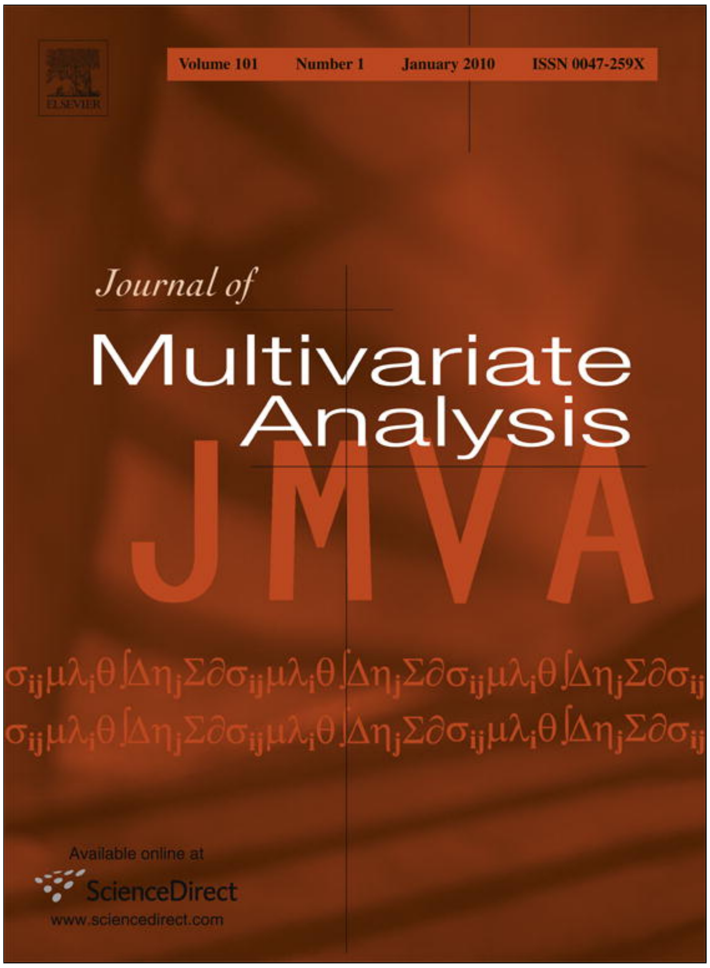
Provided for non-commercial research and education use.
Not for reproduction, distribution or commercial use.



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

http://www.elsevier.com/copyright

Author's personal copy

Journal of Multivariate Analysis 101 (2010) 77–100

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Representations of SO(3) and angular polyspectra

D. Marinucci [a,*], G. Peccati [b]

[a] Dipartimento di Matematica, Universita' di Roma Tor Vergata, via della Ricerca Scientifica, 1, 00133 Roma, Italy

[b] Centre de Recherche Modal'X, Université Paris Ouest Nanterre la Défense, 200, Avenue de la République, 92000 Nanterre, France

ARTICLE INFO

Article history:
Received 7 October 2008
Available online 5 May 2009

AMS 2000 classifications:
60G10
60G35
20C12
20C35

Keywords:
Group representations
Isotropy
Polyspectra
Spherical random fields

ABSTRACT

We characterize the angular polyspectra, of arbitrary order, associated with isotropic fields defined on the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Our techniques rely heavily on group representation theory, and specifically on the properties of Wigner matrices and Clebsch–Gordan coefficients. The findings of the present paper constitute a basis upon which one can build formal procedures for the statistical analysis and the probabilistic modelization of the Cosmic Microwave Background radiation, which is currently a crucial topic of investigation in cosmology. We also outline an application to random data compression and "simulation" of Clebsch–Gordan coefficients.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

The connection between probability theory and group representation theory has led to a long tradition of fruitful interactions. A well-known reference is provided by [1]; see e.g. [2, Section 40–41], [3–8], and the references therein, for other relevant contributions. In this paper we shall focus in particular on the connection between the probabilistic notion of *isotropy*, i.e. invariance in law under the action of a group, and the representation theory of the group itself. One instance of this connection is well-known, i.e. the celebrated Peter–Weyl Theorem, which allows the construction of spectral representations for isotropic random fields on homogeneous spaces of general compact groups, see [9] for a general construction and [10,11] for examples related, respectively, to the torus and the sphere. Our aim here is to use these representations in order to characterize random fields by means of a higher order spectral theory; in particular, one of our main goals will be to establish the link between the so-called *polyspectra* (or higher order spectra) and alternative (tensor product and direct sum) representations of the underlying isotropy group. In particular, we shall provide a general expression for higher order spectra of isotropic spherical random fields in terms of convolutions of Clebsch–Gordan or Wigner coefficients. The latter were introduced in Mathematics in the XIX century for the analysis of Algebraic Invariants; they have since then played a crucial role in the development of Quantum Physics in the XX century (see for instance [12] for a comprehensive reference); their role in Group Representation theory will be discussed below, while more details can be found for instance in [13].

Our analysis may have an intrinsic mathematical interest, but it is also strongly motivated by applications to Physics and Cosmology. Concerning the latter, the analysis of higher order spectra for isotropic spherical random fields is currently at the core of several research efforts which are related to the analysis of Cosmic Microwave Background (CMB) radiation

* Corresponding author.
E-mail address: marinucc@mat.uniroma2.it (D. Marinucci).

