

Mean-Square Continuity on Homogeneous Spaces of Compact Groups*

D.Marinucci and G.Peccati

Department of Mathematics, University of Rome Tor Vergata
and Luxembourg University, Unité de Recherche en Mathématiques

January 25, 2013

Abstract

We show that any finite-variance, isotropic random field on a compact group is necessarily mean-square continuous, under standard measurability assumptions. The result extends to isotropic random fields defined on homogeneous spaces where the group acts continuously.

- **Keywords and Phrases:** Random Processes; Isotropy; Mean-Square Continuity.
- **AMS Classification:** 60G05, 60G60

1 Introduction

The analysis of the spectral representations of stationary and isotropic finite-variance random fields on (subsets of) \mathbb{R}^d is a classical topic of probability theory, presented in many standard textbooks in the area (see for instance [1, 2, 13, 22]). Such representations are sometimes based on Karhunen-Loëve constructions (specially under Gaussianity assumptions), realized by first computing the eigenfunctions associated with the covariance kernel, and then by expanding the field into these orthogonal components (see for instance [2]). In other cases, the argument proceeds from the construction of an isometry between an L^2 space of deterministic square integrable functions, and some space of finite-variance random variables, with inner product defined in terms of the covariance function of the process to be represented (see for instance [13, 22]). In all these approaches, mean-square continuity is assumed as a necessary condition to ensure that the spectral representation holds pointwise.

More recently, considerable attention has been drawn to the case where the process at hand is defined on the homogenous space of a compact group (including the group itself). In this context, one of the most relevant examples for applications is the sphere S^2 , which is well-known to be isomorphic to the quotient space $SO(3)/SO(2)$, where $SO(d)$ denotes as usual the special group of rotations in \mathbb{R}^d . Under these circumstances, spectral representation results take a particularly neat form, as they can be viewed as stochastic versions of the celebrated Peter-Weyl Theorem (see [8, Section 4.6] or [14, Section 2.5]); the latter ensures that the matrix coefficients of the irreducible representations of a compact group G provide an orthonormal basis for the space $L^2(G)$ of square-integrable functions on the group, endowed with the Haar measure. This is the standpoint adopted for instance by [14, 16] – see also [4]. It should be noted that random fields on the unit sphere S^2 play now an extremely important role in many applied fields, for instance in Cosmology – see [6, 9], [14] for an overview.

As we shall point out in the sections to follow, the argument based on the group-theoretic point of view does not only provide an alternative proof for classical results, but yields also an unexpected bonus: the assumption of mean-square continuity turns out to be no longer necessary for the spectral representation to hold. More than that, mean-square continuity follows as a necessary consequence of

*We thank S. Trapani for a crucial input to the content of Section 5; we are also grateful to P.Baldi, O. El-Dakkak, N.Leonenko and H.-J.Starkloff for useful discussions. Research by DM is supported by the ERC Grant 277742 Pascal.

the spectral representation, under standard measurability conditions. The aim of this short note is then to highlight this result, which we can state concisely as follows:

Let T be a measurable finite-variance isotropic random field defined on the homogeneous space of a compact group acting continuously. Then, T is necessarily mean-square continuous.

Our main findings are contained in the statements of Theorem 2 (for fields defined on compact groups) and of Theorem 4 (for fields defined on homogeneous spaces). Section 2 and Section 3 contain preliminary results, respectively on group representations and random fields. Some historical remarks are provided in Section 6.

2 Preliminaries on group representations

We now provide a brief overview of the results about group representations that are used in this note. For any unexplained definition or result, the reader is referred to [14, Chapter 2], as well as to the classic reference [8].

A *topological group* is a pair (G, \mathbb{G}) , where G is a group and \mathbb{G} is a topology such that the following three conditions are satisfied: (i) G is a Hausdorff topological space, (ii) the multiplication $G \times G \rightarrow G : (g, h) \mapsto gh$ is continuous, (iii) the inversion $G \rightarrow G : g \mapsto g^{-1}$ is continuous. In what follows, we use the symbol G to denote a topological group (the topology \mathbb{G} being implicitly defined) which is also *compact* and such that \mathbb{G} has a countable basis. We will denote by $C(G)$ the class of continuous, complex-valued functions on G ; \mathcal{G} is the (Borel) σ -field generated by \mathbb{G} ; dg denotes the normalized *Haar measure* of G (in particular, $\int_G dg = 1$). We shall denote by $L^2(G, \mathcal{G}, dg) = L^2(G)$ the Hilbert space of complex-valued functions on G that are square-integrable with respect to dg . Plainly, the space $L^2(G)$ is endowed with the usual inner product $\langle f_1, f_2 \rangle_G = \int_G f_1(g) \overline{f_2(g)} dg$; also, $\|\cdot\|_G = \langle \cdot, \cdot \rangle_G^{1/2}$.

