Paul Malliavin (10 September 1925–3 June 2010)

Anton Thalmaier (Université du Luxembourg)

Malliavin as a young boy of around 12 years

Paul Malliavin at the Hammamet conference in stochastic analysis, Tunesia, November 2009

On 3 June 2010, Paul Malliavin passed away at the American Hospital in Paris. Less than four weeks prior to his death many of his colleagues and friends came together at an international conference with 250 participants at the Chinese Academy of Science in Beijing, honouring him and his scientific work. Probably no one at this meeting anticipated that this would be the last opportunity to experience Paul Malliavin talking in public about mathematics. Malliavin seemed to be a timeless figure. Being in his 80s, his intellect was sharper than ever; his curiosity, passion and enthusiasm for mathematics was without limitation. His personality seemed to be untouchable by physical conditions; age could not bend or slow down this man. Still giving four talks within ten days in China, his health however deteriorated after returning to Paris. Despite suffering from pulmonary fibrosis for some years, his death came unexpectedly to everyone who knew him.

Born in 1925 in Neuilly-sur-Seine, Malliavin's way into mathematics was by no means straightforward. He had strong interests in other fields as well, including law, history and literature, which made the decision between law and mathematics a difficult choice; he began his university studies by taking courses in both fields. To say it in his own words: "*I was born into a family of intellectuals who were deeply involved in politics for several generations, either by writing books or by exercising political responsibilities at a national level in France. I have the highest respect for the fighting life of my parents, uncles, and grandparents; I have often seen their disillusions after fighting for carefully planned political proposals that were finally withdrawn. One of my reasons for choosing mathematics has been that as soon as truth is discovered, it enters immediately into reality.*" (From *Mathematicians: An Outer View of an Inner World*, 2009.)

The early years: Malliavin as a harmonic analyst

Paul Malliavin finished his graduate studies in mathematics at Sorbonne University in Paris in 1946. He had the chance to take courses taught by the great masters of the French school of the beginning of the 20th century: Émile Borel for integration and Élie Cartan for geometry. He was deeply influenced by Jean Leray and Szolem Mandelbrojt, later his thesis advisor, both of whom had returned to France after the war. Szolem Mandelbrojt advised Malliavin to read his joint Comptes Rendus note with Norbert Wiener, which was devoted to the characterization of the set of zeros of some Laplace transforms. He asked him the question of what could be said about the set of real zeros of a holomorphic function in the right half-space satisfying a certain growth condition – a question which had its origin in this joint work. Malliavin detected in this problem of complex analysis of one variable a certain infinite dimensional non-linear duality, to which the Banach-Baire principle could be applied in order to prove the needed uniform estimate. He came back to Mandelbrojt with a complete and definitive answer to the question which resulted in his thesis published in *Acta Mathematica* and brought him, with the recommendation of Jean Leray, an invitation by Marston Morse to come as a postdoctoral fellow to the Princeton Institute for Advanced Study (IAS) in 1954–55. The IAS was at this time a unique gathering place of mathematicians from all over the world.

At Princeton he shared an office with Alberto Calderón for one year. Calderón, who had just finished his work with Zygmund on singular integrals, was renewing the theory of partial differential equations with the introduction of pseudo-differential operators. The contact with Calderón marked the beginning of a lifelong friendship, and opened up Malliavin's vision of Fourier analysis. Calderón showed him his forthcoming paper where he proved the localization of Littlewood-Paley theory for Fourier series of one variable, a method which Malliavin later used for his own work in collaboration with his wife.

In 1954 Arne Beurling, a visionary mathematician, had settled at the Institute for Advanced Study. From him Malliavin learned about the "spectral synthesis problem": *Is it true that in the normed ring of functions with absolutely convergent Fourier series, any closed ideal is the intersection of maximal ideals containing it?*" Four years later Malliavin noticed that an appropriate extension of the Wiener–Gelfand analytic symbolic calculus could be used to give a negative answer to the spectral synthesis problem on the real line. Malliavin's complete and definitive solution of the problem has been the subject of many lectures and related works; it brought him instant recognition. His final proof in 1959 that *spectral analysis fails for any non-discrete, locally compact, abelian group* made Malliavin's name famous. It nevertheless killed the field and marked the end of an era.