0047-259X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2009.04.017

Author's personal copy

data, see for instance [14,15] for a general introduction and [16–19] for some references on the bi- and trispectrum. A general characterization of the theoretical properties of higher order angular power spectra can yield several insights into the statistical analysis of the massive datasets that are or will be made available by satellite experiments such as *WMAP* or *Planck*. For instance, the current understanding of the behaviour of the bispectrum for some simple physical models has already led to many applications (see [20–22]), aiming at obtaining constraints on nonlinearity parameters of utmost physical significance; needless to say, a proper understanding of higher order spectra can lead to more efficient statistical procedures and better constraints, which may help to solve some of the important scientific issues at stake in CMB analysis (primarily a proper understanding of the Big Bang *inflationary* dynamics, which is tightly linked with the CMB nonlinear structure, see [15,23–25]).

The relevance of the current results need not be limited to cosmological applications. Indeed, the analysis of spherical random fields has currently led to remarkable developments in the Geophysical and Planetary Sciences, and even in Medical Imaging (see e.g. [26–28]). Moreover, we shall show below how the relationships established in this paper lead very naturally to some numerical algorithms for the estimation of Clebsch–Gordan and Wigner coefficients. The latter represent probability amplitudes of quantum interactions: as such, a rich literature in Mathematical Physics has been concerned with recipes for their numerical estimation. Our procedure lends itself to easy implementation and can be simply extended to very general compact groups, although in this paper we focus solely on SO(3).

The plan of this paper is as follows: in Section 2 we introduce our general probabilistic setting and provide some preliminary notation and background material. In Section 3 we discuss basic facts on representation theory, while in Sections 4 and 5 we obtain our main results, including the aforementioned explicit characterization of polyspectra. These results are applied in Section 6 to derive explicit expressions in some important cases (such as χ^2 random fields). Section 7 is devoted to further issues that we see as the seeds for future research: they concern, in particular, the connection with the representation theory of the symmetric group, and the Monte Carlo estimation of Clebsch–Gordan coefficients.

In the subsequent sections, every random element is defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$.

2. General setting

In this paper, we focus on real-valued, centered, square-integrable and isotropic random fields on the sphere $S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$. A centered and square integrable random field T on S^2 is just a collection of random variables of the type $T = \left\{T(x) : x \in S^2\right\}$ such that, for every $x \in S^2$, $ET(x) = 0$ and $ET^2(x) < \infty$. In the following, whenever we write that T is a field on S^2, we will implicitly assume that T is real-valued, centered and square-integrable. From now on, we shall distinguish between two notions of isotropy, which we name *strong isotropy* and *weak isotropy of order n* ($n \geq 2$).

Strong isotropy— The field T is said to be *strongly isotropic* if, for every $k \in \mathbb{N}$, every $x_1, \ldots, x_k \in S^2$ and every $g \in SO(3)$ (the group of rotations in $\mathbb{R}^3$) we have

$$\{T(x_1), \ldots, T(x_k)\} \stackrel{d}{=} \{T(gx_1), \ldots, T(gx_k)\}, \tag{2.1}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Weak isotropy— The field T is said to be *n-weakly isotropic* ($n \geq 2$) if $E|T(x)|^n < \infty$ for every $x \in S^2$, and if, for every $x_1, \ldots, x_n \in S^2$ and every $g \in SO(3)$,

$$E\left[T(x_1) \times \cdots \times T(x_n)\right] = E\left[T(gx_1) \times \cdots \times T(gx_n)\right].$$

The following statement, whose proof is elementary, indicates some relations between the two notions of isotropy described above.

Proposition 1. 1. *A strongly isotropic field with finite moments of some order $n \geq 2$ is also n-weakly isotropic.*
2. *Suppose that the field T is n-weakly isotropic for every $n \geq 2$ (in particular, $E|T(x)|^n < \infty$ for every $n \geq 2$ and every $x \in S^2$) and that, for every $k \geq 1$ and every $(x_1, \ldots, x_k)$, the law of the vector $\{T(x_1), \ldots, T(x_k)\}$ is determined by its moments. Then, T is also strongly isotropic.*

Now suppose that T is a strongly isotropic field, and denote by dx the Lebesgue measure on S^2. Since the variance $ET(x)^2$ is finite and independent of x (by isotropy), one deduces immediately that

$$E\left[\int_{S^2} T(x)^2 \mathrm{d}x\right] < \infty,$$

from which one infers that the random path $x \longmapsto T(x)$ is a.s. square integrable with respect to the Lebesgue measure. Then, it is a standard result that the following spectral representation holds:

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x), \quad \text{where } a_{lm} \triangleq \int_{S^2} T(x)\overline{Y_{lm}(x)}\mathrm{d}x, \tag{2.2}$$

Author's personal copy

and where the complex-valued functions $\{Y_{lm} : l \geq 0, m = -l, \ldots, l\}$ are the so-called *spherical harmonics*, to be defined below. The spectral representation (2.2) must be understood in the $L^2(\Omega \times S^2)$ sense, i.e.

