit disk and denote by $\widetilde{\Psi}^r_{x,t}$ the corresponding flow on the disk. Outside the origin, $\widetilde{\Psi}^r_{x,t}$ is a C^∞ flow; solving the Beltrami equation leads to $F^r_{x,t}$. We have the key fact that the covariance of $\bar\partial \widetilde{Z}^r_{x,t}$ satisfies uniformly in $r \to 1$ decreasing estimates in terms of the hyperbolic distance.

Now fix r and vary another deformation parameter: the time t along the evolution of the stochastic flow $\widetilde{\Psi}^r_{x^\varepsilon,t}$. For the differential calculus along the time variable, we use the transfer principle (see [22, Chapter VIII]) and proceed by smoothening the Brownian motion. To a Brownian motion x_k, a mollifier a, that is a C^∞ function with compact support contained in the interval $[-1, 0]$, and a number $\varepsilon > 0$, we associate the smoothened Brownian motion x^ε_k defined by

$$x^\varepsilon_k(t) = \int_{-\varepsilon}^{0} x_k(t-s)\, a\left(\frac{s}{\varepsilon}\right) \frac{\mathrm{d}s}{\varepsilon}.$$

By definition, the paths of $x^\varepsilon(\cdot)$ are C^∞ functions. As in Eq. (3.6), let

$$Z^r_{x^\varepsilon,t} := V^r_{x^\varepsilon,t} + \mathrm{i} W^r_{x^\varepsilon,t},$$

where

$$V^r_{x^\varepsilon,t} = \sum_{n>1} \frac{r^n}{\sqrt{n^3-n}} \widehat{K}(yn) \bigl(x^\varepsilon_{2n}(t)\cos n\theta + x^\varepsilon_{2n+1}(t)\sin n\theta\bigr),$$
$$W^r_{x^\varepsilon,t} = -6 \sum_{n>1} \frac{r^n}{\sqrt{n^3-n}} \widehat{K}'(yn) \bigl(x^\varepsilon_{2n+1}(t)\cos n\theta - x^\varepsilon_{2n}(t)\sin n\theta\bigr).$$

Let $z = \exp(-y + \mathrm{i}\theta)$. The vector field $\widetilde{Z}^r_{x^\varepsilon,t}(z) := \mathrm{i} z Z^r_{x^\varepsilon,t}(-\mathrm{i}\log z)$ depends smoothly on t; the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Z}^r_{x^\varepsilon,t}$ with respect to t defines a non-autonomous C^2 vector field on the unit disk D. We consider the associated flow

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\Psi}^r_{x^\varepsilon,t}(z)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Z}_{x^\varepsilon,t}\right)\big(\widetilde{\Psi}_{x^\varepsilon,t}(z)\big),\quad \widetilde{\Psi}^r_{x^\varepsilon,0}(z)=z. \tag{6.1a}$$

It results from the limit theorem that $\lim_{\varepsilon\to 0}\widetilde{\Psi}^r_{x^\varepsilon,t}(z)=\widetilde{\Psi}^r_{x,t}(z)$. Letting $\Theta^r_{x^\varepsilon,t}:=(\widetilde{\Psi}^r_{x^\varepsilon,t})^{-1}$, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Theta^r_{x^\varepsilon,t}\right)(z)=-(\Theta^r_{x^\varepsilon,t})'(z)\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Z}^r_{x^\varepsilon,t}(z)\right). \tag{6.1b}$$

Following Eqs. (5.1)–(5.5), we denote

$$f^r_{x^\varepsilon,t}=F^r_{x^\varepsilon,t}\circ\big(\widetilde{\Psi}^r_{x^\varepsilon,t}\big)^{-1}.$$

6.1. An infinitesimal Beltrami equation

Theorem 6.1 (Loewner's equation along a stochastic flow). *The infinitesimal increments*

$$\delta_{t,\varepsilon}F^r:=\left(\frac{\mathrm{d}}{\mathrm{d}t}F^r_{x^\varepsilon,t}\right)\circ(F^r_{x^\varepsilon,t})^{-1} \tag{6.2}$$

satisfy

$$\bar\partial(\delta_{t,\varepsilon}F^r)=A^r_{x^\varepsilon,t},\quad A^r_{x^\varepsilon,t}:=\left[\frac{\partial f^r_{x^\varepsilon,t}}{\overline{\partial f^r_{x^\varepsilon,t}}}\bar\partial\left(\frac{\mathrm{d}\widetilde{Z}^r_{x^\varepsilon,t}}{\mathrm{d}t}\right)\right]\circ(f^r_{x^\varepsilon,t})^{-1} \tag{6.3}$$

and are given by the formula:

$$(\delta_{t,\varepsilon}F^r)(z)=\dot\Gamma_{x^\varepsilon,t}:=\frac{1}{2\pi\mathrm{i}}\int\limits_{f^r_{x^\varepsilon,t}(D_r)}\frac{1}{z-z'}A^r_{x^\varepsilon,t}(z')\,\mathrm{d}z'\wedge\mathrm{d}\bar z',\quad z\in\mathbb{C}. \tag{6.4}$$

Proof. Differentiating the identity $\bar\partial(f^r_{x^\varepsilon,t})=0$ and taking (6.1) into account, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}f^r_{x^\varepsilon,t}=\delta_tF\circ f^r_{x^\varepsilon,t}-(\partial f^r_{x^\varepsilon,t})\circ\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{Z}^r_{x^\varepsilon,t};\\&\bar\partial\left\{\delta_tF\circ f^r_{x^\varepsilon,t}-(\partial f^r_{x^\varepsilon,t})\circ\frac{\mathrm{d}}{\mathrm{d}t}Z_{x^\varepsilon,t}\right\}=0.\end{aligned} \tag{6.5}$$

We recall the rule of change of variables for holomorphic and anti-holomorphic derivatives which can be found in [1]. Denote $\zeta=f(z)$, $f_z=\partial f$, $f_{\bar z}=\bar\partial f$; notice the identities $\overline{\partial f}=\bar\partial\bar f$ and $\overline{\bar\partial f}=\partial\bar f$. We have

$$\bar\partial(g\circ f)=\big((\bar\partial g)\circ f\big)\big(\overline{\partial f}\big)+\big((\partial g)\circ f\big)(\bar\partial f). \tag{6.6}$$

In fact,

$$\mathrm{d}f = (\partial f)\,\mathrm{d}z + (\bar\partial f)\,\mathrm{d}\bar z,$$
$$\mathrm{d}g = (\partial g)\,\mathrm{d}\zeta + (\bar\partial g)\,\mathrm{d}\bar\zeta,$$
$$\mathrm{d}(g\circ f) = (\partial_\zeta g)(\partial_z f\,\mathrm{d}z + \partial_{\bar z} f\,\mathrm{d}\bar z) + (\partial_{\bar\zeta} g)\overline{(\partial_z f\,\mathrm{d}z + \partial_{\bar z} f\,\mathrm{d}\bar z)}.$$

The second equation of (6.5) follows from the first one since $\bar\partial f^r_{x^\varepsilon} = 0$. Next exploit (6.6), the vector fields being considered as infinitesimal transformations: for the first term we take $f = f^r_{x^\varepsilon,t}$ and $g = \exp(\eta\delta_t F)$; as this choice gives $\bar\partial f = 0$, we get

$$\bar\partial\{\delta_t F\circ f^r_{x^\varepsilon,t}\} = (\bar\partial\delta_t F)\overline{\partial(f^r_{x^\varepsilon,t})}.$$

Taking $f = \exp(-\eta\frac{\mathrm{d}}{\mathrm{d}t}Z_{x^\varepsilon,t})$ and $g = f_{x^\varepsilon,t}$, we have

$$\bar\partial\left\{-f^r_{x^\varepsilon,t}\circ\frac{\mathrm{d}}{\mathrm{d}t}\widetilde Z^r_{x^\varepsilon,t}\right\} = -\partial f^r_{x^\varepsilon,t}\times\bar\partial\frac{\mathrm{d}}{\mathrm{d}t}\widetilde Z^r_{x^\varepsilon,t},$$

which proves (6.3). To obtain the integrated form (6.4), we use the solution of the $\bar\partial$ equation. Define

$$H_1(z) := \frac{1}{2\pi\mathrm{i}}\int\limits_{f^r_{x,t}(D)}\frac{1}{z-z'}A^r_{x,t}(z')\,\mathrm{d}z'\wedge\mathrm{d}\bar z',\quad z\in\mathbb{C}. \tag{6.7}$$

Let $H_2 := (H_1 - \delta_t F)$, then $\bar\partial H_2 = 0$; therefore H_2 is a holomorphic vector field on the Riemann sphere which vanishes according to the boundary conditions. □

Remark 5. After a change of variables in the integral (6.4), we obtain

$$(\delta_{t,\varepsilon}F^r)(z) = \frac{1}{2\pi\mathrm{i}}\int\limits_{D_r}\frac{1}{z-f^r_{x^\varepsilon,t}(u)}|\partial f^r_{x^\varepsilon,t}|^2A^r_{x^\varepsilon,t}\big(f^r_{x^\varepsilon,t}(u)\big)\,\mathrm{d}u\wedge\mathrm{d}\bar u,\quad u\in\mathbb{C}. \tag{6.8}$$

From the expression for $A^r_{x^\varepsilon,t}$ in (6.3) we deduce

$$(\delta_{t,\varepsilon}F^r)(z) = \frac{1}{2\pi\mathrm{i}}\int\limits_{D_r}\frac{1}{z-f^r_{x^\varepsilon,t}(u)}(\partial f^r_{x^\varepsilon,t})^2\bar\partial\left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde Z^r\right)\mathrm{d}u\wedge\mathrm{d}\bar u,\quad u\in\mathbb{C}. \tag{6.9}$$

Consider the C^∞ Jordan curve given as image of the unit circle under $F^r_{x,t}$. We want to establish a uniform estimate in r of the modulus of Hölder continuity for the map

$$\psi^r_{x,t}(\theta) := F^r_{x,t}(\cos\theta + \mathrm{i}\sin\theta). \tag{6.10}$$

Note that $\psi^r_{x,t}$ is a function of θ only, which maps the circle into the plane. Thus it is not a stochastic flow in the usual sense where the flow provides a correspondence between two spaces of the same dimension.

The criterion for Hölder continuity in [10] is given for flows in Itô form. Consider the classical ODE

$$\frac{\mathrm{d}F^r_{x^\varepsilon,t}}{\mathrm{d}t}(\zeta) = (\dot{\Gamma}_{x^\varepsilon,t})\big(F^r_{x^\varepsilon,t}(\zeta)\big) \tag{6.11}$$

where the vector field $\dot{\Gamma}_{x^\varepsilon,t}$ is defined by (6.2)–(6.4). When $\varepsilon \to 0$, the flow (6.11) converges to the Stratonovich flow, see [13],

$$\delta_t F^r_{x,t}(\zeta) = (\delta_t \Gamma_{x,t})\big(F^r_{x,t}(\zeta)\big). \tag{6.12}$$

Passing from the Stratonovich to the Itô flow is not straightforward in this case and not studied in the present work. The goal would be to show that the stochastic contraction is bounded when ζ tends to the boundary of $F_{x,t}(D)$, which would come from the estimate by the hyperbolic distance of $|\mathcal{D}(z_1, z_2)|$ given in (3.16)–(3.20).

The purpose of the next paragraph is to obtain uniform estimates in r for the maps $F^r_{x,t}$ and to deduce estimates for the univalent function $\varphi_{x,t}$.

6.2. *The area of the image $F^r_{x,t}$*

Theorem 6.2. *Denote $a^r_x(t) := \operatorname{area}(F^r_{x,t}(D_r))$. There exist constants c_1, c_2, c_3, independent of $r < 1$, such that for any $R > 0$,*

$$\operatorname{Prob}\Big(\sup_{t\in[0,T]} \log a^r_x(t) - c_1 T > c_2 + R\Big) \leqslant \exp\Big(-c_3 \frac{R^2}{T}\Big). \tag{6.13}$$

Proof. As $a^r_x(0) = \pi r^2$, the estimate can be reduced to the study of the evolution in t. Since the flow is orientation preserving, the area of $F^r_{x,t}(D_r)$ is the determinant of the Jacobian of the flow integrated over the domain D_r. The Jacobian is calculated by solving the linear stochastic differential equation obtained by differentiating the stochastic differential $\delta_t F^r_{x,t}$ in (6.12) with respect to ζ. We pass to the real coordinate system $z = x + \mathrm{i}y$, $\mathrm{d}z \wedge \mathrm{d}\bar{z} = -2\mathrm{i}\,\mathrm{d}x\,\mathrm{d}y$ and $\delta_t F^r_{x,t} =: u + \mathrm{i}v$. The determinant of the Jacobian of the flow is obtained by integrating the trace

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2\operatorname{Re}\partial(\delta_t F^r_{x,t}).$$

We get

$$a^r_x(t) = \int_{D_r} \det(\partial F^r_{x,t})(z)\,\mathrm{d}x\,\mathrm{d}y = \frac{\mathrm{i}}{2}\int_{D_r} \exp\Bigg(\int_0^t 2\operatorname{Re}(\partial\delta_s F^r_{x,t})\big(z_x(s)\big)\Bigg)\,\mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $\int_0^t 2\operatorname{Re}(\partial\delta_s F^r_{x,t})(z_x(s))$ is the stochastic integral of the stochastic differential considered along $z_x(s) := F^r_{x,s}(z)$. Thus

$$
\begin{aligned}
\mathrm{d}_t a_x^r(t) &= \frac{\mathrm{i}}{2}\int_{D_r} \det(\partial F_{x,t}^r)(z) 2\,\mathrm{Re}(\partial\delta_t F_{x,t}^r)\big(F_{x,t}(z)\big)\,\mathrm{d}z\wedge\mathrm{d}\bar{z} \\
&= \frac{\mathrm{i}}{2}\int_{F_{x,t}^r(D_r)} 2\,\mathrm{Re}(\partial\delta_t F_{x,t}^r)(z)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}. \qquad (6.14)
\end{aligned}
$$

With [1, p. 87] and (6.3), we have

$$
(\partial\delta_t F_{x,t}^r)(z) = \frac{1}{2\mathrm{i}\pi}\int_{F_{x,t}^r(D_r)} \frac{A_{x,t}^r(z')}{(z'-z)^2}\,\mathrm{d}z'\wedge\mathrm{d}\bar{z}'
$$

and

$$
2\,\mathrm{Re}(\partial\delta_t F_{x,t}^r)(z) = \frac{1}{\pi}\,\mathrm{Im}\int_{F_{x,t}^r(D_r)} \frac{A_{x,t}^r(z')}{(z'-z)^2}\,\mathrm{d}z'\wedge\mathrm{d}\bar{z}'.
$$

Substitution in (6.14) gives

$$
\mathrm{d}_t a_x^r(t) = \frac{\mathrm{i}}{2\pi}\,\mathrm{Im}\int_{F_{x,t}^r(D_r)\times F_{x,t}^r(D_r)} \frac{1}{(z-z')^2} A_{x,t}^r(z')\,\mathrm{d}z'\wedge\mathrm{d}\bar{z}'\,\mathrm{d}z\wedge\mathrm{d}\bar{z}. \qquad (6.15)
$$

Since Calderon convolution by the kernel $1/(\pi\lambda^2)$ is an isometry on $L^2(\mathbb{R}^2)$, we get by means of Hölder's inequality,

$$
\begin{aligned}
\big|\mathrm{d}_t a_x^r(x)\big|^2 &\leqslant \frac{1}{\pi^2}\bigg|\int_{F_{x,t}^r(D_r)\times F_{x,t}^r(D_r)} \frac{1}{(z-z')^2} A_{x,t}^r(z')\,\mathrm{d}z'\wedge\mathrm{d}\bar{z}'\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\bigg|^2 \\
&\leqslant \|A^r\|^2_{L^2(F_{x,t}^r(D_r))} \times \|1_{F_{x,t}^r(D_r)}\|^2 \leqslant c a_x(t)^2\,\mathrm{d}t,
\end{aligned}
$$

where c is some constant. The last inequality is a consequence of (3.4) and (6.3). Let $\alpha_x^r(t) := \log a_x^r(t)$. By Itô calculus, $\alpha_x^r(t)$ is a semi-martingale; its stochastic differential is given by

$$
\mathrm{d}_t\alpha_x^r(t) = \gamma_1\,\mathrm{d}\beta + \gamma_2\,\mathrm{d}t \qquad (6.16)
$$

where β is a Brownian motion and γ_i are bounded functions because of (3.4), (6.3). Using the exponential martingale inequality, we deduce inequality (6.13) from (6.16), see [24]. □

Corollary 6.3. *Let $a_x^1(t)$ be the area of $\varphi_{x,t}(D)$, then $a_x^1(t)$ satisfies estimate* (6.13).

