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# INVARIANCE PRINCIPLES FOR HOMOGENEOUS SUMS: UNIVERSALITY OF GAUSSIAN WIENER CHAOS

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We compute explicit bounds in the normal and chi-square approximations of multilinear homogenous sums (of arbitrary order) of general centered independent random variables with unit variance. In particular, we show that chaotic random variables enjoy the following form of *universality*: (a) the normal and chi-square approximations of any homogenous sum can be completely characterized and assessed by first switching to its Wiener chaos counterpart, and (b) the simple upper bounds and convergence criteria available on the Wiener chaos extend almost verbatim to the class of homogeneous sums.

## 1. Introduction.

1.1. *Overview.* The aim of this paper is to study and characterize the normal and chi-square approximations of the laws of *multilinear homogeneous sums* involving general independent random variables. We shall perform this task by implicitly combining three probabilistic techniques, namely: (i) the *Lindeberg invariance principle* (in a version due to Mossel et al. [10]), (ii) *Stein's method* for the normal and chi-square approximations (see, e.g., [1, 25, 29, 30]), and (iii) the *Malliavin calculus of variations* on a Gaussian space (see, e.g., [8, 18]). Our analysis reveals that the Gaussian Wiener chaos (see Section 2 below for precise definitions) enjoys the following properties: (a) the normal and chi-square approximations of any multilinear homogenous sum are completely characterized and assessed by those of its Wiener chaos counterpart, and (b) the strikingly simple upper bounds and convergence criteria available on the Wiener chaos (see [11–13, 16, 17, 20]) extend almost verbatim to the class of homogeneous sums. Our findings partially rely on the notion of "low influences" (see again [10]) for real-valued functions defined on product spaces. As indicated by the title, we regard the two properties (a) and (b) as an instance of the *universality phenomenon*, according to which most information about large random systems (such as the "distance to Gaussian" of nonlinear functionals of large samples of independent random variables) does not depend on the particular distribution of the components. Other recent examples of the universality phenomenon appear in the already quoted paper [10], as well as in the Tao–Vu proof of the circular law for random matrices,

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as detailed in [31] (see also the Appendix to [31] by Krishnapur). Observe that, in Section 7, we will prove analogous results for the multivariate normal approximation of *vectors* of homogenous sums of possibly different orders. In a further work by the first two authors (see [14]) the results of the present paper are applied in order to deduce universal Gaussian fluctuations for traces associated with non-Hermitian matrix ensembles.

1.2. *The approach.* In what follows, every random object is defined on a suitable (common) probability space $(\Omega, \mathscr{F}, P)$. The symbol $E$ denotes expectation with respect to $P$. We start by giving a precise definition of the main objects of our study.

DEFINITION 1.1 (Homogeneous sums). Fix some integers $N, d \geq 2$ and write $[N] = \{1, \dots, N\}$. Let $\mathbf{X} = \{X_i : i \geq 1\}$ be a collection of centered independent random variables, and let $f : [N]^d \to \mathbb{R}$ be a *symmetric function vanishing on diagonals* [i.e., $f(i_1, \dots, i_d) = 0$ whenever there exist $k \neq j$ such that $i_k = i_j$]. The random variable

$$
\begin{aligned}
Q_d(N, f, \mathbf{X}) = Q_d(\mathbf{X}) &= \sum_{1 \leq i_1, \dots, i_d \leq N} f(i_1, \dots, i_d) X_{i_1} \cdots X_{i_d} \\
&= d! \sum_{\{i_1, \dots, i_d\} \subset [N]^d} f(i_1, \dots, i_d) X_{i_1} \cdots X_{i_d} \qquad (1.1) \\
&= d! \sum_{1 \leq i_1 < \cdots < i_d \leq N} f(i_1, \dots, i_d) X_{i_1} \cdots X_{i_d}
\end{aligned}
$$

is called the *multilinear homogeneous sum*, of order $d$, based on $f$ and on the first $N$ elements of $\mathbf{X}$.

As in (1.1), and when there is no risk of confusion, we will drop the dependence on $N$ and $f$ in order to simplify the notation. Plainly, $E[Q_d(\mathbf{X})] = 0$ and also, if $E(X_i^2) = 1$ for every $i$, then $E[Q_d(\mathbf{X})^2] = d! \| f \|_d^2$, where we use the notation $\| f \|_d^2 = \sum_{1 \leq i_1, \dots, i_d \leq N} f^2(i_1, \dots, i_d)$ (here and for the rest of the paper). In the following, we will systematically use the expression "homogeneous sum" instead of "multilinear homogeneous sum."

Objects such as (1.1) are sometimes called "polynomial chaoses," and play a central role in several branches of probability theory and stochastic analysis. When $d = 2$, they are typical examples of quadratic forms. For general $d$, homogeneous sums are, for example, the basic building blocks of the Wiener, Poisson and Walsh chaoses (see, e.g., [24]). Despite the almost ubiquitous nature of homogeneous sums, results concerning the normal approximation of quantities such as (1.1) in the nonquadratic case (i.e., when $d \geq 3$) are surprisingly scarce: indeed, to our knowledge, the only general statements in this respect are contained in references [3, 4], both by P. de Jong (as discussed below), and in a different direction, general criteria allowing to assess the proximity of the laws of homogenous sums based on different independent sequences are obtained in [10, 27, 28].

In this paper we are interested in controlling objects of the type $d_{\mathscr{H}}\{Q_d(\mathbf{X}); Z\}$, where: (i) $Q_d(\mathbf{X})$ is defined in (1.1), (ii) $Z$ is either a standard Gaussian $\mathscr{N}(0,1)$ or a centered chi-square random variable, and (iii) the distance $d_{\mathscr{H}}\{F; G\}$, between the laws of two random variables $F$ and $G$, is given by

$$d_{\mathscr{H}}\{F; G\} = \sup\{|E[h(F)] - E[h(G)]| : h \in \mathscr{H}\} \tag{1.2}$$

with $\mathscr{H}$ some suitable class of real-valued functions. Even with some uniform control on the components of $\mathbf{X}$, the problem of directly and generally assessing $d_{\mathscr{H}}\{Q_d(\mathbf{X}); Z\}$ looks very arduous. Indeed, any estimate comparing the laws of $Q_d(\mathbf{X})$ and $Z$ capriciously depends on the kernel $f$, and on the way in which the analytic structure of $f$ interacts with the specific "shape" of the distribution of the random variables $X_i$. One revealing picture of this situation appears if one tries to evaluate the moments of $Q_d(\mathbf{X})$ and to compare them with those of $Z$; see, for example, [22] for a discussion of some associated combinatorial structures. In the specific case where $Z$ is Gaussian, one should also observe that $Q_d(\mathbf{X})$ is a *completely degenerate $U$-statistic*, as $E[f(i_1,\dots,i_d)X_{i_1}x_{i_2}\cdots x_{i_d}] = 0$ for all $x_{i_2},\dots,x_{i_d}$, so that the standard results for the normal approximation of $U$-statistics do not apply.

The main point developed in the present paper is that one can successfully overcome these difficulties by implementing the following strategy: first (I) measure the distance $d_{\mathscr{H}}\{Q_d(\mathbf{X}); Q_d(\mathbf{G})\}$, between the law of $Q_d(\mathbf{X})$ and the law of the random variable $Q_d(\mathbf{G})$, obtained by replacing $\mathbf{X}$ with a centered standard i.i.d. Gaussian sequence $\mathbf{G} = \{G_i : i \geq 1\}$; then (II) assess the distance $d_{\mathscr{H}}\{Q_d(\mathbf{G}); Z\}$; and finally (III) use the triangle inequality in order to write

$$d_{\mathscr{H}}\{Q_d(\mathbf{X}); Z\} \leq d_{\mathscr{H}}\{Q_d(\mathbf{X}); Q_d(\mathbf{G})\} + d_{\mathscr{H}}\{Q_d(\mathbf{G}); Z\}. \tag{1.3}$$

We will see in the subsequent sections that the power of this approach resides in the following two facts.

FACT 1. The distance evoked at Point (I) can be effectively controlled by means of the techniques developed in [10], where the authors have produced a general theory allowing to estimate the distance between homogeneous sums constructed from different sequences of independent random variables. A full discussion of this point is presented in Section 4 below. In Theorem 4.1 we shall observe that, under the assumptions that $E(X_i^2) = 1$ and that the moments $E(|X_i|^3)$ are uniformly bounded by some constant $\beta > 0$ (recall that the $X_i$'s are centered), one can deduce from [10] (provided that the elements of $\mathscr{H}$ are sufficiently smooth) that

$$d_{\mathscr{H}}\{Q_d(\mathbf{X}); Q_d(\mathbf{G})\} \leq C \times \sqrt{\max_{1\leq i\leq N} \sum_{\{i_2,\dots,i_d\}\in[N]^{d-1}} f^2(i,i_2,\dots,i_d)}, \tag{1.4}$$

where $C$ is a constant depending only on $d$, $\beta$ and on the class $\mathscr{H}$. The quantity

$$\begin{aligned}\operatorname{Inf}_i(f) &:= \sum_{\{i_2,\dots,i_d\}\in[N]^{d-1}} f^2(i,i_2,\dots,i_d)\\ &= \frac{1}{(d-1)!}\sum_{1\le i_2,\dots,i_d\le N} f^2(i,i_2,\dots,i_d)\end{aligned} \tag{1.5}$$

is called the *influence* of the variable $i$, and roughly quantifies the contribution of $X_i$ to the overall configuration of the homogenous sum $Q_d(\mathbf{X})$. Influence indices already appear (under a different name) in the papers by Rotar' [27, 28].

FACT 2. The random variable $Q_d(\mathbf{G})$ is an element of the $d$th *Wiener chaos* associated with $\mathbf{G}$ (see Section 2 for definitions). As such, the distance between $Q_d(\mathbf{G})$ and $Z$ (in both the normal and the chi-square cases) can be assessed by means of the results appearing in [11–13, 16, 17, 19, 20, 23], which are in turn based on a powerful interaction between standard Gaussian analysis, Stein's method and the Malliavin calculus on variations. As an example, Theorem 3.1 of Section 3 proves that, if $Q_d(\mathbf{G})$ has variance one and $Z$ is standard Gaussian, then

$$d_{\mathscr{H}}\{Q_d(\mathbf{G});Z\}\le C\sqrt{|E[Q_d(\mathbf{G})^4]-E(Z^4)|}=C\sqrt{|E[Q_d(\mathbf{G})^4]-3|}, \tag{1.6}$$

where $C>0$ is some finite constant depending only on $\mathscr{H}$ and $d$.

1.3. *Universality.* Bounds such as (1.4) and (1.6) only partially account for the term "universality" appearing in the title of the present paper. Our techniques allow indeed to prove the following statement, involving vectors of homogeneous sums of possibly different orders; see also Theorem 7.5 for a more general statement.

THEOREM 1.2 (Universality of Wiener chaos). *Let $\mathbf{G}=\{G_i:i\ge 1\}$ be a standard centered i.i.d. Gaussian sequence, and fix integers $m\ge 1$ and $d_1,\dots,d_m\ge 2$. For every $j=1,\dots,m$, let $\{(N_n^{(j)},f_n^{(j)}):n\ge 1\}$ be a sequence such that $\{N_n^{(j)}:n\ge 1\}$ is a sequence of integers going to infinity, and each function $f_n^{(j)}:[N_n^{(j)}]^{d_j}\to\mathbb{R}$ is symmetric and vanishes on diagonals. Define $Q_{d_j}(N_n^{(j)},f_n^{(j)},\mathbf{G})$, $n\ge 1$, according to (1.1) and assume that, for every $j=1,\dots,m$, the sequence $E[Q_{d_j}(N_n^{(j)},f_n^{(j)},\mathbf{G})^2]$, $n\ge 1$, is bounded. Let $V$ be a $m\times m$ nonnegative symmetric matrix whose diagonal elements are different from zero, and let $\mathscr{N}_m(0,V)$ indicate a centered Gaussian vector with covariance $V$. Then, as $n\to\infty$, the following conditions* (1) *and* (2) *are equivalent*: (1) *The vector $\{Q_{d_j}(N_n^{(j)},f_n^{(j)},\mathbf{G}):j=1,\dots,m\}$ converges in law to $\mathscr{N}_m(0,V)$*; (2) *for every sequence $\mathbf{X}=\{X_i:i\ge 1\}$ of independent centered random variables, with unit variance and such that $\sup_i E|X_i|^3<\infty$, the law of the vector $\{Q_{d_j}(N_n^{(j)},f_n^{(j)},\mathbf{X}):j=1,\dots,m\}$ converges to the law of $\mathscr{N}_m(0,V)$ in the Kolmogorov distance.*

REMARK 1.3. 1. Given random vectors $F = (F_1, \ldots, F_m)$ and $H = (H_1, \ldots, H_m)$, $m \geq 1$, the *Kolmogorov distance* between the law of $F$ and the law of $H$ is defined as

$$d_{\mathrm{Kol}}(F, H) = \sup_{(z_1,\ldots,z_m)\in\mathbb{R}^m} |P(F_1 \leq z_1, \ldots, F_m \leq z_m) - P(H_1 \leq z_1, \ldots, H_m \leq z_m)|. \tag{1.7}$$

Recall that the topology induced by $d_{\mathrm{Kol}}$ on the class of all probability measures on $\mathbb{R}^m$ is strictly stronger than the topology of convergence in distribution.

2. Note that, in the statement of Theorem 1.2, we do not require that the matrix $V$ is *positively definite*, and we do *not* introduce any assumption on the asymptotic behavior of influence indices.

3. Due to the matching moments up to second order, one has that

$$\begin{aligned} &E[Q_{d_i}(N_n^{(i)}, f_n^{(i)}, \mathbf{G}) \times Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G})] \\ &\quad = E[Q_{d_i}(N_n^{(i)}, f_n^{(i)}, \mathbf{X}) \times Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{X})] \end{aligned}$$

for every $i, j = 1, \ldots, m$ and every sequence $\mathbf{X}$ as in Theorem 1.2.

Theorem 1.2 basically ensures that any statement concerning the asymptotic normality of (vectors of) general homogeneous sums can be proved by simply focusing on the elements of a Gaussian Wiener chaos. Since central limit theorems (CLTs) on Wiener chaos are by now completely characterized (thanks to the results proved in [20]), this fact represents a clear methodological breakthrough. As explained later in the paper, and up to the restriction on the third moments, we regard Theorem 1.2 as the first exact equivalent—for homogeneous sums—of the usual CLT for linear functionals of i.i.d. sequences. The proof of Theorem 1.2 is achieved in Section 7.

REMARK 1.4. When dealing with the multidimensional case, our way to use the techniques developed in [9] makes it unavoidable to require a uniform bound on the third moments of $\mathbf{X}$. However, one advantage is that we easily obtain convergence in the Kolmogorov distance, as well as explicit upper bounds on the rates of convergence. We will see below (see Theorem 1.10 for a precise statement) that in the one-dimensional case one can simply require a bound on the moments of order $2 + \varepsilon$, for some $\varepsilon > 0$. Moreover, still in the one-dimensional case and when the sequence $\mathbf{X}$ is i.i.d., one can alternatively deduce convergence in distribution from a result by Rotar' ([28], Proposition 1), for which the existence of moments of order greater than 2 is not required.

1.4. *The role of contractions*. The universality principle stated in Theorem 1.2 is based on [10], as well as on general characterizations of (possibly multidimensional) CLTs on a fixed Wiener chaos. Results of this kind have been first proved in [20] (for the one-dimensional case) and [23] (for the multidimensional case), and make an important use of the notion of "contraction" of a given deterministic kernel. When studying homogeneous sums, one is naturally led to deal with contractions defined on discrete sets of the type $[N]^d$, $N \geq 1$. In this section we shall briefly explore these discrete objects, in particular, by pointing out that discrete contractions are indeed the key element in the proof of Theorem 1.2. More general statements, as well as complete proofs, are given in Section 3.

DEFINITION 1.5. Fix $d, N \geq 2$. Let $f : [N]^d \to \mathbb{R}$ be a symmetric function vanishing of diagonals. For every $r = 0, \dots, d$, the *contraction* $f \star_r f$ is the function on $[N]^{2d-2r}$ given by

$$\begin{aligned} & f \star_r f(j_1, \dots, j_{2d-2r}) \\ &\quad = \sum_{1 \leq a_1, \dots, a_r \leq N} f(a_1, \dots, a_r, j_1, \dots, j_{d-r}) f(a_1, \dots, a_r, j_{d-r+1}, \dots, j_{2d-2r}). \end{aligned}$$

Observe that $f \star_r f$ is not necessarily symmetric and does not necessarily vanish on diagonals. The symmetrization of $f \star_r f$ is written $f \widetilde{\star}_r f$. The following result, whose proof is achieved in Section 7 as a special case of Theorem 7.5, is based on the findings of [20, 23].

