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STEIN'S METHOD AND NORMAL APPROXIMATION OF POISSON FUNCTIONALS

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We combine Stein's method with a version of Malliavin calculus on the Poisson space. As a result, we obtain explicit Berry–Esséen bounds in Central limit theorems (CLTs) involving multiple Wiener–Itô integrals with respect to a general Poisson measure. We provide several applications to CLTs related to Ornstein–Uhlenbeck Lévy processes.

1. Introduction. In a recent series of papers, Nourdin and Peccati [13, 14], Nourdin, Peccati and Réveillac [15] and Nourdin, Peccati and Reinert [16] have shown that one can effectively combine Malliavin calculus on a Gaussian space (see, e.g., [17]) and Stein's method (see, e.g., [3, 28]) in order to obtain explicit bounds for the normal and nonnormal approximation of smooth functionals of Gaussian fields.

The aim of the present paper is to extend the analysis initiated in [13] to the framework of the normal approximation (in the Wasserstein distance) of regular functionals of Poisson measures defined on abstract Borel spaces. As in the Gaussian case, the main ingredients of our analysis are the following:

1. A set of *Stein differential equations*, relating the normal approximation in the Wasserstein distance to first-order differential operators.
2. A (Hilbert space-valued) *derivative operator* D, acting on real-valued square-integrable random variables.
3. An *integration by parts formula*, involving the adjoint operator of D.
4. A "pathwise representation" of D which, in the Poisson case, involves standard *difference operators*.

As a by-product of our analysis, we obtain substantial generalizations of the Central limit theorems (CLTs) for functionals of Poisson measures (for instance, for sequences of single and double Wiener–Itô integrals) recently proved in [22–24] (see also [4, 21] for applications of these results to Bayesian nonparametric statistics). In particular, one of the main results of the present paper (see Theorem 5.1

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below) is a CLT for sequences of multiple Wiener–Itô integrals of arbitrary (fixed) order with respect to a general Poisson measure. Our conditions are expressed in terms of *contraction operators*, and can be seen as a Poisson counterpart to the CLTs on Wiener space proved by Nualart and Peccati [19] and Nualart and Ortiz-Latorre [18]. The reader is referred to Decreusefond, Joulin and Savy [5] for other applications of Stein-type techniques and Malliavin operators to the assessment of Rubinstein distances on configuration spaces.

The remainder of the paper is organized as follows. In Section 2, we discuss some preliminaries, involving multiplication formulae, Malliavin operators, limit theorems and Stein's method. In Section 3, we derive a general inequality concerning the Gaussian approximation of regular functionals of Poisson measures. Section 4 is devoted to upper bounds for the Wasserstein distance, and Section 5 to CLTs for multiple Wiener–Itô integrals of arbitrary order. Section 6 deals with sums of a single and a double integral. In Section 7, we apply our results to nonlinear functionals of Ornstein–Uhlenbeck Lévy processes.

2. Preliminaries.

2.1. *Poisson measures.* Throughout the paper, $(Z, \mathcal{Z}, \mu)$ indicates a measure space such that Z is a Borel space and μ is a σ-finite nonatomic Borel measure. We define the class $\mathcal{Z}_\mu$ as $\mathcal{Z}_\mu = \{B \in \mathcal{Z} : \mu(B) < \infty\}$. The symbol $\widehat{N} = \{\widehat{N}(B) : B \in \mathcal{Z}_\mu\}$ indicates a *compensated Poisson random measure* on $(Z, \mathcal{Z})$ with control μ. This means that $\widehat{N}$ is a collection of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of $\mathcal{Z}_\mu$, and such that: (i) for every $B, C \in \mathcal{Z}_\mu$ such that $B \cap C = \varnothing$, $\widehat{N}(B)$ and $\widehat{N}(C)$ are independent, (ii) for every $B \in \mathcal{Z}_\mu$,

$$\widehat{N}(B) \stackrel{\text{law}}{=} \mathfrak{P}(B) - \mu(B),$$

where $\mathfrak{P}(B)$ is a Poisson random variable with parameter $\mu(B)$. Note that properties (i) and (ii) imply, in particular, that $\widehat{N}$ is an *independently scattered* (or *completely random*) measure (see, e.g., [24]).

REMARK 2.1. According to [27], Section 1, and due to the assumptions on the space $(Z, \mathcal{Z}, \mu)$, it is always possible to define $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that

$$\Omega = \left\{ \omega = \sum_{j=0}^{n} \delta_{z_j}, n \in \mathbb{N} \cup \{\infty\}, z_j \in Z \right\}, \tag{2.1}$$

where δ_z denotes the Dirac mass at z, and $\widehat{N}$ is the *compensated canonical mapping* given by

$$\widehat{N}(B)(\omega) = \omega(B) - \mu(B), \qquad B \in \mathcal{Z}_\mu. \tag{2.2}$$

Also, in this case one can take $\mathcal{F}$ to be the $\mathbb{P}$-completion of the σ-field generated by the mapping (2.2). Note that, under these assumptions, one has that

$$\begin{aligned} &\mathbb{P}\{\omega : \omega(B) < \infty, \forall B \text{ s.t. } \mu(B) < \infty\} = 1 \quad \text{and} \\ &\mathbb{P}\big\{\omega : \exists z \text{ s.t. } \omega(\{z\}) > 1\big\} = 0. \end{aligned} \tag{2.3}$$

REMARK 2.2. From now on, and for the rest of the paper, the hypotheses (2.2) and (2.3), on the structure of Ω and $\widehat{N}$, are implicitly satisfied. In particular, $\mathcal{F}$ is the $\mathbb{P}$-completion of the σ-field generated by the mapping in (2.2).

For every deterministic function $h \in L^2(Z, \mathcal{Z}, \mu) = L^2(\mu)$, we write $\widehat{N}(h) = \int_Z h(z)\widehat{N}(dz)$ to indicate the Wiener–Itô integral of h with respect to $\widehat{N}$ (see, e.g., [9] or [29]). We recall that, for every $h \in L^2(\mu)$, the random variable $\widehat{N}(h)$ has an infinitely divisible law, with Lévy–Khinchine exponent (see again [29]) given by

$$\psi(h, \lambda) = \log \mathbb{E}\big[e^{i\lambda\widehat{N}(h)}\big] = \int_Z \big[e^{i\lambda h(z)} - 1 - i\lambda h(z)\big]\mu(dz), \qquad \lambda \in \mathbb{R}. \tag{2.4}$$

Recall also the isometric relation: for every $g, h \in L^2(\mu)$, $\mathbb{E}[\widehat{N}(g)\widehat{N}(h)] = \int_Z h(z)g(z)\mu(dz)$.

Fix $n \geq 2$. We denote by $L^2(\mu^n)$ the space of real-valued functions on Z^n that are square-integrable with respect to μ^n, and we write $L_s^2(\mu^n)$ to indicate the subspace of $L^2(\mu^n)$ composed of symmetric functions. For every $f \in L_s^2(\mu^n)$, we denote by $I_n(f)$ the *multiple Wiener–Itô integral* of order n, of f with respect to $\widehat{N}$. Observe that, for every $m, n \geq 2$, $f \in L_s^2(\mu^n)$ and $g \in L_s^2(\mu^m)$, one has the isometric formula (see, e.g., [33]):

$$\mathbb{E}[I_n(f)I_m(g)] = n!\langle f, g\rangle_{L^2(\mu^n)}\mathbf{1}_{(n=m)}. \tag{2.5}$$

Now fix $n \geq 2$ and $f \in L^2(\mu^n)$ (not necessarily symmetric) and denote by $\tilde{f}$ the canonical symmetrization of f: for future use we stress that, as a consequence of Jensen's inequality,

$$\|\tilde{f}\|_{L^2(\mu^n)} \leq \|f\|_{L^2(\mu^n)}. \tag{2.6}$$

The Hilbert space of random variables of the type $I_n(f)$, where $n \geq 1$ and $f \in L_s^2(\mu^n)$ is called the nth *Wiener chaos* associated with $\widehat{N}$. We also use the following conventional notation: $I_1(f) = \widehat{N}(f)$, $f \in L^2(\mu)$; $I_n(f) = I_n(\tilde{f})$, $f \in L^2(\mu^n)$, $n \geq 2$ [this convention extends the definition of $I_n(f)$ to nonsymmetric functions f]; $I_0(c) = c$, $c \in \mathbb{R}$. The following proposition, whose content is known as the *chaotic representation property* of $\widehat{N}$, is one of the crucial results used in this paper. (See, e.g., [20] or [34] for a proof.)

PROPOSITION 2.3 (Chaotic decomposition). *Every random variable* $F \in L^2(\mathcal{F}, \mathbb{P}) = L^2(\mathbb{P})$ *admits a* (*unique*) *chaotic decomposition of the type*

$$F = \mathbb{E}(F) + \sum_{n \geq 1}^{\infty} I_n(f_n), \tag{2.7}$$

where the series converges in L^2 *and, for each* $n \geq 1$, *the kernel* f_n *is an element of* $L_s^2(\mu^n)$.

2.2. *Contractions, stars and products*. We now recall a useful version of the *multiplication formula* for multiple Poisson integrals. To this end, we define, for $q, p \geq 1$, $f \in L_s^2(\mu^p)$, $g \in L_s^2(\mu^q)$, $r = 0, \ldots, q \wedge p$ and $l = 1, \ldots, r$, the (contraction) kernel on $Z^{p+q-r-l}$, which reduces the number of variables in the product fg from $p+q$ to $p+q-r-l$ as follows: r variables are identified and, among these, l are integrated out. This contraction kernel is formally defined as follows:

$$\begin{aligned} &f \star_r^l g(\gamma_1, \ldots, \gamma_{r-l}, t_1, \ldots, t_{p-r}, s_1, \ldots, s_{q-r}) \\ &\quad = \int_{Z^l} f(z_1, \ldots, z_l, \gamma_1, \ldots, \gamma_{r-l}, t_1, \ldots, t_{p-r}) \\ &\qquad\quad \times g(z_1, \ldots, z_l, \gamma_1, \ldots, \gamma_{r-l}, s_1, \ldots, s_{q-r}) \mu^l(dz_1 \cdots dz_l) \end{aligned} \tag{2.8}$$

and, for $l = 0$,

$$\begin{aligned} &f \star_r^0 g(\gamma_1, \ldots, \gamma_r, t_1, \ldots, t_{p-r}, s_1, \ldots, s_{q-r}) \\ &\quad = f(\gamma_1, \ldots, \gamma_r, t_1, \ldots, t_{p-r}) g(\gamma_1, \ldots, \gamma_r, s_1, \ldots, s_{q-r}), \end{aligned} \tag{2.9}$$

so that $f \star_0^0 g(t_1, \ldots, t_p, s_1, \ldots, s_q) = f(t_1, \ldots, t_p) g(s_1, \ldots, s_q)$. For example, if $p = q = 2$,

$$\begin{aligned} f \star_1^0 g(\gamma, t, s) &= f(\gamma, t) g(\gamma, s), \\ f \star_1^1 g(t, s) &= \int_Z f(z, t) g(z, s) \mu(dz), \end{aligned} \tag{2.10}$$

$$f \star_2^1 g(\gamma) = \int_Z f(z, \gamma) g(z, \gamma) \mu(dz), \tag{2.11}$$

$$f \star_2^2 g = \int_Z \int_Z f(z_1, z_2) g(z_1, z_2) \mu(dz_1) \mu(dz_2). \tag{2.12}$$

The quite suggestive "star-type" notation is standard, and has been first used by Kabanov in [8] (but see also Surgailis [33]). Plainly, for some choice of f, g, r, l the contraction $f \star_r^l g$ may not exist, in the sense that its definition involves integrals that are not well defined. On the positive side, the contractions of the following three types are well defined (although possibly infinite) for every $q \geq 2$ and every kernel $f \in L_s^2(\mu^q)$:

(a) $f \star_r^0 f(z_1, \dots, z_{2q-r})$, where $r = 0, \dots, q$, as obtained from (2.9), by setting $g = f$;
(b) $f \star_q^l f(z_1, \dots, z_{q-l}) = \int_{Z^l} f^2(z_1, \dots, z_{q-l}, \cdot)\, d\mu^l$, for every $l = 1, \dots, q$;
(c) $f \star_r^r f$, for $r = 1, \dots, q-1$.

In particular, a contraction of the type $f \star_q^l f$, where $l = 1, \dots, q-1$ may equal $+\infty$ at some point $(z_1, \dots, z_{q-l})$. The following (elementary) statement ensures that any kernel of the type $f \star_r^r g$ is square-integrable.