Let X be a topological space. A *continuous left action* of G on X is a jointly measurable mapping $A : G \times X : (g, x) \mapsto A(g, x) := g \cdot x$, satisfying the following properties: (i) $g \cdot (h \cdot x) = (gh) \cdot x$, (ii) if e is the identity of G , then $e \cdot x = x$ for every $x \in X$, and (iii) the mapping $(g, x) \mapsto g \cdot x$ is jointly continuous. Right actions are defined analogously and will not be directly considered in this note, albeit every result concerning left actions proved below extends trivially to right actions. The space X is called a *G-homogeneous space* if G acts transitively on X , that is: for every $y, x \in X$, there exists $g \in G$ such that $y = g \cdot x$. Group representations (as described in the next paragraph) are distinguished examples of group representations.

Let V be a normed finite-dimensional vector space over \mathbb{C} . A (finite-dimensional) *representation* of G in V is an homomorphism π , from G into $\mathbf{GL}(V)$ (the set of complex isomorphisms of V into itself), such that the mapping $G \times V \rightarrow V : (g, v) \mapsto \pi(g)(v)$ is continuous. Using e.g. [14, Proposition 2.25], one sees that it is always possible to endow V with an inner product $\langle \cdot, \cdot \rangle_V$ such that π is *unitary* with respect to it, that is: for every $g \in G$ and every $u, v \in V$, $\langle \pi(g)u, \pi(g)v \rangle_V = \langle u, v \rangle_V$. Note that $\langle \cdot, \cdot \rangle_V$ can be chosen in such a way that the associated norm preserves the topology of V (see [8, Corollary 4.2.2]). The *dimension* d_π of a representation π is defined to be the dimension of V . A representation π of G in V is *irreducible*, if the only closed $\pi(G)$ -invariant subspaces of V are $\{0\}$ and V . It is well-known that unitary irreducible representations are defined up to equivalence classes (see [14, p. 25]). We will denote by $[\pi]$ the equivalence class of a given unitary irreducible representation π ; the set of equivalence classes of unitary irreducible representations of G is written \widehat{G} , and it is called the *dual* of G . We recall that, according e.g. to [8, Theorem 4.3.4 (v)], since G is second countable (and therefore metrizable) \widehat{G} is necessarily countable.

To every $[\pi] \in \widehat{G}$ we associate a finite-dimensional subspace $M_\pi \subseteq L^2(G)$ in the following way. Select an element $\pi : G \mapsto \mathbf{GL}(V)$ in $[\pi]$, as well as an orthonormal basis $\mathbf{e} = \{e_1, \dots, e_{d_\pi}\}$ of V . The space M_π is defined as the finite-dimensional complex vector space spanned by the functions

$$g \mapsto \pi_{i,j}(g) := \langle \pi(g)e_j, e_i \rangle_V, \quad i, j = 1, \dots, d_\pi.$$

Note that such a definition is well given, since M_π does not depend on the choices of the representative element of $[\pi]$ and of the orthonormal basis of V . The following three facts are relevant for the subsequent discussion: (i) $\{\sqrt{d_\pi}\pi_{ij} : i, j = 1, \dots, d_\pi\}$ is an orthonormal system of $L^2(G)$ (see [14, p. 34]); (ii) $\dim M_\pi = d_\pi^2$, and (iii) $M_\pi \subseteq C(G)$, for every $[\pi] \in \widehat{G}$.