PROPOSITION 1.6 (CLT for chaotic sums). *Let the assumptions and notation of Theorem 1.2 prevail, and suppose, moreover, that, for every $i, j = 1, \dots, m$ (as $n \to \infty$),*

$$E\big[Q_{d_i}(N_n^{(i)}, f_n^{(i)}, \mathbf{G}) \times Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G})\big] \to V(i, j), \tag{1.8}$$

*where $V$ is a nonnegative symmetric matrix. Then, the following three conditions* (1)–(3) *are equivalent, as $n \to \infty$*: (1) *The vector $\{Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G}) : j = 1, \dots, m\}$ converges in law to a centered Gaussian vector with covariance matrix $V$*; (2) *for every $j = 1, \dots, m$, $E[Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G})^4] \to 3V(i, i)^2$*; (3) *for every $j = 1, \dots, m$ and every $r = 1, \dots, d_j - 1$, $\| f_n^{(j)} \star_r f_n^{(j)} \|_{2d_j - 2r} \to 0$.*

REMARK 1.7. Strictly speaking, the results of [23] only deal with the case where $V$ is positive definite. The needed general result will be obtained in Section 7 by means of Malliavin calculus.

Let us now briefly sketch the proof of Theorem 1.2. Suppose that the sequence $E[Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G})^2]$ is bounded and that the vector $\{Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G}) : j = 1, \dots, m\}$ converges in law to $\mathscr{N}_m(0, V)$. Then, by uniform integrability (using

Proposition 2.6), the convergence (1.8) is satisfied and, according to Proposition 1.6, we have $\|f_n^{(j)} \star_{d_j-1} f_n^{(j)}\|_2 \to 0$. The crucial remark is now that

$$
\begin{aligned}
\|f_n^{(j)} \star_{d_j-1} f_n^{(j)}\|_2^2 &\geq \sum_{1\leq i\leq N_n^{(j)}} \Bigg[ \sum_{1\leq i_2,\ldots,i_{d_j}\leq N_n^{(j)}} f_n^{(j)}(i,i_2,\ldots,i_{d_j})^2 \Bigg]^2 \\
&\geq \max_{1\leq i\leq N_n^{(j)}} \Bigg[ \sum_{1\leq i_2,\ldots,i_{d_j}\leq N_n^{(j)}} f_n^{(j)}(i,i_2,\ldots,i_{d_j})^2 \Bigg]^2 \\
&= \Big[(d_j-1)! \max_{1\leq i\leq N_n^{(j)}} \mathrm{Inf}_i\big(f_n^{(j)}\big)\Big]^2
\end{aligned} \tag{1.9}
$$

[recall formula (1.5)], from which one immediately obtains that, as $n \to \infty$,

$$
\max_{1\leq i\leq N_n^{(j)}} \mathrm{Inf}_i\big(f_n^{(j)}\big) \to 0 \qquad \text{for every } j = 1,\ldots,m. \tag{1.10}
$$

The proof of Theorem 1.2 is concluded by using Theorem 7.1, which is a statement in the same vein as the results established in [9], that is, a multidimensional version of the findings of [10]. Indeed, this result will imply that, if (1.10) is verified, then, for every sequence $\mathbf{X}$ as in Theorem 1.2, the distance between the law of $\{Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{G}) : j = 1,\ldots,m\}$ and the law of $\{Q_{d_j}(N_n^{(j)}, f_n^{(j)}, \mathbf{X}) : j = 1,\ldots,m\}$ necessarily tends to zero and, therefore, the two sequences must converge in distribution to the same limit.

As proved in [11], contractions play an equally important role in the chi-square approximation of the laws of elements of a fixed chaos of even order. Recall that a random variable $Z_\nu$ has a *centered chi-square distribution* with $\nu \geq 1$ degrees of freedom [noted $Z_\nu \sim \chi^2(\nu)$] if $Z_\nu \stackrel{\text{Law}}{=} \sum_{i=1}^{\nu}(G_i^2 - 1)$, where $(G_1,\ldots,G_\nu)$ is a vector of i.i.d. $\mathscr{N}(0,1)$ random variables. Note that $E(Z_\nu^2) = 2\nu$, $E(Z_\nu^3) = 8\nu$ and $E(Z_\nu^4) = 12\nu^2 + 48\nu$.

THEOREM 1.8 (Chi-square limit theorem for chaotic sums, [11]). *Let* $\mathbf{G} = \{G_i : i \geq 1\}$ *be a standard centered i.i.d. Gaussian sequence, and fix an even integer* $d \geq 2$. *Let* $\{N_n, f_n : n \geq 1\}$ *be a sequence such that* $\{N_n : n \geq 1\}$ *is a sequence of integers going to infinity, and each* $f_n : [N_n]^d \to \mathbb{R}$ *is symmetric and vanishes on diagonals. Define* $Q_d(N_n, f_n, \mathbf{G})$, $n \geq 1$, *according to* (1.1), *and assume that, as* $n \to \infty$, $E[Q_d(N_n, f_n, \mathbf{G})^2] \to 2\nu$. *Then, as* $n \to \infty$, *the following conditions* (1)–(3) *are equivalent*: (1) $Q_d(N_n, f_n, \mathbf{G}) \stackrel{\text{Law}}{\to} Z_\nu \sim \chi^2(\nu)$; (2) $E[Q_d(N_n, f_n, \mathbf{G})^4] - 12E[Q_d(N_n, f_n, \mathbf{G})^3] \to E[Z_\nu^4] - 12E[Z_\nu^3] = 12\nu^2 - 48\nu$; (3) $\|f_n \widetilde{\star}_{d/2} f_n - c_d \times f_n\|_d \to 0$ *and* $\|f_n \star_r f_n\|_{2d-2r} \to 0$ *for every* $r = 1,\ldots,d-1$ *such that* $r \neq d/2$, *where* $c_d := 4(d/2)!^3 d!^{-2}$.

1.5. *Example*: *Revisiting de Jong's criterion.* To further clarify the previous discussion, we provide an illustration of how one can use our results in order to refine a remarkable result by de Jong, originally proved in [4].

THEOREM 1.9 (See [4]). *Let $\mathbf{X} = \{X_i : i \geq 1\}$ be a sequence of independent centered random variables such that $E(X_i^2) = 1$ and $E(X_i^4) < \infty$ for every $i$. Fix $d \geq 2$, and let $\{N_n, f_n : n \geq 1\}$ be a sequence such that $\{N_n : n \geq 1\}$ is a sequence of integers going to infinity, and each $f_n : [N_n]^d \to \mathbb{R}$ is symmetric and vanishes on diagonals. Define $Q_d(n, \mathbf{X}) = Q_d(N_n, f_n, \mathbf{X})$, $n \geq 1$, according to* (1.1). *Assume that $E[Q_d(n, \mathbf{X})^2] = 1$ for all $n$. Suppose that, as $n \to \infty$:* (i) $E[Q_d(n, \mathbf{X})^4] \to 3$, *and* (ii) $\max_{1 \leq i \leq N_n} \mathrm{Inf}_i(f_n) \to 0$. *Then, $Q_d(n, \mathbf{X})$ converges in law to $Z \sim \mathscr{N}(0, 1)$.*

In the original proof given in [4], assumption (i) in Theorem 1.9 appears as a convenient (and mysterious) way of reexpressing the asymptotic "lack of interaction" between products of the type $X_{i_1} \cdots X_{i_d}$, whereas assumption (ii) plays the role of a usual Lindeberg-type assumption. In the present paper, under the slightly stronger assumption that $\sup_i E(X_i^4) < \infty$, we will be able to produce bounds neatly indicating the exact roles of both assumptions (i) and (ii). To see this, define $d_{\mathscr{H}}$ according to (1.2), and set $\mathscr{H}$ to be the class of thrice differentiable functions whose first three derivatives are bounded by some finite constant $B > 0$. In Section 5, in the proof of Theorem 5.1, we will show that there exist universal, explicit, finite constants $C_1, C_2, C_3 > 0$, depending only on $\beta$, $d$ and $B$, such that (writing $\mathbf{G}$ for an i.i.d. centered standard Gaussian sequence)

$$d_{\mathscr{H}}\{Q_d(n, \mathbf{X}); Q_d(n, \mathbf{G})\} \leq C_1 \times \sqrt{\max_{1 \leq i \leq N_n} \mathrm{Inf}_i(f_n)}, \tag{1.11}$$

$$d_{\mathscr{H}}\{Q_d(n, \mathbf{G}); Z\} \leq C_2 \times \sqrt{|E[Q_d(n, \mathbf{G})^4] - 3|}, \tag{1.12}$$

$$|E[Q_d(n, \mathbf{X})^4] - E[Q_d(n, \mathbf{G})^4]| \leq C_3 \times \sqrt{\max_{1 \leq i \leq N_n} \mathrm{Inf}_i(f_n)}. \tag{1.13}$$

In particular, the estimates (1.11) and (1.13) show that assumption (ii) in Theorem 1.9 ensures that both the laws and the fourth moments of $Q_d(n, \mathbf{X})$ and $Q_d(n, \mathbf{G})$ are asymptotically close: this fact, combined with assumption (i), implies that the LHS of (1.12) converges to zero, hence so does $d_{\mathscr{H}}\{Q_d(n, \mathbf{X}); Z\}$. This gives an alternate proof of Theorem 1.9 in the case of uniformly bounded fourth moments.

Also, by combining the universality principle stated in Theorem 1.2 with (1.12) (or, alternatively, with Proposition 1.6 in the case $m = 1$), one obtains the following "universal version" of de Jong's criterion.

THEOREM 1.10. *Let* $\mathbf{G} = \{X_i : i \geq 1\}$ *be a centered i.i.d. Gaussian sequence with unit variance. Fix* $d \geq 2$, *and let* $\{N_n, f_n : n \geq 1\}$ *be a sequence such that* $\{N_n : n \geq 1\}$ *is a sequence of integers going to infinity, and each* $f_n : [N_n]^d \to \mathbb{R}$ *is symmetric and vanishes on diagonals. Define* $Q_d(n, \mathbf{G}) = Q_d(N_n, f_n, \mathbf{G})$, $n \geq 1$, *according to* (1.1). *Assume that* $E[Q_d(n, \mathbf{G})^2] \to 1$ *as* $n \to \infty$. *Then, the following four properties are equivalent as* $n \to \infty$:

(1) *The sequence* $Q_d(n, \mathbf{G})$ *converges in law to* $Z \sim \mathscr{N}(0, 1)$.
(2) $E[Q_d(n, \mathbf{G})^4] \to 3$.
(3) *For every sequence* $\mathbf{X} = \{X_i : i \geq 1\}$ *of independent centered random variables with unit variance and such that* $\sup_i E|X_i|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *the sequence* $Q_d(n, \mathbf{X})$ *converges in law to* $Z \sim \mathscr{N}(0, 1)$ *in the Kolmogorov distance.*
(4) *For every sequence* $\mathbf{X} = \{X_i : i \geq 1\}$ *of independent and identically distributed centered random variables with unit variance*, *the sequence* $Q_d(n, \mathbf{X})$ *converges in law to* $Z \sim \mathscr{N}(0, 1)$ (*not necessarily in the Kolmogorov distance*).

REMARK 1.11. 1. Note that at point (4) of the above statement we *do not* require the existence of moments of order greater than 2. We will see that the equivalence between (1) and (4) is partly a consequence of Rotar's results (see [28], Proposition 1).

2. Theorem 1.10 is a particular case of Theorem 1.2, and can be seen as refinement of de Jong's Theorem 1.9, in the sense that: (i) since several combinatorial devices are at hand (see, e.g., [22]), it is in general easier to evaluate moments of multilinear forms of Gaussian sequences than of general sequences, and (ii) when the $\{X_i\}$ are not identically distributed, we only need existence (and uniform boundedness) of the moments of order $2 + \varepsilon$.

In Section 7 we will generalize the content of this section to multivariate Gaussian approximations. By using Proposition 1.8 and [28], Proposition 1, one can also obtain the following universal chi-square limit result.

THEOREM 1.12. *We let the notation of Theorem* 1.10 *prevail*, *except that we now assume that* $d \geq 2$ *is an even integer and* $E[Q_d(n, \mathbf{G})^2] \to 2\nu$, *where* $\nu \geq 1$ *is an integer. Then*, *the following four conditions* (1)–(4) *are equivalent as* $n \to \infty$: (1) *The sequence* $Q_d(n, \mathbf{G})$ *converges in law to* $Z_\nu \sim \chi^2(\nu)$; (2) $E[Q_d(n, \mathbf{G})^4] - 12E[Q_d(n, \mathbf{G})^3] \to E(Z_\nu^4) - 12E(Z_\nu^3) = 12\nu^2 - 48\nu$; (3) *for every sequence* $\mathbf{X} = \{X_i : i \geq 1\}$ *of independent centered random variables with unit variance and such that* $\sup_i E|X_i|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$, *the sequence* $Q_d(n, \mathbf{X})$ *converges in law to* $Z_\nu$; (4) *for every sequence* $\mathbf{X} = \{X_i : i \geq 1\}$ *of independent and identically distributed centered random variables with unit variance*, *the sequence* $Q_d(n, \mathbf{X})$ *converges in law to* $Z_\nu$.

1.6. *Two counterexamples.*

*"There is no universality for sums of order one".* One striking feature of Theorems 1.2 and 1.10 is that *they do not have any equivalent for sums of order* $d = 1$. To see this, consider an array of real numbers $\{f_n(i) : 1 \leq i \leq n\}$ such that $\sum_{i=1}^n f_n^2(i) = 1$. Let $\mathbf{G} = \{G_i : i \geq 1\}$ and $\mathbf{X} = \{X_i : i \geq 1\}$ be, respectively, a centered i.i.d. Gaussian sequence with unit variance, and a sequence of independent random variables with zero mean and unit variance. Then, $Q_1(n, \mathbf{G}) := \sum_{i=1}^n f_n(i) G_i \sim \mathscr{N}(0, 1)$ for every $n$, but it is in general *not true* that $Q_1(n, \mathbf{X}) := \sum_{i=1}^n f_n(i) X_i$ converges in law to a Gaussian random variable [just take $X_1$ to be non-Gaussian, $f_n(1) = 1$ and $f_n(j) = 0$ for $j > 1$]. As it is well known, to ensure that $Q_1(n, \mathbf{X})$ has a Gaussian limit, one customarily adds the Lindeberg-type requirement that $\max_{1 \leq i \leq n} |f_n(i)| \to 0$. A closer inspection indicates that the fact that no Lindeberg conditions are required in Theorems 1.2 and 1.10 is due to the implication (1) $\Rightarrow$ (3) in Proposition 1.6, as well as to the inequality (1.9).

*"Walsh chaos is not universal".* One cannot replace the Gaussian sequence $\mathbf{G}$ with a Rademacher one in the statements of Theorems 1.2 and 1.10. Let $\mathbf{X} = \{X_i : i \geq 1\}$ be an i.i.d. Rademacher sequence, and fix $d \geq 2$. For every $N \geq d$, consider the homogeneous sum $Q_d(N, \mathbf{X}) = X_1 X_2 \cdots X_{d-1} \sum_{i=d}^N \frac{X_i}{\sqrt{N-d+1}}$. It is easily seen that each $Q_d(N, \mathbf{X})$ can be written in the form (1.1), for some symmetric $f = f_N$ vanishing on diagonals and such that $d! \|f_N\|_d^2 = 1$. Since $X_1 X_2 \cdots X_{d-1}$ is a random sign independent of $\{X_i : i \geq d\}$, a simple application of the central limit theorem yields that, as $N \to \infty$, $Q_d(N, \mathbf{X}) \overset{\text{Law}}{\to} \mathscr{N}(0, 1)$. On the other hand, if $\mathbf{G} = \{G_i : i \geq 1\}$ is a i.i.d. standard Gaussian sequence, one sees that $Q_d(N, \mathbf{G}) \overset{\text{Law}}{=} G_1 \cdots G_d$, for every $N \geq 2$. Since (for $d \geq 2$) the random variable $G_1 \cdots G_d$ is not Gaussian, this yields that $Q_d(N, \mathbf{G}) \overset{\text{Law}}{\nrightarrow} \mathscr{N}(0, 1)$ as $n \to \infty$.

REMARK 1.13. 1. In order to enhance the readability of the forthcoming material, we decided not to state some of our findings in full generality. In particular: (i) It will be clear later on that the results of this paper easily extend to the case of *infinite* homogeneous sums [obtained by putting $N = +\infty$ in (1.1)]. This requires, however, a somewhat heavier notation, as well as some distracting digressions about convergence. (ii) Our findings do not hinge at all on the fact that $\mathbb{N}$ is an ordered set: it follows that our results exactly apply to homogeneous sums of random variables indexed by a general finite set.

2. As discussed below, the results of this paper are tightly related with a series of recent findings concerning the normal and Gamma approximation of the law of nonlinear functionals of Gaussian fields, Poisson measures and Rademacher sequences. In this respect, the most relevant references are the following. In [12], Stein's method and Malliavin calculus have been combined for the first time, in

the framework of the one-dimensional normal and Gamma approximations on Wiener space. The findings of [12] are extended in [13] and [17], dealing respectively with lower bounds and multidimensional normal approximations. Reference [16] contains applications of the results of [12] to the derivation of second-order Poincaré inequalities. References [21] and [15] use appropriate versions of the non-Gaussian Malliavin calculus in order to deal with the one-dimensional normal approximation, respectively, of functionals of Poisson measures and of functionals of infinite Rademacher sequences. Note that all the previously quoted references deal with the normal and Gamma approximation of functionals of Gaussian fields, Poisson measure and Rademacher sequences. The theory developed in the present paper represents the first extension of the above quoted criteria to a possibly non-Gaussian, non-Poisson and non-Rademacher framework.