$$\lim_{L\to\infty} E \left\| T - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm} \right\|_{L^2(S^2)}^2 = 0,$$

where $L^2(S^2)$ is the complex Hilbert space of functions on S^2, which are square-integrable with respect to dx. If, moreover, the trajectories of $T(x)$ are a.s. continuous, then the representation (2.2) holds pointwise, i.e.

$$\lim_{L\to\infty} \left\{ T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \right\} = 0 \quad \text{for all } x \in S^2, \text{ a.s.-} P,$$

see for instance [29] or [8]. The spherical harmonics $\{Y_{lm}\}_{m=-l,\ldots,l}$ are the eigenfunctions of the Laplace-Beltrami operator on the sphere, denoted by Δ_{S^2}, satisfying the relation $\Delta_{S^2} Y_{lm} = -l(l+1)Y_{lm}$. These functions can be represented by means of spherical coordinates $x = (\theta, \varphi)$ as follows:

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) \exp(im\varphi), \quad \text{for } m > 0,$$

$$Y_{lm}(\theta, \varphi) = (-1)^m \overline{Y_{l,-m}}(\theta, \varphi), \quad \text{for } m < 0, 0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi,$$

where $P_{lm}(\cos\theta)$ denotes the associated Legendre polynomial of degree l, m, i.e.

$$P_{lm}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \qquad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l, \quad m = 0, 1, 2, \ldots, l, l = 0, 1, 2, 3, \ldots.$$

The random spherical harmonics coefficients $\{a_{lm}\}$ appearing in (2.2) form a triangular array of zero-mean and square-integrable random variables, which are complex-valued for $m \neq 0$ and such that $E a_{lm} \overline{a_{l'm'}} = \delta_l^{l'} \delta_m^{m'} C_l$, the bar denoting complex conjugation. Here, and for the rest of the paper, the symbol δ_b^a is equal to one if $a = b$ and zero otherwise. We also write $C_l = E|a_{lm}|^2$, $l \geq 0$, to indicate the *angular power spectrum* of T (we stress that the quantity C_l does not depend on m — see e.g. [30] for a proof of this fact). Observe that, by definition of the spherical harmonics, $a_{lm} = (-1)^m \overline{a_{l-m}}$. Note also that a convenient route to derive (2.2) is by means of an appropriate version of the *stochastic Peter–Weyl theorem* — see for instance [31] or [9], as well as Section 3.1.

Observe that the representation (2.2) still holds for fields $\{T(x)\}$ that are not necessarily isotropic, but such that the random path $x \longmapsto T(x)$ is P-a.s. square integrable with respect to the Lebesgue measure dx. Indeed, if the last property holds, then one has that, P-almost surely,

$$\lim_{L\to\infty} \int_{S^2} \left(T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \right)^2 dx = 0. \tag{2.3}$$

In this case, however, none of the previously stated properties on the array $\{a_{lm}\}$ holds in general. By an argument similar to those displayed above, a sufficient condition to have that $x \longmapsto T(x)$ is P-a.s. Lebesgue-square integrable is that $\sup_{x\in S^2} ET(x)^2 < \infty$.

The next result, that we record for future reference, is proved in [30].

Proposition 2. *Let T be a centered, square-integrable and strongly isotropic random field. Let the coefficients $\{a_{lm}\}$ be defined according to (2.2). Then, for every l, m, one has that $E|a_{lm}|^2 < \infty$. Moreover, for every $l \geq 1$, the coefficients $\{a_{l0}, \ldots, a_{ll}\}$ are independent if and only if they are Gaussian. If the vector $\{a_{l0}, \ldots, a_{ll}\}$ is Gaussian, one also has that $\Re(a_{lm})$ and $\Im(a_{lm})$ are independent and identically distributed for every fixed $m = 1, \ldots, l$ ($\Re(z)$ and $\Im(z)$ stand, respectively, for the real and imaginary parts of z).*

The following result formalizes the fact that, in general, one cannot deduce strong isotropy from weak isotropy. The proof makes use of Proposition 1.

Proposition 3. *For every $n \geq 2$, there exists a n-weakly isotropic field T such that T is not strongly isotropic.*

Proof. Fix $l \geq 1$, and consider a vector

$$b_m, \quad m = -l, \ldots, l,$$

of centered complex-valued random variables such that: (i) b_0 is real, (ii) $b_{-m} = (-1)^m \overline{b_m}$ ($m = 1, \ldots, l$), (iii) the vector $\{b_0, \ldots, b_l\}$ is not Gaussian and is composed of independent random variables, (iv) for every $k = 1, \ldots, n$, the (possibly mixed) moments of order k of the variables $\{b_0, \ldots, b_l\}$ coincide with those of a vector $\{a_0, \ldots, a_l\}$ of independent, centered and complex-valued Gaussian random variables with common variance C_l and such that a_0 is real and, for every $m = 1, \ldots, l$,

Author's personal copy

the real and imaginary parts of a_m are independent and identically distributed (the existence of a vector such as $\{b_0, \ldots, b_l\}$ is easily proved). Now define the two fields

$$T(x) = \sum_{m=-l}^{l} b_m Y_{lm}(x) \quad \text{and} \quad T^*(x) = \sum_{m=-l}^{l} a_m Y_{lm}(x).$$