Proof. It is sufficient to prove that $b^r := \mathrm{area}(\varphi(D_r))$ satisfies the same estimate for any $r<1$; as the boundary of $\varphi(D_r)$ is a C^∞ Jordan curve, we have by means of (5.1) that $b^r < a_{x,t}^\rho + \varepsilon$ for ρ sufficiently close to 1. □

Lemma 6.4 (Comparison lemma for the hyperbolic metric). *Let G_1, G_2 be two simply connected domains such that $G_1 \subset G_2$ and denote by ds_i^2 the corresponding Poincaré metrics. Then*

$$ds_1^2 \geqslant ds_2^2.$$

Proof. Consider the Poincaré distance given by the infinitesimal Harnack inequality: for every positive harmonic function bounded by 1, the length of its gradient is $\leqslant 1$, where the upper bound is reached for the Poincaré metric. It is sufficient to prove the inequality at the center of the disk. As harmonic functions, which are restrictions to G_1 of harmonic functions on G_2, constitute a subspace of the harmonic functions on G_2, the result follows. □

Given a Jordan curve Γ, consider the two domains D^+, D^- limited by this curve, D^+ being compact. Given $\zeta_0 \in \Gamma$ then $\log(\zeta_0 - \zeta)$ is a holomorphic map of Θ of D^+ into a strip of width at most 2π. Any point of $\Theta(D^+)$ is contained in the middle of a strip of length 4π. Therefore, using the comparison Lemma for hyperbolic metric, we deduce:

Lemma 6.5. *There exists a positive constant c_0, independent of ζ_1, ζ_2, such that*

$$d_{\mathcal{P}}^{D^+}(\zeta_1, \zeta_2) > c_0 \log\left|\frac{\zeta_0 - \zeta_1}{\zeta_0 - \zeta_2}\right|. \tag{6.17}$$

6.3. Uniqueness of the welding

We adopt the point of view of [16, p. 304]. The circle S^1 is the boundary of the two closed hemispheres of the Riemann sphere. Let S^1_+ be the northern hemisphere and S^1_- be the southern hemisphere. Given $h \in \mathrm{Homeo}(S^1)$, we define on $S^1_+ \oplus S^1_-$ an equivalence relation where equivalence classes are constituted by single points with the exception of the boundaries $\partial S^1_\pm$ which are identified using h; the set of equivalence classes has the structure of a topological manifold Δ_h. A continuous function Φ on Δ_h is given by the data of a couple of continuous function $\Phi_\pm$ defined on the closed hemispheres such that $\Phi_+(s) = \Phi_-(h(s))$ on the equator. This family of functions forms an algebra $\mathcal{A}_h$; an equivalent definition is to define Δ_h as the Gelfand spectrum of $\mathcal{A}_h$.

The welding problem is equivalent to the following question:

Does there exist a conformal structure on Δ_h which restricted to each of the open hemispheres coincides with the given conformal structure on the hemisphere?

For such a conformal structure $\mathcal{C}$, we denote by $\Delta_h^{\mathcal{C}}$ the corresponding Riemann surface. By Poincaré's uniformization theorem, up a homeomorphism, there is a unique conformal structure on the sphere; this means that there is a homeomorphism Θ carrying Δ_h onto Δ_{Id}. The image of the equator $\Theta(\partial S^1_+)$ is a Jordan curve $\Gamma_h^{\mathcal{C}}$. We define a *Hölder Jordan curve* as a curve which is parametrizable by a univalent function φ such that φ and φ^{-1} are Hölder continuous.

Theorem 6.6. *Assume that there exists a welding conformal structure $\mathcal{C}_0$ such that $\Gamma_h^{\mathcal{C}_0}$ is a Hölder Jordan curve, then every welding structure $\mathcal{C}$ coincides with $\mathcal{C}_0$.*

Proof. Let Θ_0, Θ be the corresponding homeomorphisms of Δ_h. Then $v := \Theta \circ \Theta_0^{-1}$ defines a new conformal structure on the complement of $\Gamma_h^{\mathcal{C}_0}$. By [15, Corollaries 2 and 4, pp. 267–268], there is a unique conformal structure which coincides with the trivial one on the complement of $\Gamma_h^{\mathcal{C}_0}$; therefore $\mathcal{C} = \mathcal{C}_0$. □

Theorem 6.7. *To a univalent function f defined on $\bar{D}$ we associate the Jordan curve*

$$\Phi(f) = f\left(S^1\right).$$

Let $\varphi_{x,t}$ be the univalent function constructed in Theorem 5.2 *as solution of the stochastic welding problem* (5.19)*. Then $t \mapsto \Phi(\varphi_{x,t})$ defines a Markov processes with values in $\mathcal{J}$.*

Proof. The map $\varphi_{x,t} \mapsto g_{x,t}$ is injective up to a rotation: this operation preserves the law of $g_{x,t}$. The passage from univalent functions to Jordan curves is obtained by taking the quotient of the space $\mathcal{M}$ of univalent functions by the group of Poincaré automorphisms of D (see [2]). □

Appendix A. The canonical Brownian motion on Diff(S^1) is not quasi-symmetric

The classical theory of conformal welding is developed for diffeomorphism of the circle in case where the diffeomorphisms have a quasi-conformal extension to the unit disk. The class of diffeomorphisms h preserving the point at infinity and admitting a quasi-conformal extension to the half plane $\mathcal{P}$ is fully characterized by the Beurling–Ahlfors quasi-symmetry property:

$$\sup_{\theta,\psi \in S^1} \frac{h(\theta+\psi) - h(\theta)}{h(\theta) - h(\theta-\psi)} < \infty. \tag{A.1}$$

If by coincidence the canonical Brownian motion on $\mathrm{Diff}(S^1)$ should satisfy condition (A.1), the results of the present paper would follow more easily without the indirect approach we have followed.

The purpose of this Appendix is to show that, almost surely, $g_{x,t}$ does not satisfy condition (A.1).

A.1. Study of the three point motion

Consider the covariance

$$\mathrm{d}_t z_{x,t}(\theta) * \mathrm{d}_t z_{x,t}(\theta') = \mathcal{B}(\theta - \theta')\,\mathrm{d}t, \quad \mathcal{B}(\psi) = \sum_{k=2}^{\infty} \frac{1}{k^3 - k} \cos(k\psi),$$

by [5, Lemma 3.2] or [9]

$$\mathcal{B}(\psi) = \mathcal{B}(0)\big(1 - \gamma(\psi)\big), \quad \text{where } \gamma(\psi) \simeq c\psi^2 \log^- |\psi|, \text{ as } \psi \to 0,$$

c being a numerical constant. Given a finite subset σ of S^1, we denote by $\sigma(t) = g_{x,t}(\sigma)$ its image under $g_{x,t}$ and by $d := |\sigma(t)|$ its constant cardinality. The function $\mathcal{B}$ is positive definite; the $d \times d$ symmetric matrix

$$Q_{\sigma(t)} := \big(\mathcal{B}(\sigma_i - \sigma_j)\big), \quad \sigma_i, \sigma_j \in \sigma(t),$$

is therefore positive definite. Through $Q_{\sigma(t)}$ we define an Euclidean metric on $\mathbb{R}^\sigma \simeq T((S^1)^{\sigma(t)})$. Then the motion $\sigma(t)$ is Markovian and given as solution of the Itô SDE,

$$\mathrm{d}\sigma(t) = \sqrt{Q_{\sigma(t)}}\,\mathrm{d}x(t), \tag{A.2}$$

where $x(t)$ is an $\mathbb{R}^d$ valued Brownian motion.

We fix three points enumerated in increasing order by $\sigma^-, \sigma, \sigma^+$ and let

$$\rho^+ = \sigma^+ - \sigma, \qquad \rho^- = \sigma - \sigma^-.$$

The covariance matrix of the triplet $\sigma^-, \sigma, \sigma^+$, divided by $\mathcal{B}(0)$, takes the form:

$$\begin{pmatrix} 1 & 1-\gamma(\rho^-) & 1-\gamma(\rho^+ + \rho^-) \\ 1-\gamma(\rho^-) & 1 & 1-\gamma(\rho^+) \\ 1-\gamma(\rho^+ + \rho^-) & 1-\gamma(\rho^+) & 1 \end{pmatrix}.$$

We restrict the corresponding quadratic form to the two dimensional space spanned by ρ^+, ρ^-. The variance of a random variable of the type

$$\alpha\rho^+ + \beta\rho^- = \alpha(\sigma^+ - \sigma) + \beta(\sigma - \sigma^-)$$

is a quadratic form with the coefficients:

$$\begin{aligned} &\mathbb{E}\big[(\sigma^+ - \sigma)^2\big] = 2\gamma(\rho^+), \\ &\mathbb{E}\big[(\sigma^+ - \sigma)(\sigma - \sigma^-)\big] = -\gamma(\rho^+) + \gamma(\rho^+ + \rho^-) - \gamma(\rho^-), \\ &\mathbb{E}\big[(\sigma - \sigma^-)^2\big] = 2\gamma(\rho^-). \end{aligned}$$

We use again that the Itô invariants factorize through ρ^+, ρ^- which means that *the projected process is a Markov process associated to the elliptic operator*

$$\mathcal{L} := \gamma(\rho^+)\frac{\partial^2}{\partial(\rho^+)^2} + 2\big(\gamma(\rho^+ + \rho^-) - \gamma(\rho^-) - \gamma(\rho^+)\big)\frac{\partial^2}{\partial\rho^+\partial\rho^-} + \gamma(\rho^-)\frac{\partial^2}{\partial(\rho^-)^2},$$

and apply Itô calculus to the function

$$\phi := \log^- \rho^+ - \log^- \rho^-.$$

As ρ^+, ρ^- play symmetric roles, the properties of the random variable ϕ are stable under the symmetry $\eta \mapsto -\eta$. Since $\mathcal{L}(\rho^+) = \mathcal{L}(\rho^-) = 0$, we get

$$\begin{aligned}
&\mathcal{L}(\log^- \rho^\pm) = \frac{1}{(\rho^\pm)^2}\gamma(\rho^\pm),\\
&\mathcal{L}\phi = -\frac{\gamma(\rho^-)}{(\rho^-)^2} + \frac{\gamma(\rho^+)}{(\rho^+)^2} \simeq \phi,\\
&\frac{1}{2}\|\nabla\phi\|^2 = \frac{\gamma(\rho^+)}{(\rho^+)^2} - \frac{\gamma(\rho^+ + \rho^-) - \gamma(\rho^+) - \gamma(\rho^-)}{\rho^+\rho^-} + \frac{\gamma(\rho^-)}{(\rho^-)^2}.
\end{aligned} \tag{A.3}$$

Observe that $\rho^+ = \rho^- \mathrm{e}^\phi$ and

$$\begin{aligned}
\frac{1}{2}\|\nabla\phi\|^2 = {} & \log^- \rho^- - \phi - (1 + \mathrm{e}^\phi)^2 \mathrm{e}^{-\phi}\big(\log^- \rho^- - \log(1 + \mathrm{e}^\phi)\big)\\
& + \mathrm{e}^\phi(\log^- \rho^- - \phi) + \mathrm{e}^{-\phi}\log^- \rho^- + \log^- \rho^-.
\end{aligned}$$

We note the remarkable fact that the coefficient of $\log^- \rho^-$ vanishes: $2 - \mathrm{e}^{-\phi} - 2 - \mathrm{e}^\phi + \mathrm{e}^\phi + \mathrm{e}^{-\phi} = 0$. Thus we get

$$\frac{1}{2}\|\nabla\phi\|^2 \simeq |\phi|, \quad \phi \to \pm\infty. \tag{A.4}$$

Fixing $h_s = 2^{-s}$ and $\theta_k = k2^{\lambda s}$, the corresponding function $\phi_{k,s}$, taking into account (A.2) and (A.3), admits as comparison SDE in the sense of Ikeda–Watanabe the following SDE

$$\mathrm{d}\tilde{y} = \sqrt{|\tilde{y}|}\,\mathrm{d}b(t) + \tilde{y}\,\mathrm{d}t, \quad \tilde{y} = 0, \tag{A.5}$$

where b is a scalar-valued Brownian motion which can be written as a deterministic function of the $\{x_k\}$, i.e., $b(t) = \sum_k \int_0^t \alpha^k(s)\,\mathrm{d}x_k(s)$ with α^k adapted such that $\sum_k \alpha^k = 1$. According to [12], we have

$$\sup_{t\in[0,T]} \big|\phi_{k,s}(t)\big| \simeq \sup_{t\in[0,T]} \big|\tilde{y}(t)\big|. \tag{A.6}$$

A remarkable fact is that the comparison Eq. (A.5) is independent of the scale s.

Theorem A.1. *Almost surely, the canonical Brownian motion on the group of diffeomorphism of S^1 does not satisfy the quasi-symmetry condition* (A.1).