To conclude, we recall that, if two representations π and π' are not equivalent, then M_π and $M_{\pi'}$ are orthogonal in $L^2(G)$. One crucial element of our discussion is the celebrated *Peter-Weyl Theorem* (see [14, Section 2.5]), stating that

$$L^2(G) = \bigoplus_{[\pi] \in \widehat{G}} M_\pi, \quad (2.1)$$

that is: the family of finite-dimensional spaces $\{M_\pi : [\pi] \in \widehat{G}\}$ constitutes an orthogonal decomposition of $L^2(G)$. In particular, the class $\{\sqrt{d_\pi}\pi_{ij} : [\pi] \in \widehat{G}, i, j = 1, \dots, d_\pi\}$ is an orthonormal basis of $L^2(G)$. Plainly, the orthogonal projection of a given function $f \in L^2(G)$ on the space M_π is given by the mapping

$$g \mapsto f^\pi(g) = \sum_{i,j=1}^{d_\pi} d_\pi \int_G f(h)\bar{\pi}_{i,j}(h)dh \times \pi_{i,j}(g) = d_\pi \int_G f(h)\chi_\pi(h^{-1}g)dh, \quad (2.2)$$

where $\chi_\pi(g) = \text{Trace } \pi(g)$ is the *character* of π . See [14, Section 2.4.5] for a discussion of the basic properties of group characters; in particular, one has that two equivalent representations have the same character, in such a way that the projection formula (2.2) is well-defined, in the sense that it does not depend on the choice of the representative element of the equivalence class $[\pi]$.

From now on, we shall fix a topological compact group (G, \mathbb{G}) , and freely use the notation and terminology introduced above.

3 General setting and spectral decompositions

Let $T = \{T(g) : g \in G\}$ be a finite variance, isotropic random field on G , by which we mean that T is a real-valued random mapping on G verifying the following properties (a)–(c).

- (a) (*Joint measurability*) The field T is defined on a probability space (Ω, \mathcal{F}, P) , and the mapping $T : G \times \Omega \rightarrow \mathbb{R} : (g, \omega) \mapsto T(g, \omega)$ is $\mathcal{G} \otimes \mathcal{F}$ -measurable, where (as before) \mathcal{G} denotes the Borel σ -field associated with (G, \mathbb{G}) .
- (b) (*Isotropy*) The distribution of T is invariant in law with respect to the action of G on itself. This means that, for every $h \in G$, $T(hg) \stackrel{d}{=} T(g)$, where “ $\stackrel{d}{=}$ ” indicates equality in distribution in the sense of stochastic processes, that is: for every $d \geq 1$ and every $g_1, \dots, g_d \in G$, the two vectors $(T(g_1), \dots, T(g_d))$ and $(T(hg_1), \dots, T(hg_d))$ have the same distribution. Note that, since the application $g \mapsto hg$ is continuous, then the mapping $(g, \omega) \mapsto T(hg, \omega)$ is jointly measurable, in the sense of Point (a) above.
- (c) (*Finite variance*) The field T has finite variance, i.e.: $E[T^2(g)] = \int T^2(g, \omega)dP(\omega) < \infty$, for every $g \in G$. For notational simplicity, and without loss of generality, we will assume in the sequel that $E[T(g)] = 0$.

Under the previous assumptions and by virtue of the invariance properties of Haar measures, one has that, for every fixed $g_0 \in G$, $E[T^2(g_0)] = E[\int_G T^2(g)dg] < \infty$. This implies that there exists a \mathcal{F} -measurable set Ω' of P -probability 1 such that, for every $\omega \in \Omega'$, the mapping $T(\cdot, \omega) : g \mapsto T(g, \omega)$ is an element of $L^2(G)$. For every $[\pi] \in \widehat{G}$, we now define the quantity $T^\pi(g, \omega)$ according to (2.2), whenever $\omega \in \Omega'$, and we set $T^\pi(g, \omega) = 0$ otherwise. It is easily checked that, for every $[\pi] \in \widehat{G}$, the mapping $(g, \omega) \mapsto T^\pi(g, \omega)$ is $\mathcal{G} \otimes \mathcal{F}$ measurable.

According to the results discussed in [14, Section 5.2.1], the following two facts take place.