By Proposition 2, T^* is strongly isotropic, and also n-weakly isotropic by Proposition 1. By construction, one also has that T is n-weakly isotropic. However, T cannot be strongly isotropic, since this would violate Proposition 2 (indeed, if T was isotropic, one would have an example of an isotropic field whose harmonic coefficients $\{b_0, \ldots, b_l\}$ are independent and non-Gaussian). ■

In what follows, we use the symbol $A \otimes B$ to indicate the *Kronecker product* between two matrices A and B. Given $n \geq 2$, we denote by $\Pi(n)$ the class of partitions of the set $\{1, \ldots, n\}$. Given an element $\pi \in \Pi(n)$, we write $\pi = \{b_1, \ldots, b_k\}$ to indicate that the sets $b_j \subseteq \{1, \ldots, n\}$, $j = 1, \ldots, k$, are the *blocks* of π. The blocks of a partition are always listed according to the lexicographic order, that is: the block b_1 always contains 1, the block b_2 contains the least element of $\{1, \ldots, n\}$ not contained in b_1, and so on. Also the elements within each block b_j are written in increasing order. For instance, if a partition π of $\{1, \ldots, 5\}$ is composed of the blocks $\{1, 3\}$, $\{5, 4\}$ and $\{2\}$, we will write π in the form $\pi = \{\{1, 3\}, \{2\}, \{4, 5\}\}$.

Definition A. (A1) Let the field T admit the representation (2.2), and suppose that, for some $n \geq 2$, one has that $E|a_{lm}|^n < \infty$ for every l, m. Then, T is said to have *finite spectral moments* of order n.

(A2) Suppose that T has finite spectral moments of order $n \geq 2$, and, for $l \geq 0$, use the notation

$$a_{l.} = (a_{l-l}, \ldots, a_{l0}, \ldots, a_{ll}). \tag{2.4}$$

The *polyspectrum of order* $n-1$, associated with T, is given by the collection of vectors

$$S_{l_1 \ldots l_n} = E\left[a_{l_1.} \otimes a_{l_2.} \otimes \cdots \otimes a_{l_n.}\right], \tag{2.5}$$

where $0 \leq l_1, l_2, \ldots, l_n$. Note that the vector $S_{l_1 \ldots l_n}$ appearing in (2.5) has dimension $(2l_1 + 1) \times \cdots \times (2l_n + 1)$.

(A3) Suppose that T has finite spectral moments of order $n \geq 2$. The (mixed) *cumulant polyspectrum of order* $n-1$, associated with T, is given by the vectors

$$S^c_{l_1 \ldots l_n} = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! E\left[\otimes_{i \in b_1} a_{l_i.}\right] \otimes \cdots \otimes E\left[\otimes_{i \in b_k} a_{l_i.}\right], \tag{2.6}$$

where $0 \leq l_1, l_2, \ldots, l_n$, and, for every block $b_j = \{i_1, \ldots, i_p\}$, we use the notation

$$E\left[\otimes_{i \in b_j} a_{l_i.}\right] = E\left[a_{l_{i_1}.} \otimes \cdots \otimes a_{l_{i_p}.}\right]$$

(recall that we always list the elements of b_j in such a way that $i_1 \leq \cdots \leq i_p$). Plainly, the vector $S^c_{l_1 \ldots l_n}$ in (2.6) has also dimension $(2l_1 + 1) \times \cdots \times (2l_n + 1)$.

Remark. Suppose that T has finite spectral moments of order $n \geq 2$. Then, by selecting frequencies $l_1 = l_2 = \cdots = l_3 = l \geq 0$, one obtains that

$$S^c_{\underbrace{l \ldots l}_{n \text{ times}}} := S^c_{l \ldots l}(n) = \sum_{\pi = \{b_1, \ldots, b_k\} \in \Pi(n)} (-1)^{k-1} (k-1)! E\left[(a_{l.})^{\otimes |b_1|}\right] \otimes \cdots \otimes E\left[(a_{l.})^{\otimes |b_k|}\right] \tag{2.7}$$

where $|b_j|$ stands for the size of the block b_j, and we use the notation

$$(a_{l.})^{\otimes |b_j|} = \underbrace{a_{l.} \otimes \cdots \otimes a_{l.}}_{|b_j| \text{ times}}.$$

3. Preliminary material

3.1. *Representation theory for* SO(3)

We start by reviewing some background material on the special group of rotations SO(3), i.e. the space of 3×3 real matrices A such that $A'A = I_3$ (the three-dimensional identity matrix) and $\det(A) = 1$. We first recall that each element $g \in SO(3)$ can be parametrized by the set $(\varphi, \vartheta, \psi)$ of the so-called *Euler angles* ($0 \leq \varphi < 2\pi$, $0 \leq \vartheta \leq \pi$, $0 \leq \psi < 2\pi$); indeed each rotation in $\mathbb{R}^3$ can be realized sequentially as

$$A = A(g) = R(\psi, \vartheta, \varphi) = R_z(\varphi) R_x(\vartheta) R_z(\psi) \tag{3.8}$$

Author's personal copy

where $R_z(\varphi), R_x(\vartheta), R_z(\psi) \in \mathrm{SO}(3)$ can be expressed by means of the following general definitions, valid for every angle α,

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

The representation (3.8) is unique except for $\vartheta = 0$ or $\vartheta = \pi$, in which case only the sum $\varphi + \psi$ is determined. In words, the rotation is realized by rotating first by ψ around the axis z, then rotating around the initial x axis by ϑ, then rotating by φ around the initial z axis. It is clear that the last two rotations identify one point on the sphere, so the whole operation could be also interpreted as rotating by ψ the tangent plane at the North Pole, and then moving the latter to a location in S^2.