Proof. We shall follow a Dubois–Reymond type methodology of condensation of singularities. Consider the sequence $s_q = 2^q$ and take θ_{k_q} such that $|\theta_{k_q} - \theta_{k_{q'}}| > |q - q'|^{-3}$, $q \neq q'$. Then the sequence of processes $\{w_q\}$ defined by $w_q(t) := \phi_{k_q, s_q}(t)$ is asymptotically independent: in the realm of Gaussian variables asymptotical independence is equivalent to asymptotical orthogonality. Considering the six point motion covariance matrix, we remark that

$$\mathrm{d}_t w_q * \mathrm{d}_t w_{q'} = \mathrm{O}\left(\frac{\log(|q - q'|)}{q \wedge q'}\right).$$

As each of these processes is governed by the same comparison Eq. (A.5), and as the process driven by (A.5) is not uniformly bounded, the Borel–Cantelli lemma shows that a.s.

$$\limsup_{q \to \infty} w_q(t) = \infty. \qquad \square$$

Appendix B. Asymptotic expansion of some inverse Fourier transforms

Our objective is to compare the series defining the covariance functions with the Poincaré distance in the half plane. We give two integral representations of the series, one as a convolution double integral (Proposition B.3 and Appendix B.2); the other one is a one time integrated form of the first representation and is expressed in terms of a single integral (Appendix B.2). Both representations provide a mean for studying the series.

In Appendix B.1, using the double integral representation of the series, we prove the estimates obtained in Theorems B.1 and B.2 below. In Appendices B.2 and B.3, we discuss how to obtain the integral representations of the series.

In Appendix B.3, we construct the kernel functions of Proposition B.3 below. In Appendix B.3, we express the series $\phi, \psi, \alpha, \beta, \gamma$ of (B.3)–(B.4) as h-transforms, that is in the form

$$\sum_{i,j} \int_0^1 P_{ij}(y_1, y_2, s) h\big((\varepsilon_i y_1 + \varepsilon_j y_2)s + \theta\big)\, \mathrm{d}s, \quad \varepsilon_i = -1, 0, 1,$$

where P_{ij} is a polynomial in s and a rational function *homogeneous of degree* 0 in (y_1, y_2); if $y_1 = y_2 = y$, then $P_{ij}(y, y, s)$ does not depend on y. The study of P_{ij} permits to study the behaviour of the functions α, β, γ. We give explicit expressions for the polynomials P_{ij} and obtain asymptotic expansions for the covariance functions.

In the half plane $\zeta = \theta + \mathrm{i}y$, $y > 0$, the Poincaré metric is given by $|\mathrm{d}\zeta|/y$. Let d be the Poincaré distance; then, in particular,

$$d\big((\theta, y), (\theta', y)\big) = \min_f \int_\theta^{\theta'} \frac{\sqrt{1 + f'(t)^2}}{f(t)}\, \mathrm{d}t \sim \frac{1}{y}|\theta - \theta'|$$

with $f(\theta) = f(\theta') = y$ and $d((\theta, y), (\theta, y')) = |\log(y/y')|$. With the triangle inequality for the distance, we can deduce from the two previous estimates a domination for $d((\theta, y), (\theta', y'))$.

Let $\rho = d((\theta, y), (\theta', y'))$, we have (see for example [14])

$$\cosh \rho = \frac{|\theta - \theta'|^2 + y^2 + (y')^2}{2yy'}. \tag{B.1}$$

When $y \to 0$, the behaviour of the covariances of the vector field $\bar{\partial} Z$ is estimated in terms of the hyperbolic distance $d(\zeta_1, \zeta_2)$. See Eqs. (3.16)–(3.20).

Theorem B.1. *Letting $\mathcal{Q}(y) = \widehat{K}(y) + 7\widehat{K}'(y) + 6\widehat{K}''(y)$ as in Eq.* (2.5), *we have*

$$\mathcal{Q}(y) = 2\int_0^1 (1-s)\bigl(1-6s^2\bigr)\cos(ys)\,\mathrm{d}s - 14\int_0^1 (1-s)s\sin(ys)\,\mathrm{d}s.$$

Denoting $\mathcal{D}(\zeta_1, \zeta_2)\,\mathrm{d}t = \mathbb{E}[\mathrm{d}_t \bar{\partial} Z_{x,t}(\zeta_1, \zeta_2) * \overline{\mathrm{d}_t \bar{\partial} Z_{x,t}(\zeta_1, \zeta_2)}]$, then for $\zeta_j = \theta_j + \mathrm{i}y_j$, $j = 1, 2$, and $\theta = \theta_1 - \theta_2$,

$$\mathcal{D}(\zeta_1, \zeta_2) = \sum_{|n|>1} \frac{|n|}{n^2 - 1} \mathcal{Q}(y_1 n)\mathcal{Q}(y_2 n) \exp\bigl(\mathrm{i}n(\theta_1 - \theta_2)\bigr).$$

If $\theta \neq 0$, there exist constants c_1, c_2, c_3 such that for $\sup(y_1, y_2) \leqslant c_3|\theta| < c_2$, we have the estimate

$$\bigl|\mathcal{D}(\zeta_1, \zeta_2)\bigr| \leqslant \exp\bigl(-c_1 d(\zeta_1, \zeta_2)\bigr).$$

If $\theta = 0$, then

$$\bigl|\mathcal{D}(y_1, y_2)\bigr| \leqslant \frac{\inf(y_1, y_2)}{\sup(y_1, y_2)}.$$

To handle the different cases in a unified way, we consider a function h of the type

$$h(t) = \sum_{n \geqslant 1} \lambda(n) \cos nt. \tag{B.2}$$

Assume that h is a C^∞ function defined on $]0, 2\pi[$ and moreover that there exist $c > 0$ and $\delta > 0$ such that $h(t) \geqslant c$ for $0 < t < \delta$. Assume furthermore that for $y > 0$, the integral $\int_0^1 h(ys)\,\mathrm{d}s$ is finite. Consider the series

$$\begin{aligned}\phi(y_1, y_2, \theta) &= \sum_{n\geqslant 1} \lambda(n)\widehat{K}(y_1 n)\widehat{K}(y_2 n)\cos n\theta,\\ \psi(y_1, y_2, \theta) &= \sum_{n\geqslant 1} \lambda(n)\widehat{K}'(y_1 n)\widehat{K}(y_2 n)\sin n\theta\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}\alpha(y_1, y_2, \theta) &= \sum_{n\geqslant 1} \lambda(n)\big(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\big)\big(6\widehat{K}''(y_2 n) + \widehat{K}(y_2 n)\big)\cos n\theta,\\ \beta(y_1, y_2, \theta) &= \sum_{n\geqslant 1} \lambda(n)\widehat{K}'(y_1 n)\widehat{K}'(y_2 n)\cos n\theta,\\ \gamma(y_1, y_2, \theta) &= \sum_{n\geqslant 1} \lambda(n)\big(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\big)\widehat{K}'(y_2 n)\sin n\theta,\end{aligned} \tag{B.4}$$

where

$$\begin{aligned}\widehat{K}(x) &= 2\int_0^1 (1-s)\cos(sx)\,\mathrm{d}s = \frac{2(1-\cos x)}{x^2} = 1 - \frac{x^2}{12} + \frac{x^4}{360} - \cdots,\\ \widehat{K}'(x) &= -\frac{4}{x^3}(1-\cos x) + \frac{2\sin x}{x^2} = -2\int_0^1 s(1-s)\sin(sx)\,\mathrm{d}s\\ &= -\frac{x}{6} + \frac{x^3}{90} - \frac{x^5}{3360} + \cdots,\\ \widehat{K}''(x) &= \frac{12}{x^4}(1-\cos x) - \frac{8\sin x}{x^3} + \frac{2\cos x}{x^2} = -2\int_0^1 s^2(1-s)\cos(sx)\,\mathrm{d}s;\end{aligned} \tag{B.5}$$

$$\begin{aligned}6\widehat{K}''(\xi) + \widehat{K}(\xi) &= \left(\frac{2}{\xi^2} + \frac{72}{\xi^4}\right)(1-\cos\xi) - \frac{48}{\xi^3}\sin\xi + \frac{12}{\xi^2}\cos\xi\\ &= \frac{7}{60}\xi^2 - \frac{31}{5040}\xi^4 + \cdots\\ &= 2\int_0^1 (1-s)\big(1-6s^2\big)\cos(s\xi)\,\mathrm{d}s,\end{aligned} \tag{B.6a}$$

and since $\int_0^1 (1-s)(1-6s^2)ds = 0$, for any differentiable function g,

$$\int_0^1 (1-s)\big(1-6s^2\big)g(s)\,\mathrm{d}s = \frac{1}{2}\int_0^1 u(1-u)^2(2+3u)g'(u)\,\mathrm{d}u, \tag{B.6b}$$

$$\begin{aligned}\widehat{K}'(x)^2 &= \frac{32}{x^6}(1-\cos x) + \left(\frac{2}{x^4}-\frac{8}{x^6}\right)(1-\cos 2x) - \frac{16\sin x}{x^5} + \frac{8\sin 2x}{x^5}\\ &= \frac{16}{15}\int_0^1 (1-t)^3\big(1-2t-4t^2\big)\cos(2xt)\,\mathrm{d}t - \frac{2}{15}\int_0^1 (1-t)^4(3+2t)\,\mathrm{d}t. \end{aligned}\tag{B.6c}$$

Theorem B.2. *Consider the series given by* (B.4)*, where* $h(t)=\sum_{n\geqslant 1}\frac{1}{n}\cos nt$.
If $\theta\neq 0$*, there exist constants* $c_1, c_2, c_3 > 0$*, a neighbourhood*

$$V = \big\{0 < \sup(y_1, y_2) \leqslant c_3|\theta| \leqslant c_2\big\}$$

such that if $(\theta, y_1, y_2) \in V$*, then*

$$\big|\alpha(y_1,y_2,\theta)\big| + \big|\beta(y_1,y_2,\theta)\big| + \big|\gamma(y_1,y_2,\theta)\big| \leqslant \frac{c_1}{\cosh(\rho)}$$

where $\rho = d((y_1,\theta),(y_2,0))$ *is the hyperbolic distance defined in* (B.1).
If $\theta = 0$*, there exist* c_1, c_2 *such that for* $\sup(y_1,y_2) < c_1$*,*

$$\big|\alpha(y_1,y_2,0)\big| + \big|\beta(y_1,y_2,0)\big| \leqslant \frac{c_2}{\cosh(\rho)}$$

where $\rho = |\log(y_1/y_2)|$.

For the proofs of Theorems B.1 and B.2 (see Appendix B.1), we use the following proposition.

Proposition B.3. *There exist kernels* $N_\phi, N_\psi, N_\alpha, N_\beta, N_\gamma$ *which are functions of* (y_1, y_2, θ) *such that*

$$\begin{aligned}\phi(y_1,y_2,\theta) &= \int_0^1\int_0^1 (1-s)(1-u)N_\phi(sy_1, uy_2, \theta)\,\mathrm{d}s\,\mathrm{d}u,\\ \psi(y_1,y_2,\theta) &= \int_0^1\int_0^1 s(1-s)(1-u)N_\psi(sy_1, uy_2, \theta)\,\mathrm{d}s\,\mathrm{d}u;\end{aligned}\tag{B.7}$$

$$\begin{aligned}
\alpha(y_1, y_2, \theta) &= \int_0^1 (1-s)(1-t)\big(1-6s^2\big)\big(1-6t^2\big) N_\alpha(sy_1, ty_2, \theta)\,\mathrm{d}s\,\mathrm{d}t, \\
\beta(y_1, y_2, \theta) &= -\int_0^1\int_0^1 s(1-s)t(1-t) N_\beta(sy_1, ty_2, \theta)\,\mathrm{d}s\,\mathrm{d}t, \\
\gamma(y_1, y_2, \theta) &= \int_0^1\int_0^1 (1-s)\big(1-6s^2\big)t(1-t) N_\gamma(sy_1, ty_2, \theta)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{B.8}$$

We consider the following choices for h:

- When $\lambda(n) = 1/n$, $n \geqslant 1$,

$$h(t) = \sum_{n\geqslant 1} \frac{\cos nt}{n} = -\log\left|2\sin\left(\frac{t}{2}\right)\right| \quad \text{for } 0 < t < 2\pi. \tag{B.9}$$

- For $\lambda(1) = 0$ and

$$\lambda(n) = \begin{cases} n/(n^2-1), & n \geqslant 2, \\ 1/(n^3-n), & n \geqslant 2, \end{cases}$$

or since

$$\frac{1}{n^3-n} = -\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)},$$

we get

$$\begin{aligned}
h(t) &= \sum_{n\geqslant 2} \frac{n\cos nt}{n^2-1} = \left[\sum_{n\geqslant 1} \frac{\cos nt}{n}\cos t\right] - \frac{1}{2} - \frac{1}{4}\cos t, \quad \text{respectively,} \\
h(t) &= \sum_{n\geqslant 2} \frac{\cos nt}{n^3-n} = \left[\sum_{n\geqslant 1} \frac{\cos nt}{n}(\cos t - 1)\right] - \frac{1}{2} + \frac{3}{4}\cos t.
\end{aligned} \tag{B.10}$$

If $\lambda(n)$ is an even function of n, then

$$\begin{aligned}
\sum_{n\neq 0} \lambda(n)\mathcal{Q}(ny_1)\mathcal{Q}(ny_2)\mathrm{e}^{\mathrm{i}n\theta} &= 2\alpha(y_1, y_2, \theta) + 2\beta(y_1, y_2, \theta) \\
&\quad + 2\mathrm{i}\big[\gamma(y_1, y_2, \theta) + \gamma(y_2, y_1, \theta)\big],
\end{aligned} \tag{B.11}$$

where $\mathcal{Q}(y) = (\widehat{K} + 7\widehat{K}' + 6\widehat{K}'')(y)$.