- The class $\{T, T^\pi : [\pi] \in \widehat{G}\}$ is an isotropic (possibly infinite-dimensional) square-integrable centered field over G , that is: every T^π is centered and square-integrable, and for every $m, d \geq 1$, for every $[\pi_1], \dots, [\pi_m] \in \widehat{G}$ and every $h, g_1, \dots, g_d \in G$, the $(m+1)d$ -dimensional vector

$$(T(g_1), \dots, T(g_d); T^{\pi_i}(g_1), \dots, T^{\pi_i}(g_d), i = 1, \dots, m)$$

has the same distribution as

$$(T(hg_1), \dots, T(hg_d); T^{\pi_i}(hg_1), \dots, T^{\pi_i}(hg_d), i = 1, \dots, m)$$

- Let $\{[\pi_k] : k = 1, 2, \dots\}$ be any enumeration of \widehat{G} . Then, for every fixed $g_0 \in G$ one has that

$$\lim_{n \rightarrow \infty} E \left\{ \left| T(g_0) - \sum_{k=1}^n T^{\pi_k}(g_0) \right|^2 \right\} = \lim_{n \rightarrow \infty} E \left\{ \int_G \left| T(g) - \sum_{k=1}^n T^{\pi_k}(g) \right|^2 dg \right\} = 0, \quad (3.3)$$

in other words: the sequence $\sum_{k=1}^n T^{\pi_k}$, $n \geq 1$, approximates T in the $L^2(P)$ sense, both for every fixed element of G , and in the sense of the space $L^2(G)$. Note that the equality in formula (3.3) is a consequence of the invariance and finiteness properties of the Haar measure dg and of the isotropy of $\{T, T^\pi : [\pi] \in \widehat{G}\}$, yielding

$$E \left\{ \left| T(g_0) - \sum_{k=1}^n T^{\pi_k}(g_0) \right|^2 \right\} = E \left\{ \int_G \left| T(gg_0) - \sum_{k=1}^n T^{\pi_k}(gg_0) \right|^2 dg \right\} = E \left\{ \int_G \left| T(g) - \sum_{k=1}^n T^{\pi_k}(g) \right|^2 dg \right\}.$$

The proof of the isotropy of $\{T^\pi : [\pi] \in \widehat{G}\}$ is given in [14], see Proposition 5.3 on pages 116-117. It should be noted that the proof of this Proposition implicitly exploits the fact that, under isotropy, for every $h \in G$ the scalar products of $T(\cdot)$ and $T^h(\cdot) := T(h \cdot)$ with any continuous function have necessarily the same distribution (we thank P. Baldi for raising this point). This result is trivial under mean-square continuity, but in the present general circumstances it is a bit less obvious, and hence we report here a proof for completeness.

Lemma 1 *Let T be an a.s. square-integrable invariant random field on \mathcal{G} and define, for $f \in L^2(\mathcal{X})$,*

$$T(f) := \int_G T(x) \overline{f(x)} dx \quad (3.4)$$

Then, for every $h \in G$ and every $f_1, \dots, f_m \in L^2(G)$, the two random variables

$$(T(f_1), \dots, T(f_m)) \text{ and } (T^h(f_1), \dots, T^h(f_m))$$

have the same distribution.

Proof. For the sake of simplicity, we shall only deal with the case $m = 1$; the general case follows along similar lines. For every $n > 0$ let $T_n = T \wedge n \vee (-n)$, which is a bounded random field, itself invariant. Now, if $f \in L^2(G)$, by dominated convergence,

$$T_n(f) \xrightarrow[n \rightarrow \infty]{a.s.} T(f) \quad \text{and} \quad T_n^g(f) \xrightarrow[n \rightarrow \infty]{a.s.} T^g(f),$$

and therefore the convergence takes place also in distribution. Hence it is sufficient to prove the statement under the additional assumption that T is bounded. But in this case the two r.v.'s $T(f)$, $T^h(f)$ are themselves bounded and in order to prove that they are equi-distributed it is sufficient to show that they have the same moments, i.e. that, for every choice of integers $p, q \geq 0$,

$$E[(\operatorname{Re} T(f))^p (\operatorname{Im} T(f))^q] = E[(\operatorname{Re} T^h(f))^p (\operatorname{Im} T^h(f))^q].$$

Now

$$E[(\text{Re}T(f))^p(\text{Im}T(f))^q]$$

$$= E \left(\int_G \text{Re}(T(x_1) \overline{f(x_1)}) dx_1 \dots \int_G \text{Re}(T(x_p) \overline{f(x_p)}) \int_G \text{Im}(T(y_1) \overline{f(y_1)}) dy_1 \dots \int_G \text{Re}(T(y_q) \overline{f(y_q)}) dy_q \right).$$