LEMMA 2.4. *Let $p, q \geq 1$, and let $f \in L_s^2(\mu^q)$ and $g \in L_s^2(\mu^p)$. Fix $r = 0, \dots, q \wedge p$. Then, $f \star_r^r g \in L^2(\mu^{p+q-2r})$.*

PROOF. Just use equation (2.8) in the case $l = r$, and deduce the conclusion by a standard use of the Cauchy–Schwarz inequality. □

We also record the following identity (which is easily verified by a standard Fubini argument), valid for every $q \geq 1$, every $p = 1, \dots, q$:

$$\int_{Z^{2q-p}} (f \star_p^0 f)^2\, d\mu^{2q-p} = \int_{Z^p} (f \star_q^{q-p} f)^2\, d\mu^p \tag{2.13}$$

for every $f \in L_s^2(\mu^q)$. The forthcoming product formula for two Poisson multiple integrals is proved, for example, in [8] and [33].

PROPOSITION 2.5 (Product formula). *Let $f \in L_s^2(\mu^p)$ and $g \in L_s^2(\mu^q)$, $p, q \geq 1$, and suppose moreover that $f \star_r^l g \in L^2(\mu^{p+q-r-l})$ for every $r = 0, \dots, p \wedge q$ and $l = 1, \dots, r$ such that $l \neq r$. Then*

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} \sum_{l=0}^{r} \binom{r}{l} I_{q+p-r-l}(\widetilde{f \star_r^l g}), \tag{2.14}$$

where the tilde "$\sim$" stands for symmetrization, that is,

$$\widetilde{f \star_r^l g}(x_1, \dots, x_{q+p-r-l}) = \frac{1}{(q+p-r-l)!} \sum_\sigma f \star_r^l g\big(x_{\sigma(1)}, \dots, x_{\sigma(q+p-r-l)}\big),$$

where σ runs over all $(q+p-r-l)!$ permutations of the set $\{1, \dots, q+p-r-l\}$.

REMARK 2.6 (Multiple integrals and Lévy processes). In the multiple Wiener–Itô integrals introduced in this section, the integrators are compensated Poisson measures of the type $\widehat{N}(dz)$, defined on some abstract Borel space. It is well known that one can also build similar objects in the framework of Lévy processes indexed by the real line. Suppose indeed that we are given a cadlag Lévy process $X = \{X_t, t \geq 0\}$ (that is, X has stationary and independent increments, X is continuous in probability and $X_0 = 0$), defined on a complete probability space

$(\Omega, \mathcal{F}, P)$, and with Lévy triplet (γ, σ^2, ν) where $\gamma \in \mathbb{R}, \sigma \geq 0$ and ν is a Lévy measure on $\mathbb{R}$ (see, e.g., [29]). The process X admits a Lévy–Itô representation

$$
\begin{aligned}
X_t &= \gamma t + \sigma W_t + \iint_{(0,t]\times\{|x|>1\}} x\, dN(s,x) \\
&\quad + \lim_{\varepsilon \downarrow 0} \iint_{(0,t]\times\{\varepsilon<|x|\leq 1\}} x\, d\widetilde{N}(s,x),
\end{aligned}
\tag{2.15}
$$

where: (i) $\{W_t, t \geq 0\}$ is a standard Brownian motion, (ii)

$$
N(B) = \#\{t : (t, \Delta X_t) \in B\}, \qquad B \in \mathcal{B}\big((0,\infty) \times \mathbb{R}_0\big),
$$

is the jump measure associated with X (where $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $\Delta X_t = X_t - X_{t-}$ and $\#A$ denotes the cardinal of a set A) and (iii)

$$
d\widetilde{N}(t,x) = dN(t,x) - dt\, d\nu(x)
$$

is the compensated jump measure. The convergence in (2.15) is a.s. uniform (in t) on every bounded interval. Following Itô [7], the process X can be extended to an independently scattered random measure M on $(\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}))$ as follows. First, consider the measure $d\mu^*(t,x) = \sigma^2\, dt\, d\delta_0(x) + x^2\, dt\, d\nu(x)$, where δ_0 is the Dirac measure at point 0, and dt is the Lebesgue measure on $\mathbb{R}$. This means that, for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$
\mu^*(E) = \sigma^2 \int_{E(0)} dt + \iint_{E'} x^2\, dt\, d\nu(x),
$$

where $E(0) = \{t \in \mathbb{R}_+ : (t,0) \in E\}$ and $E' = E - \{(t,0) \in E\}$; this measure is continuous (see Itô [7], page 256). Now, for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu^*(E) < \infty$, define

$$
M(E) = \sigma \int_{E(0)} dW_t + \lim_n \iint_{\{(t,x)\in E : 1/n<|x|<n\}} x\, d\widetilde{N}(t,x)
$$

[with convergence in $L^2(\Omega)$], that is, M is a centered independently scattered random measure such that

$$
E[M(E_1)M(E_2)] = \mu^*(E_1 \cap E_2)
$$

for $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $\mu^*(E_1) < \infty$ and $\mu^*(E_2) < \infty$. Now write $L_n^2 = L^2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mu^{*\otimes n})$. For every $f \in L_n^2$, one can now define a multiple stochastic integral $I_n(f)$, with integrator M, by using the same procedure as in the Poisson case (the first construction of this type is due to Itô—see [7]). Since M is defined on a product space, in this scenario, we can separate the time and the jump sizes. If the Lévy process has no Brownian components, then M has the representation $M(dz) = M(ds, dx) = x\widetilde{N}(ds, dx)$, that is, M is obtained by integrating a factor x with respect to the underlying compensated Poisson measure of intensity $dt\, \nu(dx)$. Therefore, in order to define multiple integrals of order

n with respect to M, one should consider kernels that are square-integrable with respects the measure $(x^2\,dt\,\nu(dx))^{\otimes n}$. The product formula of multiple integrals in this different context would be analogous to the one given in Proposition 2.5, but in the definitions of the contraction kernels one has to to take into account the supplementary factor x (see, e.g., [10] for more details on this point).

2.3. *Four Malliavin-type operators.* In this section, we introduce four operators of the Malliavin-type, that are involved in the estimates of the subsequent sections. Each of these operators is defined in terms of the chaotic expansions of the elements in its domain, thus implicitly exploiting the fact that the chaotic representation property (2.7) induces an isomorphism between $L^2(\mathbb{P})$ and the symmetric Fock space canonically associated with $L^2(\mu)$. The reader is referred to [20] or [30] for more details on the construction of these operators, as well as for further relations with chaotic expansions and Fock spaces. One crucial fact in our analysis is that the derivative operator D (to be formally introduced below) admits a neat characterization in terms of a usual difference operator (see the forthcoming Lemma 2.8).

For later reference, we recall that the space $L^2(\mathbb{P}; L^2(\mu)) \simeq L^2(\Omega \times Z, \mathcal{F} \otimes \mathcal{Z}, P \otimes \mu)$ is the space of the measurable random functions $u : \Omega \times Z \to \mathbb{R}$ such that

$$\mathbb{E}\left[\int_Z u_z^2 \mu(dz)\right] < \infty.$$

In what follows, given $f \in L_s^2(\mu^q)$ $(q \geq 2)$ and $z \in Z$, we write $f(z, \cdot)$ to indicate the function on Z^{q-1} given by $(z_1, \dots, z_{q-1}) \to f(z, z_1, \dots, z_{q-1})$.

(i) *The derivative operator D.* The derivative operator, denoted by D, transforms random variables into random functions. Formally, the domain of D, written $\operatorname{dom} D$, is the set of those random variables $F \in L^2(\mathbb{P})$ admitting a chaotic decomposition (2.7) such that

$$\sum_{n \geq 1} n n! \|f_n\|^2_{L^2(\mu^n)} < \infty. \tag{2.16}$$

If F verifies (2.16) (that is, if $F \in \operatorname{dom} D$), then the random function $z \mapsto D_z F$ is given by

$$D_z F = \sum_{n \geq 1} n I_{n-1}(f(z, \cdot)), \qquad z \in Z. \tag{2.17}$$

For instance, if $F = I_1(f)$, then $D_z F$ is the nonrandom function $z \mapsto f(z)$. If $F = I_2(f)$, then $D_z F$ is the random function

$$z \mapsto 2I_1(f(z, \cdot)). \tag{2.18}$$

By exploiting the isometric properties of multiple integrals, and thanks to (2.16), one sees immediately that $DF \in L^2(\mathbb{P}; L^2(\mu))$, for every $F \in \operatorname{dom} D$.

REMARK 2.7. Strictly speaking, the random function appearing in (2.17) provides a *version* of the Malliavin derivative of F. In particular, any $u \in L^2(\mathbb{P}; L^2(\mu))$ such that $u(\omega, z)$ equals the right-hand side of (2.17) for $d\mathbb{P} \times d\mu$-almost every (ω, z) can be used as a definition of DF.

Now recall that, in this paper, the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that Ω is the collection of discrete measures given by (2.1). Fix $z \in Z$; given a random variable $F : \Omega \to \mathbb{R}$, we define F_z to be the random variable obtained by adding the Dirac mass δ_z to the argument of F. Formally, for every $\omega \in \Omega$, we set

$$F_z(\omega) = F(\omega + \delta_z). \tag{2.19}$$

A version of the following result, whose content is one of the staples of our analysis, is proved, for example, in [20], Theorem 6.2 (see also [30], Proposition 5.1). One should also note that the proof given in [20] applies to the framework of a Radon control measure μ: however, the arguments extend immediately to the case of a Borel control measure considered in the present paper, and we do not reproduce them here (see, however, the remarks below). It relates F_z to $D_z F$ and provides a representation of D as a difference operator.

LEMMA 2.8. *For every* $F \in \operatorname{dom} D$,

$$D_z F = F_z - F, \qquad \text{a.e.-}\mu(dz). \tag{2.20}$$

REMARK 2.9.

1. In the Itô-type framework detailed in Remark 2.6, we could alternatively use the definition of the Malliavin derivative introduced in [30], that is,

$$D_{(t,x)} F = \frac{F_{(t,x)} - F}{x}, \qquad \text{a.e.-}x^2\, dt\, \nu(dx).$$

   See also [11].
2. When applied to the case $F = I_1(f)$, formula (2.20) is just a consequence of the straightforward relation

$$\begin{aligned} F_z - F &= \int_Z f(x)\big(\widehat{N}(dx) + \delta_z(dx)\big) - \int_Z f(x)\widehat{N}(dx) \\ &= \int_Z f(x)\delta_z(dx) = f(z). \end{aligned}$$

3. To have an intuition of the proof of (2.20) in the general case, consider for instance a random variable of the type $F = \widehat{N}(A) \times \widehat{N}(B)$, where the sets $A, B \in \mathcal{Z}_\mu$ are disjoint. Then one has that $F = I_2(f)$ where $f(x, y) = 2^{-1}\{\mathbf{1}_A(x)\mathbf{1}_B(y) + \mathbf{1}_A(y)\mathbf{1}_B(x)\}$. Using (2.18), we have that

$$D_z F = \widehat{N}(B)\mathbf{1}_A(z) + \widehat{N}(A)\mathbf{1}_B(z), \qquad z \in Z, \tag{2.21}$$

and

$$F_z - F = \{\widehat{N} + \delta_z\}(A) \times \{\widehat{N} + \delta_z\}(B) - \widehat{N}(A)\widehat{N}(B), \qquad z \in Z.$$

Now fix $z \in Z$. There are three possible cases: (i) $z \notin A$ and $z \notin B$, (ii) $z \in A$, and (iii) $z \in B$. If z is as in (i), then $F_z = F$. If z is as in (ii) [resp., as in (iii)], then

$$F_z - F = \{\widehat{N} + \delta_z\}(A) \times \widehat{N}(B) - \widehat{N}(A)\widehat{N}(B) = \widehat{N}(B)$$

[resp.,

$$F_z - F = \widehat{N}(A) \times \{\widehat{N} + \delta_z\}(B) - \widehat{N}(A)\widehat{N}(B) = \widehat{N}(A)].$$

As a consequence, one deduces that $F_z - F = \widehat{N}(B)\mathbf{1}_A(z) + \widehat{N}(A)\mathbf{1}_B(z)$, so that relation (2.20) is obtained from (2.21).

4. Observe also that Lemma 2.8 yields that, if $F, G \in \operatorname{dom} D$ are such that $FG \in \operatorname{dom} D$, then

$$D(FG) = FDG + GDF + DGDF, \tag{2.22}$$

(see [20], Lemma 6.1, for a detailed proof of this fact).