B.1. Proofs of Theorems B.1 *and* B.2, $h(t)=\sum_{n\geqslant 1}\frac{\cos nt}{n}$

Proof of Theorem B.2. If $h(t)$ is given by (B.9), we get

$$
\begin{aligned}
N_\alpha &= -\log\left|\sin^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{y_1s+y_2t}{2}\right)\right| - \log\left|\sin^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{y_1s-y_2t}{2}\right)\right|,\\
\frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t}N_\alpha &= y_1y_2\left[\frac{1-\cos\theta\cos(y_1s+y_2t)}{(\cos\theta-\cos(y_1s+y_2t))^2}-\frac{1-\cos\theta\cos(y_1s-y_2t)}{(\cos\theta-\cos(y_1s-y_2t))^2}\right]\\
&= 2y_1y_2\sin(y_1s)\sin(y_2t)\left[\frac{A}{B}\right]
\end{aligned}
\tag{B.12}
$$

with

$$
\begin{aligned}
A &= \cos^3\theta-\cos\theta\big(1+\cos^2(y_1s)+\cos^2(y_2t)\big)+2\cos(y_1s)\cos(y_2t)\\
&= (1-\cos\theta)\big[2\cos(y_1s)\cos(y_2t)-\cos\theta(1+\cos\theta)\big]-\cos\theta\big(\cos(y_1s)-\cos(y_2t)\big)^2\\
&= (1-\cos\theta)\big[\cos^2(y_1s)+\cos^2(y_2t)\big]-\big[\cos(y_1s)-\cos(y_2t)\big]^2-\cos\theta\sin^2\theta
\end{aligned}
\tag{B.13}
$$

and

$$
\sqrt{B}=(1-\cos\theta)\big[2\cos(y_1s)\cos(y_2t)-(1+\cos\theta)\big]+\big(\cos(y_1s)-\cos(y_2t)\big)^2. \tag{B.14}
$$

If $0<\theta\leqslant 1$ and $y_1\leqslant y_2\leqslant\theta/2$, then $\cos(y_1)\geqslant\cos(y_2)$ and

$$
\begin{aligned}
&2\cos(y_1s)\cos(y_2t)\geqslant 2\cos(y_1)\cos(y_2)\geqslant 1+\cos\theta,\\
&\big(\cos(y_1s)-\cos(y_2t)\big)^2\leqslant(1-\cos\theta)^2.
\end{aligned}
\tag{B.15}
$$

From the first estimate in (B.15), we obtain

$$
\begin{aligned}
\sqrt{B}&\geqslant(1-\cos\theta)\big[2\cos(y_1s)\cos(y_2t)-(1+\cos\theta)\big]\\
&\geqslant(1-\cos\theta)\big[2\cos(y_1)\cos(y_2)-(1+\cos\theta)\big],
\end{aligned}
\tag{B.16}
$$

whereas the second inequality in (B.15) gives

$$
\sqrt{B}\leqslant 2(1-\cos\theta). \tag{B.17}
$$

From Eq. (B.13) and the second inequality in (B.15), we obtain

$$
A\geqslant(1-\cos\theta)\big(\cos^2(y_1s)+\cos^2(y_2t)\big)-(1-\cos\theta)^2-\cos\theta\sin^2\theta.
$$

This gives

$$
A\geqslant(1-\cos\theta)\big[\cos^2(y_1s)+\cos^2(y_2t)-1-\cos^2\theta\big];
$$

on the other hand, $y_1 \leqslant y_2 \leqslant \theta/2$ implies that

$$\cos^2(y_1 s) + \cos^2(y_2 t) \geqslant 1 + \cos^2\theta,$$

which together proves that $A \geqslant 0$. Thus

$$\frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t} N_\alpha \geqslant 0.$$

On the other hand, because of (B.6), if $\theta \neq 0$, the series α is equal to

$$-\int_0^1\int_0^1 (1-s)\big(1-6s^2\big)(1-t)\big(1-6t^2\big) \times \log\left|\left(1 - \frac{\sin^2((y_1 s + y_2 t)/2)}{\sin^2(\theta/2)}\right)\left(\frac{1-\sin^2((y_1 s - y_2 t)/2)}{\sin^2\big(\theta/2\big)}\right)\right| \mathrm{d}s\,\mathrm{d}t.$$

Using (B.6b), this equals

$$\alpha = \int_0^1\int_0^1 f(s) f(t) \frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t} N_\alpha\,\mathrm{d}s\,\mathrm{d}t, \quad \text{where } f(s) \geqslant 0 \text{ for } 0 \leqslant s \leqslant 1. \tag{B.18}$$

We integrate the last integral and obtain that $|\alpha|$ is dominated by

$$\int_0^1\int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t} N_\alpha\,\mathrm{d}s\,\mathrm{d}t = -\log\left|\frac{[\sin^2(\theta/2) - \sin^2((y_1+y_2)/2)][\sin^2(\theta/2) - \sin^2((y_1-y_2)/2)]}{[\sin^2(\theta/2) - \sin^2(y_1/2)]^2[\sin^2(\theta/2) - \sin^2(y_2/2)]^2}\right|. \tag{B.19}$$

Theorem B.1 for α is a consequence of the previous lemma. □

Lemma B.4. *For* $0 < \theta \leqslant 1$ *and* $\sup(y_1, y_2) < \theta/2$*, let*

$$\alpha = \sum_{n \geqslant 1} \frac{1}{n}\big(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\big)\big(6\widehat{K}''(y_2 n) + \widehat{K}(y_2 n)\big)\cos n\theta.$$

There exist two positive constants c_1, c_2*, such that*

$$c_1 \frac{y_1^2 y_2^2(\theta^2 - y_1^2 - y_2^2)}{\theta^6} \leqslant |\alpha| \leqslant c_2 \frac{y_1^2 y_2^2}{\theta^4}. \tag{B.20}$$

Proof. The lower estimate follows from

$$\frac{\cos^2(y_1)+\cos^2(y_2)-1-\cos^2\theta}{4(1-\cos\theta)^3}\leqslant\frac{A}{B}. \tag{B.21}$$

The upper bound is a consequence of (B.19). □

Remark 6. From the second equation in (B.13), we deduce that

$$A\leqslant(1-\cos\theta)\bigl[2\cos(y_1s)\cos(y_2t)-\cos\theta\bigl(1+\cos\theta\bigr)\bigr]. \tag{B.22}$$

If $y_1=y_2=y$, it gives

$$\begin{aligned}&\alpha(y,y,\theta)\\&\quad=-\int_0^1\int_0^1(1-s)\bigl(1-6s^2\bigr)(1-t)\bigl(1-6t^2\bigr)\log\left|1-\frac{\sin^2(y(s+t)/2)}{\sin^2(\theta/2)}\right|\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{B.23}$$

Thus, for $0<y\leqslant 2\theta$, we have $\alpha(y,y,\theta)\leqslant\mathrm{const}y^2/(\theta^2+y^2)$.

The series β can be handled in the same way,

$$\begin{aligned}N_\beta(y_1,y_2,\theta)&=\log\left|\frac{\sin((y_1+y_2+\theta)/2)\sin((y_1+y_2-\theta)/2)}{\sin((y_1-y_2+\theta)/2)\sin((y_1-y_2-\theta)/2)}\right|\\&=-\log\left|\frac{\mathrm{e}^{-\mathrm{i}y_2}-\mathrm{e}^{-\mathrm{i}(y_1+\theta)}}{1-\mathrm{e}^{-\mathrm{i}(y_1+\theta)}\mathrm{e}^{-\mathrm{i}y_2}}\right|-\log\left|\frac{\mathrm{e}^{-\mathrm{i}y_2}-\mathrm{e}^{-\mathrm{i}(y_1-\theta)}}{1-\mathrm{e}^{-\mathrm{i}(y_1-\theta)}\mathrm{e}^{-\mathrm{i}y_2}}\right|.\end{aligned}$$

It simplifies to

$$N_\beta=-\log\left|\frac{\sin^2(\theta/2)-\sin^2((y_1s+y_2t)/2)}{\sin^2(\theta/2)-\sin^2((y_1s-y_2t)/2)}\right|.$$

In particular, if $y_1=y_2=y$, the estimate $\beta(y,y,\theta)\leqslant\mathrm{const}y^2/(\theta^2+y^2)$ holds for

$$\beta(y_1,y_2,\theta)=-\int_0^1\int_0^1 s(1-s)t(1-t)\log\left|\frac{\sin^2(\theta/2)-\sin^2((y_1s+y_2t)/2)}{\sin^2(\theta/2)-\sin^2((y_1s-y_2t)/2)}\right|\mathrm{d}s\,\mathrm{d}t.$$

For N_γ, the estimate in Theorem B.2 is a consequence of the following lemma. It can also be deduced from Proposition B.16 below along with (B.43).

Lemma B.5. *Let $h(t)$ be given by* (B.9). *Assume that* $\sup(y_1, y_2) \leqslant \theta/4$, *then there exists a constant c such that*

$$\left|\gamma(y_1, y_2, \theta)\right| \leqslant c\frac{y_1^2 y_2}{\theta^3}. \tag{B.24}$$

Proof. If $h(t)$ is given by (B.9), then

$$\begin{aligned} N_\gamma(y_1, y_2, \theta) &= -\log\left|\frac{\sin((y_1 + y_2 + \theta)/2)\sin((y_1 - y_2 - \theta)/2)}{\sin((y_1 + y_2 - \theta)/2)\sin((y_1 - y_2 + \theta)/2)}\right| \\ &= -\log\left|\frac{\sin^2(y_1/2) - \sin^2((y_2 + \theta)/2)}{\sin^2(y_1/2) - \sin^2((y_2 - \theta)/2)}\right| \end{aligned}$$

and

$$\begin{aligned} \gamma(y_1, y_2, \theta) &= -\int_0^1\int_0^1 s(1-s)(1-t)\left(1 - 6t^2\right)\log\left|\frac{\sin^2(y_1 t/2) - \sin^2((y_2 s + \theta)/2)}{\sin^2(y_1 t/2) - \sin^2((y_2 s - \theta)/2)}\right| \mathrm{d}s\,\mathrm{d}t \\ &= \int_0^1\int_0^1 \frac{s(1-s)}{2} t(1-t)^2(2+3t) H(ty_1, sy_2, \theta)\,\mathrm{d}s\,\mathrm{d}t \end{aligned}$$

with

$$\begin{aligned} H(ty_1, sy_2, \theta) &= \frac{\mathrm{d}}{\mathrm{d}t}\log\left|\frac{\sin^2(y_1 t/2) - \sin^2((y_2 s + \theta)/2)}{\sin^2(y_1 t/2) - \sin^2((y_2 s - \theta)/2)}\right| \\ &= y_1\left[\frac{\sin(y_1 t)}{\cos(y_2 s + \theta) - \cos(y_1 t)} - \frac{\sin(y_1 t)}{\cos(y_2 s - \theta) - \cos(y_1 t)}\right]. \end{aligned}$$

Since $s(1-s)t(1-t)^2(2+3t) \geqslant 0$ for $0 \leqslant s \leqslant 1$, we have

$$\left|\gamma(y_1, y_2, \theta)\right| \leqslant \text{const}\int_0^1\int_0^1 \left|H(ty_1, sy_2, \theta)\right| \mathrm{d}s\,\mathrm{d}t \leqslant c\frac{y_1^2 y_2}{\theta^3}. \qquad \square$$

Corollary B.6.

$$\left|\gamma(y_1, y_2, \theta) + \gamma(y_2, y_1, \theta)\right| \leqslant c\left[\frac{y_1^2 y_2}{\theta^3} + \frac{y_1 y_2^2}{\theta^3}\right] \leqslant c\frac{y_1 y_2}{\theta^2}.$$

Proof. Since $y_1 + y_2 \leqslant \theta/2$. $\square$

B.2. The series as double integrals. The kernel functions

The function $h(t)$ given by (B.2) is even. For the series α and β, we consider the following even function of θ,

$$h_1(y,\theta) = h(y+\theta) + h(y-\theta). \tag{B.25}$$

Since $h(t)$ is even, $h_1(y,\theta)$ is obviously even in y. Define the kernel:

$$N_\beta(y_1,y_2,\theta) = h_1(y_1+y_2,\theta) - h_1(y_1-y_2,\theta). \tag{B.26}$$

Then $N_\beta(y_1,y_2,\theta)$ is odd in y_1 and odd in y_2; it will be an appropriate choice to study the series β. In the same way,

$$N_\alpha(y_1,y_2,\theta) = h_1(y_1+y_2,\theta) + h_1(y_1-y_2,\theta) \tag{B.27}$$

is even in y_1, even in y_2 and even in θ, thus it is appropriate to study the series α. For the series γ, we consider the function $h_2(y,\theta)$ which is odd in θ and odd in y,

$$h_2(y,\theta) = h(y+\theta) - h(y-\theta). \tag{B.28}$$

The kernel

$$N_\gamma(y_1,y_2,\theta) = h_2(y_1+y_2,\theta) - h_2(y_1-y_2,\theta) \tag{B.29}$$

is even in y_1 and odd in y_2 and will be used for the series γ. Let

$$\begin{aligned} A &= h(y_1+y_2+\theta), \qquad & B &= h(y_1+y_2-\theta), \\ C &= h(y_1-y_2+\theta), \qquad & D &= h(-y_1+y_2+\theta), \end{aligned}$$

then

$$N_\alpha = A+B+C+D, \qquad N_\beta = A+B-C-D, \qquad N_\gamma = A-B-C+D.$$

Lemma B.7. *Let $\mu(t)$ such that $\mu''(t) = h(t)$. Then for $y_1 \neq 0$, $y_2 \neq 0$,*

$$\int_0^1\int_0^1 h(y_1 s + y_2 t + \theta)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{y_1 y_2}\big[\mu(y_1+y_2+\theta) - \mu(y_1+\theta) - \mu(y_2+\theta) + \mu(\theta)\big].$$

Proof. We start with the identity

$$J = \int_0^1\int_0^1 \mu''(y_1 s + y_2 t + \theta)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{y_1}\int_0^1\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\big[\mu'\big(y_1 s + y_2 t + \theta\big)\big]\,\mathrm{d}s\,\mathrm{d}t$$

and evaluate the integral with respect to s to obtain

$$J = \frac{1}{y_1}\int_0^1 \bigl[\mu'(y_1 + y_2 t + \theta) - \mu'(y_2 t + \theta)\bigr]\,\mathrm{d}t$$
$$= \frac{1}{y_1 y_2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\bigl[\mu(y_1 + y_2 t + \theta) - \mu(y_2 t + \theta)\bigr]\,\mathrm{d}t.$$

This proves the lemma. □

As a particular case, let

$$\mu(t) = \frac{t^2 \log|t|}{2!} - \frac{3}{4}t^2.$$

Since $\mu''(t) = \log|t|$, we get for $y_1 \neq 0$, $y_2 \neq 0$,

$$\int_0^1\int_0^1 \log|y_1 s + y_2 t + \theta|\,\mathrm{d}s\,\mathrm{d}t$$
$$= \frac{1}{y_1 y_2}\bigl[\mu(y_1 + y_2 + \theta) - \mu(y_1 + \theta) - \mu(y_2 + \theta) + \mu(\theta)\bigr]. \tag{B.30}$$

In the same manner, see (B.10), we obtain $\phi(t) = \mu''_\phi(t)$, where

$$\phi(t) = \sum_{n\geqslant 2} \frac{n\cos nt}{n^2 - 1} = \left[\sum_{n\geqslant 1} \frac{\cos nt}{n}\cos t\right] - \frac{1}{2} - \frac{1}{4}\cos t,$$
$$\mu_\phi(t) = 2\sin^2\left(\frac{t}{2}\right)\log\left|2\sin\left(\frac{t}{2}\right)\right| + \frac{5}{4}\cos t + \frac{1}{8}\cos 2t - \frac{t^2}{2}.$$

Proof of Proposition B.3.