Developing the real and imaginary parts and applying Fubini's theorem, the previous expectation reduces to a sum of terms of the form

$$\int_G \dots \int_G E [b_1(x_1) \dots b_p(x_p) c_1(y_1) \dots c_q(y_q)] d_1(x_1) \dots d_p(x_p) e_1(y_1) \dots e_q(y_q) dx_1 \dots dx_m dy_1 \dots dy_k$$

where $b_i(x_i)$ (resp. $c_j(y_j)$) can be equal to $\text{Re}T(x_i)$ or $\text{Im}T(x_i)$ (resp. $\text{Re}T(y_j)$ or $\text{Im}T(y_j)$) and $d_i(x_i)$ (resp. $e_j(y_j)$) can be equal to $\pm \text{Re}f(x_i)$ or $\pm \text{Im}f(x_i)$ (resp $\pm \text{Re}f(y_j)$ or $\pm \text{Im}f(y_j)$). Now just remark that the quantity $E [b_1(x_1) \dots b_p(x_p) c_1(y_1) \dots c_q(y_q)]$ does not change if the random field T is replaced by its rotated version T^h , as such an expectation only depends on the joint distribution; this concludes the proof. ■

4 Mean-square continuity on G

In the next statement we show that isotropic fields such as those of the previous section are necessarily mean-square continuous.

Theorem 2 *Let T be a square-integrable centered isotropic field on the topological compact group G , verifying properties (a)-(c) of Section 3. Then, T is mean-square continuous: for every $g \in G$*

$$\lim_{h \rightarrow g} E |T(g) - T(h)|^2 = 0. \quad (4.5)$$

Remark 3 *Since (G, \mathbb{G}) is a topological space, equation (4.5) has to be interpreted in the following sense: for every net $\{h_i\} \subset G$ converging to g , one has that $E |T(g) - T(h_i)|^2 \rightarrow 0$. See [7, p. 28ff] for details on these notions.*

Proof of Theorem 2. Let $\{[\pi_k] : k \geq 1\}$ be an arbitrary enumeration of \widehat{G} . In view of (3.3), for every $\epsilon > 0$, there exists $n \geq 1$ such that

$$\sup_{g \in G} E \left\{ \left| T(g) - \sum_{k=1}^n T^{\pi_k}(g) \right|^2 \right\} \leq \frac{\epsilon}{6},$$

in such a way that, for every $h, g \in G$,

$$\begin{aligned} E |T(g) - T(h)|^2 &\leq 3 \left\{ E \left[\left| T(g) - \sum_{k=1}^n T^{\pi_k}(g) \right|^2 \right] + E \left[\left| T(h) - \sum_{k=1}^n T^{\pi_k}(h) \right|^2 \right] \right\} \\ &\quad + 3E \left[\left| \sum_{k=1}^n (T^{\pi_k}(g) - T^{\pi_k}(h)) \right|^2 \right] \\ &\leq \epsilon + 3E \left[\left| \sum_{k=1}^n (T^{\pi_k}(g) - T^{\pi_k}(h)) \right|^2 \right] = \epsilon + 3 \sum_{k=1}^n E [|T^{\pi_k}(g) - T^{\pi_k}(h)|^2], \end{aligned}$$

where in the last relation we have used the fact that, for $k \neq k'$, the two fields T^{π_k} and $T^{\pi_{k'}}$ are uncorrelated (see [14, Proposition 5.4]). Now, for every $[\pi] \in \widehat{G}$, we define $\widehat{T}_{i,j}^{\pi} := \int_G T(h) \pi_{i,j}(h) dh$ (it is

easily seen that $\widehat{T}_{i,j}^\pi$ is a square-integrable random variable). Using (2.2), we therefore deduce that, for every $k \geq 1$,

$$E \left[|T^{\pi_k}(g) - T^{\pi_k}(h)|^2 \right] \leq d_{\pi_k}^2 \sum_{i,j=1}^{d_{\pi_k}} E[|\widehat{T}_{i,j}^{\pi_k}|^2] \times |\pi_{i,j}(g) - \pi_{i,j}(h)|^2.$$