Beurling invited Malliavin again to the Institute in 1961, where he also met Lennart Carleson with whom he established another lifelong relation. The collaboration with Beurling turned out to be extremely fruitful; within one year they solved two hard open problems in analysis. Malliavin liked to tell anecdotes about Beurling's perfectionist style, not wanting to publish results which he thought were not yet in an ultimate and definitive form. As a consequence, their second joint *Acta Mathematica* paper did not appear until 1967, although the authors knew the results as early as 1961.

The later years: Malliavin as a probabilist

Malliavin's first steps into probability theory were anything but streamlined. Around the age of 40, Malliavin started working on functions of several complex variables. One of his objectives was a generalization of Blaschke's theorem from one to several complex variables. He realised that certain asymptotic estimates of the Green kernel near the boundary of the unit ball in several complex variables, along with subsequent results of Henri Skoda and Guennadi Henkin, would allow him to solve Blaschke's problem. Stuck in his effort to prove the necessary estimate, Malliavin had his first encounter with Itô's theory of stochastic differential equations. He had met Kiyosi Itô at the Institute for Advanced Study already in 1954, and had grasped from him the basics of Itô calculus. Concerning his problem, Malliavin observed that substituting Brownian motion associated to the natural Kähler metric on the unit ball into the corresponding Kähler potential and developing the resulting one-dimensional diffusion by means of Itô's formula leads to a process which, using easy geometric estimates, can be dominated by a diffusion on the real line, with a simple Sturm–Liouville type operator as generator. This "comparison lemma" published in 1972 did not only give the desired estimate; it turned out to be the first application of the later Ikeda–Watanabe comparison theory for stochastic differential equations. It also marks the starting point of Malliavin using probabilistic arguments in analysis and geometry, for which he would develop an unequalled mastership.

A turning point in Malliavin's career was Kiyosi Itô's talk at the ICM at Stockholm in 1962, where he showed that the Levi–Civita parallel transport of tensors on a Riemannian manifold can be done along the trajectories of Brownian motion. Taught by Élie Cartan, who had written two books on the method of moving frames, Malliavin immediately recognised the importance of this construction, which allows one to globalise the local Itô construction in the context of the bundle of orthonormal frames. This was the starting point of a new field: stochastic differential geometry, formed by mixing Élie Cartan's geometry on the frame bundle with Kiyosi Itô's theory of diffusion processes.

Malliavin saw right from the beginning what was later called "Malliavin's transfer principle", namely that every construction in differential geometry which can be done with smooth curves can also be done with paths of diffusions if the classical derivatives are interpreted in the sense of Stratonovich differentials. Since the tangent bundle of the orthonormal frame bundle over a Riemannian manifold is trivial (it is trivialised by the standard horizontal and vertical vector fields on the frame bundle), one can construct a diffusion associated to Bochner's horizontal Laplacian by solving a canonical stochastic differential equation on the frame bundle. The projection of this process down to the manifold then gives an intrinsic construction of Brownian motion associated to the Levi–Civita Laplacian. This elegant geometric method, developed by Eells–Elworthy and Malliavin, of constructing random processes on curved spaces by rolling the manifold along the paths of a flat Brownian motion in the tangent space, transfers the classical Cartan development of differentiable curves to the probabilistic world; it provides at the same time Brownian motion together with an intrinsic notion of parallel transport along its paths. Malliavin used the new method in 1975 for a probabilistic Feynman–Kac representation of the de Rham–Hodge semigroup on differential forms which allowed him to prove Bochner–Kodaira type vanishing theorems for the cohomology of the manifold.

At that time it already became clear that Brownian motion might serve as a tool to interpolate between the local and global geometry of a manifold: for small time Brownian motion is governed by the local geometry, while for large times it captures its global structure. Jean-Michel Bismut quickly absorbed the new ideas and used them later in his stochastic proof of the Atiyah-Singer index theorem for Dirac operators. Here one investigates the small time asymptotics of a certain deformed parallel transport in a Clifford bundle along Brownian loops. The local index density is then calculated as the expectation of the supertrace of this random holonomy under contraction of the Brownian loops to constant loops. The advantage of this method is that all relevant calculations can be done under the expectation at the level of random functionals; the evaluation of the supertrace is reduced to elementary linear algebra, and the so-called "fantastic cancellations" become fully transparent.