(ii) *The Skorohod integral* δ. Observe that, due to the chaotic representation property of $\widehat{N}$, every random function $u \in L^2(\mathbb{P}, L^2(\mu))$ admits a (unique) representation of the type

$$u_z = \sum_{n \geq 0} I_n(f_n(z, \cdot)), \qquad z \in Z, \tag{2.23}$$

where, for every z, the kernel $f_n(z, \cdot)$ is an element of $L_s^2(\mu^n)$. The domain of the Skorohod integral operator, denoted by $\operatorname{dom}\delta$, is defined as the collections of those $u \in L^2(\mathbb{P}, L^2(\mu))$ such that the chaotic expansion (2.23) verifies the condition

$$\sum_{n \geq 0} (n+1)! \|f_n\|^2_{L^2(\mu^{n+1})} < \infty. \tag{2.24}$$

If $u \in \operatorname{dom}\delta$, then the random variable $\delta(u)$ is defined as

$$\delta(u) = \sum_{n \geq 0} I_{n+1}(\tilde{f}_n), \tag{2.25}$$

where $\tilde{f}_n$ stands for the canonical symmetrization of f (as a function in $n+1$ variables). For instance, if $u(z) = f(z)$ is a deterministic element of $L^2(\mu)$, then $\delta(u) = I_1(f)$. If $u(z) = I_1(f(z, \cdot))$, with $f \in L_s^2(\mu^2)$, then $\delta(u) = I_2(f)$ (we stress that we have assumed f to be symmetric). The following classic result, proved, for example, in [20], provides a characterization of δ as the adjoint of the derivative D.

LEMMA 2.10 (Integration by parts formula). *For every* $G \in \operatorname{dom} D$ *and every* $u \in \operatorname{dom} \delta$, *one has that*

$$\mathbb{E}[G\delta(u)] = \mathbb{E}\big[\langle DG, u\rangle_{L^2(\mu)}\big], \tag{2.26}$$

where

$$\langle DG, u\rangle_{L^2(\mu)} = \int_Z D_z G \times u(z)\mu(dz).$$

(iii) *The Ornstein–Uhlenbeck generator* L. The domain of the Ornstein–Uhlenbeck generator (see [17], Chapter 1), written $\operatorname{dom} L$, is given by those $F \in L^2(\mathbb{P})$ such that their chaotic expansion (2.7) verifies

$$\sum_{n\geq 1} n^2 n! \|f_n\|^2_{L^2(\mu^n)} < \infty.$$

If $F \in \operatorname{dom} L$, then the random variable LF is given by

$$LF = -\sum_{n\geq 1} n I_n(f_n). \tag{2.27}$$

Note that $\mathbb{E}(LF) = 0$, by definition. The following result is a direct consequence of the definitions of D, δ and L.

LEMMA 2.11. *For every* $F \in \operatorname{dom} L$, *one has that* $F \in \operatorname{dom} D$ *and* $DF \in \operatorname{dom} \delta$. *Moreover*,

$$\delta DF = -LF. \tag{2.28}$$

PROOF. The first part of the statement is easily proved by applying the definitions of $\operatorname{dom} D$ and $\operatorname{dom} \delta$ given above. In view of Proposition 2.3, it is now enough to prove (2.28) for a random variable of the type $F = I_q(f)$, $q \geq 1$. In this case, one has immediately that $D_z F = q I_{q-1}(f(z,\cdot))$, so that $\delta DF = q I_q(f) = -LF$. $\square$

(iv) *The inverse of* L. The domain of L^{-1}, denoted by $L_0^2(\mathbb{P})$, is the space of *centered* random variables in $L^2(\mathbb{P})$. If $F \in L_0^2(\mathbb{P})$ and $F = \sum_{n\geq 1} I_n(f_n)$ (and thus is centered) then

$$L^{-1}F = -\sum_{n\geq 1} \frac{1}{n} I_n(f_n). \tag{2.29}$$

2.4. *Normal approximation in the Wasserstein distance, via Stein's method.* We shall now give a short account of Stein's method, as applied to normal approximations in the Wasserstein distance. We denote by Lip(1) the class of real-valued Lipschitz functions, from $\mathbb{R}$ to $\mathbb{R}$, with Lipschitz constant less or equal to one, that is, functions h that are absolutely continuous and satisfy the relation $\|h'\|_\infty \leq 1$. Given two real-valued random variables U and Y, the *Wasserstein distance* between the laws of U and Y, written $d_W(U, Y)$ is defined as

$$d_W(U, Y) = \sup_{h \in \mathrm{Lip}(1)} |\mathbb{E}[h(U)] - \mathbb{E}[h(Y)]|. \tag{2.30}$$

We recall that the topology induced by d_W on the class of probability measures over $\mathbb{R}$ is strictly stronger than the topology of weak convergence (see, e.g., [6]). We will denote by $\mathscr{N}(0, 1)$ the law of a centered standard Gaussian random variable.

Now let $X \sim \mathscr{N}(0, 1)$. Consider a real-valued function $h : \mathbb{R} \to \mathbb{R}$ such that the expectation $\mathbb{E}[h(X)]$ is well defined. The *Stein equation* associated with h and X is classically given by

$$h(x) - \mathbb{E}[h(X)] = f'(x) - xf(x), \qquad x \in \mathbb{R}. \tag{2.31}$$

A solution to (2.31) is a function f depending on h which is Lebesgue a.e.-differentiable, and such that there exists a version of f' verifying (2.31) for every $x \in \mathbb{R}$. The following result is basically due to Stein [31, 32]. The proof of point (i) (whose content is usually referred as *Stein's lemma*) involves a standard use of the Fubini theorem (see, e.g., [3], Lemma 2.1). Point (ii) is proved, for example, in [2], Lemma 4.3.

LEMMA 2.12. (i) *Let W be a random variable. Then $W \stackrel{\mathrm{Law}}{=} X \sim \mathscr{N}(0, 1)$ if, and only if,*

$$\mathbb{E}[f'(W) - Wf(W)] = 0 \tag{2.32}$$

for every continuous and piecewise continuously differentiable function f verifying the relation $\mathbb{E}|f'(X)| < \infty$.

(ii) *If h is absolutely continuous with bounded derivative, then (2.31) has a solution f_h which is twice differentiable and such that $\|f_h'\|_\infty \leq \|h'\|_\infty$ and $\|f_h''\|_\infty \leq 2\|h'\|_\infty$.*

Let $\mathscr{F}_W$ denote the class of twice differentiable functions, whose first derivative is bounded by 1 and whose second derivative is bounded by 2. If $h \in \mathrm{Lip}(1)$ and thus $\|h'\|_\infty \leq 1$, then the solution f_h, appearing in Lemma 2.12(ii), satisfies $\|f_h'\|_\infty \leq 1$ and $\|f_h''\|_\infty \leq 2$, and hence $f_h \in \mathscr{F}_W$.

Now consider a Gaussian random variable $X \sim \mathscr{N}(0, 1)$, and let Y be a generic random variable such that $\mathbb{E}Y^2 < \infty$. By integrating both sides of (2.31) with respect to the law of Y and by using (2.30), one sees immediately that point (ii) of

Lemma 2.12 implies that

$$d_W(Y,X) \le \sup_{f\in\mathscr{F}_W} \bigl|\mathbb{E}\bigl(f'(Y) - Yf(Y)\bigr)\bigr|. \tag{2.33}$$

Note that the square-integrability of Y implies that the quantity $\mathbb{E}[Yf(Y)]$ is well defined (recall that f is Lipschitz). In the subsequent sections, we will show that one can effectively bound the quantity appearing on the right-hand side of (2.33) by means of the operators introduced in Section 2.3.

3. A general inequality involving Poisson functionals. The following estimate, involving the normal approximation of smooth functionals of $\widehat{N}$, will be crucial for the rest of the paper. We use the notation introduced in the previous section.

THEOREM 3.1. *Let $F \in \operatorname{dom} D$ be such that $\mathbb{E}(F) = 0$. Let $X \sim \mathscr{N}(0,1)$. Then,*

$$d_W(F,X) \le \mathbb{E}\bigl[\bigl|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\bigr|\bigr] + \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz) \tag{3.1}$$

$$\le \sqrt{\mathbb{E}\bigl[\bigl(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\bigr)^2\bigr]} + \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz), \tag{3.2}$$

where we used the standard notation

$$\langle DF, -DL^{-1}F\rangle_{L^2(\mu)} = -\int_Z [D_zF \times D_zL^{-1}F]\mu(dz).$$

Moreover, if F has the form $F = I_q(f)$, where $q \ge 1$ and $f \in L^2_s(\mu^q)$, then

$$\langle DF, -DL^{-1}F\rangle_{L^2(\mu)} = q^{-1}\|DF\|^2_{L^2(\mu)}, \tag{3.3}$$

$$\int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz) = q^{-1}\int_Z \mathbb{E}[|D_zF|^3]\mu(dz). \tag{3.4}$$

PROOF. By virtue of the Stein-type bound (2.33), it is sufficient to prove that, for every function f such that $\|f'\|_\infty \le 1$ and $\|f''\|_\infty \le 2$ (that is, for every $f \in \mathscr{F}_W$), the quantity $|\mathbb{E}[f'(F) - Ff(F)]|$ is smaller than the right-hand side of (3.2). To see this, fix $f \in \mathscr{F}_W$ and observe that, by using (2.19), for every ω and every z one has that $D_zf(F)(\omega) = f(F)_z(\omega) - f(F)(\omega) = f(F_z)(\omega) - f(F)(\omega)$. Now use twice Lemma 2.8, combined with a standard Taylor expansion, and write

$$\begin{aligned} D_zf(F) &= f(F)_z - f(F) = f(F_z) - f(F) \\ &= f'(F)(F_z - F) + R(F_z - F) = f'(F)(D_zF) + R(D_zF), \end{aligned} \tag{3.5}$$

where (due to the fact that $\|f''\|_\infty \le 2$) the mapping $y \mapsto R(y)$ is such that $|R(y)| \le y^2$ for every $y \in \mathbb{R}$. We can therefore apply (in order) Lemmas 2.10 and 2.11 [in the case $u = DL^{-1}F$ and $G = f(F)$] and infer that

$$\begin{aligned}\mathbb{E}[Ff(F)] &= \mathbb{E}[LL^{-1}Ff(F)] = \mathbb{E}[-\delta(DL^{-1}F)f(F)] \\ &= \mathbb{E}\big[\langle Df(F), -DL^{-1}F\rangle_{L^2(\mu)}\big].\end{aligned}$$

According to (3.5), one has that

$$\begin{aligned}\mathbb{E}\big[\langle Df(F), -DL^{-1}F\rangle_{L^2(\mu)}\big] &= \mathbb{E}\big[f'(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big] \\ &\quad + \mathbb{E}\big[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}\big].\end{aligned}$$

It follows that

$$\begin{aligned}|\mathbb{E}[f'(F) - Ff(F)]| &\le \big|\mathbb{E}\big[f'(F)\big(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big)\big]\big| \\ &\quad + \big|\mathbb{E}\big[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}\big]\big|.\end{aligned}$$

By the fact that $\|f'\|_\infty \le 1$ and by Cauchy–Schwarz,

$$\begin{aligned}\big|\mathbb{E}\big[f'(F)\big(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big)\big]\big| &\le \mathbb{E}\big[\big|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big|\big] \\ &\le \sqrt{\mathbb{E}\big[\big(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big)^2\big]}.\end{aligned}$$

On the other hand, one sees immediately that

$$\begin{aligned}\big|\mathbb{E}\big[\langle R(DF), -DL^{-1}F\rangle_{L^2(\mu)}\big]\big| &\le \int_Z \mathbb{E}[|R(D_zF)D_zL^{-1}F|]\mu(dz) \\ &\le \int_Z \mathbb{E}[|D_zF|^2|D_zL^{-1}F|]\mu(dz).\end{aligned}$$

Relations (3.3) and (3.4) are immediate consequences of the definition of L^{-1} given in (2.29). The proof is complete. □

REMARK 3.2. Note that, in general, the bounds in formulae (3.1) and (3.2) can be infinite. In the forthcoming Sections 4–7, we will exhibit several examples of random variables in dom D such that the bounds in the statement of Theorem 3.1 are finite.

REMARK 3.3. Let G be an isonormal Gaussian process (see [17]) over some separable Hilbert space $\mathfrak{H}$, and suppose that $F \in L^2(\sigma(G))$ is centered and differentiable in the sense of Malliavin. Then, the Malliavin derivative of F, noted DF, is a $\mathfrak{H}$-valued random element, and in [13] it is proved that one has the following upper bound on the Wasserstein distance between the law of F and the law of $X \sim \mathscr{N}(0, 1)$:

$$d_W(F, X) \le \mathbb{E}|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}|,$$

where L^{-1} is the inverse of the Ornstein–Uhlenbeck generator associated with G.

Now fix $h \in L^2(\mu)$. It is clear that the random variable $I_1(h) = \widehat{N}(h)$ is an element of $\operatorname{dom} D$. One has that $D_z\widehat{N}(h) = h(z)$; moreover $-L^{-1}\widehat{N}(h) = \widehat{N}(h)$. Thanks to these relations, one deduces immediately from Theorem 3.1 the following refinement of Part A of Theorem 3 in [24].