- *The series α*. With $6\widehat{K}''(x) + \widehat{K}(x) = 2\int_0^1 (1-s)(1-6s^2)\cos(sx)\,\mathrm{d}s$, we obtain

$$\bigl(6\widehat{K}''(y_1 n) + \widehat{K}(y_1 n)\bigr)\bigl(6\widehat{K}''(y_2 n) + \widehat{K}(y_2 n)\bigr)$$
$$= 2\int_0^1\int_0^1 (1-s)\bigl(1-6s^2\bigr)(1-t)\bigl(1-6t^2\bigr)2\cos(ny_1 s)\cos(ny_2 t)\,\mathrm{d}s\,\mathrm{d}t$$

$$= 2\int_0^1\int_0^1 (1-s)\big(1-6s^2\big)(1-t)\big(1-6t^2\big) \times \big[\cos(ny_1s+ny_2t)+\cos(ny_1s-ny_2t)\big]\,\mathrm{d}s\,\mathrm{d}t$$

and

$$\big(6\widehat{K}''(y_1n)+\widehat{K}(y_1n)\big)\big(6\widehat{K}''(y_2n)+\widehat{K}(y_2n)\big)\cos n\theta = \int_0^1\int_0^1 (1-s)\big(1-6s^2\big)(1-t)\big(1-6t^2\big) \times \big[2\cos(ny_1s+ny_2t)\cos n\theta + 2\cos(ny_1s-ny_2t)\cos n\theta\big]\,\mathrm{d}s\,\mathrm{d}t.$$

- *The series* β. Since $\widehat{K}(x)=2\int_0^1(1-s)\cos(xs)\,\mathrm{d}s$, by taking the derivative with respect to x, $\widehat{K}'(x)=-2\int_0^1 s(1-s)\sin(xs)\,\mathrm{d}s$, we get

$$\begin{aligned}\widehat{K}'(y_1n)\widehat{K}'(y_2n) &= 2\int_0^1\int_0^1 s(1-s)t(1-t)\times 2\sin(ny_1s)\sin(ny_2t)\,\mathrm{d}s\,\mathrm{d}t\\ &= -2\int_0^1\int_0^1 s(1-s)t(1-t)\big[\cos(ny_1s+ny_2t)-\cos(ny_1s-ny_2t)\big]\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

and thus

$$\begin{aligned}\widehat{K}'(y_1n)\widehat{K}'(y_2n)\cos n\theta &= -\int_0^1\int_0^1 s(1-s)t(1-t)\big[2\cos(ny_1s+ny_2t)\cos n\theta - 2\cos(ny_1s-ny_2t)\cos n\theta\big]\,\mathrm{d}s\,\mathrm{d}t\\ &= -\int_0^1\int_0^1 s(1-s)t(1-t)\big[\cos(ny_1s+ny_2t+n\theta)+\cos(ny_1s+ny_2t-n\theta)\\ &\qquad -\cos(ny_1s-ny_2t+n\theta)\\ &\qquad -\cos(ny_1s-ny_2t-n\theta)\big]\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

- *For the series* γ the proof is similar. □

Remark 7. We can calculate $N(sy_1, ty_2, \theta)$ in terms of y_1+y_2 and y_1-y_2 with the identities

$$sy_1 + ty_2 = \frac{1}{2}\big[(s+t)(y_1+y_2) + (s-t)(y_1-y_2)\big],$$
$$sy_1 - ty_2 = \big[(s+t)(y_1-y_2) + (s-t)(y_1+y_2)\big].$$

B.3. The h-transform. Extended cos *and* sin *functions*

This method differs basically from the previous one by the fact that we linearize the three products

$$\big(6\widehat{K}''(y_1) + \widehat{K}(y_1)\big)\big(6\widehat{K}''(y_2) + \widehat{K}(y_2)\big), \qquad \widehat{K}'(y_1)\widehat{K}'(y_2),$$
$$\big(6\widehat{K}''(y_1) + \widehat{K}(y_1)\big)\widehat{K}'(y_2),$$

before taking the integral transform. The interest of this second representation is that we get immediately only one integral. Of course, using Taylor formula with integral remainder, and after several integrations by parts, we can recover the double integral of the representation of Section B.2, and conversely with a change of variables in the double integral of Section B.2 and an integration, we find the integral representation of Section B.3. The function $6\widehat{K}''(y) + \widehat{K}(y)$ is a cos transform and $\widehat{K}'(y)$ a sin transform: Let $\phi(x_1, x_2, \theta)$ a function of the variables x_1, x_2, θ. We say that $\phi(x_1, x_2, \theta)$ is a *homogeneous* cos transform in x_1, x_2, if

$$\phi(x_1, x_2, \theta) = \sum_{i,j} \int_0^1 P_{ij}(x_1, x_2, s) \cos\big((\varepsilon_i x_1 + \varepsilon_j x_2)s + \theta\big)\,\mathrm{d}s, \quad \varepsilon_i = 0, \pm 1, \tag{B.31}$$

where P_{ij} is a polynomial in s and a rational function *homogeneous of degree* 0 in (x_1, x_2); if $x_1 = x_2 = x$, the function $P_{ij}(x, x, s)$ does not depend on x. To define *homogeneous* sin transforms, we put sin instead of cos. For example,

$$\int_0^1 \cos(x_1 s)\,\mathrm{d}s \int_0^1 \cos(x_2 t)\,\mathrm{d}t = \frac{1}{2}\Bigg[\frac{(x_1+x_2)^2}{x_1x_2}\int_0^1 (1-s)\cos\big((x_1+x_2)s\big)\,\mathrm{d}s - \frac{(x_1-x_2)^2}{x_1x_2}\int_0^1 (1-s)\cos\big((x_1-x_2)s\big)\,\mathrm{d}s\Bigg]$$

is a homogeneous cos transform. We have

$$\sum_{n\geqslant 1} \lambda(n)\phi(nx_1, nx_2, n\theta) = \sum_{i,j}\int_0^1 P_{ij}(x_1, x_2, s) h\big((\varepsilon_i x_1 + \varepsilon_j x_2)s + \theta\big)\,\mathrm{d}s. \tag{B.32}$$

We say that the series (B.32) is a homogeneous h-transform.

We shall see that the products

$$\widehat{K}(y_1)\widehat{K}(y_2), \qquad \bigl(6\widehat{K}''(y_1)+\widehat{K}(y_1)\bigr)\bigl(6\widehat{K}''(y_2)+\widehat{K}(y_2)\bigr), \qquad \widehat{K}'(y_1)\widehat{K}'(y_2)$$

are *homogeneous* cos transforms whereas $(6\widehat{K}''(y_1)+\widehat{K}(y_1))\widehat{K}'(y_2)$ is a *homogeneous* sin transform. In particular, we have

$$\widehat{K}(x)^2=\frac{4}{3}\int_0^1(1-s)^3\bigl(4\cos(2xs)-\cos(xs)\bigr)\,\mathrm{d}s. \tag{B.33}$$

Thus

$$\begin{gathered}\bigl[\bigl(6\widehat{K}''(y_1)+\widehat{K}(y_1)\bigr)\bigl(6\widehat{K}''(y_2)+\widehat{K}(y_2)\bigr)\bigr]\cos\theta,\\ \bigl[\widehat{K}'(y_1)\widehat{K}'(y_2)\bigr]\cos\theta \quad \text{and} \quad \bigl[\bigl(6\widehat{K}''(y_1)+\widehat{K}(y_1)\bigr)\widehat{K}'(y_2)\bigr]\sin\theta\end{gathered}$$

are all *homogeneous* cos transforms.

Lemma B.8. *For any integer p,*

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_0^1 y^p\frac{(1-s)^{p-1}}{(p-1)!}f(ys)\,\mathrm{d}s=\int_0^1 y^{p-1}\frac{(1-s)^{p-2}}{(p-2)!}f(ys)\,\mathrm{d}s. \tag{B.34}$$

Proof. We have, for $p>1$, p and q integers,

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_0^1 y^q\frac{(1-s)^{p-1}}{(p-1)!}f(ys)\,\mathrm{d}s=\int_0^1 y^{p-1}\frac{(1-s)^{p-2}}{(p-2)!}\left[\frac{(q-1)}{(p-1)}(1-s)+s\right]f(ys)\,\mathrm{d}s. \qquad \square$$

Proposition B.9. *Let $h(t)$ be given by* (B.2)*. Then*

$$\phi(y_1,y_2,\theta)=\frac{1}{2}\bigl[F_h(y_1,y_2,\theta)+F_h(y_1,y_2,-\theta)\bigr],$$

where

$$\begin{aligned}F_h(y_1,y_2,\theta)=-4\int_0^1\frac{(1-s)^3}{3!}&\left[\frac{y_1^2}{y_2^2}h(y_1s+\theta)+\frac{y_2^2}{y_1^2}h(y_2s+\theta)\right.\\&-\frac{1}{2}\frac{(y_1+y_2)^4}{y_1^2y_2^2}h\bigl((y_1+y_2)s+\theta\bigr)\\&\left.-\frac{1}{2}\frac{(y_1-y_2)^4}{y_1^2y_2^2}h\bigl((y_1-y_2)s+\theta\bigr)\right]\mathrm{d}s\end{aligned} \tag{B.35}$$

and

$$\psi(y_1, y_2, \theta) = \frac{1}{2}\big[G_h(y_1, y_2, \theta) - G_h(y_1, y_2, -\theta)\big]$$

with

$$\begin{aligned} G_h(y_1, y_2, \theta) = 4\int_0^1 \frac{y_1^2}{y_2^2}\left[2\frac{(1-s)^4}{4!} - \frac{(1-s)^3}{3!}\right] h(y_1 s + \theta)\,\mathrm{d}s \\ + 8\int_0^1 \frac{y_2^3}{y_1^3}\frac{(1-s)^4}{4!} h(y_2 s + \theta)\,\mathrm{d}s + \int_0^1 L(y_1, y_2, s) h\big((y_1 + y_2)s + \theta\big)\,\mathrm{d}s \\ + \int_0^1 L(y_1, -y_2, s) h\big((y_1 - y_2)s + \theta\big)\,\mathrm{d}s \end{aligned} \tag{B.36}$$

and

$$L(y_1, y_2, s) = -4\frac{(y_1 + y_2)^5}{y_1^3 y_2^2}\frac{(1-s)^4}{4!} + 2\frac{(y_1 + y_2)^4}{y_1^2 y_2^2}\frac{(1-s)^3}{3!}.$$

The expressions (B.35) and (B.36) can be used for any integrable function h; if we take $h(t) = c$, where c is a constant, we find that $F_h(y_1, y_2, \theta)) = c$. If $h(t) = ct$, then

$$\frac{1}{2}\big[G_h(y_1, y_2, \theta) + G_h(y_1, y_2, -\theta)\big] = \frac{c y_1}{6}.$$

Corollary B.10. *We have*

$$\sum_{n \geqslant 1} \lambda(n) \widehat{K}(yn)^2 \cos n\theta = -8\int_0^1 \frac{(1-s)^3}{3!}\big[h(ys + \theta) - 4h(2ys + \theta)\big]\,\mathrm{d}s.$$

If $\theta \neq 0$, *then*

$$\sum_{n \geqslant 1} \frac{1}{n} \widehat{K}(yn)^2 \cos n\theta \sim h(\theta), \quad \textit{as } y \to 0. \tag{B.37a}$$

If $\theta = 0$, *then*

$$\sum_{n \geqslant 1} \frac{1}{n} \widehat{K}(yn)^2 \sim -\log|y|, \quad \textit{as } y \to 0. \tag{B.37b}$$

Corollary B.11. *We have*

$$\sum_{n\geqslant 1}\lambda(n)\widehat{K}(yn)^2\sin n\theta$$
$$=4\int_0^1\frac{s(1-s)^3}{3!}\Big[8\big(h(2ys+\theta)-h(2ys-\theta)\big)-\big(h(ys+\theta)-h(ys-\theta)\big)\Big]\,\mathrm{d}s,$$

and

$$\sum_{n\geqslant 1}\frac{1}{n}\widehat{K}(yn)^2\sin n\theta\sim \mathrm{const}\frac{\theta}{y}\quad \text{as } \theta\to 0 \text{ and } y \text{ is fixed.}\tag{B.38}$$

We can study in the same way the functions α, β and γ.

Proposition B.12. *Let $h(t)$ be given by* (B.2)*. Then*

$$\alpha(y_1,y_2,\theta)=\frac{1}{2}\big[A_h(y_1,y_2,\theta)+A_h(y_1,y_2,-\theta)\big]$$

and

$$\begin{aligned}A_h(y_1,y_2,\theta)&=\int_0^1\left[P_1(s)\frac{y_1^4}{y_2^4}+P_2(s)\frac{y_1^2}{y_2^2}\right]h(y_1s+\theta)\,\mathrm{d}s\\&\quad+\int_0^1\left[P_1(s)\frac{y_2^4}{y_1^4}+P_2(s)\frac{y_2^2}{y_1^2}\right]h(y_2s+\theta)\,\mathrm{d}s\\&\quad+\int_0^1 R(y_1,y_2,s)h\big((y_1+y_2)s+\theta\big)\,\mathrm{d}s\\&\quad+\int_0^1 R(y_1,-y_2,s)h\big((y_1-y_2)s+\theta\big)\,\mathrm{d}s,\end{aligned}\tag{B.39}$$

where

$$P_1(s)=-72\times 72\times\frac{(1-s)^7}{7!}+48\times 72\times\frac{(1-s)^6}{6!}-10\times 72\times\frac{(1-s)^5}{5!},$$
$$P_2(s)=72\times 2\times\frac{(1-s)^5}{5!}-48\times 2\times\frac{(1-s)^4}{4!}+20\times\frac{(1-s)^3}{3!},$$

$$R(y_1,y_2,s) = -\frac{P_1(s)}{2}\frac{(y_1+y_2)^8}{y_1^4y_2^4} + 6\times 72\times\frac{(y_1+y_2)^6}{y_1^3y_2^3}\frac{(1-s)^5}{5!}$$
$$- 24\times 10\times\frac{(y_1+y_2)^6}{y_1^3y_2^3}\frac{(1-s)^4}{4!} + 50\times\frac{(y_1+y_2)^4}{y_1^2y_2^2}\frac{(1-s)^3}{3!}.$$

We remark that if we put $h(t)=1$ in (B.39), then everything cancels, we have

$$\int_0^1\left[P_1(s)\frac{y_1^4}{y_2^4} + P_2(s)\frac{y_1^2}{y_2^2} + P_1(s)\frac{y_2^4}{y_1^4} + P_2(s)\frac{y_2^2}{y_1^2} + R(y_1,y_2,s) + R(y_1,-y_2,s)\right]\mathrm{d}s = 0.$$

Thus, (B.39) remains true if we replace $h(ys+\theta)$ by $h(ys+\theta)-h(\theta)$. For $h(t)=\cos t$, which is a particular case of (B.2), we obtain with (B.39) a representation of

$$\bigl(6\widehat{K}''(y_1)+\widehat{K}(y_1)\bigr)\bigl(6\widehat{K}''(y_2)+\widehat{K}(y_2)\bigr)\cos\theta$$

as an integral transform.

Corollary B.13. *We have*

$$\alpha(y,y,\theta) = \int_0^1\bigl(P_1(s)+P_2(s)\bigr)\bigl[h(ys+\theta)+h(ys-\theta)\bigr]\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^1 R(s)\bigl[h(2ys+\theta)+h(2ys-\theta)\bigr]\,\mathrm{d}s,$$

where $R(s)=R(y,y,s)$.