In these coordinates, a complete set of irreducible matrix representations for SO(3) is provided by the Wigner's D matrices $D^l(\psi, \vartheta, \varphi) = \left\{D^l_{mn}(\psi, \vartheta, \varphi)\right\}_{m,n=-l,\dots,l}$, of dimensions $(2l+1) \times (2l+1)$ for $l = 0, 1, 2, \dots$; we refer to classical textbooks, such as [13,2] or [1], for any unexplained definition or result concerning group representation theory. An analytic expression for the elements of Wigner's D matrices is provided by

$$D^l_{mn}(\psi, \vartheta, \varphi) = \mathrm{e}^{-\mathrm{i}n\psi} d^l_{mn}(\vartheta) \mathrm{e}^{\mathrm{i}m\varphi}, \quad m, n = -(2l+1), \dots, 2l+1$$

where the indices m, n indicate, respectively, columns and rows, and

$$d^l_{mn}(\vartheta) = (-1)^{l-n} \left[(l+m)!(l-m)!(l+n)!(l-n)!\right]^{1/2} \sum_k (-1)^k \frac{\left(\cos\frac{\vartheta}{2}\right)^{m+n+2k} \left(\sin\frac{\vartheta}{2}\right)^{2l-m-n-2k}}{k!(l-m-k)!(l-n-k)!(m+n+k)!},$$

and the sum runs over all k such that the factorials are non-negative; see [12, Chapter 4] for a huge collection of alternative expressions. Here we simply recall that the elements of $D^l(\psi, \vartheta, \varphi)$ are related to the spherical harmonics by the relationship

$$D^l_{0m}(\varphi, \vartheta, \psi) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y_{l-m}(\vartheta, \varphi) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\vartheta, \varphi). \tag{3.9}$$

In other words, the spherical harmonics correspond (up to a constant) to the elements of the "central" column in the Wigner's D matrix. Such matrices operate irreducibly and equivalently on $(2l+1)$ spaces (the so-called isotypical spaces), each of them spanned by a different column n of the matrix representation itself. The elements of column n correspond to the so-called *spin n* spherical harmonics, which enjoy great importance in particle physics and in harmonic expansions for tensor valued random fields, see [32]. In this paper, we restrict our attention only to the usual $n = 0$ spherical harmonics, which correspond to usual scalar functions.

Remark. By exploiting relation (3.9), it is not difficult to show that the usual spectral representation for random fields on the sphere, as given in (2.2), is just the stochastic Peter–Weyl Theorem on the quotient space $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$. Indeed, by the stochastic Peter–Weyl Theorem (see e.g. [9]) we obtain, for any square integrable and isotropic random field $\{T(g) : g \in \mathrm{SO}(3)\}$,

$$T(g) = T(\varphi, \vartheta, \psi) = \sum_l \sum_{m,n} a_{lmn} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{mn}(\varphi, \vartheta, \psi),$$

where $\mathrm{d}g$ is the Haar (uniform) measure on SO(3) with total mass $8\pi^2$. Now if we consider the restriction of $T(g)$ to $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, denoted by $T_{S^2}(\varphi, \vartheta)$, we deduce that

$$\begin{aligned} a_{lmn} &= \int_{\mathrm{SO}(3)} T_{S^2}(g) \sqrt{\frac{2l+1}{8\pi^2}} \overline{D}^l_{mn}(g) \mathrm{d}g \\ &= \int_{S^2} T_{S^2}(\varphi, \vartheta) \left\{ \int_0^{2\pi} \mathrm{e}^{\mathrm{i}n\psi} \mathrm{d}\psi \right\} \sqrt{\frac{2l+1}{8\pi^2}} d^l_{mn}(\vartheta) \mathrm{e}^{-\mathrm{i}m\varphi} \sin\vartheta \mathrm{d}\varphi \mathrm{d}\vartheta, \\ &= \int_{S^2} T_{S^2}(\varphi, \vartheta) \delta^0_n (2\pi) \sqrt{\frac{2l+1}{8\pi^2}} d^l_{mn}(\vartheta) \mathrm{e}^{-\mathrm{i}m\varphi} \sin\vartheta \mathrm{d}\varphi \mathrm{d}\vartheta, \end{aligned}$$

the second equality following from the fact that $T_{S^2}(g)$ is constant with respect to ψ. We can thus conclude that

$$a_{lmn} = \begin{cases} 0 & \text{for } n \neq 0 \\ \sqrt{2\pi}\, a_{lm} & \text{for } n = 0, \end{cases}$$

where the array $\{a_{lm}\}$ is defined by (2.2).