**2. Wiener chaos.** In this section we briefly introduce the notion of (Gaussian) *Wiener chaos*, and point out some of its crucial properties. The reader is referred to [18], Chapter 1, or [6], Chapter 2, for any unexplained definition or result. Let $\mathbf{G} = \{G_i : i \geq 1\}$ be a sequence of i.i.d. centered Gaussian random variables with unit variance.

DEFINITION 2.1. 1. The Hermite polynomials $\{H_q : q \geq 0\}$ are defined as $H_q = \delta^q \mathbf{1}$, where $\mathbf{1}$ is the function constantly equal to 1, and $\delta$ is the divergence operator, acting on smooth functions as $\delta f(x) = xf(x) - f'(x)$. For instance, $H_0 = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, and so on. Recall that the class $\{(q!)^{-1/2} H_q : q \geq 0\}$ is an orthonormal basis of $L^2(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2}\, dx)$.

2. A *multi-index* $q = \{q_i : i \geq 1\}$ is a sequence of nonnegative integers such that $q_i \neq 0$ only for a finite number of indices $i$. We also write $\Lambda$ to indicate the class of all multi-indices, and use the notation $|q| = \sum_{i \geq 1} q_i$, for every $q \in \Lambda$.

3. For every $d \geq 0$, the $d$th *Wiener chaos* associated with $\mathbf{G}$ is defined as follows: $C_0 = \mathbb{R}$, and, for $d \geq 1$, $C_d$ is the $L^2(P)$-closed vector space generated by random variables of the type $\Phi(q) = \prod_{i=1}^{\infty} H_{q_i}(G_i)$, $q \in \Lambda$ and $|q| = d$.

EXAMPLE 2.2. (i) The first Wiener chaos $C_1$ is the Gaussian space generated by $\mathbf{G}$, that is, $F \in C_1$ if and only if $F = \sum_{i=1}^{\infty} \lambda_i G_i$ for some sequence $\{\lambda_i : i \geq 1\} \in \ell^2$.

(ii) Fix $d, N \geq 2$ and let $f : [N]^d \to \mathbb{R}$ be symmetric and vanishing on diagonals. Then, an element of $C_d$ is, for instance, the following $d$-homogeneous sum:

$$
\begin{aligned}
Q_d(\mathbf{G}) &= d! \sum_{\{i_1,\ldots,i_d\} \subset [N]^d} f(i_1, \ldots, i_d) G_{i_1} \cdots G_{i_d} \\
&= \sum_{1 \leq i_1,\ldots,i_d \leq N} f(i_1, \ldots, i_d) G_{i_1} \cdots G_{i_d}.
\end{aligned}
\tag{2.1}
$$

It is easily seen that two random variables belonging to a Wiener chaos of different orders are orthogonal in $L^2(P)$. Moreover, since linear combinations of polynomials are dense in $L^2(P, \sigma(\mathbf{G}))$, one has that $L^2(P, \sigma(\mathbf{G})) = \bigoplus_{d \geq 0} C_d$, that is, any square integrable functional of $\mathbf{G}$ can be written as an infinite sum, converging in $L^2$ and such that the $d$th summand is an element of $C_d$ [the *Wiener–Itô chaotic decomposition* of $L^2(P, \sigma(\mathbf{G}))$]. It is often useful to encode the properties of random variables in the spaces $C_d$ by using increasing tensor powers of Hilbert spaces (see, e.g., [6], Appendix E, for a collection of useful facts about tensor products). To do this, introduce an (arbitrary) real separable Hilbert space $\mathfrak{H}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and, for $d \geq 2$, denote by $\mathfrak{H}^{\otimes d}$ (resp. $\mathfrak{H}^{\odot d}$) the $d$th tensor power (resp. symmetric tensor power) of $\mathfrak{H}$; write, moreover, $\mathfrak{H}^{\otimes 0} = \mathfrak{H}^{\odot 0} = \mathbb{R}$ and $\mathfrak{H}^{\otimes 1} = \mathfrak{H}^{\odot 1} = \mathfrak{H}$. Let $\{e_j : j \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$. With every multi-index $q \in \Lambda$, we associate the tensor $e(q) \in \mathfrak{H}^{\otimes |q|}$ given by $e(q) = e_{i_1}^{\otimes q_{i_1}} \otimes \cdots \otimes e_{i_k}^{\otimes q_{i_k}}$, where $\{q_{i_1}, \dots, q_{i_k}\}$ are the nonzero elements of $q$. We also denote by $\tilde{e}(q) \in \mathfrak{H}^{\odot |q|}$ the canonical symmetrization of $e(q)$. It is well known that, for every $d \geq 2$, the collection $\{\tilde{e}(q) : q \in \Lambda, |q| = d\}$ defines a complete orthogonal system in $\mathfrak{H}^{\odot d}$. For every $d \geq 1$ and every $h \in \mathfrak{H}^{\odot d}$ with the form $h = \sum_{q \in \Lambda, |q| = d} c_q \tilde{e}(q)$, we define $I_d(h) = \sum_{q \in \Lambda, |q| = d} c_q \Phi(q)$. We also recall that, for every $d \geq 1$, the mapping $I_d : \mathfrak{H}^{\odot d} \to C_d$ is onto, and provides an isomorphism between $C_d$ and the Hilbert space $\mathfrak{H}^{\odot d}$, endowed with the norm $\sqrt{d!} \| \cdot \|_{\mathfrak{H}^{\otimes d}}$. In particular, for every $h, h' \in \mathfrak{H}^{\odot d}$, $E[I_d(h) I_d(h')] = d! \langle h, h' \rangle_{\mathfrak{H}^{\otimes d}}$. If $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$, with $\mu$ $\sigma$-finite and nonatomic, then the operators $I_d$ are indeed (multiple) Wiener–Itô integrals.

EXAMPLE 2.3. By definition, $G_i = I_1(e_i)$, for every $i \geq 1$. Moreover, the random variable $Q_d(\mathbf{G})$ defined in (2.1) is such that

$$
\begin{aligned}
& Q_d(\mathbf{G}) = I_d(h) \\
& \qquad \text{where } h = d! \sum_{\{i_1, \dots, i_d\} \subset [N]^d} f(i_1, \dots, i_d) e_{i_1} \otimes \cdots \otimes e_{i_d} \in \mathfrak{H}^{\odot d}.
\end{aligned} \tag{2.2}
$$

The notion of "contraction" is the key to prove the general bounds stated in the forthcoming Section 3.

DEFINITION 2.4 (Contractions). Let $\{e_i : i \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$, so that, for every $m \geq 2$, $\{e_{j_1} \otimes \cdots \otimes e_{j_m} : j_1, \dots, j_m \geq 1\}$ is a complete orthonormal system in $\mathfrak{H}^m$. Let $f = \sum_{j_1, \dots, j_p} a(j_1, \dots, j_p) e_{j_1} \otimes \cdots \otimes e_{j_p} \in \mathfrak{H}^{\odot p}$ and $g = \sum_{k_1, \dots, k_q} b(k_1, \dots, k_q) e_{k_1} \otimes \cdots \otimes e_{k_q} \in \mathfrak{H}^{\odot q}$, with $\sum_{j_1, \dots, j_p} a(j_1, \dots, j_p)^2 < \infty$ and $g = \sum_{k_1, \dots, k_q} b(k_1, \dots, k_q)^2 < \infty$ (note that $a$ and $b$ need not vanish on diagonals). For every $r = 0, \dots, p \wedge q$, the $r$th *contrac-*

*tion* of $f$ and $g$ is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined as

$$f \otimes_r g = \sum_{j_1,\dots,j_{p-r}=1}^{\infty} \sum_{k_1,\dots,k_{q-r}=1}^{\infty} a \star_r b(j_1,\dots,j_{p-r},k_1,\dots,k_{q-r}) \\ \times e_{j_1} \otimes \cdots \otimes e_{j_{p-r}} \otimes e_{k_1} \otimes \cdots \otimes e_{k_{q-r}} \\ = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}},$$

where the kernel $a \star_r b$ is defined according to Definition 1.5, by taking $N = \infty$.

Plainly, $f \otimes_0 g = f \otimes g$ equals the tensor product of $f$ and $g$ while, for $p = q$, $f \otimes_p g = \langle f, g \rangle_{\mathfrak{H}^{\otimes p}}$. Note that, in general (and except for trivial cases), the contraction $f \otimes_r g$ is *not* a symmetric element of $\mathfrak{H}^{\otimes(p+q-2r)}$. The canonical symmetrization of $f \otimes_r g$ is written $f \widetilde{\otimes}_r g$. Contractions appear in multiplication formulae like the following one:

PROPOSITION 2.5 (Multiplication formulae). *If $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then* $I_p(f) I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g)$.

Note that the previous statement implies that multiple integrals admit finite moments of every order. The next result (see [6], Theorem 5.10) establishes a more precise property, namely, that random variables living in a finite sum of Wiener chaos are hypercontractive.

PROPOSITION 2.6 (Hypercontractivity). *Let $d \geq 1$ be a finite integer and assume that $F \in \bigoplus_{k=0}^{d} C_k$. Fix reals $2 \leq p \leq q < \infty$. Then $E[|F|^q]^{1/q} \leq (q-1)^{d/2} E[|F|^p]^{1/p}$.*

**3. Normal and chi-square approximation on Wiener chaos.** Starting from this section, and for the rest of the paper, we adopt the following notation for distances between laws of real-valued random variables. The symbol $d_{\mathrm{TV}}(F,G)$ indicates the *total variation distance* between the law of $F$ and $G$, obtained from (1.2) by taking $\mathscr{H}$ equal to the class of all indicators of the Borel subsets of $\mathbb{R}$. The symbol $d_{\mathrm{W}}(F,G)$ denotes the *Wasserstein distance*, obtained from (1.2) by choosing $\mathscr{H}$ as the class of all Lipschitz functions with Lipschitz constant less than or equal to 1. The symbol $d_{\mathrm{BW}}(F,G)$ stands for the *bounded Wasserstein distance* (or *Fortet–Mourier distance*), deduced from (1.2) by choosing $\mathscr{H}$ as the class of all Lipschitz functions that are bounded by 1, and with Lipschitz constant less than or equal to 1. While $d_{\mathrm{Kol}}(F,G) \leq d_{\mathrm{TV}}(F,G)$ and $d_{\mathrm{BW}}(F,G) \leq d_{\mathrm{W}}(F,G)$, in general, $d_{\mathrm{TV}}(F,G)$ and $d_{\mathrm{W}}(F,G)$ are not comparable.

In what follows, we consider as given an i.i.d. centered standard Gaussian sequence $\mathbf{G} = \{G_i : i \geq 1\}$, and we shall adopt the Wiener chaos notation introduced in Section 2.

3.1. *Central limit theorems.* In the recent series of papers [12, 13, 16], it has been shown that one can effectively combine Malliavin calculus with Stein's method, in order to evaluate the distance between the law of an element of a fixed Wiener chaos, say, $F$, and a standard Gaussian distribution. In this section we state several refinements of these results, by showing, in particular, that all the relevant bounds can be expressed in terms of the fourth moment of $F$. The proof of the following theorem involves the use of Malliavin calculus and is deferred to Section 8.3.

THEOREM 3.1 (Fourth moment bounds). *Fix $d \geq 2$. Let $F = I_d(h)$, $h \in \mathfrak{H}^{\odot d}$, be an element of the $d$th Gaussian Wiener chaos $C_d$ such that $E(F^2) = 1$, let $Z \sim \mathscr{N}(0,1)$, and write*

$$T_1(F) := \sqrt{d^2 \sum_{r=1}^{d-1} (r-1)!^2 \binom{d-1}{r-1}^4 (2d-2r)! \|h \widetilde{\otimes}_r h\|^2_{\mathfrak{H}^{\otimes 2(d-r)}}},$$

$$T_2(F) := \sqrt{\frac{d-1}{3d} |E(F^4) - 3|}.$$

*We have $T_1(F) \leq T_2(F)$. Moreover, $d_{\mathrm{TV}}(F, Z) \leq 2T_1(F)$ and $d_{\mathrm{W}}(F, Z) \leq T_1(F)$. Finally, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function such that $\|\varphi'''\|_\infty < \infty$. Then, one has that $|E[\varphi(F)] - E[\varphi(Z)]| \leq C_* \times T_1(F)$, with*

$$\begin{aligned} C_* &= 4\sqrt{2}(1 + 5^{3d/2}) \\ &\quad \times \max\left\{ \frac{3}{2}|\varphi''(0)| + \frac{\|\varphi'''\|_\infty}{3} \frac{2\sqrt{2}}{\sqrt{\pi}}; 2|\varphi'(0)| + \frac{1}{3}\|\varphi'''\|_\infty \right\}. \end{aligned} \tag{3.1}$$

REMARK 3.2. If $E(F) = 0$ and $F$ has a finite fourth moment, then the quantity $\kappa_4(F) = E(F^4) - 3E(F^2)^2$ is known as the *fourth cumulant* of $F$. One can also prove (see, e.g., [20]) that, if $F$ is a nonzero element of the $d$th Wiener chaos of a given Gaussian sequence ($d \geq 2$), then $\kappa_4(F) > 0$.

Now fix $d \geq 2$, and consider a sequence of random variables of the type $F_n = I_d(h_n)$, $n \geq 1$, such that, as $n \to \infty$, $E(F_n^2) = d!\|h_n\|^2_{\mathfrak{H}^{\otimes d}} \to 1$. In [20] it is proved that the following double implication holds: as $n \to \infty$,

$$\begin{aligned} &\|h_n \widetilde{\otimes}_r h_n\|_{\mathfrak{H}^{\otimes(2d-2r)}} \to 0 \qquad \forall r = 1, \dots, d-1 \\ &\quad \Leftrightarrow \quad \|h_n \otimes_r h_n\|_{\mathfrak{H}^{\otimes(2d-2r)}} \to 0 \qquad \forall r = 1, \dots, d-1. \end{aligned} \tag{3.2}$$

Theorem 3.1, combined with (3.3), allows therefore to recover the following characterization of CLTs on Wiener chaos. It has been first proved (by other methods) in [20].

THEOREM 3.3 (See [19, 20]). *Fix* $d \geq 2$, *and let* $F_n = I_d(h_n)$, $n \geq 1$ *be a sequence in the* $d$*th Wiener chaos of* $\mathbf{G}$. *Assume that* $\lim_{n\to\infty} E(F_n^2) = 1$. *Then, the following three conditions* (1)–(3) *are equivalent, as* $n \to \infty$: (1) $F_n$ *converges in law to* $Z \sim \mathscr{N}(0,1)$; (2) $E(F_n^4) \to E(Z^4) = 3$; (3) *for every* $r = 1, \dots, d-1$, $\|h_n \otimes_r h_n\|_{\mathfrak{H}^{\otimes 2(d-r)}} \to 0$.

PROOF. Since $\sup_n E(F_n^2) < \infty$, one deduces from Proposition 2.6 that, for every $M > 2$, one has $\sup_n E|F_n|^M < \infty$. By uniform integrability, it follows that, if (1) is in order, then necessarily $E(F_n^4) \to E(Z^4) = 3$. The rest of the proof is a consequence of the bounds in Theorem 3.1. $\square$

The following (elementary) result is one of the staples of the present paper. We state it in a form which is also useful for the chi-square approximation of Section 3.2.

LEMMA 3.4. *Fix* $d \geq 2$, *and suppose that* $h \in \mathfrak{H}^{\odot d}$ *is given by* (2.2), *with* $f : [N]^d \to \mathbb{R}$ *symmetric and vanishing on diagonals. Then, for* $r = 1, \dots, d-1$, $\|h \otimes_r h\|_{\mathfrak{H}^{\otimes(2d-2r)}} = \|f \star_r f\|_{2d-2r}$, *where we have used the notation introduced in Definition* 1.5. *Also, if* $d$ *is even, then, for every* $\alpha_1, \alpha_2 \in \mathbb{R}$, $\|\alpha_1(h \otimes_{d/2} h) + \alpha_2 h\|_{\mathfrak{H}^{\otimes d}} = \|\alpha_1(f \star_{d/2} f) + \alpha_2 f\|_d$.