If $\theta\neq 0$, then $\alpha(y,y,\theta)\sim \mathrm{const}\,h(\theta)$ as $y\to 0$. If $\theta\neq 0$, as $y\to 0$,

$$\sum_{n\geqslant 1}\frac{1}{n}\bigl[6\widehat{K}''(ny)+\widehat{K}(ny)\bigr]^2\sim \mathrm{const}\log y.$$

Moreover, there exist a constant $c_1>0$ independent of y, θ and a neighbourhood $V=\{|\theta|+y<c_2\}$ such that for $(\theta,y)\in V$, $\theta\neq 0$, $y>0$,

$$\sum_{n\geqslant 1}\frac{1}{n}\bigl[6\widehat{K}''(ny)+\widehat{K}(ny)\bigr]^2\cos n\theta\leqslant\frac{c}{\cosh(\rho)},$$

where $\rho=d((y,0),(y,\theta))$ is given by (B.1).

In the same manner, we have

Proposition B.14. *For $h(t)$ given by* (B.2),

$$\beta(y_1,y_2,\theta) = \sum_{n\geqslant 1}\lambda(n)\widehat{K}'(ny_1)\widehat{K}'(ny_2)\cos n\theta = \frac{1}{2}\bigl[B_h(y_1,y_2,\theta)+B_h(y_1,y_2,-\theta)\bigr]$$

with

$$\begin{aligned}B_h(y_1,y_2,\theta)={}&\int_0^1\left(\frac{16(1-s)^5}{5!}-\frac{8(1-s)^4}{4!}\right)\left(\frac{y_1^3}{y_2^3}h(y_1s+\theta)+\frac{y_2^3}{y_1^3}h(y_2s+\theta)\right)\mathrm{d}s\\&+\int_0^1 Q(y_1,y_2,s)h\big((y_1+y_2)s+\theta\big)\,\mathrm{d}s\\&-\int_0^1 Q(y_1,-y_2,s)h\big((y_1-y_2)s+\theta\big)\,\mathrm{d}s,\end{aligned}$$

where

$$Q(y_1,y_2,s)=\left(\frac{4(1-s)^4}{4!}-\frac{8(1-s)^5}{5!}\right)\frac{(y_1+y_2)^6}{y_1^3y_2^3}-\frac{2(1-s)^3}{3!}\frac{(y_1+y_2)^4}{y_1^2y_2^2}. \tag{B.40}$$

Setting $h(t)=1$ in B_h, we obtain

$$\int_0^1\left\{\left(\frac{16(1-s)^5}{5!}-\frac{8(1-s)^4}{4!}\right)\left(\frac{y_1^3}{y_2^3}+\frac{y_2^3}{y_1^3}\right)+Q(y_1,y_2,s)-Q(y_1,-y_2,s)\right\}\mathrm{d}s=0.$$

In particular

$$\int_0^1\big[Q(y_1,y_2,s)-Q(y_1,-y_2,s)\big]\,\mathrm{d}s=c\left(\frac{y_1^3}{y_2^3}+\frac{y_2^3}{y_1^3}\right)$$

where c is a constant (there is no term in y_1/y_2 or in y_2/y_1). For the coefficient of $y_1/y_2+y_2/y_1$, the last statement follows from $(4/5!-8/6!)\times 15-2/4!\times 4=0$. Thus, as in Proposition B.14, we can replace $h(ys+\theta)$ by $h(ys+\theta)-h(\theta)$ in (B.40) and the formula stays valid.

Corollary B.15. *We have*

$$\begin{aligned}\sum_{n\geqslant 1}\lambda(n)\widehat{K}'(ny)^2\cos n\theta={}&\int_0^1\left(16\frac{(1-s)^5}{5!}-8\frac{(1-s)^4}{4!}\right)\big[h(ys+\theta)+h(ys-\theta)\big]\,\mathrm{d}s\\&+\frac{1}{2}\int_0^1 Q(y,y,s)\big[h(2ys+\theta)+h(2ys-\theta)\big]\,\mathrm{d}s.\end{aligned}$$

Moreover,

$$\sum_{n\geqslant 1}\frac{1}{n}\widehat{K}'(ny)^2\cos n\theta \leqslant \frac{c}{\cosh(\rho)}.$$

For the series γ, we obtain the following result.

Proposition B.16. *We have*

$$\begin{aligned}\gamma(y_1,y_2,\theta) &= \sum_{n\geqslant 1}\lambda(n)\bigl(6\widehat{K}''(y_1 n)+\widehat{K}(y_1 n)\bigr)\widehat{K}'(y_2 n)\sin n\theta\\ &= \frac{1}{2}\bigl[F(y_1,y_2,\theta)-F(y_1,y_2,-\theta)\bigr]\end{aligned}$$

where

$$\begin{aligned}F(y_1,y_2,\theta) = {}& \frac{y_1^3}{y_2^3}\int_0^1 P_1(s)h(y_1 s)\,\mathrm{d}s + \frac{y_2^4}{y_1^4}\int_0^1 R_1(s)h(y_2 s)\,\mathrm{d}s + \frac{y_2^2}{y_1^2}\int_0^1 R_2(s)h(y_2 s)\,\mathrm{d}s\\ &+ \int_0^1 Q(y_1,y_2,s)h\bigl((y_1+y_2)s\bigr)\,\mathrm{d}s\\ &+ \int_0^1 T(y_1,y_2,s)h\bigl((y_1-y_2)s\bigr)\,\mathrm{d}s \end{aligned}\tag{B.41}$$

with

$$\begin{aligned}P_1(s) &= 72\times 4\frac{(1-s)^6}{6!} - 48\times 4\frac{(1-s)^5}{5!} + 10\times 4\frac{(1-s)^4}{4!},\\ R_1(s) &= 72\times 4\frac{(1-s)^6}{6!} - 72\times 2\frac{(1-s)^5}{5!},\\ R_2(s) &= -8\frac{(1-s)^4}{4!} + 4\frac{(1-s)^3}{3!},\\ Q(y_1,y_2,s) &= -144\frac{(y_1+y_2)^7}{y_1^4 y_2^3}\frac{(1-s)^6}{6!} + \left(\frac{72}{y_1^4 y_2^2}+\frac{96}{y_1^3 y_2^3}\right)(y_1+y_2)^6\frac{(1-s)^5}{5!}\\ &\quad - \left(\frac{48}{y_1^3 y_2^2}+\frac{20}{y_1^2 y_2^3}\right)(y_1+y_2)^5\frac{(1-s)^4}{4!} + \frac{10}{y_1^2 y_2^2}(y_1+y_2)^4\frac{(1-s)^3}{3!},\end{aligned}$$

$$T(y_1,y_2,s) = -144\frac{(y_1-y_2)^7}{y_1^4y_2^3}\frac{(1-s)^6}{6!} - \left(\frac{72}{y_1^4y_2^2} - \frac{96}{y_1^3y_2^3}\right)(y_1-y_2)^6\frac{(1-s)^5}{5!}$$
$$+\left(\frac{48}{y_1^3y_2^2} - \frac{20}{y_1^2y_2^3}\right)(y_1-y_2)^5\frac{(1-s)^4}{4!} - \frac{10}{y_1^2y_2^2}(y_1-y_2)^4\frac{(1-s)^3}{3!}.$$

Corollary B.17. *If $h(t)$ is given by* (B.9) *or* (B.10)*, then when $\theta \neq 0$ is small,*

$$\gamma(y,y,\theta) \sim \text{const}\frac{y^3}{\theta^3} \quad \textit{when } y \to 0. \tag{B.42}$$

Corollary B.18. *There exist constants $c_1 > 0$, $c_2 > 0$, independent of (θ, y), and a neighbourhood $V = \{(\theta,y) \mid 0 < y < |\theta|/2 \leqslant c_2\}$ such that for $(\theta,y) \in V$,*

$$\sum_{n\geqslant 1}\frac{1}{n}\big(6\widehat{K}''(ny) + \widehat{K}(ny)\big)\widehat{K}'(ny)\sin(n\theta) \leqslant \frac{c}{\cosh(\rho)} \tag{B.43}$$

where $\rho = d((y,0),(y,\theta))$ is given by (B.1).

B.3.1. The series β

Proof of Proposition B.14. Let

$$C_2(x) = x^6\int_0^1 \frac{(1-s)^5}{5!}\cos(xs)\,\mathrm{d}s = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cos x,$$
$$xS_2(x) = -x^6\int_0^1 \frac{(1-s)^4}{4!}\cos(xs)\,\mathrm{d}s = x\left(x - \frac{x^3}{3!} - \sin x\right),$$
$$C_1(x) = -x^4\int_0^1 \frac{(1-s)^3}{3!}\cos(xs)\,\mathrm{d}s = 1 - \frac{x^2}{2} - \cos x;$$

then

$$\widehat{K}'(x_1)\widehat{K}'(x_2) = \frac{16}{x_1^3x_2^3}\left(C_2(x_1) + C_2(x_2) - \frac{1}{2}C_2(x_1+x_2) - \frac{1}{2}C_2(x_1-x_2)\right)$$
$$+\frac{8}{x_1^3x_2^3}\left(x_1S_2(x_1) + x_2S_2(x_2) - \frac{1}{2}(x_1+x_2)S_2(x_1+x_2)\right.$$
$$\left.- \frac{1}{2}(x_1-x_2)S_2(x_1-x_2)\right)$$
$$+\frac{2}{x_1^2x_2^2}\big(C_1(x_1+x_2) - C_1(x_1-x_2)\big).$$

Replacing C_1, C_2, S_2 by their integral representations, we obtain

$$\widehat{K}'(x_1)\widehat{K}'(x_2) = \int_0^1 \frac{(1-s)^5}{5!}\frac{16}{x_1^3x_2^3}\phi_6(x_1,x_2,s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^4}{4!}\frac{8}{x_1^3x_2^3}\phi_6(x_1,x_2,s)\,\mathrm{d}s$$
$$- \int_0^1 \frac{(1-s)^3}{3!}\frac{2}{x_1^2x_2^2}\psi_4(x_1,x_2,s)\,\mathrm{d}s, \tag{B.44}$$

where

$$\phi_6(x_1,x_2,s) = x_1^6\cos(x_1s) + x_2^6\cos(x_2s) - \frac{1}{2}(x_1+x_2)^6\cos\bigl((x_1+x_2)s\bigr)$$
$$- \frac{1}{2}(x_1-x_2)^6\cos\bigl((x_1-x_2)s\bigr),$$
$$\psi_4(x_1,x_2,s) = (x_1+x_2)^4\cos\bigl((x_1+x_2)s\bigr) - (x_1-x_2)^4\cos\bigl((x_1-x_2)s\bigr). \tag{B.45}$$

If $x_1 = x_2$, we recover formula (2.5) for $(\widehat{K}'(x))^2$.

Using the identity $2\cos(nx)\cos n\theta = \cos(n(x+\theta)) + \cos(n(\theta - x))$, we obtain

$$\beta(x_1,x_2,\theta) = \sum_{n\geqslant 1}\lambda(n)\widehat{K}'(nx_1)\widehat{K}'(nx_2)\cos n\theta = \frac{1}{2}\bigl[B(x_1,x_2,\theta) + B(x_1,x_2,-\theta)\bigr]$$

with

$$B(x_1,x_2,\theta) = \int_0^1 \frac{(1-s)^5}{5!}\frac{16}{x_1^3x_2^3}\tilde{\phi}_6(x_1,x_2,\theta,s)\,\mathrm{d}s - \int_0^1 \frac{(1-s)^4}{4!}\frac{8}{x_1^3x_2^3}\tilde{\phi}_6(x_1,x_2,\theta,s)\,\mathrm{d}s$$
$$- \int_0^1 \frac{(1-s)^3}{3!}\frac{2}{x_1^2x_2^2}\tilde{\psi}_4(x_1,x_2,\theta,s)\,\mathrm{d}s \tag{B.46}$$

where

$$\tilde{\phi}_6(x_1,x_2,\theta,s) = x_1^6h(x_1s+\theta) + x_2^6h(x_2s+\theta) - \frac{1}{2}(x_1+x_2)^6h\bigl((x_1+x_2)s+\theta\bigr)$$
$$- \frac{1}{2}(x_1-x_2)^6h\bigl((x_1-x_2)s+\theta\bigr),$$
$$\tilde{\psi}_4(x_1,x_2,s) = (x_1+x_2)^4h\bigl((x_1+x_2)s+\theta\bigr) - (x_1-x_2)^4h\bigl((x_1-x_2)s+\theta\bigr)$$

and $h(x) = \sum_{n\geqslant 1}\lambda(n)\cos(nx)$. After rearranging the terms in (B.46), we obtain Proposition B.14. □

Corollary B.19. *Assume that $0 < x_1 < \pi/3$ and that x_1 is fixed, then in a neighbourhood of $x_2 = 0$,*

$$B(x_1, x_2, 0) = \frac{2}{3}\frac{x_2}{x_1}\int_0^1 (1-s)h(x_1 s)\,\mathrm{d}s + \varepsilon\big((x_2/x_1)^2\big). \tag{B.47}$$

Proof. If $x_1 \neq 0$, $x_1 > 0$ is fixed, we have $\tilde{\phi}_6(x_1, 0, s) = 0$ and $\tilde{\psi}_4(x_1, 0, s) = 0$, this gives $B(x_1, 0) = 0$. If $x_1 \neq 0$, the function $x_2 \to B(x_1, x_2)$ is C^2 in the variable x_2. Moreover, it is an odd function of x_2. We use Taylor expansions of $x_2 \mapsto B(x_1, x_2)$, $x_2 \mapsto \tilde{\phi}_6(x_1, x_2, s)$ and $x_2 \to \tilde{\psi}_4(x_1, x_2, s)$ at $x_2 = 0$. By means of Taylor's formula, we get

$$\begin{aligned}
\tilde{\phi}_6(x_1, x_2, s) &= x_1^4\big(-30h(x_1 s) - 12x_1 s h'(x_1 s) - x_1^2 s^2 h''(x_1 s)\big)\frac{x_2^2}{2!} \\
&\quad + x_1^2\frac{x_2^4}{4!}\big(-360h(x_1 s) - 480x_1 s h'(x_1 s) - 180x_1^2 s^2 h''(x_1 s) \\
&\qquad - 24x_1^3 s^3 h^{(3)}(x_1 s) - x_1^4 s^4 h^{(4)}(x_1 s)\big) + \varepsilon\big(x_2^5\big); \\
\tilde{\psi}_4(x_1, x_2, s) &= x_1^3\big(8h(x_1 s) + 2x_1 s h'(x_1 s)\big)x_2 \\
&\quad + x_1\big(48h(x_1 s) + 72x_1 s h'(x_1 s) + 24x_1^2 s^2 h''(x_1 s) + 2x_1^3 s^3 h'''(x_1 s)\big)\frac{x_2^3}{3!} \\
&\quad + \varepsilon\big(x_2^3\big).
\end{aligned}$$