COROLLARY 3.4. *Let $h \in L^2(\mu)$ and let $X \sim \mathscr{N}(0,1)$. Then the following bound holds*:

$$d_W(\widehat{N}(h), X) \le \left|1 - \|h\|^2_{L^2(\mu)}\right| + \int_Z |h(z)|^3 \mu(dz). \tag{3.6}$$

As a consequence, *if $\mu(Z) = \infty$ and if a sequence $\{h_k\} \subset L^2(\mu) \cap L^3(\mu)$ verifies*, *as $k \to \infty$*,

$$\|h_k\|_{L^2(\mu)} \to 1 \quad \textit{and} \quad \|h_k\|_{L^3(\mu)} \to 0, \tag{3.7}$$

one has the CLT

$$\widehat{N}(h_k) \xrightarrow{\text{law}} X \tag{3.8}$$

and inequality (3.6) *provides an explicit upper bound in the Wasserstein distance.*

In the following section, we will use Theorem 3.1 in order to prove general bounds for the normal approximation of multiple integrals of arbitrary order. As a preparation, we now present two simple applications of Corollary 3.4.

EXAMPLE 3.5. Consider a centered Poisson measure $\widehat{N}$ on $Z = \mathbb{R}^+$, with control measure μ equal to the Lebesgue measure. Then, the random variable $k^{-1/2}\widehat{N}([0,k]) = \widehat{N}(h_k)$, where $h_k = k^{-1/2}\mathbf{1}_{[0,k]}$, is an element of the first Wiener chaos associated with $\widehat{N}$. Plainly, since the random variables $\widehat{N}((i-1,i])$ ($i = 1,\dots,k$) are i.i.d. centered Poisson with unitary variance, a standard application of the central limit theorem yields that, as $k \to \infty$, $\widehat{N}(h_k) \overset{\text{law}}{\to} X \sim \mathscr{N}(0,1)$. Moreover, $\widehat{N}(h_k) \in \operatorname{dom} D$ for every k, and $D\widehat{N}(h_k) = h_k$. Since

$$\|h_k\|^2_{L^2(\mu)} = 1 \quad \text{and} \quad \int_Z |h_k|^3 \mu(dz) = \frac{1}{k^{1/2}},$$

one deduces from Corollary 3.4 that

$$d_W(\widehat{N}(h_k), X) \le \frac{1}{k^{1/2}},$$

which is consistent with the usual Berry–Esséen estimates.

EXAMPLE 3.6. Fix $\lambda > 0$. We consider the *Ornstein–Uhlenbeck Lévy process* given by

$$Y_t^\lambda = \sqrt{2\lambda} \int_{-\infty}^t \int_{\mathbb{R}} u \exp\bigl(-\lambda(t-x)\bigr)\widehat{N}(du, dx), \qquad t \ge 0, \tag{3.9}$$

where $\widehat{N}$ is a centered Poisson measure over $\mathbb{R} \times \mathbb{R}$, with control measure given by $\mu(du, dx) = \nu(du)\, dx$, where $\nu(\cdot)$ is positive, nonatomic and normalized in such a way that $\int_{\mathbb{R}} u^2\, d\nu = 1$. We assume also that $\int_{\mathbb{R}} |u|^3\, d\nu < \infty$. Note that Y_t^λ is a stationary *moving average Lévy process*. According to [24], Theorem 4, one has that, as $T \to \infty$,

$$A_T^\lambda = \frac{1}{\sqrt{2T/\lambda}} \int_0^T Y_t^\lambda\, dt \overset{\text{law}}{\to} X \sim \mathscr{N}(0, 1).$$

We shall show that Corollary 3.4 implies the existence of a finite constant $q_\lambda > 0$, depending uniquely on λ, such that, for every $T > 0$

$$d_W(A_T^\lambda, X) \le \frac{q_\lambda}{T^{1/2}}. \tag{3.10}$$

To see this, first use a Fubini theorem in order to represent A_T^λ as an integral with respect to $\widehat{N}$, that is, write

$$A_T^\lambda = \int_{-\infty}^T \int_{\mathbb{R}} u \biggl[\biggl(\frac{2\lambda}{2T/\lambda} \biggr)^{1/2} \int_{x \vee 0}^T \exp\bigl(-\lambda(t-x)\bigr)\, dt \biggr] \widehat{N}(du, dx) := \widehat{N}(h_T^\lambda).$$

Clearly, $A_T^\lambda \in \operatorname{dom} D$ and $DA_T^\lambda = h_T^\lambda$. By using the computations contained in [24], proof of Theorem 4, one sees immediately that there exists a constant β_λ, depending uniquely on λ and such that

$$\int_{\mathbb{R}\times\mathbb{R}} |h_T^\lambda(u, x)|^3 \nu(du)\, dx \le \frac{\beta_\lambda}{T^{1/2}}.$$

Since one has also that $|\|h_T^\lambda\|^2_{L^2(\mu)} - 1| = O(1/T)$, the estimate (3.10) is immediately deduced from Corollary 3.4.

4. Bounds on the Wasserstein distance for multiple integrals of arbitrary order. In this section, we establish general upper bounds for multiple Wiener–Itô integrals of arbitrary order $q \ge 2$, with respect to the compensated Poisson measure $\widehat{N}$. Our techniques hinge on the forthcoming Theorem 4.2, which uses the product formula (2.14) and inequality (3.2).

4.1. *The operators G_p^q and $\widehat{G}_p^q$.* Fix $q \ge 2$ and let $f \in L_s^2(\mu^q)$. The operator G_p^q transforms the function f, of q variables, into a function $G_p^q f$ of p variables, where p can be as large as $2q$. When $p = 0$, we set

$$G_0^q f = q! \|f\|^2_{L^2(\mu^q)}$$

and, for every $p = 1, \dots, 2q$, we define the function $(z_1, \dots, z_p) \to G_p^q f(z_1, \dots, z_p)$, from Z^p into $\mathbb{R}$, as follows:

$$G_p^q f(z_1, \dots, z_p) = \sum_{r=0}^q \sum_{l=0}^r \mathbf{1}_{\{2q-r-l=p\}} r! \binom{q}{r}^2 \binom{r}{l} \widetilde{f \star_r^l f}(z_1, \dots, z_p), \tag{4.1}$$

where the "star" contractions have been defined in formulae (2.8) and (2.9), and the tilde "$\sim$" indicates a symmetrization. The notation (4.1) is mainly introduced in order to give a more compact representation of the right-hand side of (2.14) when $g = f$. Indeed, suppose that $f \in L_s^2(\mu^q)$ $(q \geq 2)$, and that $f \star_r^l f \in L^2(\mu^{2q-r-l})$ for every $r = 0, \dots, q$ and $l = 1, \dots, r$ such that $l \neq r$; then, by using (2.14) and (4.1), one deduces that

$$I_q(f)^2 = \sum_{p=0}^{2q} I_p(G_p^q f), \tag{4.2}$$

where $I_0(G_0^q f) = G_0^q f = q!\|f\|^2_{L^2(\mu^q)}$. Note that the advantage of (4.2) [over (2.14)] is that the square $I_q(f)^2$ is now represented as an *orthogonal* sum of multiple integrals. As before, given $f \in L_s^2(\mu^q)$ $(q \geq 2)$ and $z \in Z$, we write $f(z, \cdot)$ to indicate the function on Z^{q-1} given by $(z_1, \dots, z_{q-1}) \to f(z, z_1, \dots, z_{q-1})$. To simplify the presentation of the forthcoming results, we now introduce a further assumption.

ASSUMPTION A. For the rest of the paper, whenever considering a kernel $f \in L_s^2(\mu^q)$, we will implicitly assume that every contraction of the type

$$(z_1, \dots, z_{2q-r-l}) \longrightarrow |f| \star_r^l |f|(z_1, \dots, z_{2q-r-l})$$

is *well defined and finite* for every $r = 1, \dots, q$, every $l = 1, \dots, r$ and every $(z_1, \dots, z_{2q-r-l}) \in Z^{2q-r-l}$.

Assumption A ensures, in particular, that the following relations are true for every $r = 0, \dots, q-1$, every $l = 0, \dots, r$ and every $(z_1, \dots, z_{2(q-1)-r-l}) \in Z^{2(q-1)-r-l}$:

$$\int_Z [f(z, \cdot) \star_r^l f(z, \cdot)]\mu(dz) = \{f \star_{r+1}^{l+1} f\}$$

and

$$\int_Z [\widetilde{f(z, \cdot) \star_r^l f}(z, \cdot)]\mu(dz) = \{\widetilde{f \star_{r+1}^{l+1} f}\}$$

(note that the symmetrization on the left-hand side of the second equality *does not* involve the variable z).

The notation $G_p^{q-1} f(z, \cdot)$ $[p = 0, \dots, 2(q-1)]$ stands for the action of the operator G_p^{q-1} [defined according to (4.1)] on $f(z, \cdot)$, that is,

$$\begin{aligned} &G_p^{q-1} f(z, \cdot)(z_1, \dots, z_p) \\ &\quad = \sum_{r=0}^{q-1} \sum_{l=0}^{r} \mathbf{1}_{\{2(q-1)-r-l=p\}} r! \binom{q-1}{r}^2 \binom{r}{l} \\ &\qquad \times \widetilde{f(z, \cdot) \star_r^l f}(z, \cdot)(z_1, \dots, z_p). \end{aligned} \tag{4.3}$$

For instance, if $f \in L_s^2(\mu^3)$, then

$$G_1^2 f(z,\cdot)(a) = 4 \times f(z,\cdot) \star_2^1 f(z,\cdot)(a) = 4 \int_Z f(z,a,u)^2 \mu(du).$$

Note also that, for a fixed $z \in Z$, the quantity $G_0^{q-1} f(z,\cdot)$ is given by the following constant

$$G_0^{q-1} f(z,\cdot) = (q-1)! \int_{Z^{q-1}} f^2(z,\cdot)\, d\mu^{q-1}. \tag{4.4}$$

Finally, for every $q \geq 2$ and every $f \in L_s^2(\mu^q)$, we set

$$\widehat{G}_0^q f = \int_Z G_0^{q-1} f(z,\cdot)\mu(dz) = q^{-1} G_0^q f = (q-1)! \|f\|^2_{L^2(\mu^q)}$$

and, for $p = 1, \ldots, 2(q-1)$, we define the function $(z_1, \ldots, z_p) \to \widehat{G}_p^q f(z_1, \ldots, z_p)$, from Z^p into $\mathbb{R}$, as follows:

$$\widehat{G}_p^q f(\cdot) = \int_Z G_p^{q-1} f(z,\cdot)\mu(dz) \tag{4.5}$$

or, more explicitly,

$$\begin{aligned}
&\widehat{G}_p^q f(z_1, \ldots, z_p) \\
&\quad = \int_Z \sum_{r=0}^{q-1} \sum_{l=0}^{r} \mathbf{1}_{\{2(q-1)-r-l=p\}} r! \binom{q-1}{r}^2 \binom{r}{l} \\
&\qquad\qquad \times \widetilde{f(z,\cdot) \star_r^l f(z,\cdot)}(z_1, \ldots, z_p)\mu(dz) && (4.6) \\
&\quad = \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2q-t-s=p\}} (t-1)! \binom{q-1}{t-1}^2 \binom{t-1}{s-1} \\
&\qquad\qquad \times \widetilde{f \star_t^s f}(z_1, \ldots, z_p), && (4.7)
\end{aligned}$$

where in (4.7) we have used the change of variables $t = r+1$ and $s = l+1$, as well as the fact that $p \geq 1$. We stress that the symmetrization in (4.6) does not involve the variable z.

4.2. *Bounds on the Wasserstein distance.* When $q \geq 2$ and $\mu(Z) = \infty$, we shall focus on kernels $f \in L_s^2(\mu^q)$ verifying the following technical condition: for every $p = 1, \ldots, 2(q-1)$,

$$\int_Z \left[\sqrt{\int_{Z^p} \{G_p^{q-1} f(z,\cdot)\}^2 \, d\mu^p} \right] \mu(dz) < \infty. \tag{4.8}$$

As will become clear from the subsequent discussion, the requirement (4.8) is used to justify a Fubini argument.

REMARK 4.1.