PROOF. Fix $r = 1, \dots, d-1$. Using (2.2) and the fact that $\{e_j : j \geq 1\}$ is an orthonormal basis of $\mathfrak{H}$, one infers that

$$
\begin{aligned}
h \otimes_r h &= \sum_{1 \leq i_1, \dots, i_d \leq N} \sum_{1 \leq j_1, \dots, j_d \leq N} f(i_1, \dots, i_d) f(j_1, \dots, j_d) \\
&\qquad \times [e_{i_1} \otimes \cdots \otimes e_{i_d}] \otimes_r [e_{j_1} \otimes \cdots \otimes e_{j_d}] \\
&= \sum_{1 \leq a_1, \dots, a_r \leq N} \sum_{1 \leq k_1, \dots, k_{2d-2r} \leq N} f(a_1, \dots, a_r, k_1, \dots, k_{d-r}) \\
&\qquad \times f(a_1, \dots, a_r, k_{d-r+1}, \dots, k_{2d-2r}) \\
&\qquad \times e_{k_1} \otimes \cdots \otimes e_{k_{2d-2r}} \\
&= \sum_{1 \leq k_1, \dots, k_{2d-2r} \leq N} f \star_r f(k_1, \dots, k_{2d-2r}) e_{k_1} \otimes \cdots \otimes e_{k_{2d-2r}}.
\end{aligned} \tag{3.3}
$$

Since the set $\{e_{k_1} \otimes \cdots \otimes e_{k_{2d-2r}} : k_1, \dots, k_{2d-2r} \geq 1\}$ is an orthonormal basis of $\mathfrak{H}^{\otimes(2d-2r)}$, one deduces immediately $\|h \otimes_r h\|_{\mathfrak{H}^{\otimes(2d-2r)}} = \|f \star_r f\|_{2d-2r}$. The proof of the other identity is analogous. $\square$

REMARK 3.5. Theorem 3.3 and Lemma 3.4 yield immediately a proof of Proposition 1.6 in the case $m = 1$.

3.2. *Chi-square limit theorems.* As demonstrated in [11, 12], the combination of Malliavin calculus and Stein's method also allows to estimate the distance between the law of an element $F$ of a fixed Wiener chaos and a (centered) chi-square distribution $\chi^2(\nu)$ with $\nu$ degrees of freedom. Analogously to the previous section for Gaussian approximations, we now state a number of refinements of the results proved in [11, 12]. In particular, we will show that all the relevant bounds can be expressed in terms of a specific linear combination of the third and fourth moments of $F$. The proof is deferred to Section 8.4.

THEOREM 3.6 (Third and fourth moment bounds). *Fix an even integer $d \geq 2$ as well as an integer $\nu \geq 1$. Let $F = I_d(h)$ be an element of the $d$th Gaussian chaos $C_d$ such that $E(F^2) = 2\nu$, let $Z_\nu \sim \chi^2(\nu)$, and write*

$$T_3(F) := \Bigg[4d!\left\|h - \frac{d!^2}{4(d/2)!^3} h \widetilde{\otimes}_{d/2} h\right\|^2_{\mathfrak{H}^{\otimes d}} + d^2 \sum_{\substack{r=1,\ldots,d-1\\ r\neq d/2}} (r-1)!^2 \binom{d-1}{r-1}^4 (2d-2r)!\|h \widetilde{\otimes}_r h\|^2_{\mathfrak{H}^{\otimes 2(d-r)}}\Bigg]^{1/2},$$

$$T_4(F) := \sqrt{\frac{d-1}{3d}|E(F^4) - 12E(F^3) - 12\nu^2 + 48\nu|}.$$

*Then $T_3(F) \leq T_4(F)$ and $d_{\mathrm{BW}}(F, Z_\nu) \leq \max\{\sqrt{\frac{2\pi}{\nu}}, \frac{1}{\nu} + \frac{2}{\nu^2}\} T_3(F)$.*

Now fix an even integer $d \geq 2$, and consider a sequence of random variables of the type $F_n = I_d(h_n)$, $n \geq 1$, such that, as $n \to \infty$, $E(F_n^2) = d!\|h_n\|^2_{\mathfrak{H}^{\otimes d}} \to 2\nu$. In [11] it is proved that the following double implication holds: as $n \to \infty$,

$$\|h_n \widetilde{\otimes}_r h_n\|_{\mathfrak{H}^{\otimes 2(d-r)}} \to 0 \qquad \forall r = 1, \ldots, d-1, r \neq d/2$$
$$\Longleftrightarrow \quad \|h_n \otimes_r h_n\|_{\mathfrak{H}^{\otimes 2(d-r)}} \to 0 \qquad \forall r = 1, \ldots, d-1, r \neq d/2. \tag{3.4}$$

Theorem 3.6, combined with (3.4), allows therefore to recover the following characterization of chi-square limit theorems on Wiener chaos. Note that this is a special case of a "noncentral limit theorem"; one usually calls "noncentral limit theorem" any result involving convergence in law to a non-Gaussian distribution.

THEOREM 3.7 (See [11]). *Fix an even integer $d \geq 2$, and let $F_n = I_d(h_n)$, $n \geq 1$ be a sequence in the $d$th Wiener chaos of* $\mathbf{G}$. *Assume that* $\lim_{n\to\infty} E(F_n^2) = 2\nu$. *Then, the following three conditions* (1)–(3) *are equivalent, as $n \to \infty$:* (1) *$F_n$ converges in law to $Z_\nu \sim \chi^2(\nu)$;* (2) $E(F_n^4) - 12E(F_n^3) \to E(Z_\nu^4) - 12E(Z_\nu^3) = 12\nu^2 - 48\nu$; (3) $\|h_n \widetilde{\otimes}_{d/2} h_n - 4(d/2)!^3 d!^{-2} \times h_n\|_{\mathfrak{H}^{\otimes d}} \to 0$ *and, for every* $r = 1, \ldots, d-1$ *such that* $r \neq d/2$, $\|h_n \otimes_r h_n\|_{\mathfrak{H}^{\otimes 2(d-r)}} \to 0$.

PROOF. Since $\sup_n E(F_n^2) < \infty$, one deduces from Proposition 2.6 that, for every $M > 2$, one has $\sup_n E|F_n|^M < \infty$. By uniform integrability, it follows that, if (1) holds, then necessarily $E(F_n^4) - 12E(F_n^3) \to E(Z_\nu^4) - 12E(Z_\nu^3) = 12\nu^2 - 48\nu$. The rest of the proof is a consequence of Theorem 3.6. □

REMARK 3.8. By using the second identity in Lemma 3.4 in the case $\alpha_1 = 1$ and $\alpha_2 = -4(\frac{d}{2})!^3 d!^{-2}$, Theorem 3.7 yields an immediate proof of Proposition 1.8.

**4. Low influences and proximity of homogeneous sums.** We now turn to some remarkable invariance principles by Rotar' [28] and Mossel, O'Donnell and Oleszkiewicz [10]. As already discussed, the results proved in [28] yield sufficient conditions in order to have that the laws of homogeneous sums (or, more generally, polynomial forms) that are built from two different sequences of independent random variables are asymptotically close, whereas in [10] one can find explicit upper bounds on the distance between these laws. Since in this paper we adopt the perspective of deducing general convergence results from limit theorems on a Gaussian space, we will state the results of [28] and [10] in a slightly less general form, namely, by assuming that one of the sequences is i.i.d. Gaussian. See also Davydov and Rotar' [2], and the references therein, for some general characterizations of the asymptotic proximity of probability distributions.

THEOREM 4.1 (See [10]). *Let $\mathbf{X} = \{X_i, i \geq 1\}$ be a collection of centered independent random variables with unit variance, and let $\mathbf{G} = \{G_i : i \geq 1\}$ be a collection of standard centered i.i.d. Gaussian random variables. Fix $d \geq 1$, and let $\{N_n, f_n : n \geq 1\}$ be a sequence such that $\{N_n : n \geq 1\}$ is a sequence of integers going to infinity, and each $f_n : [N_n]^d \to \mathbb{R}$ is symmetric and vanishes on diagonals. Define $Q_d(N_n, f_n, \mathbf{X})$ and $Q_d(N_n, f_n, \mathbf{G})$ according to (1.1). Recall the definition (1.5) of $\mathrm{Inf}_i(f_n)$.*

1. *If $\sup_{i\geq 1} E[|X_i|^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$ and if $\max_{1\leq i\leq N_n} \mathrm{Inf}_i(f_n) \to 0$ as $n \to \infty$, then $\sup_{z\in\mathbb{R}} |P[Q_d(N_n, f_n, \mathbf{X}) \leq z] - P[Q_d(N_n, f_n, \mathbf{G}) \leq z]| \to 0$ as $n \to \infty$.*
2. *If the random variables $X_i$ are identically distributed and if*
$$\max_{1\leq i\leq N_n} \mathrm{Inf}_i(f_n) \to 0 \qquad \text{as } n \to \infty,$$
*then $|E[\psi(Q_d(N_n, f_n, \mathbf{X}))] - E[\psi(Q_d(N_n, f_n, \mathbf{G}))]| \to 0$ as $n \to \infty$, for every continuous bounded function $\psi : \mathbb{R} \to \mathbb{R}$.*
3. *If $\beta := \sup_{i\geq 1} E[|X_i|^3] < \infty$, then, for all thrice differentiable $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\|\varphi'''\|_\infty < \infty$ and for every fixed $n$, $|E[\varphi(Q_d(N_n, f_n, \mathbf{X}))] - E[\varphi(Q_d(N_n, f_n, \mathbf{G}))]| \leq \|\varphi'''\|_\infty (30\beta)^d d! \sqrt{\max_{1\leq i\leq N_n} \mathrm{Inf}_i(f_n)}$.*

PROOF. Point 1 is Theorem 2.2 in [10]. Point 2 is Proposition 1 in [28]. Point 3 is Theorem 3.18 (under Hypothesis H2) in [10]. Note that our polynomials $Q_d$ relate to polynomials $d!Q$ in [10], hence the extra factor of $d!$ in the bound. □

In the sequel, we will also need the following technical lemma, which follows directly by combining Propositions 3.11, 3.12 and 3.16 in [10].

LEMMA 4.2. *Let* $\mathbf{X} = \{X_i, i \geq 1\}$ *be a collection of centered independent random variables with unit variance. Assume, moreover, that* $\gamma := \sup_{i\geq 1} E[|X_i|^q] < \infty$ *for some* $q > 2$. *Fix* $N, d \geq 1$, *and let* $f : [N]^d \to \mathbb{R}$ *be a symmetric function* (*here, observe that we do not require that* $f$ *vanishes on diagonals*). *Define* $Q_d(\mathbf{X}) = Q_d(N, f, \mathbf{X})$ *by* (1.1). *Then* $E[|Q_d(\mathbf{X})|^q] \leq \gamma^d (2\sqrt{q-1})^{qd} \times E[Q_d(\mathbf{X})^2]^{q/2}$.

As already evoked in the Introduction, one of the key elements in the proof of Theorem 4.1 given in [10] is the use of an elegant probabilistic technique, which is in turn inspired by the well-known Lindeberg's proof of the central limit theorem. We will now state and prove a useful lemma, concerning moments of homogeneous sums. We stress that the proof of the forthcoming Lemma 4.3 could be directly deduced from the general Lindeberg-type results developed in [10] (basically, by representing powers of homogeneous sums as linear combinations of homogeneous sums, and then by exploiting hypercontractivity). However, this would require the introduction of some more notation (in order to take into account different powers of the same random variable), and we prefer to provide a direct proof, which also serves as an illustration of some of the crucial techniques of [10].

LEMMA 4.3. *Let* $\mathbf{X} = \{X_i : i \geq 1\}$ *and* $\mathbf{Y} = \{Y_i : i \geq 1\}$ *be two collections of centered independent random variables with unit variance. Fix some integers* $N$, $d \geq 1$, *and let* $f : [N]^d \to \mathbb{R}$ *be a symmetric function vanishing on diagonals. Define* $Q_d(\mathbf{X}) = Q_d(N, f, \mathbf{X})$ *and* $Q_d(\mathbf{Y}) = Q_d(N, f, \mathbf{Y})$ *according to* (1.1).

1. *Suppose* $k \geq 2$ *is such that*: (a) $X_i$ *and* $Y_i$ *belong to* $L^k(\Omega)$ *for all* $i \geq 1$; (b) $E(X_i^l) = E(Y_i^l)$ *for all* $i \geq 1$ *and* $l \in \{2, \dots, k\}$. *Then* $Q_d(\mathbf{X})$ *and* $Q_d(\mathbf{Y})$ *belong to* $L^k(\Omega)$, *and* $E[Q_d(\mathbf{X})^l] = E[Q_d(\mathbf{Y})^l]$ *for all* $l \in \{2, \dots, k\}$.
2. *Suppose* $m > k \geq 2$ *are such that*: (a) $\alpha := \max\{\sup_{i\geq 1} E|X_i|^m, \sup_{i\geq 1} E|Y_i|^m\} < \infty$; (b) $E(X_i^l) = E(Y_i^l)$ *for all* $i \geq 1$ *and* $l \in \{2, \dots, k\}$. *Assume, moreover,* (*for simplicity*) *that*: (c) $E[Q_d(\mathbf{X})^2]^{1/2} \leq M$ *for some finite constant* $M \geq 1$. *Then* $Q_d(\mathbf{X})$ *and* $Q_d(\mathbf{Y})$ *belong to* $L^m(\Omega)$ *and, for all* $l \in \{k+1, \dots, m\}$, $|E(Q_d(\mathbf{X})^l) - E(Q_d(\mathbf{Y})^l)| \leq c_{d,l,m,\alpha} \times M^{l-k+1} \times \max_{1\leq i\leq N}\{\max[\mathrm{Inf}_i(f)^{k-1/2}; \mathrm{Inf}_i(f)^{l/2-1}]\}$, *where* $c_{d,l,m,\alpha} = 2^{l+1}(d-1)!^{-1} \times \alpha^{dl/m}(2\sqrt{l-1})^{(2d-1)l} d!^{l-1}$.

PROOF. While Point 1 could be verified by a direct (elementary) computation, we will obtain the same conclusion as the by-product of a more sophisticated construction which will also lead to the proof of Point 2. We shall assume, without loss of generality, that the two sequences $\mathbf{X}$ and $\mathbf{Y}$ are stochastically independent. For $i = 0, \ldots, N$, let $\mathbf{Z}^{(i)}$ denote the sequence $(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_N)$. Fix a particular $i \in \{1, \ldots, N\}$, and write

$$U_i = \sum_{\substack{1 \le i_1, \ldots, i_d \le N \\ \forall k : i_k \neq i}} f(i_1, \ldots, i_d) Z_{i_1}^{(i)} \cdots Z_{i_d}^{(i)},$$

$$V_i = \sum_{\substack{1 \le i_1, \ldots, i_d \le N \\ \exists k : i_k = i}} f(i_1, \ldots, i_d) Z_{i_1}^{(i)} \cdots \widehat{Z_i^{(i)}} \cdots Z_{i_d}^{(i)},$$

where $\widehat{Z_i^{(i)}}$ means that this particular term is dropped (observe that this notation bears no ambiguity: indeed, since $f$ vanishes on diagonals, each string $i_1, \ldots, i_d$ contributing to the definition of $V_i$ contains the symbol $i$ exactly once). Note that $U_i$ and $V_i$ are independent of the variables $X_i$ and $Y_i$, and that $Q_d(\mathbf{Z}^{(i-1)}) = U_i + X_i V_i$ and $Q_d(\mathbf{Z}^{(i)}) = U_i + Y_i V_i$. By using the independence of $X_i$ and $Y_i$ from $U_i$ and $V_i$ [as well as the fact that $E(X_i^l) = E(Y_i^l)$ for all $i$ and all $1 \le l \le k$], we infer from the binomial formula that, for $l \in \{2, \ldots, k\}$,

$$\begin{aligned} E[(U_i + X_i V_i)^l] &= \sum_{j=0}^{l} \binom{l}{j} E(U_i^{l-j} V_i^j) E(X_i^j) \\ &= \sum_{j=0}^{l} \binom{l}{j} E(U_i^{l-j} V_i^j) E(Y_i^j) = E[(U_i + Y_i V_i)^l]. \end{aligned} \tag{4.1}$$

That is, $E[Q_d(\mathbf{Z}^{(i-1)})^l] = E[Q_d(\mathbf{Z}^{(i)})^l]$ for all $i \in \{1, \ldots, N\}$ and $l \in \{2, \ldots, k\}$. The desired conclusion of Point 1 follows by observing that $Q_d(\mathbf{Z}^{(0)}) = Q_d(\mathbf{X})$ and $Q_d(\mathbf{Z}^{(N)}) = Q_d(\mathbf{Y})$. To prove Point 2, let $l \in \{k+1, \ldots, m\}$. Using (4.1) and then Hölder's inequality, we can write

$$\begin{aligned} &\left| E[Q_d(\mathbf{Z}^{(i-1)})^l] - E[Q_d(\mathbf{Z}^{(i)})^l] \right| \\ &\quad = \left| \sum_{j=k+1}^{l} \binom{l}{j} E(U_i^{l-j} V_i^j) \big(E(X_i^j) - E(Y_i^j)\big) \right| \\ &\quad \le \sum_{j=k+1}^{l} \binom{l}{j} (E|U_i|^l)^{1-j/l} (E|V_i|^l)^{j/l} (E|X_i|^j + E|Y_i|^j). \end{aligned}$$