Since $\pi_{i,j} \in C(G)$, this last relation together with the previous estimates implies that

$$\limsup_{h \rightarrow g} E |T(g) - T(h)|^2 \leq \epsilon,$$

and the conclusion follows from the fact that ϵ is arbitrary. ■

5 Mean-square continuity on homogeneous spaces

We now fix a topological compact group G , and consider a topological space X which is also a G -homogeneous space, where G acts transitively and continuously from the left. Let $T = \{T(x) : x \in X\}$ be a centered, finite-variance isotropic random field on X . This means that the following three properties are verified: **(i)** the field T is defined on a probability space (Ω, \mathcal{F}, P) , and the mapping $T : X \times \Omega \rightarrow \mathbb{R} : (x, \omega) \mapsto T(x, \omega)$ is $\mathcal{X} \otimes \mathcal{F}$ -measurable, with \mathcal{X} denoting the Borel σ -field associated with X ; **(ii)** for every $h \in G$, $T(hx) \stackrel{d}{=} T(x)$, where “ $\stackrel{d}{=}$ ” indicates as before equality in distribution in the sense of stochastic processes; **(iii)** $ET^2(x) = \int T^2(x, \omega) dP(\omega) < \infty$ and $ET(x) = 0$, for every $x \in X$. Plainly isotropic fields as those introduced in Section 3 are special cases of the above class, obtained by taking $X = G$. To simplify the discussion, in what follows we implicitly assume that both X and G are metric spaces. The following result shows that the content of Theorem 2 extends to random fields defined on X .

Theorem 4 *Let T be a square-integrable centered isotropic field on the G -homogeneous space X , verifying properties **(i)**–**(iii)** above. Then, T is mean-square continuous: for every $x \in X$,*

$$\lim_{y \rightarrow x} E |T(y) - T(x)|^2 = 0. \quad (5.6)$$

The proof of Theorem 4 is based on the following lemma.

Lemma 5 *Let $\{x_n\} \subset X$ be a sequence converging to $x_0 \in X$ in the topology of the homogeneous space (written $x_n \rightarrow_X x_0$). Then, there exists a subsequence $\{x'_n\} \subset \{x_n\}$ verifying the following property: there exists a sequence $\{g_n\} \subset G$ such that $g_n \cdot x_0 = x'_n$ for every n , and $g_n \rightarrow_G e$, where e denotes the identity element of G .*

Proof. By transitivity, there exists a sequence $\{g_n''\}$ such that $g_n'' \cdot x_0 \equiv x_n$ for every n . Moreover, because the group is compact, this sequence admits a subsequence $\{g'_n\} \subset \{g_n''\}$ such that $g'_n \rightarrow_G g_0$. Note that g_0 need not be the identity (otherwise the proof would be finished), but it does need to belong to the isotropy group of x_0 , written $g_0 \in Iso(x_0)$, meaning that $g_0 \cdot x_0 = x_0$. Write $x'_n = g'_n \cdot x_0$. We claim that there exists a sequence $\{h_n\} \subset G$ such that the sequence $g_n := h_n g'_n$, $n \geq 1$, satisfies the following two properties:

$$(\mathbf{A}) : \quad g_n \cdot x_0 = h_n g'_n \cdot x_0 = g'_n \cdot x_0 = x'_n \quad (\text{that is, } h_n g'_n \in Iso(x'_n))$$

and

$$(\mathbf{B}) : \quad g_n = h_n g'_n \rightarrow_G e.$$

Such a sequence is given by $h_n := g'_n g_0^{-1} (g'_n)^{-1}$. Indeed, it is immediate to see that

$$\begin{aligned} h_n g'_n \cdot x_0 &= g'_n g_0^{-1} (g'_n)^{-1} g'_n \cdot x_0 = g'_n g_0^{-1} \cdot x_0 \\ &= g'_n \cdot x_0 = x'_n, \end{aligned}$$

where we have exploited the trivial fact that g_0^{-1} is an element of $\text{Iso}(x_n)$, because g_0 is. Hence, (A) is fulfilled. Moreover, since $g'_n \rightarrow_G g_0$, by continuity one infers that

$$h_n \rightarrow_G g_0^{-1}, \text{ and consequently } h_n g'_n \rightarrow_G e,$$

yielding (B). It follows that the sequence $\{x'_n\}$ satisfies the requirements in the statement, and the proof is concluded. ■