1. When $q = 2$, one deduces from (4.3) that $G_1^1 f(z,\cdot)(x) = f(z,x)^2$ and $G_2^1 f(z,\cdot)(x,y) = f(z,x)f(z,y)$. It follows that, in this case, condition (4.8) is verified if, and only if, the following relation holds:

$$\int_Z \sqrt{\int_Z f(z,a)^4 \mu(da)}\mu(dz) < \infty. \tag{4.9}$$

Indeed, since f is square-integrable, the additional relation

$$\int_Z \sqrt{\int_{Z^2} f(z,a)^2 f(z,b)^2 \mu(da)\mu(db)}\mu(dz) < \infty \tag{4.10}$$

is always satisfied, since

$$\int_Z \sqrt{\int_{Z^2} f(z,a)^2 f(z,b)^2 \mu(da)\mu(db)}\mu(dz) = \|f\|^2_{L^2(\mu^2)}.$$

2. From relation (4.3), one deduces immediately that (4.8) is implied by the following (stronger) condition: for every $p = 1, \dots, 2(q-1)$ and every (r,l) such that $2(q-1) - r - l = p$

$$\int_Z \left[\sqrt{\int_{Z^p} \{f(z,\cdot) \star_r^l f(z,\cdot)\}^2 \, d\mu^p} \right] \mu(dz) < \infty. \tag{4.11}$$

3. When $\mu(Z) < \infty$ and

$$\int_Z \left[\int_{Z^p} \{G_p^{q-1} f(z,\cdot)\}^2 \, d\mu^p \right] \mu(dz) < \infty,$$

condition (4.8) is automatically satisfied (to see this, just apply the Cauchy–Schwarz inequality).
4. Arguing as in the previous point, a sufficient condition for (4.8) to be satisfied, is that the support of the symmetric function f is contained in a set of the type $A \times \cdots \times A$ where A is such that $\mu(A) < \infty$.

THEOREM 4.2 (Wasserstein bounds on a fixed chaos). *Fix $q \ge 2$ and let $X \sim \mathscr{N}(0,1)$. Let $f \in L_s^2(\mu^q)$ be such that*:

(i) *whenever $\mu(Z) = \infty$, condition (4.8) is satisfied for every $p = 1, \dots, 2(q-1)$*;

(ii) *for $d\mu$-almost every $z \in Z$, every $r = 1, \dots, q-1$ and every $l = 0, \dots, r-1$, the kernel $f(z,\cdot) \star_r^l f(z,\cdot)$ is an element of $L^2(\mu^{2(q-1)-r-l})$.*

Denote by $I_q(f)$ the multiple Wiener–Itô integral, of order q, of f with respect to $\widehat{N}$. Then the following bound holds:

$$d_W(I_q(f), X) \tag{4.12}$$
$$\leq \sqrt{\left(1 - q!\|f\|^2_{L^2(\mu^q)}\right)^2 + q^2 \sum_{p=1}^{2(q-1)} p! \int_{Z^p} \{\widehat{G}^q_p f\}^2 \, d\mu^p}$$

$$+ q^2 \sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}} \tag{4.13}$$
$$\times \sqrt{\sum_{p=0}^{2(q-1)} p! \int_Z \left\{ \int_{Z^p} G^{q-1}_p f(z,\cdot)^2 \, d\mu^p \right\} \mu(dz)},$$

where the notation $\widehat{G}^q_p f$ and $G^{q-1}_p f(z,\cdot)$ are defined, respectively, in (4.5)–(4.7) *and* (4.3). *Moreover, the bound appearing on the right-hand side of* (4.12) *and* (4.13) *can be assessed by means of the following estimate*:

$$\sqrt{\left(1 - q!\|f\|^2_{L^2(\mu^q)}\right)^2 + q^2 \sum_{p=1}^{2(q-1)} p! \int_{Z^p} \{\widehat{G}^q_p f\}^2 \, d\mu^p} \tag{4.14}$$
$$\leq \left|1 - q!\|f\|^2_{L^2(\mu^q)}\right|$$

$$+ q \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2 \leq t+s \leq 2q-1\}} (2q-t-s)!^{1/2} (t-1)! \tag{4.15}$$
$$\times \binom{q-1}{t-1}^2 \binom{t-1}{s-1} \|f \star^s_t f\|_{L^2(\mu^{2q-t-s})}.$$

Also, if one has that

$$f \star^{q-b}_q f \in L^2(\mu^b) \qquad \forall b = 1, \dots, q-1, \tag{4.16}$$

then

$$\sqrt{\sum_{p=0}^{2(q-1)} p! \int_Z \left\{ \int_{Z^p} G^{q-1}_p f(z,\cdot)^2 \, d\mu^p \right\}^2 \mu(dz)} \tag{4.17}$$

$$\leq \sum_{b=1}^{q} \sum_{a=0}^{b-1} \mathbf{1}_{\{1 \leq a+b \leq 2q-1\}} (a+b-1)!^{1/2} (q-a-1)! \tag{4.18}$$
$$\times \binom{q-1}{q-1-a}^2 \binom{q-1-a}{q-b} \|f \star^a_b f\|_{L^2(\mu^{2q-a-b})}.$$

REMARK 4.3.

1. There are contraction norms in (4.18) that do not appear in the previous formula (4.15), and vice versa. For example, in (4.18) one has $\|f \star_b^0 f\|_{L^2(\mu^{2q-b})}$, where $b = 1, \ldots, q$ (this corresponds to the case $b \in \{1, \ldots, q\}$ and $a = 0$). By using formula (2.13), these norms can be computed as follows:

$$\|f \star_b^0 f\|_{L^2(\mu^{2q-b})} = \|f \star_q^{q-b} f\|_{L^2(\mu^b)}, \qquad b = 1, \ldots, q-1; \tag{4.19}$$

$$\|f \star_q^0 f\|_{L^2(\mu^q)} = \sqrt{\int_{Z^q} f^4 \, d\mu^q}. \tag{4.20}$$

   We stress that $f^2 = f \star_q^0 f$, and therefore $f \star_q^0 f \in L^2(\mu^q)$ if, and only if, $f \in L^4(\mu^q)$.
2. One should compare Theorem 4.2 with Proposition 3.2 in [13], which provides upper bounds for the normal approximation of multiple integrals with respect to an *isonormal Gaussian process*. The bounds in [13] are also expressed in terms of contractions of the underlying kernel.

EXAMPLE 4.4 (Double integrals). We consider a double integral of the type $I_2(f)$, where $f \in L_s^2(\mu^2)$ satisfies (4.9) [according to Remark 4.1(1), this implies that (4.8) is satisfied]. We suppose that the following three conditions are satisfied: (a) $\mathbb{E}I_2(f)^2 = 2\|f\|^2_{L^2(\mu^2)} = 1$, (b) $f \star_2^1 f \in L^2(\mu^1)$ and (c) $f \in L^4(\mu^2)$. Since $q = 2$ here, one has $f(z, \cdot) \star_1^0 f(z, \cdot)(a) = f(z, a)^2$ which is square-integrable, and hence assumption (ii) and condition (4.16) in Theorem 4.2 are verified. Using relations (4.19) and (4.20), we can deduce the following bound on the Wasserstein distance between the law of $I_2(f)$ and the law of $X \sim \mathscr{N}(0, 1)$:

$$\begin{aligned} d_W(I_2(f), X) &\le \sqrt{8}\|f \star_1^1 f\|_{L^2(\mu^2)} + \{2 + \sqrt{8}(1 + \sqrt{2})\}\|f \star_2^1 f\|_{L^2(\mu^2)} \\ &\quad + \sqrt{8\int_{Z^2} f^4 \, d\mu^2}. \end{aligned} \tag{4.21}$$

To obtain (4.21), observe first that, since assumption (a) above is in order, then relations (4.12) and (4.13) in the statement of Theorem 4.2 yield that

$$d_W(I_2(f), X) \le 0 + (4.15) + 2^{3/2} \times (4.18).$$

To conclude, observe that

$$(4.15) = 2\{2^{1/2}\|f \star_1^1 f\|_{L^2(\mu^2)} + \|f \star_2^1 f\|_{L^2(\mu)}\}$$

and

$$(4.18) = \|f \star_2^1 f\|_{L^2(\mu)}\{2^{1/2} + 1\} + \|f\|^2_{L^4(\mu^2)},$$

since $\|f \star_2^1 f\|_{L^2(\mu)} = \|f \star_1^0 f\|_{L^2(\mu^3)}$. A general statement, involving random variables of the type $F = I_1(g) + I_2(h)$ is given in Theorem 6.1.

EXAMPLE 4.5 (Triple integrals). We consider a random variable of the type $I_3(f)$, where $f \in L_s^2(\mu^3)$ verifies (4.8) [for instance, according to Remark 4.1(4), we may assume that f has support contained in some rectangle of finite μ^3-measure]. We shall also suppose that the following three conditions are satisfied: (a) $\mathbb{E}I_3(f)^2 = 3!\|f\|^2_{L^2(\mu^3)} = 1$, (b) for every $r = 1, \dots, 3$, and every $l = 1, \dots, r \wedge 2$, one has that $f \star_r^l f \in L^2(\mu^{6-r-l})$, and (c) $f \in L^4(\mu^3)$. One can check that all the assumptions in the statement of Theorem 4.2 (in the case $q = 3$) are satisfied. In view of (4.19) and (4.20), we therefore deduce (exactly as in the previous example and after some tedious bookkeeping) the following bound on the Wasserstein distance between the law of $I_3(f)$ and the law of $X \sim \mathscr{N}(0, 1)$:

$$
\begin{aligned}
d_W&(I_3(f), X)\\
&\le 3\sqrt{24}\|f \star_1^1 f\|_{L^2(\mu^4)} + \bigl(4\sqrt{54} + 12\sqrt{6}\bigr)\|f \star_2^1 f\|_{L^2(\mu^3)}\\
&\quad + 12\sqrt{2}\|f \star_2^2 f\|_{L^2(\mu^2)} + \{12\sqrt{2} + 4\sqrt{27}\}\|f \star_3^1 f\|_{L^2(\mu^2)}\\
&\quad + \{12 + \sqrt{27}(1 + \sqrt{24})\}\|f \star_3^2 f\|_{L^2(\mu^1)}\\
&\quad + 2\sqrt{54}\sqrt{\int_{Z^3} f^4\, d\mu^3}.
\end{aligned} \tag{4.22}
$$

PROOF OF THEOREM 4.2. First, observe that, according to Theorem 3.1, we have that

$$
\begin{aligned}
d_W(I_q(f), X) \le{}& \sqrt{\mathbb{E}\biggl[\biggl(1 - \frac{1}{q}\|DI_q(f)\|^2_{L^2(\mu)}\biggr)^2\biggr]}\\
&+ \frac{1}{q}\int_Z \mathbb{E}|D_z I_q(f)|^3 \mu(dz).
\end{aligned}
$$

The rest of the proof is divided in four steps:

(S1) Proof of the fact that $\sqrt{\mathbb{E}[(1 - \frac{1}{q}\|DI_q(f)\|^2_{L^2(\mu)})^2]}$ is less or equal to the quantity appearing at the line (4.12).
(S2) Proof of the fact that $\frac{1}{q}\int_Z \mathbb{E}|D_z I_q(f)|^3 \mu(dz)$ is less or equal to the quantity appearing at the line (4.13).
(S3) Proof of the estimate displayed in formulae (4.14) and (4.15).
(S4) Proof of the estimate in formulae (4.17) and (4.18), under the assumption (4.16).

Step (S1). Use (2.17) to write the Malliavin derivative $D_z I_q(f)(\omega) = q \times I_{q-1}(f(z, \cdot))(\omega)$, and recall that $DI_q(f)$ is uniquely defined up to subsets of $\Omega \times Z$ with $d\mathbb{P} \times d\mu$-measure zero. By selecting an appropriate version of $DI_q(f)$, thanks to the multiplication formulae (2.14) and (4.2) [and by adopting the notation

(4.1)] one has that, a.e.-$d\mathbb{P}\times d\mu$,

$$\{D_z I_q(f)\}^2 = qq!\int_{Z^{q-1}} f^2(z,\cdot)\,d\mu^{q-1} + q^2\sum_{p=1}^{2(q-1)} I_p(G_p^{q-1} f(z,\cdot)), \tag{4.23}$$

where the stochastic integrals are set equal to zero for every z belonging to the exceptional set where assumption (ii) in the statement is not verified. Since (4.8) is in order, one has that, for every $p=1,\dots,2(q-1)$,

$$\begin{aligned}&\mathbb{E}\int_Z |I_p(G_p^{q-1} f(z,\cdot))|\mu(dz)\\ &\quad\le \int_Z \sqrt{p!\int_{Z^p} G_p^{q-1} f(z,\cdot)^2\,d\mu^p}\,\mu(dz)<\infty,\end{aligned}\tag{4.24}$$

where we have used the Cauchy–Schwarz inequality combined with the isometric properties of multiple integrals. Relations (4.23) and (4.24) yield that one can write

$$\begin{aligned}&\frac{1}{q}\|DI_q(f)\|^2_{L^2(\mu)} - 1\\ &\quad = \frac{1}{q}\int_Z \{D_z I_q(f)\}^2\mu(dz) - 1\end{aligned}\tag{4.25}$$

$$\quad = q!\|f\|^2_{L^2(\mu^q)} - 1 + q\sum_{p=1}^{2(q-1)}\int_Z I_p(G_p^{q-1} f(z,\cdot))\mu(dz). \tag{4.26}$$