Author's personal copy

3.2. The Clebsch–Gordan matrices

It follows from standard representation theory that we can exploit the family $\{D^l\}_{l=0,1,,2,...}$ to build alternative (reducible) representations, either by taking the tensor product family $\{D^{l_1} \otimes D^{l_2}\}_{l_1,l_2}$, or by considering direct sums $\left\{\oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l\right\}_{l_1,l_2}$. These representations have dimensions

$$(2l_1+1)(2l_2+1) \times (2l_1+1)(2l_2+1)$$

and are unitarily equivalent, whence there exists a unitary matrix $C_{l_1l_2}$ such that

$$\left\{D^{l_1} \otimes D^{l_2}\right\} = C_{l_1l_2} \left\{ \bigoplus_{l=|l_2-l_1|}^{l_2+l_1} D^l \right\} C^*_{l_1l_2}. \tag{3.10}$$

The matrix $C_{l_1l_2}$ is a $\{(2l_1+1)(2l_2+1) \times (2l_1+1)(2l_2+1)\}$ block matrix, whose blocks, of dimensions $(2l_2+1)\times(2l+1)$, are customarily denoted by $C^l_{l_1(m_1)l_2}$, $m_1 = -l_1, \dots, l_1$; the elements of such a block are indexed by m_2 (over rows) and m (over columns; note that $m = -(2l+1), \dots, 2l+1$). More precisely,

$$C_{l_1l_2} = \left[C^{l.}_{l_1(m_1)l_2.}\right]_{m_1=-l_1,\dots,l_1;\, l=|l_2-l_1|,\dots,l_2+l_1} \tag{3.11}$$

$$C^{l.}_{l_1(m_1)l_2.} = \left\{C^{lm}_{l_1m_1l_2m_2}\right\}_{m_2=-l_2,\dots,l_2;\, m=-l,\dots,l}. \tag{3.12}$$

Remark. The fact that the two matrices $D^{l_1} \otimes D^{l_2}$ and $\oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l$ have the same dimension follows from the elementary relation (valid for any integers $l_1, l_2 \geq 0$):

$$\sum_{l=|l_2-l_1|}^{l_1+l_2} (2l+1) = (2l_1+1)(2l_2+1). \tag{3.13}$$

By induction, one also obtains that, for every $n \geq 3$,

$$\sum_{\lambda_1=|l_2-l_1|}^{l_1+l_2} \sum_{\lambda_2=|l_3-\lambda_1|}^{\lambda_1+l_3} \cdots \sum_{\lambda_{n-1}=|l_n-\lambda_{n-2}|}^{\lambda_{n-2}+l_n} (2\lambda_{n-1}+1) = \prod_{j=1}^{n} \left(2l_j+1\right), \tag{3.14}$$

for any integers $l_1, \dots, l_n \geq 0$ (relation (3.14) is needed in Section 5.2).

The *Clebsch–Gordan coefficients* for SO(3) are then defined as the collection $\left\{C^{lm}_{l_1m_1l_2m_2}\right\}$ of the the elements of the unitary matrices $C_{l_1l_2}$. These coefficients were introduced in Mathematics in the XIX century, as motivated by the analysis of invariants in Algebraic Geometry; in the 20th century, they have gained an enormous importance in the quantum theory of angular momentum, where $C^{lm}_{l_1m_1l_2m_2}$ represents the *probability amplitude* that two particles with total angular momentum l_1, l_2 and momentum projection on the z-axis m_1 and m_2 are coupled to form a system with total angular momentum l and projection m (see e.g. [33]). Their use in the analysis of isotropic random fields is much more recent, see for instance [16] and the references therein.

Remark (*More on the Structure of the Clebsch–Gordan Matrices*). To ease the reading of the subsequent discussion, we provide an alternative way of building a Clebsch–Gordan matrix $C_{l_1l_2}$, starting from any enumeration of its entries. Fix integers $l_1, l_2 \geq 0$ such that $l_1 \leq l_2$ (this is just for notational convenience), and consider the Clebsch–Gordan coefficients $\left\{C^{lm}_{l_1m_1l_2m_2}\right\}$ given in (3.11)–(3.12). According to the above discussion, we know that: (i) $-l_i \leq m_i \leq l_i$ for $i = 1, 2$, (ii) $l_2 - l_1 \leq l \leq l_1 + l_2$, (iii) $-l \leq m \leq l$, and (iv) the symbols (l_1, m_1, l_2, m_2) label rows, whereas the pairs (l, m) are attached to columns. Now introduce the total order $\prec_c$ on the "column pairs" (l, m), by setting that $(l, m) \prec_c (l', m')$, whenever either $l < l'$ or $l = l'$ and $m < m'$. Analogously, introduce a total order $\prec_r$ over the "row symbols" (l_1, m_1, l_2, m_2), by setting that $(l_1, m_1, l_2, m_2) \prec_r (l'_1, m'_1, l'_2, m'_2)$, if either $m_1 < m'_1$, or $m_1 = m'_1$ and $m_2 < m'_2$ (recall that l_1 and l_2 are fixed). One can check that the set of column pairs (resp. row symbols) can now be written as a *saturated chain*[1] with respect to $\prec_c$ (resp. $\prec_r$) with a least element given by $(l_2 - l_1, -(l_2 - l_1))$ (resp. $(l_1, -l_1, l_2, -l_2)$) and a maximal element given by $(l_2 + l_1, l_2 + l_1)$

1 Given a finite set $A = \{a_j : j = 1, \dots, N\}$ and an order $\prec$ on A, one says that A is a *saturated chain* with respect to $\prec$ if there exists a permutation π of $\{1, \dots, N\}$ such that

$$a_{\pi(1)} \prec a_{\pi(2)} \prec \cdots \prec a_{\pi(N-1)} \prec a_{\pi(N)}.$$

In this case, $a_{\pi(1)}$ and $a_{\pi(N)}$ are called, respectively, the least and the maximal elements of the chain (see [34, p. 99]).