Denote

$$\tilde{\phi} = -30h(x_1 s) - 12 s x_1 h'(x_1 s) - s^2 x_1^2 h''(x_1 s) \quad \text{and} \quad \tilde{\psi} = 8h(x_1 s) + 2 s x_1 h'(x_1 s).$$

Integration by parts gives

$$\frac{1}{2!}\left[\int_0^1 \frac{(1-s)^5}{5!}16\tilde{\phi}\,\mathrm{d}s - \int_0^1 \frac{(1-s)^4}{4!}8\tilde{\phi}\,\mathrm{d}s\right] - \int_0^1 \frac{(1-s)^3}{3!}2\tilde{\psi}\,\mathrm{d}s = 0.$$

We may proceed in the same way for the term of next order in the expansion of $B(x_1, x_2)$. Denote

$$\begin{aligned}
\tilde{\phi}_* = {}& -360h(x_1 s) - 480x_1 s h'(x_1 s) - 180x_1^2 s^2 h''(x_1 s) \\
& - 24x_1^3 s^3 h^{(3)}(x_1 s) - x_1^4 s^4 h^{(4)}(x_1 s)
\end{aligned}$$

and

$$\tilde{\psi}_* = 48h(x_1 s) + 72x_1 s h'(x_1 s) + 24x_1^2 s^2 h''(x_1 s) + 2x_1^3 s^3 h'''(x_1 s).$$

We calculate by integration by parts

$$J = \frac{1}{4!}\left[\int_0^1 \frac{(1-s)^5}{5!} 16\tilde{\phi}_* \, \mathrm{d}s - \int_0^1 \frac{(1-s)^4}{4!} 8\tilde{\phi}_* \, \mathrm{d}s\right] - \frac{1}{3!}\left[\int_0^1 \frac{(1-s)^3}{3!} 2\tilde{\psi}_* \, \mathrm{d}s\right]$$

which gives

$$J = \frac{2}{3}\int_0^1 (1-s)h(x_1 s)\, \mathrm{d}s.$$

This concludes the proof. □

Corollary B.20. *For* $0 < t < 2\pi$, *let* $h(t) = \sum_{n\geqslant 1} \lambda(n)\cos nt$ *as in* (B.2). *Then*

$$\sum_{n\geqslant 1} \lambda(n)\widehat{K}'(ny)^2 = \frac{16}{15}\int_0^1 (1-t)^3\bigl(1-2t-4t^2\bigr)h(2yt)\, \mathrm{d}t - \frac{2}{15}\int_0^1 (1-t)^4(3+2t)h(yt)\, \mathrm{d}t. \tag{B.48}$$

Proof. We use expression (2.6),

$$\widehat{K}'(x)^2 = \frac{16}{15}\int_0^1 (1-t)^3\bigl(1-2t-4t^2\bigr)\cos(2xt)\, \mathrm{d}t - \frac{2}{15}\int_0^1 (1-t)^4(3+2t)\cos(xt)\, \mathrm{d}t. \qquad \square$$

Corollary B.21. *At* $y = 0$, *we have the following expansions in* y:

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{1}{n}\widehat{K}'(ny)^2 &= \frac{7-4\log 2}{45} + \frac{1}{4!18}y^2 + \cdots, \\
\sum_{n\geqslant 2} \frac{n}{n^2-1}\widehat{K}'(ny)^2 &= \frac{7-4\log 2}{45} - \widehat{K}'(y)^2\log y - \frac{1}{4}\widehat{K}'(y)^2 \\
&\quad + y^2 \times (\textit{a Taylor expansion in } y^2), \\
\sum_{n\geqslant 2} \frac{1}{n^3-n}\widehat{K}'(ny)^2 &= -\widehat{K}'(y)^2\log y + y^2 \times (\textit{a Taylor expansion in } y).
\end{aligned} \tag{B.49}$$

Proof. We use

$$\frac{2}{15}\int_0^1 (1-t)^4(3+2t)\log t\,\mathrm{d}t - \frac{16}{15}\int_0^1 (1-t)^3\big(1-2t-4t^2\big)\log t\,\mathrm{d}t = \frac{7}{45}. \qquad \square$$

B.3.2. The extended cos *and* sin *functions*

The following functions play a main role in the study of the series α, β, γ. For $k>0$, let $T(k)=\int_0^{+\infty} t^{k-1}\mathrm{e}^{-t}\,\mathrm{d}t$. If k is an integer then $k! = T(k+1)$. For $k>0$ let

$$C_k(x) = (-1)^k x^{2k+2}\int_0^1 \frac{(1-s)^{2k+1}}{T(2k+2)}\cos(xs)\,\mathrm{d}s,$$

$$S_k(x) = (-1)^{k+1} x^{2k}\int_0^1 \frac{(1-s)^{2k-1}}{T(2k)}\sin(xs)\,\mathrm{d}s. \tag{B.50}$$

We describe some of the properties of these functions when k is a positive integer. Here $C_k(x)$ is the remainder of order $2k$ in the Taylor expansion of $1-\cos x$ at $x=0$;

$$C_0(x) = 1-\cos x,$$

$$C_1(x) = 1-\frac{x^2}{2}-\cos x, \quad \ldots, \quad C_k(x) = 1-\frac{x^2}{2}+\cdots+(-1)^k\frac{x^{2k}}{(2k)!}-\cos x,$$

$$S_0(x) = -\sin x,$$

$$S_1(x) = x-\sin x, \quad \ldots, \quad S_k(x) = x-\frac{x^3}{3!}+\cdots+(-1)^{k+1}\frac{x^{2k-1}}{(2k-1)!}-\sin x,$$

$$\frac{C_k(x)}{x^{2k+1}} = (-1)^k\int_0^1 \frac{(1-s)^{2k}}{(2k)!}\sin(xs)\,\mathrm{d}s = x(-1)^k\int_0^1 \frac{(1-s)^{2k+1}}{(2k+1)!}\cos(xs)\,\mathrm{d}s,$$

$$\frac{S_k(x)}{x^{2k}} = (-1)^{k+1}\int_0^1 \frac{(1-t)^{2k-1}}{(2k-1)!}\sin(xt)\,\mathrm{d}t = x(-1)^{k+1}\int_0^1 \frac{(1-t)^{2k}}{(2k)!}\cos(xt)\,\mathrm{d}t.$$

Note that $C_k'(x) = -S_k(x)$, $C_k''(x) = -C_{k-1}(x)$ and $xS_k(x) = \int_0^1 C_k(sx)\,\mathrm{d}s$. Let

$$c := \int_0^1 \frac{1-\cos t}{t}\,\mathrm{d}t - \int_1^{+\infty} \frac{\cos t}{t}\,\mathrm{d}t$$

be the Euler constant. We deduce after integration by parts, see [5,7],

$$A_1 = \int_1^{+\infty} \frac{C_0(t)}{t^3}\,\mathrm{d}t + \int_0^1 \frac{C_1(t)}{t^3}\,\mathrm{d}t = \left(\frac{3}{2} - c\right)\frac{1}{2!},$$

$$A_2 = \int_1^{+\infty} \frac{C_1(t)}{t^5}\,\mathrm{d}t + \int_0^1 \frac{C_2(t)}{t^5}\,\mathrm{d}t = \left(c - \frac{25}{12}\right)\frac{1}{4!},$$

$$A_p = -(2p+2)(2p+1)A_{p+1} + (-1)^p \frac{4p+3}{(2p+2)!}$$

with

$$A_p = \int_1^{+\infty} \frac{C_{p-1}(t)}{t^{2p+1}}\,\mathrm{d}t + \int_0^1 \frac{C_p(t)}{t^{2p+1}}\,\mathrm{d}t = (-1)^{p+1}(\alpha_p - c)\frac{1}{(2p)!}$$

and

$$\alpha_p = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2p}.$$

There are various extended addition formulas which differ, for example, whether we take C as a sin or a cos transform,

$$\begin{aligned}
C_2(y_1)C_1(y_2) &= \left(1 - \cos y_1 - \frac{y_1^2}{2} + \frac{y_1^4}{4!}\right)\left(1 - \cos y_2 - \frac{y_2^2}{2}\right) \\
&= C_3(y_1) + C_3(y_2) - \frac{y_2^2}{2}C_2(y_1) - \frac{y_1^2}{2}C_2(y_2) + \frac{y_1^4}{4!}C_1(y_2) \\
&\quad - \frac{1}{2}C_3(y_1+y_2) - \frac{1}{2}C_3(y_1-y_2) \\
&= C_4(y_1) + C_4(y_2) - \frac{y_2^2}{2}C_3(y_1) - \frac{y_1^2}{2}C_3(y_2) + \frac{y_1^4}{4!}C_2(y_2) \\
&\quad - \frac{1}{2}C_4(y_1+y_2) - \frac{1}{2}C_4(y_1-y_2).
\end{aligned}$$

Below are some of these formulas,

$$C_0(y_1)C_0(y_2) = C_1(y_1) + C_1(y_2) - \frac{1}{2}C_1(y_1+y_2) - \frac{1}{2}C_1(y_1-y_2),$$

$$C_0(y_1)S_1(y_2) = S_2(y_2) + y_2C_1(y_1) - \frac{1}{2}S_2(y_1+y_2) + \frac{1}{2}S_2(y_1-y_2),$$

$$C_2(y_1)C_0(y_2) = C_2(y_1) + C_2(y_2) - \frac{y_1^2}{2}C_1(y_2) + \frac{y_1^4}{4!}C_0(y_2)$$

$$- \frac{1}{2}C_2(y_1+y_2) - \frac{1}{2}C_2(y_1-y_2),$$

$$S_2(y_1)C_0(y_2) = S_2(y_1) + y_1C_1(y_2) - \frac{y_1^3}{3!}C_0(y_2) - \frac{1}{2}\big[S_2(y_1+y_2) + S_2(y_1-y_2)\big],$$

$$\begin{aligned}
C_1(y_1)C_0(y_2) &= C_1(y_1) + C_1(y_2) - \frac{y_1^2}{2}C_0(y_2) - \frac{1}{2}C_1(y_1+y_2) - \frac{1}{2}C_1(y_1-y_2) \\
&= C_2(y_1) + C_2(y_2) - \frac{y_1^2}{2}C_1(y_2) - \frac{1}{2}C_2(y_1+y_2) - \frac{1}{2}C_2(y_1-y_2),
\end{aligned}$$

$$\begin{aligned}
C_2(y_1)S_1(y_2) &= \left(1 - \cos y_1 - \frac{y_1^2}{2} + \frac{y_1^4}{4!}\right)(y_2 - \sin y_2) \\
&= S_3(y_2) + y_2C_2(y_1) - \frac{y_1^2}{2}S_2(y_2) + \frac{y_1^4}{4!}S_1(y_2) \\
&\quad - \frac{1}{2}\big[S_3(y_1+y_2) - S_3(y_1-y_2)\big],
\end{aligned}$$

$$\begin{aligned}
S_2(y_1)C_1(y_2) &= \left(y_1 - \sin y_1 - \frac{y_1^3}{3!}\right)\left(1 - \cos y_2 - \frac{y_2^2}{2}\right) \\
&= S_3(y_1) + y_1C_2(y_2) - \frac{y_1^3}{3!}C_1(y_2) - \frac{1}{2}\big[S_3(y_1+y_2) + S_3(y_1-y_2)\big] \\
&\quad - \frac{y_2^2}{2}S_2(y_1),
\end{aligned}$$

$$\begin{aligned}
S_2(y_1)S_1(y_2) &= \left(y_1 - \sin y_1 - \frac{y_1^3}{3!}\right)(y_2 - \sin y_2) \\
&= y_2S_2(y_1) + y_1S_2(y_2) - \frac{y_1^3}{3!}S_1(y_2) + \frac{1}{2}\big[C_2(y_1+y_2) - C_2(y_1-y_2)\big],
\end{aligned}$$

$$\begin{aligned}
C_1(y_1)C_1(y_2) &= \left(1 - \cos y_1 - \frac{y_1^2}{2}\right)\left(1 - \cos y_2 - \frac{y_2^2}{2}\right) \\
&= C_2(y_1) + C_2(y_2) - \frac{y_1^2}{2}C_1(y_2) - \frac{y_2^2}{2}C_1(y_1) \\
&\quad - \frac{1}{2}\big[C_2(y_1+y_2) + C_2(y_1-y_2)\big] \\
&= C_3(y_1) + C_3(y_2) - \frac{y_1^2}{2}C_2(y_2) - \frac{y_2^2}{2}C_2(y_1) \\
&\quad - \frac{1}{2}\big[C_3(y_1+y_2) + C_3(y_1-y_2)\big],
\end{aligned}$$

$$\begin{aligned}C_1(y_1)S_1(y_2) &= \left(1-\cos y_1-\frac{y_1^2}{2}\right)(y_2-\sin y_2)\\&= S_2(y_2)+y_2C_1(y_1)-\frac{y_1^2}{2}S_1(y_2)-\frac{1}{2}\big[S_2(y_1+y_2)-S_2(y_1-y_2)\big]\\&= S_3(y_2)+y_2C_2(y_1)-\frac{y_1^2}{2}S_2(y_2)-\frac{1}{2}\big[S_3(y_1+y_2)-S_3(y_1-y_2)\big],\end{aligned}$$

$$\begin{aligned}S_1(y_1)S_1(y_2) &= (y_1-\sin y_1)(y_2-\sin y_2)\\&= y_1S_2(y_2)+y_2S_2(y_1)+\frac{1}{2}\big[C_2(y_1+y_2))-C_2(y_1-y_2)\big],\\S_2(y_1)S_2(y_2) &= \left(y_1-\sin y_1-\frac{y_1^3}{3!}\right)\left(y_2-\sin y_2-\frac{y_2^3}{3!}\right)\\&= y_1S_3(y_2)+y_2S_3(y_1)-\frac{y_1^3}{3!}S_2(y_2)-\frac{y_2^3}{3!}S_2(y_1)\\&\quad+\frac{1}{2}\big[C_3(y_1+y_2)-C_3(y_1-y_2)\big].\end{aligned}$$

In terms of cos transform, we have

$$\begin{aligned}S_p(y_1)S_q(y_2) &= y_1S_{p+q}(y_2)-\frac{y_1^3}{3!}S_{p+q-1}(y_2)+\cdots+(-1)^{p+1}\frac{y_1^{2p-1}}{(2p-1)!}S_{p+1}(y_2)\\&\quad+y_2S_{p+q}(y_1)-\frac{y_2^3}{3!}S_{p+q-1}(y_1)+\cdots+(-1)^{q+1}\frac{y_2^{2q-1}}{(2q-1)!}S_{q+1}(y_1)\\&\quad+\frac{1}{2}\big[C_{p+q}(y_1+y_2))-C_{p+q}(y_1-y_2)\big].\end{aligned}$$

This permits to obtain

$$\int_0^1\frac{(1-s)^p}{p!}\cos(sy_1)\,\mathrm{d}s\times\int_0^1\frac{(1-t)^q}{q!}\cos(ty_2)\,\mathrm{d}t$$

as a *homogeneous* cos transform.