By Lemma 4.2, since $E(U_i^2) \le E(Q_d(\mathbf{X})^2) \le M^2$, we have $E|U_i|^l \le \alpha^{dl/m} \times (2\sqrt{l-1})^{ld} E(U_i^2)^{l/2} \le \alpha^{dl/m} (2\sqrt{l-1})^{ld} M^l$. Similarly, since $E(V_i^2) = d!^2 \times \mathrm{Inf}_i(f)$ [see (1.5)], we have $E|V_i|^l \le \alpha^{(d-1)l/m} (2\sqrt{l-1})^{l(d-1)} E(V_i^2)^{l/2} \le \alpha^{(d-1)l/m} (2\sqrt{l-1})^{l(d-1)} d!^l (\mathrm{Inf}_i(f))^{l/2}$. Hence, since $E|Y_i|^j + E|X_i|^j \le 2\alpha^{j/m}$, we can write

$$
\begin{aligned}
&|E[Q_d(\mathbf{Z}^{(i-1)})^l] - E[Q_d(\mathbf{Z}^{(i)})^l]| \\
&\quad \le 2 \sum_{j=k+1}^{l} \binom{l}{j} \bigl(\alpha^{dl/m}(2\sqrt{l-1})^{ld} M^l\bigr)^{1-j/l} \\
&\qquad\qquad \times \bigl(\alpha^{(d-1)/m}(2\sqrt{l-1})^{d-1} d!\sqrt{\operatorname{Inf}_i(f)}\bigr)^j \alpha^{j/m} \\
&\quad \le 2^{l+1}\alpha^{dl/m}(2\sqrt{l-1})^{l(2d-1)} d!^l M^{l-k-1} \times \max\bigl[\operatorname{Inf}_i(f)^{(k+1)/2}; \operatorname{Inf}_i(f)^{l/2}\bigr].
\end{aligned}
$$

Finally, summing for $i$ over $1, \dots, N$ and using that $\sum_{i=1}^{N} \operatorname{Inf}_i(f) = \frac{\|f\|_d^2}{(d-1)!} \le \frac{M^2}{d!(d-1)!}$ yields

$$
\begin{aligned}
&|E[Q_d(\mathbf{X})^l] - E[Q_d(\mathbf{Y})^l]| \\
&\quad \le 2^{l+1}\alpha^{dl/m}(2\sqrt{l-1})^{l(2d-1)} d!^l M^{l-k-1} \\
&\qquad \times \max_{1\le i\le N}\bigl\{\max\bigl[\operatorname{Inf}_i(f)^{(k-1)/2}; \operatorname{Inf}_i(f)^{l/2-1}\bigr]\bigr\} \sum_{i=1}^{N} \operatorname{Inf}_i(f) \\
&\quad \le c_{d,l,m,\alpha} \times M^{l-k+1} \times \max_{1\le i\le N}\bigl\{\max\bigl[\operatorname{Inf}_i(f)^{(k-1)/2}; \operatorname{Inf}_i(f)^{l/2-1}\bigr]\bigr\}. \qquad \square
\end{aligned}
$$

**5. Normal approximation of homogeneous sums.** The following statement provides an explicit upper bound on the normal approximation of homogenous sums, when the test function has a bounded third derivative.

THEOREM 5.1. *Let $\mathbf{X} = \{X_i, i \ge 1\}$ be a collection of centered independent random variables with unit variance. Assume, moreover, that $\beta := \sup_i E(X_i^4) < \infty$ and let $\alpha := \max\{3; \beta\}$. Fix $N, d \ge 1$, and let $f : [N]^d \to \mathbb{R}$ be symmetric and vanishing on diagonals. Define $Q_d(\mathbf{X}) = Q_d(N, f, \mathbf{X})$ according to (1.1) and assume that $E[Q_d(\mathbf{X})^2] = 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function such that $\|\varphi'''\|_\infty \le B$. Then, for $Z \sim \mathscr{N}(0, 1)$, we have, with $C_*$ defined by (3.1),*

$$
\begin{aligned}
&|E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Z)]| \\
&\quad \le B(30\beta)^d d! \sqrt{\max_{1\le i\le N} \operatorname{Inf}_i(f)} \\
&\qquad + C_* \sqrt{\frac{d-1}{3d}} \Bigl[\sqrt{|E[Q_d(\mathbf{X})^4] - 3|} \\
&\qquad\qquad + 4\sqrt{2} \times 144^{d-1/2} \alpha^{d/2} \sqrt{d} d! \Bigl(\max_{1\le i\le N} \operatorname{Inf}_i(f)\Bigr)^{1/4}\Bigr].
\end{aligned} \tag{5.1}
$$

PROOF. Let $\mathbf{G} = (G_i)_{i\geq 1}$ be a standard centered i.i.d. Gaussian sequence. We have $|E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Z)]| \leq \delta_1 + \delta_2$, with $\delta_1 = |E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Q_d(\mathbf{G}))]|$ and $\delta_2 = |E[\varphi(Q_d(\mathbf{G}))] - E[\varphi(Z)]|$. By Theorem 4.1, we have $\delta_1 \leq B(30\beta)^d d! \sqrt{\max_{1\leq i\leq N} \mathrm{Inf}_i(f)}$. Since $E[Q_d(\mathbf{X})^2] = E[Q_d(\mathbf{G})^2] = 1$, Theorem 3.1 yields $\delta_2 \leq C_* \sqrt{\frac{d-1}{3d} |E[Q_d(\mathbf{G})^4] - 3|}$. By Lemma 4.3, Point 2 (with $M = 1$, $k = 2$ and $l = m = 4$) and since $\mathrm{Inf}_i(f) \leq 1$ for all $i$, we have $|E[Q_d(\mathbf{X})^4] - E[Q_d(\mathbf{G})^4]| \leq 32 \times 144^{2d-1} \alpha^d d d!^2 \sqrt{\max_{1\leq i\leq N} \mathrm{Inf}_i(f)}$, so that $\delta_2 \leq C_* \sqrt{\frac{d-1}{3d}} \times [\sqrt{|E[Q_d(\mathbf{X})^4] - 3|} + 4\sqrt{2} \times 144^{d-1/2} \alpha^{d/2} \sqrt{d} d! (\max_{1\leq i\leq N} \mathrm{Inf}_i(f))^{1/4}]$. □

REMARK 5.2. As a corollary of Theorem 5.1, we immediately recover de Jong's Theorem 1.9, under the additional hypothesis that $\sup_i E(X_i^4) < \infty$.

As a converse statement, we now prove a slightly stronger version of Theorem 1.10 stated in Section 1.5; an additional condition on contractions [see assumption (5) in Theorem 5.3 just below and Definition 1.5] has been added with respect to Theorem 1.10, making the criterion more easily applicable in practice.

THEOREM 5.3. *We let the notation of Theorem* 1.10 *prevail. Then, as* $n \to \infty$, *the assertions* (1)–(4) *therein are equivalent, and are also equivalent to* (5) *for all* $r = 1, \dots, d-1$, $\|f_n \star_r f_n\|_{2d-2r} \to 0$.

PROOF. The equivalences (1) ⇔ (2) ⇔ (5) are a mere reformulation of Theorem 3.3, deduced by taking into account the first identity in Lemma 3.4. On the other hand, it is trivial that each one of conditions (3) and (4) implies (1). So, it remains to prove the implication (1), (2), (5) ⇒ (3), (4). Fix $z \in \mathbb{R}$. We have $|P[Q_d(n, \mathbf{X}) \leq z] - P[Z \leq z]| \leq |P[Q_d(n, \mathbf{X}) \leq z] - P[Q_d(n, \mathbf{G}) \leq z]| + |P[Q_d(n, \mathbf{G}) \leq z] - P[Z \leq z]| =: \delta_n^{(a)}(z) + \delta_n^{(b)}(z)$. By assumption (2) and Theorem 3.1, we have $\sup_{z\in\mathbb{R}} \delta_n^{(b)}(z) \to 0$. By combining assumption (5) (for $r = d-1$) with (1.9), we get that $\max_{1\leq i\leq N_n} \mathrm{Inf}_i(f_n) \to 0$ as $n \to \infty$. Hence, Theorem 4.1 (Point 1) implies that $\sup_{z\in\mathbb{R}} \delta_n^{(a)}(z) \to 0$, and the proof of the implication (1), (2), (5) ⇒ (3) is complete. To prove that (1) ⇒ (4), one uses the same line of reasoning, the only difference being that we need to use Point 2 of Theorem 4.1 (along with the characterization of weak convergence based on continuous bounded functions) instead of Point 1. □

Our techniques allow to directly control the Wasserstein distance between the law of a homogenous sum and the law of a standard Gaussian random variable, as illustrated by the following result.

PROPOSITION 5.4. *As in Theorem 5.1, let* $\mathbf{X} = \{X_i, i \geq 1\}$ *be a collection of centered independent random variables with unit variance. Assume, moreover, that* $\beta := \sup_i E(X_i^4) < \infty$ *and note* $\alpha := \max\{3; \beta\}$. *Fix* $N, d \geq 1$, *and let* $f : [N]^d \to \mathbb{R}$ *be symmetric and vanishing on diagonals. Define* $Q_d(\mathbf{X}) = Q_d(N, f, \mathbf{X})$ *according to* (1.1) *and assume that* $E[Q_d(\mathbf{X})^2] = 1$. *Put* $B_1 = 2(30\beta)^d d!\sqrt{\max_{1\leq i\leq N} \mathrm{Inf}_i(f)}$ *and* $B_2 = 12\sqrt{2}(1 + 5^{3d/2})\sqrt{\frac{d-1}{3d}} \times [\sqrt{|E[Q_d(\mathbf{X})^4] - 3|} + 4\sqrt{2} \times 144^{d-1/2}\alpha^{d/2}\sqrt{d}d!(\max_{1\leq i\leq N} \mathrm{Inf}_i(f))^{1/4}]$. *For* $Z \sim \mathscr{N}(0, 1)$, *we then have* $d_W(Q_d(\mathbf{X}), Z) \leq 4(B_1 + B_2)^{1/3}$, *provided* $B_1 + B_2 \leq \frac{3}{4\sqrt{2}}$.

PROOF. Let $h \in \mathrm{Lip}(1)$ be a Lipschitz function with constant 1. By Rademacher's theorem, $h$ is Lebesgue-almost everywhere differentiable; if we denote by $h'$ its derivative, then $\|h'\|_\infty \leq 1$. For $t > 0$, define $h_t(x) = \int_{-\infty}^{\infty} h(\sqrt{t}y + \sqrt{1-t}x)\phi(y)\,dy$, where $\phi$ denotes the standard normal density. The triangle inequality gives

$$
\begin{aligned}
&|E[h(Q_d(\mathbf{X}))] - E[h(Z)]|\\
&\quad \leq |E[h_t(Q_d(\mathbf{X}))] - E[h_t(Z)]| + |E[h(Q_d(\mathbf{X}))] - E[h_t(Q_d(\mathbf{X}))]|\\
&\qquad + |E[h(Z)] - E[h_t(Z)]|.
\end{aligned}
$$

As $h_t''(x) = \frac{1-t}{\sqrt{t}} \int_{-\infty}^{\infty} y h'(\sqrt{t}y + \sqrt{1-t}x)\phi(y)\,dy$, for $0 < t < 1$, we may bound $\|h_t''\|_\infty \leq \frac{1-t}{\sqrt{t}}\|h'\|_\infty \int_{-\infty}^{\infty} |y|\phi(y)\,dy \leq \frac{1}{\sqrt{t}}$. For $0 < t \leq \frac{1}{2}$ (so that $\sqrt{t} \leq \sqrt{1-t}$), we have

$$
\begin{aligned}
&|E[h(Q_d(\mathbf{X}))] - E[h_t(Q_d(\mathbf{X}))]|\\
&\quad \leq \left|E\left[\int_{-\infty}^{\infty} \{h(\sqrt{t}y + \sqrt{1-t}\,Q_d(\mathbf{X})) - h(\sqrt{1-t}\,Q_d(\mathbf{X}))\}\phi(y)\,dy\right]\right|\\
&\qquad + E\left[\left|h(\sqrt{1-t}\,Q_d(\mathbf{X})) - h(Q_d(\mathbf{X}))\right|\right]\\
&\quad \leq \|h'\|_\infty \sqrt{t} \int_{-\infty}^{\infty} |y|\phi(y)\,dy + \|h'\|_\infty \frac{t}{2\sqrt{1-t}} E[|Q_d(\mathbf{X})|] \leq \frac{3}{2}\sqrt{t}.
\end{aligned}
$$

Similarly, $|E[h(Z)] - E[h_t(Z)]| \leq \frac{3}{2}\sqrt{t}$. We now apply Theorem 5.1. To bound $C_*$, we use that $|h_t'(0)| \leq 1$ and that $|h_t''(0)| \leq t^{-1/2}$; also $\|h_t'''\|_\infty \leq 2/t$ (as it can be shown by using the same arguments as above). Hence, as $2 \leq \frac{1}{t}$ and $\sqrt{2} \leq t^{-1/2}$, we have

$$
\begin{aligned}
C_* &\leq 4\sqrt{2}(1 + 5^{3d/2}) \times \max\left\{\frac{3}{2}t^{-1/2} + \frac{4\sqrt{2}}{3\sqrt{\pi}}t^{-1}; 2 + \frac{2}{3}t^{-1}\right\}\\
&\leq 4\sqrt{2}(1 + 5^{3d/2}) \times \frac{3}{t}.
\end{aligned}
$$

Due to $\|h_t'''\|_\infty \le 2/t$, Theorem 5.1 gives the bound $|E[h_t(Q_d(\mathbf{X}))] - E[h_t(Z)]| \le 3\sqrt{t} + (B_1 + B_2)\frac{1}{t}$. Minimizing $3\sqrt{t} + (B_1 + B_2)\frac{1}{t}$ in $t$ gives that $t = (\frac{2}{3}(B_1 + B_2))^{2/3}$. Plugging in the values and bounding the constant part ends the proof. $\square$

**6. Chi-square approximation of homogeneous sums.** The next result provides bounds on the chi-square approximation of homogeneous sums.

THEOREM 6.1. *Let $\mathbf{X} = \{X_i, i \ge 1\}$ be a collection of centered independent random variables with unit variance. Assume, moreover, that $\beta := \sup_i E(X_i^4) < \infty$ and note $\alpha := \max\{3; \beta\}$. Fix an* even *integer $d \ge 2$ and, for $N \ge 1$, let $f : [N]^d \to \mathbb{R}$ be symmetric and vanishing on diagonals. Define $Q_d(\mathbf{X}) = Q_d(N, f, \mathbf{X})$ according to (1.1) and assume that $E[Q_d(\mathbf{X})^2] = 2\nu$ for some integer $\nu \ge 1$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function such that $\|\varphi\|_\infty \le 1$, $\|\varphi'\|_\infty \le 1$ and $\|\varphi'''\|_\infty \le B$. Then, for $Z_\nu \sim \chi^2(\nu)$, we have*

$$
\begin{aligned}
&|E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Z_\nu)]| \\
&\quad \le B(30\beta)^d d! \sqrt{\max_{1\le i\le N} \mathrm{Inf}_i(f)} \\
&\qquad + \max\left\{\sqrt{\frac{2\pi}{\nu}}, \frac{1}{\nu} + \frac{2}{\nu^2}\right\} \\
&\qquad \times \left(\sqrt{\frac{d-1}{3d}}\left[\sqrt{|E[Q_d(\mathbf{X})^4] - 12E[Q_d(\mathbf{X})^3] - 12\nu^2 + 48\nu|}\right.\right. \\
&\qquad\qquad + 4\sqrt{d}\,d!\big(\sqrt{2} \times 144^{d-1/2}\alpha^{d/2} \\
&\qquad\qquad\qquad + \sqrt{\nu}(2\sqrt{2})^{3(2d-1)/2}\alpha^{3d/2}\big) \\
&\qquad\qquad\qquad\qquad \left.\left. \times \left(\max_{1\le i\le N} \mathrm{Inf}_i(f)\right)^{1/4}\right]\right).
\end{aligned}
$$

PROOF. We proceed as in Theorem 5.1. Let $\mathbf{G} = (G_i)_{i\ge1}$ denote a standard centered i.i.d. Gaussian sequence. We have $|E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Z_\nu)]| \le \delta_1 + \delta_2$ with $\delta_1 = |E[\varphi(Q_d(\mathbf{X}))] - E[\varphi(Q_d(\mathbf{G}))]|$ and $\delta_2 = |E[\varphi(Q_d(\mathbf{G}))] - E[\varphi(Z_\nu)]|$. By Theorem 4.1 (Point 3), we have $\delta_1 \le B(30\beta)^d d! \sqrt{\max_{1\le i\le N} \mathrm{Inf}_i(f)}$. By Theorem 3.6, we have, with $C_\# = \max\{\sqrt{\frac{2\pi}{\nu}}, \frac{1}{\nu} + \frac{2}{\nu^2}\}$, that $(\delta_2)^2 \le (C_\#)^2 \times \frac{d-1}{3d}|E[Q_d(\mathbf{G})^4] - 12E[Q_d(\mathbf{G})^3] - 12\nu^2 + 48\nu|$. Additionally to the bound for $|E[Q_d(\mathbf{X})^4] - E[Q_d(\mathbf{G})^4]|$ in Theorem 5.1, we have, by Lemma 4.3, $|E[Q_d(\mathbf{X})^3] - E[Q_d(\mathbf{G})^3]| \le 16\nu(2\sqrt{2})^{3(2d-1)}\alpha^{3d/4} d d! \sqrt{\max_{1\le i\le N} \mathrm{Inf}_i(f)}$.