Note that (4.24) ensures that each integral appearing in (4.26) is P-a.s. well defined and finite. Since assumption (4.8) is in order, one has that, for every $p=1,\dots,2(q-1)$,

$$\begin{aligned}&\mathbb{E}\Big[\Big(\int_Z I_p(G_p^{q-1} f(z,\cdot))\mu(dz)\Big)^2\Big]\\ &\quad\le \int_{Z^2}\mathbb{E}|I_p(G_p^{q-1} f(z,\cdot))I_p(G_p^{q-1} f(z',\cdot))|\mu(dz)\mu(dz')\\ &\quad\le p!\Big\{\int_Z\sqrt{\int_{Z^p}\{G_p^{q-1} f(z,\cdot)\}^2\,d\mu^p}\,\mu(dz)\Big\}^2<\infty\end{aligned}\tag{4.27}$$

and one can easily verify that, for $1\le p\ne l\le 2(q-1)$, the random variables

$$\int_Z I_p(G_p^{q-1} f(z,\cdot))\mu(dz)$$

and

$$\int_Z I_l(G_l^{q-1} f(z,\cdot))\mu(dz)$$

are orthogonal in $L^2(\mathbb{P})$. It follows that

$$\mathbb{E}\biggl[\biggl(1-\frac{1}{q}\|DI_q(f)\|^2_{L^2(\mu)}\biggr)^2\biggr] = \bigl(q!\|f\|^2_{L^2(\mu^q)}-1\bigr)^2 + q^2\sum_{p=1}^{2(q-1)}\mathbb{E}\biggl[\biggl(\int_Z I_p(G_p^{q-1}f(z,\cdot))\mu(dz)\biggr)^2\biggr],$$

so that the estimate

$$\sqrt{\mathbb{E}\biggl[\biggl(1-\frac{1}{q}\|DI_q(f)\|^2_{L^2(\mu)}\biggr)^2\biggr]} \le (4.12)$$

is proved, once we show that

$$\mathbb{E}\biggl[\biggl(\int_Z I_p(G_p^{q-1}f(z,\cdot))\mu(dz)\biggr)^2\biggr] \le p!\int_{Z^p}\{\widehat{G}_p^q f\}^2\,d\mu^p. \tag{4.28}$$

The proof of (4.28) can be achieved by using the following relations:

$$\begin{aligned}
&\mathbb{E}\biggl[\biggl(\int_Z I_p(G_p^{q-1}f(z,\cdot))\mu(dz)\biggr)^2\biggr]\\
&\quad= \int_Z\int_Z \mathbb{E}[I_p(G_p^{q-1}f(z,\cdot))I_p(G_p^{q-1}f(z',\cdot))]\mu(dz)\mu(dz')\\
&\quad= p!\int_Z\int_Z\biggl[\int_{Z^p} G_p^{q-1}f(z,\cdot)G_p^{q-1}f(z',\cdot)\,d\mu^p\biggr]\mu(dz)\mu(dz') \qquad (4.29)\\
&\quad= p!\int_{Z^p}\biggl[\int_Z G_p^{q-1}f(z,\cdot)\mu(dz)\biggr]^2 d\mu^p\\
&\quad= p!\int_{Z^p}\{\widehat{G}_p^q f\}^2\,d\mu^p.
\end{aligned}$$

Note that the use of the Fubini theorem in the equality (4.29) is justified by the chain of inequalities (4.27), which is in turn a consequence of assumption (4.8).

Step (S2). First, recall that

$$\mathbb{E}[q^{-1}\|DI_q(f)\|^2_{L^2(\mu)}] = \mathbb{E}[I_q(f)^2] = q!\|f\|^2_{L^2(\mu^q)}.$$

Now use the Cauchy–Schwarz inequality, in order to write

$$\begin{aligned}
&\frac{1}{q}\mathbb{E}\biggl[\int_Z |D_zI_q(f)|^3\mu(dz)\biggr]\\
&\quad\le \frac{1}{q}\sqrt{\mathbb{E}[\|DI_q(f)\|^2_{L^2(\mu)}]}\times\sqrt{\int_Z \mathbb{E}[(D_zI_q(f))^4]\mu(dz)} \qquad (4.30)\\
&\quad= \sqrt{(q-1)!\|f\|^2_{L^2(\mu^q)}}\times\sqrt{\int_Z \mathbb{E}[(D_zI_q(f))^4]\mu(dz)}. \qquad (4.31)
\end{aligned}$$

By using (4.4) and (4.23), one deduces immediately that

$$\{D_z I_q(f)\}^2 = q^2 \sum_{p=0}^{2(q-1)} I_p[G_p^{q-1} f(z,\cdot)].$$

As a consequence,

$$\sqrt{\int_Z \mathbb{E}[D_z I_q(f)^4]\mu(dz)} \tag{4.32}$$
$$\leq q^2 \sqrt{\sum_{p=0}^{2(q-1)} p! \int_Z \left\{\int_{Z^p} G_p^{q-1} f(z,\cdot)^2 \, d\mu^p\right\}\mu(dz)},$$

yielding the desired inequality.

Step (S3). By using several times the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ $(a, b \geq 0)$ one sees that, in order to prove (4.14) and (4.15), it is sufficient to show that

$$\sum_{p=1}^{2(q-1)} p!^{1/2} \sqrt{\int_{Z^p} \{\widehat{G}_p^q f\}^2 \, d\mu^p}$$
$$\leq \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2\leq t+s\leq 2q-1\}} (2q-t-s)!^{1/2} (t-1)!$$
$$\times \binom{q-1}{t-1}^2 \binom{t-1}{s-1} \|f \star_t^s f\|_{L^2(\mu^{2q-t-s})}.$$

To see this, use (4.7) and the fact that [by (2.6)] $\|\widetilde{f \star_t^s f}\|_{L^2(\mu^{2q-t-s})} \leq \|f \star_t^s f\|_{L^2(\mu^{2q-t-s})}$, to obtain that

$$\sum_{p=1}^{2(q-1)} p!^{1/2} \sqrt{\int_{Z^p} \{\widehat{G}_p^q f\}^2 \, d\mu^p}$$
$$\leq \sum_{p=1}^{2(q-1)} p!^{1/2} \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2q-t-s=p\}} (t-1)! \binom{q-1}{t-1}^2$$
$$\times \binom{t-1}{s-1} \|f \star_t^s f\|_{L^2(\mu^{2q-t-s})}$$

and then exploit the relation

$$\sum_{p=1}^{2(q-1)} p!^{1/2} \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2q-t-s=p\}}$$
$$= \sum_{t=1}^{q} \sum_{s=1}^{t\wedge(q-1)} \mathbf{1}_{\{2\leq t+s\leq 2q-1\}} (2q-t-s)!^{1/2}.$$

Step (S4). By using (4.3) and some standard estimates, we deduce that

$$\sum_{p=0}^{2(q-1)} p!^{1/2} \sqrt{\int_Z \left\{ \int_{Z^p} G_p^{q-1} f(z,\cdot)^2 \, d\mu^p \right\} \mu(dz)}$$

$$\leq \sum_{p=0}^{2(q-1)} p!^{1/2} \sum_{r=0}^{q-1} \sum_{l=0}^{r} \mathbf{1}_{\{2(q-1)-r-l=p\}} r! \binom{q-1}{r}^2 \binom{r}{l} \tag{4.33}$$

$$\times \sqrt{\int_Z \int_{Z^p} [f(z,\cdot) \star_r^l f(z,\cdot)]^2 \, d\mu^p \mu(dz)}. \tag{4.34}$$

We claim that, if (4.16) is satisfied, then, for every $r = 0, \ldots, q-1$ and $l = 0, \ldots, r$

$$\int_Z \int_{Z^p} [f(z,\cdot) \star_r^l f(z,\cdot)]^2 \, d\mu^p \mu(dz) = \int_{Z^{l+r+1}} [f \star_{q-l}^{q-1-r} f]^2 \, d\mu^{l+r+1}. \tag{4.35}$$

In the two "easy" cases,

(a) $r = q-1$ and $l = 1, \ldots, q-1$,
(b) $r = 1, \ldots, q-1$ and $l = 0$,

relation (4.35) can be deduced by a standard use of the Fubini theorem [assumption (4.16) is not needed here]. Now fix $p = 1, \ldots, 2q-2$, as well as $r = 1, \ldots, q-2$ and $l = 0, \ldots, r$ in such a way that $2(q-1) - r - l = p$. For every fixed $z \in Z$, write $|f(z,\cdot)| \star_r^l |f(z,\cdot)|$ to indicate the contraction of indices (r, l) obtained from the positive kernel $|f(z,\cdot)|$. Note that, for z fixed, such a contraction is a function on Z^p, and also, in general, $|f(z,\cdot)| \star_r^l |f(z,\cdot)| \geq f(z,\cdot) \star_r^l f(z,\cdot)$ and $|f(z,\cdot)| \star_r^l |f(z,\cdot)| \neq f(z,\cdot) \star_r^l f(z,\cdot)$. By a standard use of the Cauchy–Schwarz inequality and of the Fubini theorem, one sees that

$$\begin{aligned} &\int_Z \int_{Z^p} [|f(z,\cdot)| \star_r^l |f(z,\cdot)|]^2 \, d\mu^p \mu(dz) \\ &\qquad \leq \| f \star_q^{q-1-r+l} f \|_{L^2(\mu^{r-l-1})}^2 < \infty, \end{aligned} \tag{4.36}$$

where the last relation is a consequence of assumption (4.16) as well as of the fact that, by construction, $1 \leq 1 + r - l \leq q - 1$. Relation (4.36) implies that one can apply the Fubini theorem to the quantity

$$\int_Z \int_{Z^p} [f(z,\cdot) \star_r^l f(z,\cdot)]^2 \, d\mu^p \mu(dz)$$

[by first writing the contractions $f(z,\cdot) \star_r^l f(z,\cdot)$ in an explicit form], so to obtain the desired equality (4.35). By plugging (4.35) into (4.33) and by applying the change of variables $a = q - 1 - r$ and $b = q - l$, one deduces (4.17) and (4.18). This concludes the proof of Theorem 4.2. □

5. Central limit theorems. We consider here CLTs. The results of this section generalize the main findings of [24]. The following result uses Theorem 4.2 in order to establish a general CLT for multiple integrals of arbitrary order.

THEOREM 5.1 (CLTs on a fixed chaos). *Let $X \sim \mathscr{N}(0,1)$. Suppose that $\mu(Z) = \infty$, fix $q \geq 2$, and let $F_k = I_q(f_k)$, $k \geq 1$, be a sequence of multiple stochastic Wiener–Itô integrals of order q. Suppose that, as $k \to \infty$, the normalization condition $\mathbb{E}(F_k^2) = q!\|f_k\|^2_{L^2(\mu^q)} \to 1$ takes place. Assume moreover that the following three conditions hold*:

(I) *For every $k \geq 1$, the kernel f_k verifies* (4.8) *for every $p = 1, \ldots, 2(q-1)$.*

(II) *For every $r = 1, \ldots, q$ and every $l = 1, \ldots, r \wedge (q-1)$, one has that $f_k \star_r^l f_k \in L^2(\mu^{2q-r-l})$ and also $\|f_k \star_r^l f_k\|_{L^2(\mu^{2q-r-l})} \to 0$ (as $k \to \infty$).*

(III) *For every $k \geq 1$, one has that $\int_{Z^q} f_k^4 \, d\mu^q < \infty$ and, as $k \to \infty$,*

$$\int_{Z^q} f_k^4 \, d\mu^q \to 0.$$

Then, $F_k \xrightarrow{\text{law}} X$, as $k \to \infty$, and formulae (4.14)–(4.20) *provide explicit bounds in the Wasserstein distance $d_W(I_q(f_k), X)$.*

PROOF. First, note that the fact that $f_k \star_r^l f_k \in L^2(\mu^{2q-r-l})$ for every $r = 1, \ldots, q$ and every $l = 1, \ldots, r \wedge (q-1)$ [due to assumption (II)] imply that assumption (ii) in the statement of Theorem 4.2 holds for every $k \geq 1$ (with f_k replacing f), and also [by using (4.19)] that condition (4.16) is satisfied by each kernel f_k. Now observe that, if assumptions (I)–(III) are in order and if $\mathbb{E}(F_k^2) = q!\|f_k\|^2_{L^2(\mu^q)} \to 1$, then relations (4.14)–(4.20) imply that $d_W(F_k, X) \to 0$. Since convergence in the Wasserstein distance implies convergence in law, the conclusion is immediately deduced. $\square$

EXAMPLE 5.2. Consider a sequence of double integrals of the type $I_2(f_k)$, $k \geq 1$ where $f_k \in L_s^2(\mu^2)$ satisfies (4.9) [according to Remark 4.1(1), this implies that (4.8) is satisfied]. We suppose that the following three conditions are satisfied: (a) $\mathbb{E}I_2(f_k)^2 = 2\|f_k\|^2_{L^2(\mu^2)} = 1$, (b) $f_k \star_2^1 f_k \in L^2(\mu^1)$ and (c) $f_k \in L^4(\mu^2)$. Then, according to Theorem 5.1, a sufficient condition in order to have that (as $k \to \infty$)

$$I_2(f_k) \xrightarrow{\text{law}} \mathscr{N}(0,1) \tag{5.1}$$

is that

$$\|f_k\|_{L^4(\mu^2)} \to 0, \tag{5.2}$$

$$\|f_k \star_2^1 f_k\|_{L^2(\mu)} \to 0 \quad \text{and} \quad \|f_k \star_1^1 f_k\|_{L^2(\mu^2)} \to 0. \tag{5.3}$$

This last fact coincides with the content of Part 1 of Theorem 2 in [24], where one can find an alternate proof based on a decoupling technique, known as the "principle of conditioning" (see, e.g., Xue [35]). Note that an explicit upper bound for the Wasserstein distance can be deduced from relation (4.21).