B.3.3. Linearization of the products

In the following, we consider

$$\begin{aligned}\big(6\widehat{K}''+\widehat{K}\big)(y) &= \frac{72}{y^4}C_1(y)+\frac{48}{y^3}S_1(y)-\frac{10}{y^2}C_0(y),\\\widehat{K}'(y) &= -\frac{4}{y^3}C_1(y)-\frac{2}{y^2}S_1(y).\end{aligned}$$

We have

$$\big(6\widehat{K}''+\widehat{K}\big)(y)=\frac{72}{y^4}C_2(y)+\frac{48}{y^3}S_2(y)-\frac{10}{y^2}C_1(y). \tag{B.51}$$

The following identity yields (B.44),

$$\begin{aligned}\widehat{K}'(y_1)\widehat{K}'(y_2)&=\frac{16}{y_1^3y_2^3}C_1(y_1)C_1(y_2)+\frac{4}{y_1^2y_2^2}S_1(y_1)S_1(y_2)\\&\quad+\frac{8}{y_1^3y_2^2}C_1(y_1)S_1(y_2)+\frac{8}{y_1^2y_2^3}S_1(y_1)C_1(y_2).\end{aligned}$$

In the same manner, we have

$$\begin{aligned}&\big(6\widehat{K}''+\widehat{K}\big)(y_1)\big(6\widehat{K}''+\widehat{K}\big)(y_2)\\&\quad=\left[\frac{72}{y_1^4}C_1(y_1)+\frac{48}{y_1^3}S_1(y_1)-\frac{10}{y_1^2}C_0(y_1)\right]\times\left[\frac{72}{y_2^4}C_1(y_2)+\frac{48}{y_2^3}S_1(y_2)-\frac{10}{y_2^2}C_0(y_2)\right].\end{aligned}$$

Proof of Proposition B.12. From (B.51), we deduce the linearization

$$\begin{aligned}&\big(6\widehat{K}''+\widehat{K}\big)(y_1)\big(6\widehat{K}''+\widehat{K}\big)(y_2)\\&\quad=\int_0^1 P_1(s)\left[\frac{y_1^4}{y_2^4}\cos(y_1s)+\frac{y_2^4}{y_1^4}\cos(y_2s)\right]\mathrm{d}s\\&\qquad+\int_0^1 P_2(s)\left[\frac{y_1^2}{y_2^2}\cos(y_1s)+\frac{y_2^2}{y_1^2}\cos(y_2s)\right]\mathrm{d}s\\&\qquad+\int_0^1 R(y_1,y_2,s)\cos\big((y_1+y_2)s\big)\,\mathrm{d}s\\&\qquad+\int_0^1 R(y_1,-y_2,s)\cos\big((y_1-y_2)s\big)\,\mathrm{d}s\end{aligned} \tag{B.52}$$

where $P_1(s)$, $P_2(s)$ are given in (B.39) and, where

$$\begin{aligned}R(y_1,y_2,s)&=72\times 36\times\frac{(y_1+y_2)^8}{y_1^4y_2^4}\frac{(1-s)^7}{7!}-36\times 48\times\frac{(y_1+y_2)^8}{y_1^4y_2^4}\frac{(1-s)^6}{6!}\\&\quad+36\times 10\times\frac{(y_1+y_2)^6(y_1^2+y_2^2)}{y_1^4y_2^4}\frac{(1-s)^5}{5!}\end{aligned}$$

$$+ 24 \times 48 \times \frac{(y_1+y_2)^6}{y_1^3 y_2^3} \frac{(1-s)^5}{5!}$$
$$- 24 \times 10 \times \frac{(y_1+y_2)^6}{y_1^3 y_2^3} \frac{(1-s)^4}{4!} + 50 \times \frac{(y_1+y_2)^4}{y_1^2 y_2^2} \frac{(1-s)^3}{3!}.$$

If $y_1 = y_2 = y$, $R(y, y, s) = \phi_2'(s)/2$ is given by (2.7). This proves Proposition B.12. □

Corollary B.22. *If* $\theta \neq 0$, *there exist constants* c_1 *and* c_2 *such that*

$$\sum_{n\geqslant 1} \frac{1}{n}\bigl(6\widehat{K}'' + \widehat{K}\bigr)^2(ny)\cos n\theta \leqslant \frac{c_1}{\cosh(\rho)} \tag{B.53}$$

for $|\theta| + y < c_2$ *and* $y > 0$.

Proof. We have

$$\begin{aligned} h(ys+\theta) + h(ys-\theta) &= \log\left|4\sin\left(\frac{ys+\theta}{2}\right)\sin\left(\frac{ys-\theta}{2}\right)\right| \\ &= \log\bigl|2(1-\cos\theta)\bigr| + \log\left|1 - \frac{\sin^2(ys/2)}{\sin^2(\theta/2)}\right|. \end{aligned}$$

The integrals containing $\log|2(1-\cos\theta)|$ cancel and since $|\log(1-u)| \leqslant 3u$ for $0 < u < 1/10$, we obtain

$$\begin{aligned} &\sum_{n\geqslant 1} \frac{1}{n}\bigl(6\widehat{K}'' + \widehat{K}\bigr)^2(ny)\cos n\theta \\ &\quad = \int_0^1 \bigl(P_1(s) + P_2(s)\bigr)\log\left|1 - \frac{\sin^2(ys/2)}{\sin^2(\theta/2)}\right| \mathrm{d}s + \frac{1}{2}\int_0^1 R(s)\log\left|1 - \frac{\sin^2(ys)}{\sin^2(\theta/2)}\right| \mathrm{d}s. \end{aligned}$$

Thus

$$\left|\sum_{n\geqslant 1} \frac{1}{n}\bigl(6\widehat{K}'' + \widehat{K}\bigr)^2(ny)\cos n\theta\right| \leqslant \mathrm{const}\frac{\sin^2(y)}{\sin^2(\theta/2)} \leqslant \mathrm{const}\frac{y^2}{\theta^2} \leqslant \mathrm{const}\frac{2y^2}{\theta^2 + 2y^2}. \qquad \square$$

B.3.4. The series γ. *Proof of Proposition* B.16

We have

$$\begin{aligned} &\bigl[6\widehat{K}''(y_1) + \widehat{K}(y_1)\bigr]\widehat{K}'(y_2) \\ &\quad = -\frac{72 \times 4}{y_1^4 y_2^3} C_2(y_1)C_1(y_2) - \frac{72 \times 2}{y_1^4 y_2^2} C_2(y_1)S_1(y_2) - \frac{48 \times 4}{y_1^3 y_2^3} S_2(y_1)C_1(y_2) \end{aligned}$$

$$-\frac{48\times 2}{y_1^3y_2^2}S_2(y_1)S_1(y_2)+\frac{10\times 4}{y_1^2y_2^3}C_1(y_1)C_1(y_2)+\frac{10\times 2}{y_1^2y_2^2}C_1(y_1)S_1(y_2)$$

$$=\frac{y_1^3}{y_2^3}\int_0^1 P_1(s)\sin(y_1s)\,\mathrm{d}s+\frac{y_2^4}{y_1^4}\int_0^1 R_1(s)\sin(y_2s)\,\mathrm{d}s+\frac{y_2^2}{y_1^2}\int_0^1 R_2(s)\sin(y_2s)\,\mathrm{d}s$$

$$+\int_0^1 Q(y_1,y_2,s)\sin\bigl((y_1+y_2)s\bigr)\,\mathrm{d}s+\int_0^1 T(y_1,y_2,s)\sin\bigl((y_1-y_2)s\bigr)\,\mathrm{d}s$$

where the polynomials $P_1(s),\dots$ are given by (B.41). If $y_1=y_2=y$, this simplifies to

$$\bigl[6\widehat{K}''(y)+\widehat{K}(y)\bigr]\widehat{K}'(y)=\frac{2}{15}\int_0^1 s(1-s)^3\bigl(14-3s-6s^2\bigr)\sin(ys)\,\mathrm{d}s$$
$$+\frac{2^7}{5!}\int_0^1 s(1-s)^3\bigl(-11+12s+24s^2\bigr)\sin(2ys)\,\mathrm{d}s.$$

Proof of (B.42). $\sum_{n\geqslant 1}\lambda(n)(6\widehat{K}''(ny)+\widehat{K}(ny))\widehat{K}'(ny)\sin n\theta=J(y,\theta)$, where

$$J(y,\theta)=-\int_0^1 \phi(s)\bigl[h(ys+\theta)-h(ys-\theta)\bigr]\,\mathrm{d}s$$
$$-\int_0^1 \psi(s)\bigl[h(2ys+\theta)-h(2ys-\theta)\bigr]\,\mathrm{d}s \tag{B.54}$$

with

$$\begin{aligned}\phi(s)&=\frac{1}{15}s(1-s)^3\bigl(14-3s-6s^2\bigr),\\ \psi(s)&=\frac{2^6}{5!}s(1-s)^3\bigl(-11+12s+24s^2\bigr).\end{aligned}\tag{B.55}$$

If $h(t)=-\log|t|$, then $h(ys+\theta)-h(ys-\theta)=2ys/\theta+2y^3s^3/(3\theta^3)+\cdots$, as $y\to 0$. We see that

$$\int_0^1 s\phi(s)\,\mathrm{d}s+2\int_0^1 s\psi(s)\,\mathrm{d}s=\frac{9\times 2^3}{5!}\int_0^1 s^2(1-s)^3\bigl(-6+7s+14s^2\bigr)\,\mathrm{d}s=0, \tag{B.56}$$

and thus

$$J(y,\theta) \sim \mathrm{const}\frac{y^3}{\theta^3}, \quad \text{as } y \to 0. \qquad \square$$

Proof of (B.43). We have

$$h(ys+\theta) - h(ys-\theta) = -\log\left|\frac{\sin((ys+\theta)/2)}{\sin((ys-\theta)/2)}\right| = -\log\left|\frac{1+\tan(ys/2)/\tan(\theta/2)}{1-\tan(ys/2)/\tan(\theta/2)}\right|.$$

Let

$$f(u) = h(u+\theta) - h(u-\theta) + \frac{u}{\tan(\theta/2)} = -\log\left|\frac{1+\tan(u/2)/\tan(\theta/2)}{1-\tan(u/2)/\tan(\theta/2)}\right| + \frac{u}{\tan(\theta/2)}.$$

There exists a constant c such that

$$\left|f(u)\right| \leqslant c\frac{u^3}{\theta^3} \quad \text{for } 0 < u < \inf(1,\theta/2).$$

Because of (B.54) and (B.56), we have

$$J(y,\theta) = -\int_0^1 \phi(s) f(ys)\,\mathrm{d}s - \int_0^1 \psi(s) f(2ys)\,\mathrm{d}s,$$

and thus

$$\left|J(y,\theta)\right| \leqslant cM\frac{y^3}{\theta^3}$$

with $M = \int_0^1 |\phi(s)|\,\mathrm{d}s + \int_0^1 |\psi(s)|\,\mathrm{d}s$. $\square$

References

[1] L.V. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand Mathematical Studies, vol. 10, Van Nostrand, Toronto–New York–London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr.

[2] H. Airault, V. Bogachev, Realization of Virasoro unitarizing measures on the set of Jordan curves, C. R. Acad. Sci. Paris, Sér. I 336 (2003) 429–434.

[3] H. Airault, P. Malliavin, Unitarizing probability measures for representations of Virasoro algebra, J. Math. Pures Appl. 80 (2001) 627–667.

[4] H. Airault, P. Malliavin, A. Thalmaier, Support of Virasoro unitarizing measures, C. R. Math. Acad. Sci. Paris, Sér. I 335 (2002) 621–626.

[5] H. Airault, J. Ren, Modulus of continuity of the canonic Brownian motion “on the group of diffeomorphisms of the circle”, J. Funct. Anal. 196 (2002) 395–426.

[6] A. Beurling, L. Ahlfors, The boundary correspondence under quasiconformal mappings, Acta Math. 96 (1956) 125–142.

[7] G. Cao, Kai He, On a type of stochastic differential equations driven by countably many Brownian motions, J. Funct. Anal. 203 (2003) 262–285.

[8] C. Caratheodory, Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis, Math. Ann. 75 (1913) 305–320.

[9] S. Fang, Canonical Brownian motion on the diffeomorphism group of the circle, J. Funct. Anal. 196 (2002) 162–179.

[10] S. Fang, T. Zhang, A class of stochastic differential equations with non-Lipschitzian coefficients: pathwise uniqueness and no explosion, C. R. Acad. Sci. Paris 337 (2003) 737–740.

[11] D.H. Hamilton, Generalized conformal welding, Ann. Acad. Sci. Fenn. Ser. A I Math. 16 (1991) 333–343.

[12] N. Ikeda, S. Watanabe, A comparison theorem for solutions of stochastic differential equations and its applications, Osaka J. Math. 14 (1977) 619–633.

[13] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, second ed., North-Holland Mathematical Library, vol. 24, North-Holland, Amsterdam, 1989.

[14] H. Iwaniec, Spectral Methods of Automorphic Forms, second ed., Grad. Stud. Math., vol. 53, Amer. Math. Soc., Providence, RI, 2002.

[15] P.W. Jones, S.K. Smirnov, Removability theorems for Sobolev functions and quasiconformal maps, Ark. Mat. 38 (2000) 263–279.

[16] Y. Katznelson, S. Nag, D.P. Sullivan, On conformal welding homeomorphisms associated to Jordan curves, Ann. Acad. Sci. Fenn. Ser. A I Math. 15 (1990) 293–306.

[17] A.A. Kirillov, Geometric approach to discrete series of unirreps for Virasoro, J. Math. Pures Appl. 77 (1998) 735–746.

[18] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge Stud. Adv. Math., vol. 24, Cambridge Univ. Press, Cambridge, UK, 1990.

[19] G.F. Lawler, O. Schramm, W. Werner, Analyticity of intersection exponents for planar Brownian motion, Acta Math. 189 (2002) 179–201.

[20] Y. Le Jan, O. Raimond, Integration of Brownian vector fields, Ann. Probab. 30 (2002) 826–873.

[21] O. Lehto, Univalent Functions and Teichmüller Spaces, Grad. Texts in Math., vol. 109, Springer-Verlag, New York, 1987.

[22] P. Malliavin, Stochastic Analysis, Grundlehren Math. Wiss., vol. 313, Springer-Verlag, Berlin, 1997.

[23] P. Malliavin, The canonic diffusion above the diffeomorphism group of the circle, C. R. Acad. Sci. Paris, Sér. I 329 (1999) 325–329.

[24] H.P. McKean Jr., Stochastic Integrals, Probab. Math. Statist., vol. 5, Academic Press, New York, 1969.

[25] Ch. Pommerenke, Boundary Behaviour of Conformal Maps, Grundlehren Math. Wiss., vol. 299, Springer-Verlag, Berlin, 1992.

[26] J. Ren, X. Zhang, Stochastic flow for SDEs with non-Lipschitz coefficients, Bull. Sci. Math. 127 (2003) 739–754.