Hence, the proof is concluded since

$$\begin{aligned}\delta_2 \le C_\# \sqrt{\frac{d-1}{3d}} \Big[ & \sqrt{|E[Q_d(\mathbf{X})^4] - 12E[Q_d(\mathbf{X})^3] - 12\nu^2 + 48\nu|} \\ & + 4\sqrt{d}d!(\sqrt{2} \times 144^{d-1/2}\alpha^{d/2} \\ & \qquad + \sqrt{\nu}(2\sqrt{2})^{3(2d-1)/2}\alpha^{3d/2}) \\ & \qquad\qquad \times \Big(\max_{1\le i\le N} \mathrm{Inf}_i(f)\Big)^{1/4}\Big]. \qquad \square\end{aligned}$$

As an immediate corollary of Theorem 6.1, we deduce the following new criterion for the asymptotic nonnormality of homogenous sums—compare with Theorem 1.9.

COROLLARY 6.2. *Let $\mathbf{X} = \{X_i : i \ge 1\}$ be a sequence of independent centered random variables with unit variance such that $\sup_i E(X_i^4) < \infty$. Fix an* even *integer $d \ge 2$, and let $\{N_n, f_n : n \ge 1\}$ be a sequence such that $\{N_n : n \ge 1\}$ is a sequence of integers going to infinity, and each $f_n : [N_n]^d \to \mathbb{R}$ is symmetric and vanishes on diagonals. Define $Q_d(n, \mathbf{X}) = Q_d(N_n, f_n, \mathbf{X})$ according to* (1.1). *If, as $n \to \infty$,* (i) $E(Q_d(n, \mathbf{X})^2) \to 2\nu$; (ii) $E[Q_d(n, \mathbf{X})^4] - 12E[Q_d(N_n, f_n, \mathbf{X})^3] \to 12\nu^2 - 48\nu$; *and* (iii) $\max_{1\le i\le N_n} \mathrm{Inf}_i(f_n) \to 0$; *then $Q_d(n, \mathbf{X})$ converges in law to $Z_\nu \sim \chi^2(\nu)$.*

The following statement contains a universal chi-square limit theorem result: it is a general version of Theorem 1.12.

THEOREM 6.3. *We let the notation of Theorem* 1.12 *prevail. Then, as $n \to \infty$, the assertions* (1)–(4) *therein are equivalent, and are also equivalent to* (5) $\|f_n \widetilde{\star}_{d/2} f_n - 4(d/2)!^3 d!^{-2} \times f_n\|_d \to 0$ *and, for every $r = 1, \dots, d-1$ such that $r \ne d/2$, $\|f_n \star_r f_n\|_{2d-2r} \to 0$.*

PROOF. The proof follows exactly the same lines of reasoning as in Theorem 5.3. Details are left to the reader. Let us just mention that the only differences consist in the use of Theorem 3.7 instead of Theorem 3.3, and the use of Theorem 3.6 instead of Theorem 3.1. $\square$

## 7. Multivariate extensions.

7.1. *Bounds*. We recall here the standard multi-index notation. A multi-index is a vector $\alpha \in \{0, 1, \dots\}^m$. We write $|\alpha| = \sum_{j=1}^m \alpha_j$, $\alpha! = \prod_{j=1}^m \alpha_j!$, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, and $x^\alpha = \prod_{j=1}^m x_j^{\alpha_j}$. Note that, by convention, $0^0 = 1$. Also note that $|x^\alpha| = y^\alpha$, where $y_j = |x_j|$ for all $j$. Finally, for $\varphi : \mathbb{R}^m \to \mathbb{R}$ regular and $k \ge 1$, we put $\|\varphi^{(k)}\|_\infty = \max_{|\alpha|=k} \frac{1}{\alpha!} \sup_{z\in\mathbb{R}^m} |\partial^\alpha \varphi(z)|$.

The forthcoming Theorem 7.1 is a multivariate version of Theorem 4.1 (Point 3). Observe that its statement (and its proof as well) follows closely ([9], Theorem 4.1). However, the result of [9] is stated and proved under the assumption that one of the two i.i.d. sequences lives on a discrete probability space, hence, a bit more work is needed.

THEOREM 7.1. *Let $\mathbf{X} = \{X_i, i \geq 1\}$ be a collection of centered independent random variables with unit variance and such that $\beta := \sup_{i\geq 1} E[|X_i|^3] < \infty$. Let $\mathbf{G} = \{G_i : i \geq 1\}$ be a standard centered i.i.d. Gaussian sequence. Fix integers $m \geq 1$, $d_m \geq \cdots \geq d_1 \geq 1$ and $N_1, \dots, N_m \geq 1$. For every $j = 1, \dots, m$, let $f_j : [N_j]^{d_j} \to \mathbb{R}$ be a symmetric function vanishing on diagonals. Define $Q^j(\mathbf{G}) = Q_{d_j}(N_j, f_j, \mathbf{G})$ and $Q^j(\mathbf{X}) = Q_{d_j}(N_j, f_j, \mathbf{X})$ according to (1.1), and assume that $E[Q^j(\mathbf{G})^2] = E[Q^j(\mathbf{X})^2] = 1$ for all $j = 1, \dots, m$. Assume that there exists a $C > 0$ such that $\sum_{i=1}^{\max_j N_j} \max_{1\leq j\leq m} \mathrm{Inf}_i(f_j) \leq C$. Then, for all thrice differentiable $\varphi : \mathbb{R}^m \to \mathbb{R}$ with $\|\varphi'''\|_\infty < \infty$, we have*

$$
\begin{aligned}
&|E[\varphi(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[\varphi(Q^1(\mathbf{G}), \dots, Q^m(\mathbf{G}))]| \\
&\quad \leq C\|\varphi'''\|_\infty \left(\beta + \sqrt{\frac{8}{\pi}}\right) \left[\sum_{j=1}^m (16\sqrt{2}\beta)^{(d_j-1)/3} d_j!\right]^3 \\
&\qquad \times \sqrt{\max_{1\leq j\leq m} \max_{1\leq i\leq \max_j N_j} \mathrm{Inf}_i(f_j)}.
\end{aligned}
$$

Observe that, in the one-dimensional case ($m = 1$),

$$
\sum_{i=1}^{\max_j N_j} \max_{1\leq j\leq m} \mathrm{Inf}_i(f_j) = [d!(d-1)!]^{-1},
$$

so we can choose $C = [d!(d-1)!]^{-1}$. In this case, when $\beta$ is large, the bound from Theorem 7.1 essentially differs from the one in Theorem 4.1 by a constant times a factor $d$.

PROOF OF THEOREM 7.1. Abbreviate $\mathbf{Q}(\mathbf{X}) = (Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))$, and define $\mathbf{Q}(\mathbf{G})$ analogously. We proceed as for Lemma 4.3, with similar notation. For $i = 0, \dots, \max_j N_j$, let $\mathbf{Z}^{(i)}$ denote the sequence $(G_1, \dots, G_i, X_{i+1}, \dots, X_{\max_j N_j})$. Using the triangle inequality,

$$
|E[\varphi(\mathbf{Q}(\mathbf{X}))] - E[\varphi(\mathbf{Q}(\mathbf{G}))]| \leq \sum_{i=1}^{\max_j N_j} \left|E\left[\varphi\left(\mathbf{Q}\left(\mathbf{Z}^{(i-1)}\right)\right)\right] - E\left[\varphi\left(\mathbf{Q}\left(\mathbf{Z}^{(i)}\right)\right)\right]\right|.
$$

Now we can proceed as for inequality (31) in the proof of [9], Theorem 4.1 to obtain

$$\begin{aligned}
&\big|E[\varphi(\mathbf{Q}(n,\mathbf{Z}^{(i-1)}))] - E[\varphi(\mathbf{Q}(n,\mathbf{Z}^{(i)}))]\big| \\
&\quad = |E[\varphi(\mathbf{U}_i + X_i\mathbf{V}_i)] - E[\varphi(\mathbf{U}_i + G_i\mathbf{V}_i)]| \\
&\quad \le \left(\beta + \sqrt{\frac{8}{\pi}}\right)\|\varphi'''\|_\infty \sum_{|\alpha|=3} E(|\mathbf{V}_i^\alpha|).
\end{aligned}$$

While [9], Theorem 4.1, now uses hypercontractivity results for random variables on finite probability spaces, here we bound the moments directly. Abbreviate $\tau_i = \max_{1\le j\le m}\operatorname{Inf}_i(f_j)$. Next we use that, for $j = 1,\dots,m$, by Lemma 4.2 (with $q = 3$), we have $E[|V_i^{(j)}|^3] \le (16\sqrt{2}\beta)^{d_j-1}E[(V_i^{(j)})^2]^{3/2} = (16\sqrt{2}\beta)^{d_j-1}d_j!^3\tau_i^{3/2}$. Thus,

$$\begin{aligned}
\sum_{|\alpha|=3} E|(\mathbf{V}_i)^\alpha| &= \sum_{j,k,l=1}^m E\big(|V_i^{(j)}V_i^{(k)}V_i^{(l)}|\big) \\
&\le \sum_{j,k,l=1}^m E\big(|V_i^{(j)}|^3\big)^{1/3} E\big(|V_i^{(k)}|^3\big)^{1/3} E\big(|V_i^{(l)}|^3\big)^{1/3} \\
&= \left(\sum_{j=1}^m E\big(|V_i^{(j)}|^3\big)^{1/3}\right)^3 \\
&\le \left[\sum_{j=1}^m (16\sqrt{2}\beta)^{(d_j-1)/3} d_j!\right]^3 \tau_i^{3/2}.
\end{aligned}$$

Collecting the bounds, summing over $i$, and using that $\sum_{i=1}^{\max_j N_j}\tau_i \le C$ gives the desired result. $\square$

The next statement gives explicit bounds on the distance to the normal distribution for the distribution of the vector $(Q^1(\mathbf{X}),\dots,Q^m(\mathbf{X}))$.

THEOREM 7.2. *Let $\mathbf{X} = \{X_i : i \ge 1\}$ be a collection of centered independent random variables with unit variance. Assume, moreover, that $\beta := \sup_i E[|X_i|^3] < \infty$. Fix integers $m \ge 1$, $d_m \ge \cdots \ge d_1 \ge 2$ and $N_1,\dots,N_m \ge 1$. For every $j = 1,\dots,m$, let $f_j : [N_j]^{d_j} \to \mathbb{R}$ be a symmetric function vanishing on diagonals. Define $Q^j(\mathbf{X}) = Q_{d_j}(N_j, f_j, \mathbf{X})$ according to (1.1), and assume that $E[Q^j(\mathbf{X})^2] = 1$ for all $j = 1,\dots,m$. Let $V$ be the $m\times m$ symmetric matrix given by $V(i,j) = E[Q^i(\mathbf{X})Q^j(\mathbf{X})]$. Let $C$ be as in Theorem 7.1. Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a thrice differentiable function such that $\|\varphi''\|_\infty < \infty$ and $\|\varphi'''\|_\infty < \infty$. Then,* *for* $Z_V = (Z_V^1, \dots, Z_V^m) \sim \mathscr{N}_m(0, V)$ (*centered Gaussian vector with covariance matrix* $V$), *we have*

$$
\begin{aligned}
&|E[\varphi(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[\varphi(Z_V)]| \\
&\quad \le \|\varphi''\|_\infty \left( \sum_{i=1}^m \Delta_{ii} + 2 \sum_{1 \le i < j \le m} \Delta_{ij} \right) \\
&\qquad + C \|\varphi'''\|_\infty \left( \beta + \sqrt{\frac{8}{\pi}} \right) \left[ \sum_{j=1}^m (16\sqrt{2}\beta)^{(d_j-1)/3} d_j! \right]^3 \\
&\qquad \times \sqrt{\max_{1 \le j \le m} \max_{1 \le i \le N_j} \mathrm{Inf}_i(f_j)}
\end{aligned}
$$

*for* $\Delta_{ij}$ *given by*

$$
\begin{aligned}
&\frac{d_j}{\sqrt{2}} \sum_{r=1}^{d_i-1} (r-1)! \binom{d_i - 1}{r-1} \binom{d_j - 1}{r-1} \\
&\quad \times \sqrt{(d_i + d_j - 2r)!} (\|f_i \star_{d_i - r} f_i\|_{2r} + \|f_j \star_{d_j - r} f_j\|_{2r}) \qquad (7.1)\\
&\quad + \mathbf{1}_{\{d_i < d_j\}} \sqrt{d_j! \binom{d_j}{d_i} \|f_j \star_{d_j - d_i} f_j\|_{2d_i}}.
\end{aligned}
$$

PROOF. The proof is divided into four steps.

*Step 1: Reduction of the problem.* Let $\mathbf{G} = (G_i)_{i \ge 1}$ be a standard centered i.i.d. Gaussian sequence. We have $|E[\varphi(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[\varphi(Z_V)]| \le \delta_1 + \delta_2$ with $\delta_1 = |E[\varphi(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[\varphi(Q^1(\mathbf{G}), \dots, Q^m(\mathbf{G}))]|$ and $\delta_2 = |E[\varphi(Q^1(\mathbf{G}), \dots, Q^m(\mathbf{G}))] - E[\varphi(Z_V)]|$.

*Step 2: Bounding* $\delta_1$. By Theorem 7.1, we have

$$
\delta_1 \le C \|\varphi'''\|_\infty \left( \beta + \sqrt{\frac{8}{\pi}} \right) \left[ \sum_{j=1}^m (16\sqrt{2}\beta)^{(d_j-1)/3} d_j! \right]^3 \sqrt{\max_{1 \le j \le m} \max_{1 \le i \le N_j} \mathrm{Inf}_i(f_j)}.
$$

*Step 3: Bounding* $\delta_2$. We will not use the result proved in [17], since here we do not assume that the matrix $V$ is positive definite. Instead, we will rather use an interpolation technique. Without loss of generality, we assume in this step that $Z_V$ is independent of $\mathbf{G}$. By (2.2), we have that $\{Q^j(\mathbf{G})\}_{1 \le j \le m} \overset{\text{Law}}{=} \{I_{d_j}(h_j)\}_{1 \le j \le m}$ where $h_j = d_j! \sum_{\{i_1, \dots, i_{d_j}\} \subset [N_j]^{d_j}} f_j(i_1, \dots, i_{d_j}) e_{i_1} \otimes \cdots \otimes e_{i_{d_j}} \in \mathfrak{H}^{\odot d}$, with $\mathfrak{H} = L^2([0,1])$ and $\{e_j\}_{j \ge 1}$ any orthonormal basis of $\mathfrak{H}$. For $t \in [0,1]$, set $\Psi(t) = E[\varphi(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V)]$, so that $\delta_2 = |\Psi(1) - \Psi(0)| \le \sup_{t \in (0,1)} |\Psi'(t)|$. We easily see that $\Psi'(t) = \sum_{i=1}^m E[\frac{\partial \varphi}{\partial x_i}(\sqrt{1-t}(I_{d_1}(h_1), \dots,$

$I_{d_m}(h_m)) + \sqrt{t} Z_V)(\frac{1}{2\sqrt{t}} Z_V^i - \frac{1}{2\sqrt{1-t}} I_{d_i}(h_i))]$. By integrating by parts, we can write

$$E\left[\frac{\partial \varphi}{\partial x_i}(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V) Z_V^i\right]$$
$$= \sqrt{t} \sum_{j=1}^m V(i,j) E\left[\frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V)\right].$$

By using (8.1) below in order to perform the integration by parts, we can also write

$$E\left[\frac{\partial \varphi}{\partial x_i}(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V) I_{d_i}(h_i)\right]$$
$$= \frac{\sqrt{1-t}}{d_i} \sum_{j=1}^m E\left[\frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V)\right.$$
$$\left. \times \langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}}\right].$$

Hence, $\Psi'(t)$ equals

$$\frac{1}{2} \sum_{i,j=1}^m E\left[\frac{\partial^2 \varphi}{\partial x_i\, \partial x_j}(\sqrt{1-t}(I_{d_1}(h_1), \dots, I_{d_m}(h_m)) + \sqrt{t} Z_V)\right.$$
$$\left. \times \left(V(i,j) - \frac{1}{d_i}\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}}\right)\right],$$

so that we get

$$\delta_2 \le \|\varphi''\|_\infty \sum_{i,j=1}^m E\left[\left|V(i,j) - \frac{1}{d_i}\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}}\right|\right]$$
$$\le \|\varphi''\|_\infty \sum_{i,j=1}^m \sqrt{E\left[\left(V(i,j) - \frac{1}{d_i}\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}}\right)^2\right]}$$
$$= \|\varphi''\|_\infty \sum_{i,j=1}^m \frac{1}{d_i}\sqrt{\operatorname{Var}(\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}})}.$$