Author's personal copy

(resp. (l_1, l_1, l_2, l_2)). Then, (A) dispose the columns from west to east, increasingly according to $\prec_c$, (B) dispose the rows from north to south, increasingly according to $\prec_r$. For instance, by setting $l_1 = 0$ and $l_2 \geq 1$, one obtains that $C_{l_1 l_2}$ is the $(2l_2+1) \times (2l_2+1)$ square matrix $\left\{C^{l_2 m}_{00 l_2 m_2}\right\}$ with column indices $m = -(2l_2+1), \ldots, (2l_2+1)$ and row indices $m_2 = -(2l_2+1), \ldots, (2l_2+1)$ (from the subsequent discussion, one also deduces that, in general, $C^{lm}_{00l_2m_2} = \delta^{l_2}_{l}\delta^{m_2}_{m}$). By selecting $l_1 = l_2 = 1$, one sees that C_{11} is the 9×9 matrix with elements $C^{lm}_{1m_1 1m_2}$ (for $m_1, m_2 = -1, 0, 1$; $l = 0, 1, 2$, $m = -l, \ldots, l$) arranged as follows:

$$\begin{pmatrix}
C^{0,0}_{1,-1;1,-1} & C^{1,-1}_{1,-1;1,-1} & C^{10}_{1,-1;1,-1} & C^{11}_{1,-1;1,-1} & C^{2,-2}_{1,-1;1,-1} & C^{2,-1}_{1,-1;1,-1} & C^{2,0}_{1,-1;1,-1} & C^{2,1}_{1,-1;1,-1} & C^{2,2}_{1,-1;1,-1} \\
C^{0,0}_{1,-1;1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
C^{0,0}_{1,-1;1,1} & \cdots & \cdots & C^{11}_{1,-1;1,1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
C^{0,0}_{1,0;1,-1} & \cdots & \cdots & \cdots & \cdots & \cdots & C^{2,0}_{1,0;1,-1} & \cdots & \cdots \\
C^{0,0}_{1,0;1,0} & \cdots & \cdots & \cdots & C^{2,-2}_{1,0;1,0} & \cdots & \cdots & \cdots & \cdots \\
C^{0,0}_{1,0;1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & C^{2,1}_{1,0;1,1} & \cdots \\
C^{0,0}_{1,1;1,-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
C^{0,0}_{1,1;1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
C^{0,0}_{1,1;1,1} & C^{1,-1}_{1,1;1,1} & C^{1,0}_{1,1;1,1} & C^{1,1}_{1,1;1,1} & C^{2,-2}_{1,1;1,1} & C^{2,-1}_{1,1;1,1} & C^{2,0}_{1,1;1,1} & C^{2,1}_{1,1;1,1} & C^{2,2}_{1,1;1,1}
\end{pmatrix}.$$

Explicit expressions for the Clebsch–Gordan coefficients of SO(3) are known, but they are in general hardly manageable. We have for instance (see [12], expression 8.2.1.5)

$$\begin{aligned}
C^{l_3 - m_3}_{l_1 m_1 l_2 m_2} &:= (-1)^{l_1+l_3+m_2}\sqrt{2l_3+1}\left[\frac{(l_1+l_2-l_3)!(l_1-l_2+l_3)!(l_1-l_2+l_3)!}{(l_1+l_2+l_3+1)!}\right]^{1/2} \\
&\quad \times \left[\frac{(l_3+m_3)!(l_3-m_3)!}{(l_1+m_1)!(l_1-m_1)!(l_2+m_2)!(l_2-m_2)!}\right]^{1/2} \\
&\quad \times \sum_z \frac{(-1)^z(l_2+l_3+m_1-z)!(l_1-m_1+z)!}{z!(l_2+l_3-l_1-z)!(l_3+m_3-z)!(l_1-l_2-m_3+z)!},
\end{aligned}$$

where the summation runs over all z's such that the factorials are non-negative. This expression becomes much neater for $m_1 = m_2 = m_3 = 0$, where we have

$$C^{l_3 0}_{l_1 0 l_2 0} = \begin{cases} 0, & \text{for } l_1+l_2+l_3 \text{ odd} \\ (-1)^{\frac{l_1+l_2-l_3}{2}} \dfrac{\sqrt{2l_3+1}\,[(l_1+l_2+l_3)/2]!}{[(l_1+l_2-l_3)/2]!\,[(l_1-l_2+l_3)/2]!\,[(-l_1+l_2+l_3)/2]!} & \\ \quad \times \left\{\dfrac{(l_1+l_2-l_3)!(l_1-l_2+l_3)!(-l_1+l_2+l_3)!}{(l_1+l_2+l_3+1)!}\right\}^{1/2}, & \text{for } l_1+l_2+l_3 \text{ even.} \end{cases}$$