Proof of Theorem 4 Fix $x \in X$ and let $x_n \rightarrow_X x$. Using the assumptions on T , one infers that the mapping $g \mapsto T(g \cdot x) := T_x(g)$ is a centered finite-variance isotropic field on G , in the sense of Section 3. According to Lemma 5, there exist sequences $\{x'_n\} \subset \{x_n\}$ and $\{g_n\} \subset G$ such that $x'_n = g_n x$ and $g_n \rightarrow_G e$. By virtue of Theorem 2, we therefore conclude that

$$\lim_{n \rightarrow \infty} E |T(x'_n) - T(x)|^2 = \lim_{n \rightarrow \infty} E |T_x(g_n) - T_x(e)|^2 = 0.$$

This argument shows that every sequence $\{x_n\}$ converging to x in X admits a subsequence $\{x'_n\}$ such that $T(x'_n)$ converges to $T(x)$ in $L^2(P)$, and this fact is exactly equivalent to relation (5.6). ■

As already recalled, our findings apply to the important case where X equals the n dimensional unit sphere S^n , $n \geq 1$, on which the compact group $SO(n+1)$ acts transitively. Assume for notational simplicity that T is zero-mean ($ET = 0$), and write $\Gamma(x_1, x_2) := ET(x_1)T(x_2)$ for the covariance function of the random field, $\Gamma : S^n \times S^n \rightarrow \mathbb{R}$. By isotropy, there exists a function $\tilde{\Gamma}(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\Gamma(x_1, x_2) = \tilde{\Gamma}(\|x_1 - x_2\|)$, where the symbol $\|\cdot\|$ stands for Euclidean norm; it is hence straightforward (and well-known, see for instance [2, 13, 22]) that, under isotropy, mean-square continuity is equivalent to continuity of the function $\tilde{\Gamma}$ at the origin.

Of course, it is not difficult to figure out rotationally invariant, positive-definite functions which violate the continuity of $\tilde{\Gamma}$ at the origin: a simple example is provided by $\Gamma : S^n \times S^n \rightarrow \mathbb{R}$ such that

$$\Gamma(x, x) = \tilde{\Gamma}(0) = 1, \Gamma(x_1, x_2) = \tilde{\Gamma}(\|x_1 - x_2\|) = 0, \text{ for all } x_1 \neq x_2, x_1, x_2 \in S^n. \quad (5.7)$$

A consequence of Theorem 4 is that such a $\Gamma(\cdot, \cdot)$ cannot be the covariance function of a measurable isotropic process. More generally, under isotropy it is immediate to see that continuity of $\tilde{\Gamma}$ at the origin entails continuity everywhere of the covariance function Γ , indeed

$$\begin{aligned} \Gamma(x_1, y_1) - \Gamma(x_2, y_2) &= E\{T(x_1)(T(y_1) - T(y_2))\} + E\{T(y_2)(T(x_1) - T(x_2))\} \\ &\leq \sqrt{ET^2(x_1)E(T(y_1) - T(y_2))^2} + \sqrt{ET^2(y_2)E(T(x_1) - T(x_2))^2}, \end{aligned}$$

and the last term of the previous chain of inequalities converges to zero, whenever $(x_1, y_1) \rightarrow (x_2, y_2)$ and the isotropic field T is mean-square continuous (or, equivalently, $\tilde{\Gamma}$ is continuous at the origin). We can hence state the following:

Corollary 6 *The covariance function of a measurable finite-variance isotropic random field on the homogenous space of a compact group is necessarily everywhere continuous.*

6 Some Historical Remarks

Our result can be viewed as a characterization of covariance functions for random fields defined on homogenous spaces of compact groups. In a related setting, the characterization of covariance functions for stationary and isotropic random fields in \mathbb{R}^d was first considered in a celebrated paper by Schoenberg (1938), see [17]. In this reference, it was conjectured that the only form of discontinuity which could be allowed for such covariance functions would occur at the origin, i.e. given any zero-mean, finite-variance and isotropic random field $Z : \mathbb{R}^d \rightarrow \mathbb{R}$, its covariance function should be of the form

$$\begin{aligned} EZ(0, \dots, 0)Z(t_1, \dots, t_d) &= \Gamma(\|t_1, \dots, t_d\|) \\ &= \tilde{\Gamma}(t) = \tilde{\Gamma}_0(t) + \tilde{\Gamma}_1(t), \end{aligned}$$

where as before $t := \|t_1, \dots, t_d\|$ is Euclidean norm, and $\tilde{\Gamma}_0(\cdot), \tilde{\Gamma}_1(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ are such that