*Step 4: Bounding* $\operatorname{Var}(\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}})$. Assume, for instance, that $i \le j$. We have

$$\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}}$$
$$= d_i d_j \int_0^1 I_{d_i-1}(h_i(\cdot, a)) I_{d_j-1}(h_j(\cdot, a))\, da$$

$$= d_i d_j \int_0^1 \sum_{r=0}^{d_i-1} r! \binom{d_i - 1}{r} \binom{d_j - 1}{r} I_{d_i+d_j-2-2r}\big(h_i(\cdot, a) \widetilde{\otimes}_r h_j(\cdot, a)\big)\, da$$

(by Proposition 2.5)

$$= d_i d_j \sum_{r=0}^{d_i-1} r! \binom{d_i - 1}{r} \binom{d_j - 1}{r} I_{d_i+d_j-2-2r}(h_i \widetilde{\otimes}_{r+1} h_j)$$

$$= d_i d_j \sum_{r=1}^{d_i} (r-1)! \binom{d_i - 1}{r-1} \binom{d_j - 1}{r-1} I_{d_i+d_j-2r}(h_i \widetilde{\otimes}_r h_j).$$

Hence, if $d_i < d_j$, then $\mathrm{Var}(\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}})$ equals

$$d_i^2 d_j^2 \sum_{r=1}^{d_i} (r-1)!^2 \binom{d_i - 1}{r-1}^2 \binom{d_j - 1}{r-1}^2 (d_i + d_j - 2r)! \|h_i \widetilde{\otimes}_r h_j\|^2_{\mathfrak{H}^{\otimes(d_i+d_j-2r)}},$$

while, if $d_i = d_j$, it equals

$$d_i^4 \sum_{r=1}^{d_i-1} (r-1)!^2 \binom{d_i - 1}{r-1}^4 (2d_i - 2r)! \| h_i \widetilde{\otimes}_r h_j \|^2_{\mathfrak{H}^{\otimes(2d_i-2r)}}.$$

Now, let us stress the two following estimates. If $r < d_i \le d_j$, then

$$\begin{aligned}
\|h_i \widetilde{\otimes}_r h_j\|^2_{\mathfrak{H}^{\otimes(d_i+d_j-2r)}} &\le \|h_i \otimes_r h_j\|^2_{\mathfrak{H}^{\otimes(d_i+d_j-2r)}} \\
&= \langle h_i \otimes_{d_i-r} h_i, h_j \otimes_{d_j-r} h_j\rangle_{\mathfrak{H}^{\otimes 2r}} \\
&\le \|h_i \otimes_{d_i-r} h_i\|_{\mathfrak{H}^{\otimes 2r}} \|h_j \otimes_{d_j-r} h_j\|_{\mathfrak{H}^{\otimes 2r}} \\
&\le \tfrac{1}{2}(\|h_i \otimes_{d_i-r} h_i\|^2_{\mathfrak{H}^{\otimes 2r}} + \|h_j \otimes_{d_j-r} h_j\|^2_{\mathfrak{H}^{\otimes 2r}}).
\end{aligned}$$

If $r = d_i < d_j$, then $\|h_i \widetilde{\otimes}_{d_i} h_j\|^2_{\mathfrak{H}^{\otimes(d_j-d_i)}} \le \|h_i \otimes_{d_i} h_j\|^2_{\mathfrak{H}^{\otimes(d_j-d_i)}} \le \|h_i\|^2_{\mathfrak{H}^{\otimes d_i}} \|h_j \otimes_{d_j-d_i} h_j\|_{\mathfrak{H}^{\otimes 2d_i}}$. By putting all these estimates in the previous expression for $\mathrm{Var}(\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}})$, we get, using also Lemma 3.4, that $\frac{1}{d_i}\sqrt{\mathrm{Var}(\langle D[I_{d_i}(h_i)], D[I_{d_j}(h_j)]\rangle_{\mathfrak{H}})} \le \Delta_{ij}$, for $\Delta_{ij}$ defined by (7.1). This completes the proof of the theorem. □

We now translate the bound in Theorem 7.2 into a bound for indicators of convex sets.

COROLLARY 7.3. *Let the notation and assumptions from Theorem 7.2 prevail. We consider the class $\mathcal{H}(\mathbb{R}^m)$ of indicator functions of measurable convex sets in $\mathbb{R}^m$. Let $B_1 = \frac{1}{2}\sum_{i=1}^m \Delta_{ii} + \sum_{1\le i<j\le m} \Delta_{ij}$ and*

$$B_2 = C\left(\beta + \sqrt{\frac{8}{\pi}}\right)\left[\sum_{j=1}^m (16\sqrt{2}\beta)^{(d_j-1)/3} d_j!\right]^3 \sqrt{\max_{1\le j\le m} \max_{1\le i\le N_j} \mathrm{Inf}_i(f_j)}.$$

1. *Assume that the covariance matrix $V$ is the $m \times m$ identity matrix $I_m$. Then*

$$\sup_{h \in \mathcal{H}(\mathbb{R}^m)} |E[h(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[h(Z_V)]| \\ \le 8(B_1 + B_2)^{1/4} m^{3/8}.$$

2. *Assume that $V$ is of rank $k \le m$, and let $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$ be the diagonal matrix with the nonzero eigenvalues of $V$ on the diagonal. Let $B$ be a $m \times k$ column orthonormal matrix (i.e., $B^T B = I_k$ and $BB^T = I_m$), such that $V = B\Lambda B^T$, and let $b = \max_{i,j} (\Lambda^{-1/2} B^T)_{i,j}$. Then $|E[h(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[h(Z_V)]| \le 8(b^2 B_1 + b^3 B_2)^{1/4} m^{3/8}$ for all $h \in \mathcal{H}(\mathbb{R}^m)$.*

REMARK 7.4. 1. Notice that $\sup_{z \in \mathbb{R}^m} |P[(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X})) \le z] - P[Z_V \le z]| \le \sup_{h \in \mathcal{H}(\mathbb{R}^m)} |E[h(Q^1(\mathbf{X}), \dots, Q^m(\mathbf{X}))] - E[h(Z_V)]|$. Thus, Corollary 7.3 immediately gives a bound for Kolmogorov distance.

2. By using the bound for $\delta_2$ derived in the proof of Theorem 7.2 above, and following the same line of reasoning as in the proof of Corollary 7.3, we have, by keeping the notation of Theorem 7.2, that if $\Delta_{ij} \to 0$ for all $i, j = 1, \dots, m$ and $\max_{1 \le j \le m} \max_{1 \le i \le N_j} \mathrm{Inf}_i(f_j) \to 0$, then $(Q_{d_1}(N_1, f_1, \mathbf{G}), \dots, Q_{d_m}(N_m, f_m, \mathbf{G})) \to \mathscr{N}_m(0, V)$ as $N_1, \dots, N_j \to \infty$, *in the Kolmogorov distance*.

PROOF OF COROLLARY 7.3. First assume that $V$ is the identity matrix. We partially follow [26], and let $\Phi$ denote the standard normal distribution in $\mathbb{R}^m$, and $\phi$ the corresponding density function. For $h \in \mathcal{H}(\mathbb{R}^m)$, define the smoothing $h_t(x) = \int_{\mathbb{R}^m} h(\sqrt{t} y + \sqrt{1-t} x) \Phi(dy)$, $0 < t < 1$. The key result, found, for example, in [5], Lemma 2.11, is that, for any probability measure $Q$ on $\mathbb{R}^m$, for any $W \sim Q$ and $Z \sim \Phi$, and for any $0 < t < 1$, we have that $\sup_{h \in \mathcal{H}(\mathbb{R}^m)} |E[h(W)] - E[h(Z)]| \le \frac{4}{3}[\sup_{h \in \mathcal{H}(\mathbb{R}^m)} |E[h_t(W)] - E[h_t(Z)]| + 2\sqrt{m}\sqrt{t}]$. Similarly as in [7], page 24, put $u(x, t, z) = (2\pi t)^{-m/2} \times \exp(-\sum_{i=1}^m \frac{(z_i - \sqrt{1-t} x_i)^2}{2t})$, so that $h_t(x) = \int_{\mathbb{R}^m} h(z) u(x, t, z)\, dz$. Observe that $u(x, t, z)$ is the density function of the Gaussian vector $Y \sim \mathscr{N}(0, tI_m)$ taken in $z - \sqrt{1-t}x$. Because $0 \le h(z) \le 1$ for all $z \in \mathbb{R}^m$, we may bound $|\frac{\partial^2 h_t}{\partial x_i^2}(x)| \le \frac{1-t}{t} + \frac{1-t}{t^2} E[Y_i^2] = \frac{2(1-t)}{t}$. Similarly, for $i \ne j$, $|\frac{\partial^2 h_t}{\partial x_i \partial x_j}(x)| \le \frac{1-t}{t^2} E[|Y_i|] E[|Y_j|] = \frac{2(1-t)}{\pi t}$. Thus, we have $\|h_t''\|_\infty \le 1/t \le 1/t^{3/2}$. Bounding the third derivatives in a similar fashion yields, for all $i, j, k$ not necessarily distinct, that $|\frac{\partial^3 h_t}{\partial x_i \partial x_j \partial x_k}(x)|$ is less or equal than

$$\frac{(1-t)^{3/2}}{t^3} \max\{3E[|Y_i|]t + E[|Y_i|^3]; \\ E[|Y_j|]t + E[Y_i^2]E[|Y_j|]; E[|Y_i|]E[|Y_j|]E[|Y_k|]\},$$

and so $\|h_t'''\|_\infty \le 1/t^{3/2}$. With [5], Lemma 2.11, and Theorem 7.2, this gives that

$$\begin{aligned}\sup_{h\in\mathcal{H}(\mathbb{R}^m)} &|E[h(Q^1(\mathbf{X}),\dots,Q^m(\mathbf{X}))]-E[h(Z_V)]|\\ &\le \frac{4}{3}\Big[\sup_{h\in\mathcal{H}(\mathbb{R}^m)}|E[h_t(Q^1(\mathbf{X}),\dots,Q^m(\mathbf{X}))]-E[h_t(Z_V)]|+2\sqrt{m}\sqrt{t}\Big]\\ &\le \frac{8}{3}\sqrt{m}\sqrt{t}+\frac{4}{3}(B_1+B_2)t^{-3/2}.\end{aligned}$$

This function is minimized for $t=\sqrt{3(B_1+B_2)/(2\sqrt{m})}$, yielding the first assertion. For Point 2, write $W=(Q^1(\mathbf{X}),\dots,Q^m(\mathbf{X}))$ for simplicity. For $h\in\mathcal{H}(\mathbb{R}^m)$, we have

$$\begin{aligned}&E[h(W)]-E[h(Z_V)]\\ &\qquad=E[h(B\Lambda^{1/2}\times\Lambda^{-1/2}B^TW)]-E[h(B\Lambda^{1/2}\times\Lambda^{-1/2}B^TZ_V)].\end{aligned}$$

Put $g(x)=h(B\Lambda^{1/2}x)$. Then, $g\in\mathcal{H}(\mathbb{R}^k)$ and, thanks to [5], Lemma 2.11, we can write

$$\begin{aligned}\sup_{h\in\mathcal{H}(\mathbb{R}^m)}&|E[h(W)]-E[h(Z_V)]|\\ &\le \sup_{g\in\mathcal{H}(\mathbb{R}^k)}|E[g(\Lambda^{-1/2}B^TW)]-E[g(\Lambda^{-1/2}B^TZ_V)]|\\ &\le \frac{4}{3}\Big[\sup_{g\in\mathcal{H}(\mathbb{R}^k)}|E[g_t(\Lambda^{-1/2}B^TW)]-E[g_t(\Lambda^{-1/2}B^TZ_V)]|+2\sqrt{k}\sqrt{t}\Big].\end{aligned}$$

We may bound the partial derivatives of $f_t(x)=g_t(\Lambda^{-1/2}B^Tx)$ using the chain rule and the definition of $b$, to give that $\|f_t''\|_\infty\le b^2t^{-3/2}$ and $\|f_t'''\|_\infty\le b^3t^{-3/2}$. Using Theorem 7.2 and minimizing the bound in $t$ as before gives the assertion; the only changes are that $B_1$ gets multiplied by $b^2$ and $B_2$ gets multiplied by $b^3$. □

7.2. *More universality.* Here, we prove a slightly stronger version of Theorem 1.2 stated in Section 1.3. Precisely, we add the two conditions (2) and (3), making the criterion contained in Theorem 1.2 more effective for potential applications.

THEOREM 7.5. *We let the notation of Theorem 1.2 prevail. Then, as $n\to\infty$, the following four conditions* (1)–(4) *are equivalent*: (1) *The vector* $\{Q^j(n,\mathbf{G}): j=1,\dots,m\}$ *converges in law to* $\mathscr{N}_m(0,V)$; (2) *for all* $i,j=1,\dots,m$, *we have* $E[Q^i(n,\mathbf{G})Q^j(n,\mathbf{G})]\to V(i,j)$ *and* $E[Q^i(n,\mathbf{G})^4]\to 3V(i,i)^2$ *as* $n\to\infty$; (3) *for all* $i,j=1,\dots,m$, *we have* $E[Q^i(n,\mathbf{G})Q^j(n,\mathbf{G})]\to V(i,j)$ *and, for all* $1\le i\le m$ *and* $1\le r\le d_i-1$, *we have* $\|f_n^{(i)}\star_r f_n^{(i)}\|_{2d_i-2r}\to 0$; (4) *for every* *sequence* $\mathbf{X} = \{X_i : i \geq 1\}$ *of independent centered random variables, with unit variance and such that* $\sup_i E|X_i|^3 < \infty$, *the vector* $\{Q^j(n, \mathbf{X}) : j = 1, \ldots, m\}$ *converges in law to* $\mathscr{N}_m(0, V)$ *for the Kolmogorov distance.*

For the proof of Theorem 7.5, we need the following result, which consists in a collection of some of the findings contained in the papers by Peccati and Tudor [23]. Strictly speaking, the original statements contained in [23] only deal with positive definite covariance matrices: however, the extension to a nonnegative matrix can be easily achieved by using the same arguments as in Step 3 of the proof of Theorem 7.2.

THEOREM 7.6. *Fix integers* $m \geq 1$ *and* $d_m \geq \cdots \geq d_1 \geq 1$. *Let* $V = \{V(i, j) : i, j = 1, \ldots, m\}$ *be a* $m \times m$ *nonnegative symmetric matrix. For any* $n \geq 1$ *and* $i = 1, \ldots, m$, *let* $I_{d_i}(h_i^{(n)})$ *belong to the* $d_i$*th Gaussian chaos* $C_{d_i}$. *Assume that* $F^{(n)} = (F_1^{(n)}, \ldots, F_m^{(n)}) := (I_{d_1}(h_1^{(n)}), \ldots, I_{d_m}(h_m^{(n)}))$, $n \geq 1$, *is such that* $\lim_{n\to\infty} E[F_i^{(n)} F_j^{(n)}] = V(i, j)$, $1 \leq i, j \leq m$. *Then, as* $n \to \infty$, *the following four assertions* (i)–(iv) *are equivalent*: (i) *For every* $1 \leq i \leq m$, $F_i^{(n)}$ *converges in distribution to a centered Gaussian random variable with variance* $V(i, i)$; (ii) *for every* $1 \leq i \leq m$, $E[(F_i^{(n)})^4] \to 3V(i, i)^2$; (iii) *for every* $1 \leq i \leq m$ *and every* $1 \leq r \leq d_i - 1$, $\|h_i^{(n)} \otimes_r h_i^{(n)}\|_{\mathfrak{H}^{\otimes(2d_i-2r)}} \to 0$; (iv) *the vector* $F^{(n)}$ *converges in distribution to the* $d$*-dimensional Gaussian vector* $\mathscr{N}_m(0, V)$.