Paul Malliavin discussing mathematics, Kent State University, 2008

In the same way as a vector field on a manifold induces a flow, second order differential operators induce stochastic flows which however behave very irregularly in the time variable. In this sense, Brownian motion on a Riemannian manifold appears as the stochastic flow associated to the Laplace-Beltrami operator. In the 70s Malliavin became interested in the push forward of the underlying measure under such flows. Completely in the spirit of Wiener, he looked at these measures on path space as analytical objects, to which analytic methods should be applied. Wiener himself had recognised that his measure carries the same Hermitian structure as the standard Gaussian measure on the line, which led him to his famous spectral decomposition of the space of square-integrable functionals on Wiener space into subspaces of "homogeneous chaos". This decomposition can be seen as what quantum field theorists call the Fock space representation of the number operator.

Malliavin's goal was to develop a differential calculus on Wiener space which could be applied to functionals as general as those arising as solutions to Itô's stochastic differential equations. In infinite dimensions, like on path space, a function can be infinitely differentiable in the sense of Sobolev without being even well-defined at every point. Before Malliavin, differential analysis on Wiener space was mainly restricted to functions being differentiable in the classical sense of Fréchet. Based on results of R. H. Cameron and W. T. Martin, two students of Norbert Wiener, who had established quasi-invariance of the Wiener measure under translation by elements which are absolutely continuous with square integrable derivative, Malliavin chose a certain operator, known to probabilists as the Ornstein–Uhlenbeck operator, as the primary operation in his theory because it is self-adjoint and behaves well in calculations involving integration by parts. This was the starting point of a new kind of analysis in infinite dimensions which Malliavin called "Stochastic Calculus of Variations".

One of the first aims of Malliavin in this field was to give a purely probabilistic approach to Hörmander's famous hypoellipticity theorem which provides a condition for a partial differential operator, written as a sum of squares of vector fields, to be hypoelliptic. The question comes down to showing that the stochastic flow associated to this second order differential operator is sufficiently smooth and non-degenerate to guarantee that a certain induced heat kernel measure has a smooth density with respect to the Lebesgue measure. The required non-degeneracy condition can be expressed in terms of integrability conditions on the inverse determinant of the famous Malliavin matrix. This already marks a keystone of the new calculus.

Malliavin presented these ideas at the SDE Symposium in Kyoto 1976. His Japanese colleagues, particularly K. Itô and his students N. Ikeda and S. Watanabe, immediately recognised its potential and began to give it a formulation that has become standard. At the same time Dan Stroock gave a series of lectures in France on the new methods; he dubbed it "Malliavin Calculus", a term which soon became standard. The theory rapidly grew through numerous extensions, simplifications and alternative approaches. A crucial estimate which greatly simplified many calculations is due to P. A. Meyer. Over the years Malliavin calculus developed into a powerful machinery, with essential contributions from many other mathematicians like Bismut, Ikeda–Watanabe, Kusuoka–Stroock, Nualart–Zakai, Üstünel and Bouleau–Hirsch, to name just a few. A solid theory of Sobolev spaces on Wiener space was developed by Len Gross and Dan Stroock; integration by parts theorems for the measure induced by Brownian motion on path space of a manifold or by pinned Brownian motion on loop space were established by J.-M. Bismut, B. Driver, E.P. Hsu and P. Malliavin and his wife Marie-Paule. I. Shigekawa proved the Hodge decomposition on Wiener space, quasi-sure analysis was developed and an anticipative stochastic calculus was established in the 80s by Nualart–Pardoux–Zakai.

Terms like "Malliavin derivative", "Malliavin matrix" and "smooth in the sense of Malliavin" became standard vocabulary in graduate courses in probability and in conference talks. Currently more than 25 monographs on Malliavin calculus are available. Malliavin entered probability theory at the age of 45; in less than 15 years he had completely reshaped the field.