$$\tilde{\Gamma}_0(t) = \begin{cases} \gamma \geq 0, \text{ for } t = 0, \\ 0 \text{ otherwise,} \end{cases}$$

while $\tilde{\Gamma}_1(t)$ is nonnegative definite and continuous. In a later paper which went largely unnoticed, Crum (1956) ([5]) proved the conjecture to be right for $d \geq 2$. This result was drawn to the attention of the Geostatistics community by Gneiting and Sasvari in 1997 (see [10]) who argued then that isotropic random fields could be always expressed as a mean-square continuous component plus a "nugget effect", e.g. a purely discontinuous component. The fact that this latter component should be necessarily non-measurable was pointed out (for instance) in an oral presentation by Starkloff (2009) – see [18, p. 13], as well as Example 1.2.5 in [12]. Our results in this note, though, are obtained in a somewhat different setting (e.g. homogeneous spaces of compact groups), and we leave for future research a complete analysis of the relationship between measurability and mean-square continuity in non-compact circumstances.

References

- [1] R.J. Adler (1981). *The Geometry of Random Fields*, J.Wiley.
- [2] R.J. Adler and J.E. Taylor (2007). *Random Fields and Geometry*, Springer.
- [3] P. Baldi and D. Marinucci (2007). Some characterizations of the spherical harmonics coefficients for isotropic random fields. *Statistics and Probability Letters* 77 (5), 490-496.
- [4] P. Baldi, D. Marinucci and V.S. Varadarajan (2007). On the characterization of isotropic Gaussian fields on homogeneous spaces of compact groups, *Electron. Comm. Probab.* 12, 291–302.
- [5] M.M. Crum (1956). On positive-definite functions, *Proc. London Math. Soc.* (3) 6, 548-560.
- [6] S. Dodelson (2003). *Modern Cosmology*, Academic Press.
- [7] R.M. Dudley (2003). *Real Analysis and Probability* (2nd Edition). Cambridge University Press.
- [8] J.J. Duistermaat and J.A.C. Kolk (1997). *Lie groups*. Springer.
- [9] R. Durrer (2008). *The Cosmic Microwave Background*, Cambridge University Press.
- [10] T. Gneiting and Z. Sasvari (1999). The characterization problem for isotropic covariance functions, *Math. Geol.* 31 (1999), no. 1, 105–111.
- [11] G.D. James and M. Liebeck (2001). *Representations and Characters of Groups* (2nd edition). Cambridge University Press.
- [12] G. Kallianpur (1980). *Stochastic filtering theory*. Applications of Mathematics, 13. Springer.
- [13] N. Leonenko (1999). *Limit Theorems for Random Fields with Singular Spectrum*. Kluwer.
- [14] D. Marinucci, and G. Peccati (2011). *Random Fields on the Sphere: Representations, Limit Theorems and Cosmological Applications*, Cambridge University Press.
- [15] A. M. Obukhov (1947). Statistically homogeneous random fields on a sphere. *Uspehi Mat. Nauk* 2, no. 2, 196–198.
- [16] G. Peccati and J.-R. Pycke (2009). Decompositions of stochastic processes based on irreducible group representations. *Teor. Veroyatn. Primen.* 54, no. 2, 304–336; translation in *Theory Probab. Appl.* 54 (2010), no. 2, 217–245.

- [17] I.J. Schoenberg (1938). Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.* 44, no. 3, 522–536.
- [18] H.J. Starkloff (2009). Notes on Stationary Processes and Fields, Oral Presentation, available online at http://whz-cms-10.zw.fh-zwickau.de/hast/Texte/starkloff_frbg20090915.pdf
- [19] V.S.Varadarajan (1999). *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, Corrected reprint of the 1989 original, Cambridge University Press.
- [20] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii (1988). *Quantum Theory of Angular Momentum*, World Scientific Press.
- [21] N.Ja. Vilenkin and A.U. Klimyk (1991). *Representation of Lie Groups and Special Functions*, Kluwer.
- [22] M.I. Yadrenko (1983). *Spectral Theory of Random Fields*. Optimization Software, Inc.
- [23] A. M. Yaglom (1961). Second-order homogeneous random fields. In Proc. 4th Berkeley Sympos. Math. Statist. and Prob., vol. II, pp. 593–622, Berkeley, CA, Univ. California Press.