EXAMPLE 5.3. Consider a sequence of random variables of the type $I_3(f_k)$, $k \geq 1$, with unitary variance and verifying assumption (I) in the statement of Theorem 5.1. Then, according to the conclusion of Theorem 5.1, a sufficient condition in order to have that, as $k \to \infty$,

$$I_3(f_k) \xrightarrow{\text{law}} X \sim \mathscr{N}(0,1),$$

is that the following six quantities converge to zero:

$$\|f_k \star_1^1 f_k\|_{L^2(\mu^4)}, \qquad \|f_k \star_2^1 f_k\|_{L^2(\mu^3)}, \qquad \|f_k \star_2^2 f_k\|_{L^2(\mu^2)},$$
$$\|f_k \star_3^1 f_k\|_{L^2(\mu^2)}, \qquad \|f_k \star_3^2 f_k\|_{L^2(\mu^1)} \quad \text{and} \quad \|f_k\|_{L^4(\mu^3)}.$$

Moreover, an explicit upper bound in the Wasserstein distance is given by the estimate (4.22).

The following result, proved in [24], Theorem 2, represents a counterpart to the CLTs for double integrals discussed in Example 5.2.

PROPOSITION 5.4 (See [24]). *Consider a sequence $F_k = I_2(f_k)$, $k \geq 1$, of double integrals verifying assumptions* (a), (b) *and* (c) *of Example* 5.2. *Suppose moreover that* (5.2) *takes place. Then*:

1. *if $F_k \in L^4(\mathbb{P})$ for every k, a sufficient condition to have* (5.3) *is that*

$$\mathbb{E}(F_k^4) \to 3; \tag{5.4}$$

2. *if the sequence $\{F_k^4 : k \geq 1\}$ is uniformly integrable, then conditions* (5.1), (5.3) *and* (5.4) *are equivalent.*

REMARK 5.5. For the time being, it seems rather hard to prove a result analogous to Proposition 5.4 for a sequence of multiple integrals of order $q \geq 3$. The main reason for this is that the explicit computation of the fourth moments of these integrals requires the use of quite complicated (and not very tractable) *diagram formulae* (see, e.g., [25, 34]). One should compare this situation with the explicit formulae for fourth moments of multiple Gaussian integrals obtained, for example, in [19].

6. Sum of a single and a double integral. As we will see in the forthcoming Section 7, when dealing with quadratic functionals of stochastic processes built from completely random measures, one needs explicit bounds for random variables of the type $F = I_1(g) + I_2(h)$, that is, random variables that are the sum of a single and a double integral. The following result, that can be seen as a generalization of Part B of Theorem 3 in [24], provides explicit bounds for random variables of this type.

THEOREM 6.1. *Let $F = I_1(g) + I_2(h)$ be such that*:

(I) *The function g belongs to $L^2(\mu) \cap L^3(\mu)$.*
(II) *The kernel $h \in L_s^2(\mu^2)$ is such that*: (a) $h \star_2^1 h \in L^2(\mu^1)$, (b) *relation* (4.9) *is verified, with h replacing f, and* (c) $h \in L^4(\mu^2)$.

Then, one has the following upper bound on the Wasserstein distance between the law of F and the law of $X \sim \mathscr{N}(0,1)$:

$$\begin{aligned} d_W(F,X) \le{}& \big|1 - \|g\|^2_{L^2(\mu)} - 2\|h\|^2_{L^2(\mu^2)}\big| + 2\|h \star_2^1 h\|_{L^2(\mu)} \\ &+ \sqrt{8}\|h \star_1^1 h\|_{L^2(\mu^2)} + 3\|g \star_1^1 h\|_{L^2(\mu)} + 4\|g\|^3_{L^3(\mu)} \\ &+ 32\|h\|_{L^2(\mu^2)} \times \big\{\|h\|^2_{L^4(\mu^2)} + 2^{1/2}\|h \star_2^1 h\|_{L^2(\mu)}\big\}. \end{aligned} \tag{6.1}$$

The following inequality also holds:

$$\|g \star_1^1 h\|_{L^2(\mu)} \le \|g\|_{L^2(\mu)} \times \|h \star_1^1 h\|^{1/2}_{L^2(\mu^2)}. \tag{6.2}$$

PROOF. Thanks to Theorem 3.1, we know that $d_W(F,X)$ is less or equal to the right-hand side of (3.2). We also know that

$$D_z F = g(z) + 2I_1(h(z,\cdot)) \quad \text{and} \quad -D_z L^{-1} F = g(z) + I_1(h(z,\cdot)).$$

By using the multiplication formula (2.14) (in the case $p = q = 1$) as well as a Fubini argument, one easily deduces that

$$\begin{aligned} \int_Z D_z F \times (-D_z L^{-1} F)\mu(dz) ={}& \|g\|^2_{L^2(\mu)} + 2\|h\|^2_{L^2(\mu^2)} + 2I_1(h \star_2^1 h) \\ &+ 2I_2(h \star_1^1 h) + 3I_1(g \star_1^1 h). \end{aligned}$$

This last relation yields

$$\begin{aligned} &\sqrt{\mathbb{E}\big[\big(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mu)}\big)^2\big]} \\ &\quad\le \big|1 - \|g\|^2_{L^2(\mu)} - 2\|h\|^2_{L^2(\mu^2)}\big| + 2\|h \star_2^1 h\|_{L^2(\mu)} \\ &\qquad + \sqrt{8}\|h \star_1^1 h\|_{L^2(\mu^2)} + 3\|g \star_1^1 h\|_{L^2(\mu)}. \end{aligned} \tag{6.3}$$

To conclude the proof, one shall use the following relations, holding for every real a, b:

$$\begin{aligned} (a+2b)^2|a+b| &\le (|a| + 2|b|)^2(|a| + |b|) \le (|a| + 2|b|)^3 \\ &\le 4|a|^3 + 32|b|^3. \end{aligned} \tag{6.4}$$

By applying (6.4) in the case $a = g(z)$ and $b = I_1(h(z,\cdot))$, one deduces that

$$\begin{aligned} &\int_Z \mathbb{E}[|D_z F|^2 |D_z L^{-1} F|]\mu(dz) \\ &\quad\le 4\int_Z |g(z)|^3 \mu(dz) + 32\mathbb{E}\int_Z |I_1(h(z,\cdot))|^3 \mu(dz). \end{aligned} \tag{6.5}$$

By using the Cauchy–Schwarz inequality, one infers that

$$
\begin{aligned}
&32\mathbb{E}\int_Z |I_1(h(z,\cdot))|^3\mu(dz) \\
&\quad\le \sqrt{\mathbb{E}\int_Z |I_1(h(z,\cdot))|^4\mu(dz)} \times 32\|h\|_{L^2(\mu^2)}
\end{aligned}
\tag{6.6}
$$

and (6.1) is deduced from the equality

$$\mathbb{E}\int_Z |I_1(h(z,\cdot))|^4\mu(dz) = 2\|h \star_2^1 h\|^2_{L^2(\mu)} + \|h\|^4_{L^4(\mu^2)}.$$

Formula (6.2) is once again an elementary consequence of the Cauchy–Schwarz inequality. □

REMARK 6.2. Consider a sequence of vectors (F_k, H_k), $k \ge 1$, such that: (i) $F_k = I_2(f_k)$, $k \ge 1$, is a sequence of double integrals verifying assumptions (a), (b) and (c) in Example 5.2, and (ii) $H_k = I_1(h_k)$, $k \ge 1$, is a sequence of single integrals with unitary variance. Suppose moreover that the asymptotic relations in (3.7), (5.2) and (5.3) take place. Then, the estimates (6.1)–(6.2) yield that, for every $(\alpha,\beta) \ne (0,0)$, the Wasserstein distance between the law of

$$\frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha F_k + \beta H_k)$$

and the law of $X \sim \mathscr{N}(0,1)$, converges to zero as $k \to \infty$, thus implying that (F_k, H_k) converges in law to a vector (X, X') of i.i.d. standard Gaussian random variables. Roughly speaking, this last fact implies that, when assessing the asymptotic joint Gaussianity of a vector such as (F_k, H_k), one can study *separately* the one-dimensional sequences $\{F_k\}$ and $\{H_k\}$. This phenomenon coincides with the content of Part B of Theorem 3 in [24]. See [15, 18] and [26] for similar results involving vectors of multiple integrals (of arbitrary order) with respect to Gaussian random measures.

7. Applications to nonlinear functionals of Ornstein–Uhlenbeck Lévy processes. As an illustration, in this section, we focus on CLTs related to Ornstein–Uhlenbeck Lévy processes, that is, processes obtained by integrating an exponential kernel of the type

$$x \to \sqrt{2\lambda} \times e^{-\lambda(t-x)}\mathbf{1}_{\{x\le t\}}$$

with respect to an independently scattered random measure. Ornstein–Uhlenbeck Lévy processes have been recently applied to a variety of frameworks, such as finance (where they are used to model stochastic volatility—see, e.g., [1]) or nonparametric Bayesian survival analysis (where they represent random hazard rates—see, e.g., [4, 12, 21]). In particular, in the references [4] and [21] it is shown that one can use some of the CLTs of this section in the context of Bayesian prior specification.

7.1. *Quadratic functionals of Ornstein–Uhlenbeck processes.* We consider the stationary *Ornstein–Uhlenbeck Lévy process* given by

$$Y_t^\lambda = \sqrt{2\lambda}\int_{-\infty}^t \int_{\mathbb{R}} u\exp(-\lambda(t-x))\widehat{N}(du,dx), \qquad t\geq 0, \tag{7.1}$$

where $\widehat{N}$ is a centered Poisson measure over $\mathbb{R}\times\mathbb{R}$, with control measure given by $\nu(du)\,dx$, where $\nu(\cdot)$ is positive, nonatomic and σ-finite. We assume also that $\int u^j\nu(du)<\infty$ for $j=2,3,4,6$, and $\int u^2\nu(du)=1$. In particular, these assumptions yield that

$$\mathbb{E}[(Y_t^\lambda)^2] = 2\lambda\int_{-\infty}^t\int_{\mathbb{R}} u^2 e^{-2\lambda(t-x)}\nu(du)\,dx = 1.$$

The following result has been proved in [24], Theorem 5.

THEOREM 7.1 (See [24]). *For every $\lambda>0$, as $T\to\infty$,*

$$Q(T,\lambda) := \sqrt{T}\left\{\frac{1}{T}\int_0^T (Y_t^\lambda)^2\,dt - 1\right\} \xrightarrow{\text{law}} \sqrt{\frac{1}{\lambda}+c_\nu^2}\times X, \tag{7.2}$$

where $c_\nu^2 := \int u^4\nu(du)$ and $X\sim\mathscr{N}(0,1)$ is a centered standard Gaussian random variable.

By using Theorem 6.1, one can obtain the following Berry–Esséen estimate on the CLT appearing in (7.2) (compare also with Example 3.6).