PROOF OF THEOREM 7.5. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) only consist in a reformulation of the previous Theorem 7.6, by taking into account the first identity in Lemma 3.4 and the fact that (since we suppose that the sequence $E[Q^j(n, \mathbf{G})^2]$ of variances is bounded, so that an hypercontractivity argument can be applied), if Point (1) is verified, then $\lim_{n\to\infty} E[F_i^{(n)} F_j^{(n)}] = V(i, j)$ for all $1 \leq i, j \leq m$. On the other hand, it is completely obvious that (4) implies (1), since $\mathbf{G}$ is a particular case of such an $\mathbf{X}$. So, it remains to prove the implication (1), (2), (3) $\Rightarrow$ (4). Let $Z_V = (Z_V^1, \ldots, Z_V^m) \sim \mathscr{N}_m(0, V)$. We have

$$\sup_{z\in\mathbb{R}^m} |P[Q^1(n, \mathbf{X}) \leq z_1, \ldots, Q^m(n, \mathbf{X}) \leq z_m] - P[Z_V^1 \leq z_1, \ldots, Z_V^m \leq z_m]| \leq \delta_n^{(a)} + \delta_n^{(b)}$$

with

$$\begin{aligned}
\delta_n^{(a)} &= \sup_{z\in\mathbb{R}^m} |P[Q^1(n, \mathbf{X}) \leq z_1, \ldots, Q^m(n, \mathbf{X}) \leq z_m] \\
&\qquad - P[Q^1(n, \mathbf{G}) \leq z_1, \ldots, Q^m(n, \mathbf{G}) \leq z_m]|, \\
\delta_n^{(b)} &= \sup_{z\in\mathbb{R}^m} |P[Q^1(n, \mathbf{G}) \leq z_1, \ldots, Q^m(n, \mathbf{G}) \leq z_m] \\
&\qquad - P[Z_V^1 \leq z_1, \ldots, Z_V^m \leq z_m]|.
\end{aligned}$$

By assumption (3), we have that $\Delta_{ij} \to 0$ for all $i, j = 1, \dots, m$ [with $\Delta_{ij}$ defined by (7.1)]. Hence, Remark 7.4 (Point 2) implies that $\delta_n^{(b)} \to 0$. By assumption (3) (for $r = d_j - 1$) and (1.9)–(1.10), we get that $\max_{1 \le i \le N_n^{(j)}} \mathrm{Inf}_i(f_n^{(j)}) \to 0$ as $n \to \infty$ for all $j = 1, \dots, m$. Hence, Corollary 7.3 implies that $\delta_n^{(a)} \to 0$, which completes the proof. $\square$

## 8. Some proofs based on Malliavin calculus and Stein's method.

8.1. *The language of Malliavin calculus.* Let $\mathbf{G} = \{G_i : i \ge 1\}$ be an i.i.d. sequence of Gaussian random variables with zero mean and unit variance. In what follows, we will systematically use the definitions and notation introduced in Section 2. In particular, we shall encode the structure of random variables belonging to some Wiener chaos by means of increasing (tensor) powers of a fixed real separable Hilbert space $\mathfrak{H}$. We recall that the first Wiener chaos of $\mathbf{G}$ is the $L^2$-closed Hilbert space of random variables of the type $I_1(h)$, where $h \in \mathfrak{H}$. We shall denote by $L^2(\mathbf{G})$ the space of all $\mathbb{R}$-valued random elements $F$ that are measurable with respect to $\sigma\{\mathbf{G}\}$ and verify $E[F^2] < \infty$. Also, $L^2(\Omega; \mathfrak{H})$ denotes the space of all $\mathfrak{H}$-valued random elements $u$, that are measurable with respect to $\sigma\{\mathbf{G}\}$ and verify the relation $E[\|u\|_{\mathfrak{H}}^2] < \infty$. For the rest of this section, we shall use standard notation and results from Malliavin calculus: the reader is referred to [18] for a detailed presentation of these notions. In particular, $D^m$ denotes the $m$th Malliavin derivative operator, whose domain is denoted by $\mathbb{D}^{m,2}$ (we also write $D^1 = D$). An important property of $D$ is that it satisfies the following *chain rule*: if $g : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and has bounded partial derivatives, and if $(F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $g(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$ and $Dg(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(F_1, \dots, F_n) DF_i$. One can also show that the chain rule continues to hold when $(F_1, \dots, F_n)$ is a vector of multiple integrals (of possibly different orders) and $g$ is a polynomial in $n$ variables. We denote by $\delta$ the adjoint of the operator $D$, also called the *divergence operator*. If a random element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of $\delta$, noted $\mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship $E(F\delta(u)) = E\langle DF, u\rangle_{\mathfrak{H}}$, which holds for every $F \in \mathbb{D}^{1,2}$. As shown in [12], if $F = I_d(h)$, with $h \in \mathfrak{H}^{\odot d}$, then one can deduce by integrating by parts (and by an appropriate use of *Ornstein–Uhlenbeck operators*) that, for every $G \in \mathbb{D}^{1,2}$ and every continuously differentiable $g : \mathbb{R} \to \mathbb{R}$ with a bounded derivative, the following important relations hold:

$$
\begin{aligned}
&E[g(G)F] = \frac{1}{d} E[g'(G)\langle DG, DF\rangle_{\mathfrak{H}}] \quad \text{and} \\
&E[GF] = \frac{1}{d} E[\langle DG, DF\rangle_{\mathfrak{H}}].
\end{aligned} \tag{8.1}
$$

Let $h \in \mathfrak{H}^{\odot d}$ with $d \ge 2$, and let $s \ge 0$ be an integer. The following identity is obtained by taking $F = I_d(h)$ and $G = F^{s+1}$ in the second formula of (8.1), and then by applying the chain rule:

$$E[I_d(h)^{s+2}] = \frac{s+1}{d} E[I_d(h)^s \|DI_d(h)\|_{\mathfrak{H}}^2]. \tag{8.2}$$

8.2. *Relations following from Stein's method.* Originally introduced in [29, 30], *Stein's method* can be described as a collection of probabilistic techniques, allowing to compute explicit bounds on the distance between the laws of random variables by means of differential operators. The reader is referred to [25], and the references therein, for an introduction to these techniques. The following statement contains four bounds which can be obtained by means of a combination of Malliavin calculus and Stein's method. Points 1, 2 and 4 have been proved in [12], whereas the content of Point 3 is new. Our proof of such a bound gives an explicit example of the interaction between Stein's method and Malliavin calculus. We also introduce the following notation: for every $F = I_d(h)$, we set $T_0(F) = \sqrt{\mathrm{Var}(\frac{1}{d}\|DF\|_{\mathfrak{H}}^2)}$.

PROPOSITION 8.1. *Consider $F = I_d(h)$ with $d \geq 1$ and $h \in \mathfrak{H}^{\odot d}$, and let $Z$ and $Z_\nu$ have respectively a $\mathscr{N}(0,1)$ and a $\chi^2(\nu)$ distribution* ($\nu \geq 1$). *We have the following*:

1. *If $E(F^2) = 1$, then $d_{\mathrm{TV}}(F, Z) \leq 2T_0(F)$, $d_{\mathrm{W}}(F, Z) \leq T_0(F)$ and, for every thrice differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\|\varphi'''\| < \infty$, $|E[\varphi(F)] - E[\varphi(Z)]| \leq C_* \times T_0(F)$, where $C_*$ is given in* (3.1).
2. *If $E(F^2) = 2\nu$, then*

$$d_{\mathrm{BW}}(F, Z_\nu) \leq \max\left\{\sqrt{\frac{2\pi}{\nu}}, \frac{1}{\nu} + \frac{2}{\nu^2}\right\} \sqrt{E\left[\left(2\nu + 2F - \frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right)^2\right]}.$$

PROOF. Point 2 is proved in [12], Theorem 3.11. Point 1 is proved in [12], Theorem 3.1, except the bound for $|E[\varphi(F)] - E[\varphi(Z)]|$. To prove it, fix $\varphi$ as in the statement, and consider the *Stein equation* $f'(x) - xf(x) = \varphi(x) - E[\varphi(Z)]$, $x \in \mathbb{R}$. It is easily seen that a solution is given by $f(x) = f_\varphi(x) = e^{x^2/2} \int_{-\infty}^x (\varphi(y) - E[\varphi(Z)]) e^{-y^2/2}\, dy$. Set $K_* = C_* \times [4\sqrt{2}(1 + 5^{3d/2})]^{-1}$ with $C_*$ given by (3.1). According to the forthcoming Lemma 8.2, we have $|f_\varphi'(x)| \leq K_*(1 + |x| + |x|^2 + |x|^3)$. Now use (8.1) with $g = f_\varphi$ and $G = F$, as well as a standard approximation argument to take into account that $f_\varphi'$ is not necessarily bounded, in order to write

$$\begin{aligned}
&|E[\varphi(F)] - E[\varphi(Z)]| \\
&\quad = |E[f_\varphi'(F) - F f_\varphi(F)]| \\
&\quad = \left|E\left[f_\varphi'(F)\left(1 - \frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\le K_* E\left[(1+|F|+|F|^2+|F|^3)\left|1-\frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right|\right]\\
&\le 4K_* E\left[(1+|F|^3)\left|1-\frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right|\right].
\end{aligned}
$$

By applying Cauchy–Schwarz, by using $E[(1+|F|^3)^2] \le 2(1+E[F^6])$, and finally by exploiting Proposition 2.6, one infers the desired conclusion:

$$
\begin{aligned}
4K_* E\left[(1+|F|^3)\left|1-\frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right|\right] &\le C_* \sqrt{E\left[\left(1-\frac{1}{d}\|DF\|_{\mathfrak{H}}^2\right)^2\right]}\\
&= C_* T_0(F). \qquad \square
\end{aligned}
$$

LEMMA 8.2. *The function $f_\varphi$ verifies $|f'_\varphi(x)| \le K_*(1+|x|+|x|^2+|x|^3)$.*

PROOF. We want to bound the quantity $|f'_\varphi(x)|$, where $\varphi$ is such that $\varphi(x) = \varphi(0) + \varphi'(0)x + \varphi''(0)x^2/2 + R(x)$, with $|R(x)| \le \|\varphi'''\|_\infty |x|^3/6$. Let $Z \sim \mathscr{N}(0,1)$. We have $f'_\varphi(x) = A(x) + B(x)$, with $A(x) := \varphi(x) - E[\varphi(Z)]$ and $B(x) := x f_\varphi(x)$. It will become clear later on that our bounds on $|f'_\varphi(x)|$ do not depend on the sign of $x$, so that in what follows we will only focus on the case $x > 0$. Due to the assumptions on $\varphi$, we have that $A(x) = \varphi'(0)x + \frac{1}{2}\varphi''(0)x^2 + R(x) + C := ax + bx^2 + R(x) + C$, where $-C = \frac{\varphi''(0)}{2} + E[R(Z)]$ [note that the term $\varphi(0)$ simplifies]. Also, by using $E|Z|^3 = \frac{2\sqrt{2}}{\sqrt{\pi}}$ and $E|Z| = \frac{\sqrt{2}}{\sqrt{\pi}}$, we obtain $|C| \le \frac{|\varphi''(0)|}{2} + \frac{\|\varphi'''\|_\infty}{3}\frac{\sqrt{2}}{\sqrt{\pi}} := C'$ and (recall that $x > 0$) $|A(x)| \le |\varphi'(0)|x + \frac{1}{2}|\varphi''(0)|x^2 + \frac{1}{6}\|\varphi'''\|_\infty x^3 + C' = |a|x + |b|x^2 + \gamma x^3 + C'$ with $\gamma := \frac{1}{6}\|\varphi'''\|_\infty$. On the other hand, since $E[A(Z)] = 0$ by construction, $|B(x)| = xe^{x^2/2}|\int_x^{+\infty} A(y)e^{-y^2/2}\,dy| \le xe^{x^2/2}\int_x^{+\infty}[C' + |a|y + |b|y^2 + \gamma y^3]e^{-y^2/2}\,dy := Y_1(x) + Y_2(x) + Y_3(x) + Y_4(x)$. We now evaluate the four terms $Y_i$ separately (observe that each of them is positive):

$$
\begin{aligned}
Y_1(x) &= C' x e^{x^2/2}\int_x^{+\infty} e^{-y^2/2}\,dy \le C' e^{x^2/2}\int_x^{+\infty} y e^{-y^2/2}\,dy = C';\\
Y_2(x) &= x e^{x^2/2}\int_x^{+\infty} |a| y e^{-y^2/2}\,dy = |a|x;\\
Y_3(x) &= x e^{x^2/2}\int_x^{+\infty} |b| y^2 e^{-y^2/2}\,dy = |b|\left(x^2 + x e^{x^2/2}\int_x^{+\infty} e^{-y^2/2}\,dy\right)\\
&\le |b|(x^2+1);\\
Y_4(x) &= x e^{x^2/2}\int_x^{+\infty} \gamma y^3 e^{-y^2/2}\,dy = \gamma x(x^2+2) = \gamma x^3 + 2\gamma x.
\end{aligned}
$$

By combining the above bounds with $|f'_\varphi(x)| \le |A(x)| + |B(x)|$, one infers that

$$\begin{aligned}|f'_\varphi(x)| &\le 2C' + |b| + x(2|a| + 2\gamma) + x^2|b| + x^3 2\gamma \\ &\le \max\{2C' + |b|; 2|a| + 2\gamma; |b|; 2\gamma\} \times (1 + x + x^2 + x^3) \\ &= \max\{2C' + |b|; 2|a| + 2\gamma\} \times (1 + x + x^2 + x^3),\end{aligned}$$

which yields the desired conclusion. □

8.3. *Proof of Theorem* 3.1. Let $F = I_d(h)$, $h \in \mathfrak{H}^{\odot d}$. In view of Proposition 8.1, it is sufficient to show that $T_0(F) = T_1(F) \le T_2(F)$. Relation (3.42) in [12] yields that

$$\frac{1}{d}\|DF\|^2_{\mathfrak{H}} = E(F^2) + d\sum_{r=1}^{d-1}(r-1)!\binom{d-1}{r-1}^2 I_{2d-2r}(h\,\widetilde{\otimes}_r\,h), \tag{8.3}$$

which, by taking the orthogonality of multiple integrals of different orders into account, yields $\mathrm{Var}(\frac{1}{d}\|DF\|^2_{\mathfrak{H}}) = d^2\sum_{r=1}^{d-1}(r-1)!^2\binom{d-1}{r-1}^4(2d-2r)!\|h\,\widetilde{\otimes}_r\,h\|^2_{\mathfrak{H}^{\otimes 2(d-r)}}$, and so $T_0(F) = T_1(F)$. From Proposition 2.5, we get $F^2 = \sum_{r=0}^{d} r!\binom{d}{r}^2 I_{2d-2r}(h\,\widetilde{\otimes}_r\,h)$. To conclude the proof, we use (8.2) with $s = 2$, combined with the previous identities, as well as the assumption that $E(F^2) = 1$, to get that

$$\begin{aligned}E[F^4] - 3 &= \frac{3}{d}E(F^2\|DF\|^2_{\mathfrak{H}}) - 3(d!\|h\|^2_{\mathfrak{H}^{\otimes d}})^2 \\ &= 3d\sum_{r=1}^{d-1} r!(r-1)!\binom{d}{r}^2\binom{d-1}{r-1}^2(2d-2r)!\|h\,\widetilde{\otimes}_r\,h\|^2_{\mathfrak{H}^{\otimes 2(d-r)}}.\end{aligned}$$

Hence, $\mathrm{Var}(\frac{1}{d}\|DF\|^2_{\mathfrak{H}}) \le \frac{d-1}{3d}[E(F^4) - 3]$, thus yielding $T_1(F) \le T_2(F)$.

8.4. *Proof of Theorem* 3.6. Let $F = I_d(h)$, $h \in \mathfrak{H}^{\odot d}$. In view of Proposition 8.1 and since $L^{-1}F = -\frac{1}{d}F$, it is sufficient to show that

$$\sqrt{E\left[\left(2\nu + 2F - \frac{1}{d}\|DF\|^2_{\mathfrak{H}}\right)^2\right]} = T_3(F) \le T_4(F).$$

By taking into account the orthogonality of multiple integrals of different orders, relation (8.3) yields

$$\begin{aligned}&E\left[\left(2\nu + 2F - \frac{1}{d}\|DF\|^2_{\mathfrak{H}}\right)^2\right] \\ &\qquad = 4d!\left\|h - \frac{d!^2}{4(d/2)!^3}h\,\widetilde{\otimes}_{d/2}\,h\right\|^2_{\mathfrak{H}^{\otimes d}}\end{aligned}$$

$$+ d^2 \sum_{\substack{r=1,\dots,d-1 \\ r\neq d/2}} (r-1)!^2 \binom{d-1}{r-1}^4 (2d-2r)! \|h \widetilde{\otimes}_r h\|^2_{\mathfrak{H}^{\otimes(2d-2r)}},$$

and, consequently, $T_3(F) = \sqrt{E[(2\nu + 2F - \frac{1}{d}\|DF\|^2_{\mathfrak{H}})^2]}$. On the other hand, by combining (8.2) (for $s = 1$ and $s = 2$) with $F^2 = \sum_{r=0}^{d} r! \binom{d}{r}^2 I_{2d-2r}(h \widetilde{\otimes}_r h)$ [see the proof of Theorem 3.1], we get, still by taking into account the orthogonality of multiple integrals of different orders,

$$\begin{aligned}
&E[F^4] - 12E[F^3] \\
&\quad = 12\nu^2 - 48\nu + 24d! \left\| h - \frac{d!^2}{4(d/2)!^3} h \widetilde{\otimes}_{d/2} h \right\|^2_{\mathfrak{H}^{\otimes d}} \\
&\qquad + 3d \sum_{\substack{r=1,\dots,d-1 \\ r\neq d/2}} r!(r-1)! \binom{d}{r}^2 \binom{d-1}{r-1}^2 (2d-2r)! \|h \widetilde{\otimes}_r h\|^2_{\mathfrak{H}^{\otimes(2d-2r)}}.
\end{aligned}$$

It is now immediate to deduce that $T_3(F) \le T_4(F)$.

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