Around 2000 P.-L. Lions and his coworkers began to use methods from Malliavin calculus to stabilise the numerical computation of price sensitivities, so-called Greeks, in the theory of option pricing in finance. Malliavin was proud to see Malliavin calculus suddenly in the centre of such practical fields as finance; he even wrote a monograph "Stochastic Calculus of Variations in Mathematical Finance" to explain his point of view.

The aim of one of Malliavin's big projects over the last 12 years was the construction of natural measures on infinite dimensional spaces, with strong motivation from mathematical physics. Eminent examples are Brownian measures on the diffeomorphism group of the circle, on the space of univalent functions of the unit disc and on the space of Jordan curves in the complex plane. He understood that unitarizing measures for representations of Virasoro algebra can be approached as invariant measures of Brownian motion on the diffeomorphism group with a certain drift defined in terms of a Kähler potential.

Malliavin as a person

Malliavin never thought in terms of applied and pure mathematics, nor was he interested in formal generalizations; he aimed at concepts and ideas. For him mathematics was a unity and not divisible into different fields or branches. Whenever he recognised new promising ideas, even in the work of very young mathematicians or PhD students, he was extremely generous in offering his support. Many young mathematicians may have shared the potentially intimidating experience when, after a conference talk, Malliavin would come running behind them and shouting in a loud voice: "I need to talk to you…" However, such conversations usually turned out to be very encouraging and rewarding.

It was impossible to meet Malliavin without talking mathematics. When encountering him, his first question used to be: "What are you currently working on?" And then he would keep on asking questions until his curiosity was satisfied. Convinced of "the fundamental unity of mathematics", Paul Malliavin liked to characterize his career as one of "mathematical wandering", devoted to the establishing of relations between fields that seemed relatively unrelated. For him only ideas counted in mathematics and he would not start fighting with the necessary technical details before having understood a problem "from above" with a clear vision of what should be done.

The mathematical work of Paul Malliavin consists of about 200 research articles, and it would be foolish to try to go into details. He continued over the last years in a steady rhythm of publishing papers on themes as diverse as the Euler equation of deterministic incompressible fluid dynamics using tools of stochastic differential geometry, the Wiener measure on Jordan curves, unitarizing measures for a representation theory of Virasoro algebra, Stein's method for estimating the speed of convergence to Gaussian laws, numerical approximation schemes for stochastic differential equations and problems in mathematical finance, like the Fourier computation of volatilities for high frequency financial data.

Paul Malliavin in Uppsala, Sweden, 2005

To understand the person of Malliavin, one probably has to go back to his early childhood. Born as a single child into a very conservative environment – his mother couldn't have any more children after his birth – he kept a close and very emotional relationship to his parents all his life. Each year the family, together with numerous relatives, used to spend the summer months in a castle in the province of Auvergne. For birthdays of the small Paul, his grandfather ordered knights arriving on horses delivering the birthday presents. To inspire self-confidence in his grandson, the small boy had to receive the arriving delegations and the people from the village offering presents. It seems that this injection of self-confidence continued to have a lasting effect even 80 years later.

Collaborating with Malliavin has always been an exciting and challenging experience. When working on a specific problem and facing all the difficulties, one is often ready to give up, but not so Malliavin. The word "impossible" did not exist in his vocabulary. Armed with formidable technical skills, he liked such hopeless situations where he would finally turn things around by introducing new, unexpected ideas; he enjoyed it if the new approach turned out to work.

Malliavin still had many unfinished projects in mind and somehow during the last period of his life he felt that time was limited. Undeterred by technical difficulties, Malliavin pressed ahead even more than in his younger years. From his bed in hospital he still discussed mathematical projects with his collaborators. Some of his friends visiting him got worried by the alarms from the surrounding machines when he continuously lifted his breathing mask which disturbed him explaining mathematics. It is not known what the doctors in the hospital thought, when days before passing away he suggested transporting the machines necessary to prevent his lung from collapsing to his private home, as he was annoyed that without sitting at his home computer it was difficult for him to work properly.

His departure marks the end of an extraordinary career and leaves a huge gap in the community, or to say it with the words of Michèle Vergne, "… *the world without Malliavin is not quite the same*".