THEOREM 7.2. *Let $Q(T,\lambda)$, $T>0$, be defined as in (7.2), and set*

$$\tilde{Q}(T,\lambda) := Q(T,\lambda)\Big/\sqrt{\frac{1}{\lambda}+c_\nu^2}.$$

Then, there exists a constant $0<\gamma(\lambda)<\infty$, independent of T and such that

$$d_W(\tilde{Q}(T,\lambda),X) \leq \frac{\gamma(\lambda)}{\sqrt{T}}. \tag{7.3}$$

PROOF. Start by introducing the notation

$$\begin{aligned} &H_{\lambda,T}(u,x;u',x') \\ &\quad = (u\times u')\frac{\mathbf{1}_{(-\infty,T]^2}(x,x')}{T\sqrt{1/\lambda+c_\nu^2}} \\ &\qquad \times\{e^{\lambda(x+x')}(1-e^{-2T})\mathbf{1}_{(x\vee x'\leq 0)} \\ &\qquad\quad + e^{\lambda(x+x')}(e^{-2\lambda(x\vee x')}-e^{-2\lambda T})\mathbf{1}_{(x\vee x'>0)}\}, \end{aligned} \tag{7.4}$$

$$H^*_{\lambda,T}(u,x) = u^2 \frac{\mathbf{1}_{(-\infty,T]}(x)}{T\sqrt{1/\lambda + c_\nu^2}} \{e^{2\lambda x}(1-e^{-2T})\mathbf{1}_{(x\le 0)} + e^{2\lambda x}(e^{-2\lambda x} - e^{-2\lambda T})\mathbf{1}_{(x>0)}\}.$$

As a consequence of the multiplication formula (2.14) and of a standard Fubini argument, one has (see [24], proof of Theorem 5)

$$\tilde{Q}(T,\lambda) = I_1(\sqrt{T}H^*_{\lambda,T}) + I_2(\sqrt{T}H_{\lambda,T}),$$

a combination of a single and of a double integral. To apply Theorem 6.1, we use the following asymptotic relations [that one can verify by resorting to the explicit definitions of $H_{\lambda,T}$ and $H^*_{\lambda,T}$ given in (7.4)], holding for $T \to \infty$:

(a) $$\left|1 - \|\sqrt{T}H^*_{\lambda,T}\|^2_{L^2(d\nu\, dx)} - 2\|\sqrt{T}H_{\lambda,T}\|^2_{L^2((d\nu\, dx)^2)}\right| = O\left(\frac{1}{T}\right),$$

(b) $$\|\sqrt{T}H^*_{\lambda,T}\|^3_{L^3(d\nu\, dx)} \sim \frac{1}{\sqrt{T}},$$

(c) $$\|\sqrt{T}H_{\lambda,T}\|^2_{L^4((d\nu\, dx))^2} \sim \frac{1}{\sqrt{T}},$$

(d) $$\|(\sqrt{T}H_{\lambda,T}) \star_2^1 (\sqrt{T}H_{\lambda,T})\|_{L^2(d\nu\, dx)} \sim \frac{1}{\sqrt{T}},$$

(e) $$\|(\sqrt{T}H_{\lambda,T}) \star_1^1 (\sqrt{T}H_{\lambda,T})\|_{L^2((d\nu\, dx)^2)} \sim \frac{1}{\sqrt{T}},$$

(f) $$\|(\sqrt{T}H^*_{\lambda,T}) \star_1^1 (\sqrt{T}H_{\lambda,T})\|_{L^2(d\nu\, dx)} \sim \frac{1}{\sqrt{T}}.$$

The conclusion is obtained by using the estimates (6.1) and (6.2) and applying Theorem 6.1. □

7.2. *Berry–Esséen bounds for arbitrary tensor powers of Ornstein–Uhlenbeck kernels*. Let $\widehat{N}$ be a centered Poisson measure over $\mathbb{R}\times\mathbb{R}$, with control measure given by $\mu(du,dx) = \nu(du)\,dx$, where $\nu(\cdot)$ is positive, nonatomic and σ-finite. We assume that $\int u^2\nu(du) = 1$ and $\int u^4\nu(du) < \infty$. Fix $\lambda > 0$, and, for every $t \ge 0$, define the Ornstein–Uhlenbeck kernel

$$f_t(u,x) = u \times \sqrt{2\lambda}\exp\{-\lambda(t-x)\}\mathbf{1}_{\{x\le t\}}, \qquad (u,x) \in \mathbb{R}\times\mathbb{R}. \tag{7.5}$$

For every fixed $q \ge 2$, we define the qth *tensor power* of f_t, denoted by $f_t^{\otimes q}$, as the symmetric kernel on $(\mathbb{R}\times\mathbb{R})^q$ given by

$$f_t^{\otimes q}(u_1,x_1;\dots;u_q,x_q) = \prod_{j=1}^q f_t(u_j,x_j). \tag{7.6}$$

We sometimes set $y = (u, x)$. Note that, for every $t \geq 0$, one has that $\int f_t^2(y)\mu(dy) = \int f_t^2(u,x)\nu(du)\,dx = 1$, and therefore $f_t^{\otimes q} \in L_s^2(\mu^q)$; it follows that the multiple integral

$$Z_t(q) := I_q(f_t^{\otimes q})$$

is well defined for every $t \geq 0$.

REMARK 7.3. Fix $q \geq 2$, and suppose that $\int |u|^j \, d\nu < \infty$, $\forall j = 1, \dots, 2q$. Then, one can prove that $Z_t(1)^q$ is square-integrable and also that the random variable $Z_t(q)$ coincides with the projection of $Z_t(1)^q$ on the qth Wiener chaos associated with $\widehat{N}$. This fact can be easily checked when $q = 2$: indeed (using Proposition 2.5 in the case $f = g = f_t$) one has that

$$Z_t(1)^2 = I_1(f_t^2) + I_2(f_t^{\otimes 2}) = I_1(f_t^2) + Z_t(2),$$

thus implying the desired relation. The general case can be proved by induction on q.

The main result of this section is the following application of Theorems 4.2 and 5.1.

THEOREM 7.4. *Fix $\lambda > 0$ and $q \geq 2$, and define the positive constant $c = c(q, \lambda) := 2(q-1)!/\lambda$. Then, one has that, as $T \to \infty$,*

$$M_T(q) := \frac{1}{\sqrt{cT}} \int_0^T Z_t(q)\,dt \xrightarrow{\text{law}} X \sim \mathscr{N}(0,1), \tag{7.7}$$

and there exists a finite constant $\rho = \rho(\lambda, q, \nu) > 0$ such that, for every $T > 0$,

$$d_W(M_T(q), X) \leq \frac{\rho}{\sqrt{T}}. \tag{7.8}$$

PROOF. The crucial fact is that, for each T, the random variable $M_T(q)$ has the form of a multiple integral, that is, $M_T(q) = I_q(F_T)$, where $F_T \in L_s^2(\mu^q)$ is given by

$$F_T(u_1, x_1; \dots; u_q, x_q) = \frac{1}{\sqrt{cT}} \int_0^T f_t^{\otimes q}(u_1, x_1; \dots; u_q, x_q)\,dt,$$

where $f_t^{\otimes q}$ has been defined in (7.6). By using the fact that the support of F_T is contained in the set $(\mathbb{R} \times (-\infty, T])^q$ as well as the assumptions on the second and fourth moments of ν, one easily deduces that the technical condition (4.8) (with F_T replacing f) is satisfied for every $T \geq 0$. According to Theorems 4.2 and 5.1, both claims (7.7) and (7.8) are proved, once we show that, as $T \to \infty$, one has that

$$|1 - \mathbb{E}(M_T(q))^2| \sim 1/T, \tag{7.9}$$

and also that

$$\|F_T\|^2_{L^4(\mu^q)} = O\big(1/\sqrt{T}\big) \tag{7.10}$$

and

$$\|F_T \star_r^l F_T\|_{L^2(\mu^{2q-r-l})} = O(1/T) \qquad \forall r = 1, \dots, q, \forall l = 1, \dots, r \wedge (q-1) \tag{7.11}$$

[relation (7.9) improves the bounds in Theorem 4.2]. In order to prove (7.9)–(7.11), for every $t_1, t_2 \geq 0$ we introduce the notation

$$\langle f_{t_1}, f_{t_2}\rangle_\mu = \int_{\mathbb{R}\times\mathbb{R}} f_{t_1}(y) f_{t_2}(y)\mu(dy) = e^{-\lambda(t_1+t_2)} e^{2\lambda(t_1\wedge t_2)} \tag{7.12}$$

(recall that $\int u^2\,d\nu = 1$) and also, for $t_1, t_2, t_3, t_4 \geq 0$,

$$\langle f_{t_1} f_{t_2}, f_{t_3} f_{t_4}\rangle_\mu = \int_{\mathbb{R}\times\mathbb{R}} f_{t_1}(y) f_{t_2}(y) f_{t_3}(y) f_{t_4}(y)\mu(dy) \tag{7.13}$$

$$= \left[\int_{\mathbb{R}} u^4 \nu(du)\right] \times \lambda e^{-\lambda(t_1+t_2+t_3+t_4)} e^{4\lambda(t_1\wedge t_2\wedge t_3\wedge t_4)}. \tag{7.14}$$

To prove (7.9), one uses the relation (7.12) to get

$$\mathbb{E}[M_T(q)^2] = \frac{q!}{cT}\int_0^T\int_0^T \langle f_{t_1}, f_{t_2}\rangle_\mu^q \,dt_1\,dt_2 = 1 - \frac{1}{Tq\lambda}(1 - e^{-\lambda q T}).$$

In the remainder of the proof, we will write κ in order to indicate a strictly positive finite constant independent of T, that may change from line to line. To prove (7.10), one uses the fact that

$$\int_{(\mathbb{R}\times\mathbb{R})^q} F_T^4\,d\mu^q = \frac{1}{c^2T^2}\int_0^T\int_0^T\int_0^T\int_0^T \langle f_{t_1} f_{t_2}, f_{t_3} f_{t_4}\rangle_\mu^q \,dt_1\,dt_2\,dt_3\,dt_4 \leq \frac{\kappa}{T},$$

where the last relation is obtained by resorting to the explicit representation (7.14), and then by evaluating the restriction of the quadruple integral to each simplex of the type $\{t_{\pi(1)} > t_{\pi(2)} > t_{\pi(3)} > t_{\pi(4)}\}$, where π is a permutation of the set $\{1, 2, 3, 4\}$. We shall now verify the class of asymptotic relations (7.11) for $r = q$ and $l = 1, \dots, q$. With $y = (u, x)$, one has

$$F_T \star_q^l F_T(y_1, \dots, y_{q-l}) = \frac{1}{cT}\int_0^T\int_0^T \left[\prod_{i=1}^{q-l} f_{t_1}(y_i) f_{t_2}(y_i)\right]\langle f_{t_1} f_{t_2}\rangle_\mu^l \,dt_1\,dt_2$$

and hence

$$\begin{aligned}
&\|F_T \star_q^l F_T\|^2_{L^2(\mu^{q-l})} \\
&\quad = \frac{\kappa}{T^2}\int_0^T\int_0^T\int_0^T\int_0^T \langle f_{t_1} f_{t_2}, f_{t_3} f_{t_4}\rangle_\mu^{q-l} \langle f_{t_1} f_{t_2}\rangle_\mu^l \langle f_{t_3} f_{t_4}\rangle_\mu^l \,dt_1\,dt_2\,dt_3\,dt_4 \\
&\quad \leq \frac{\kappa}{T},
\end{aligned}$$

where the last relation is verified by first using (7.12)–(7.14), and then by assessing the restriction of the quadruple integral to each one of the $4! = 24$ simplexes of the type $\{t_{\pi(1)} > t_{\pi(2)} > t_{\pi(3)} > t_{\pi(4)}\}$. To deal with (7.11) in the case $r = 1, \dots, q-1$ and $l = 1, \dots, r$, one uses the fact that

$$
\begin{aligned}
&F_T \star_r^l F_T(y_1, \dots, y_{r-l}, w_1, \dots, w_{q-r}, z_1, \dots, z_{q-r}) \\
&\quad = \frac{1}{cT} \int_0^T \int_0^T \left[\prod_{i=1}^{r-l} f_{t_1}(y_i) f_{t_2}(y_i) \right] \\
&\qquad\qquad \times \big(f_{t_1}(w_1) \cdots f_{t_1}(w_{q-r}) f_{t_2}(z_1) \cdots f_{t_2}(z_{q-r}) \big) \langle f_{t_1} f_{t_2} \rangle_\mu^l \, dt_1 \, dt_2
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\| F_T \star_r^l F_T \|^2_{L^2(\mu^{2q-r-l})} \\
&\quad = \frac{\kappa}{T^2} \int_0^T \int_0^T \int_0^T \int_0^T \langle f_{t_1} f_{t_2}, f_{t_3} f_{t_4} \rangle_\mu^{r-l} \langle f_{t_1} f_{t_3} \rangle_\mu^{q-r} \\
&\qquad\qquad \times \langle f_{t_2} f_{t_4} \rangle_\mu^{q-r} \langle f_{t_1} f_{t_2} \rangle_\mu^l \langle f_{t_3} f_{t_4} \rangle_\mu^l \, dt_1 \, dt_2 \, dt_3 \, dt_4 \\
&\quad \le \frac{\kappa}{T},
\end{aligned}
$$

where the last relation is once again obtained by separately evaluating each restriction of the quadruple integral over a given simplex. This concludes the proof. $\square$

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