The coefficients, moreover, enjoy a nice set of symmetry and orthogonality properties, playing a crucial role in our results to follow. From unitary equivalence we have the two relations:

$$\sum_{m_1,m_2} C^{lm}_{l_1m_1l_2m_2} C^{l'm'}_{l_1m_1l_2m_2} = \delta^{l'}_{l}\delta^{m'}_{m}, \tag{3.15}$$

$$\sum_{l,m} C^{lm}_{l_1m_1l_2m_2} C^{lm}_{l_1m'_1l_2m'_2} = \delta^{m'_1}_{m_1}\delta^{m'_2}_{m_2}; \tag{3.16}$$

in particular, (3.15) is a consequence of the orthogonality of row vectors, whereas (3.16) comes from the orthogonality of columns. Other properties are better expressed in terms of the Wigner's coefficients, which are related to the Clebsch–Gordan coefficients by the identities (see [12], Chapter 8)

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} = (-1)^{l_3+m_3}\frac{1}{\sqrt{2l_3+1}} C^{l_3 m_3}_{l_1 -m_1 l_2 -m_2} \tag{3.17}$$

$$C^{l_3m_3}_{l_1m_1l_2m_2} = (-1)^{l_1-l_2+m_3}\sqrt{2l_3+1}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \tag{3.18}$$

The Wigner's $3j$ (and, consequently, the Clebsch–Gordan) coefficients are real-valued, they are different from zero only if $m_1+m_2+m_3 = 0$ and $l_i \leq l_j + l_k$ for all $i, j, k = 1, 2, 3$ (*triangle conditions*), and they satisfy the symmetry conditions

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{l_1+l_2+l_3}\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix},$$

Author's personal copy

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{sign(\pi)} \begin{pmatrix} l_{\pi(1)} & l_{\pi(2)} & l_{\pi(3)} \\ m_2 & m_3 & m_1 \end{pmatrix},$$

where π is a permutation of $\{1, 2, 3\}$, and *sign* (π) denotes the sign of π. It follows also that for $m_1 = m_2 = m_3 = 0$, the coefficients $C^{l_3 0}_{l_1 0 l_2 0}$ are different from zero only when the sum $l_1 + l_2 + l_3$ is even. Later in the paper, we shall also need the so-called Wigner's 6*j* coefficients, which are defined by

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} := \sum_{\substack{\alpha,\beta,\gamma \\ \varepsilon,\delta,\phi}} (-1)^{e+f+\varepsilon+\phi} \begin{pmatrix} a & b & e \\ \alpha & \beta & \varepsilon \end{pmatrix} \begin{pmatrix} c & d & e \\ \gamma & \delta & -\varepsilon \end{pmatrix} \begin{pmatrix} a & d & f \\ \alpha & \delta & -\phi \end{pmatrix} \begin{pmatrix} c & b & f \\ \gamma & \beta & \phi \end{pmatrix}, \tag{3.19}$$

see [12], chapter 9 for analytic expressions and a full set of properties; we simply recall here that the Wigner's 6*j* coefficients can themselves be given an important interpretation in terms of group representations, namely they relate different *coupling schemes* in the decomposition of tensor product into direct sum representations, see [35] for further details.

For future reference, we also recall some further standard properties of Kronecker (tensor) products and direct sums of matrices: we have

$$\bigoplus_{i=1}^{n} (A_i B_i) = \left(\bigoplus_{i=1}^{n} A_i \right) \left(\bigoplus_{i=1}^{n} B_i \right), \tag{3.20}$$

$$\left(\bigoplus_{i=1}^{n} A_i \right) \otimes B = \bigoplus_{i=1}^{n} (A_i \otimes B) \tag{3.21}$$

and, provided all matrix products are well-defined,

$$(AB \otimes C) = (A \otimes I_n)(B \otimes C). \tag{3.22}$$

Here, $\oplus_{i=1}^{n} A_i$ is defined as the block diagonal matrix *diag* $\{A_1, \dots, A_n\}$ if A_i is a set of square matrices of order $r_i \times r_i$, whereas it is defined as the stacked column vector of order $\left(\sum_{i=1}^{n} r_i\right) \times 1$ if the A_i are $r_i \times 1$ column vectors.

4. Characterization of polyspectra

4.1. Four general statements

The following result is well-known. As it is crucial in our arguments to follow and we failed to locate any explicit reference, we shall provide a short proof for the sake of completeness. Note that, in the sequel, we use the symbol $a_{l.}$ to indicate the $(2l + 1)$-dimensional complex-valued random vector defined in (2.4).

Lemma 4. *Let T be a strongly isotropic field on S^2, and let the harmonic coefficients $\{a_{lm}\}$ be defined according to (2.2). Then, for every $l \geq 0$ and every $g \in SO(3)$, we have*

$$D^l(g) a_{l.} \stackrel{d}{=} a_{l.}, \quad l = 0, 1, 2, \dots. \tag{4.23}$$

The equality (4.23) must be understood in the sense of finite-dimensional distributions for sequences of random vectors, that is, (4.23) takes place if, and only if, for every $k \geq 1$ and every $0 \leq l_1 < l_2 < \cdots < l_k$,

$$\left\{ D^{l_1}(g) a_{l_1.}, \dots, D^{l_k}(g) a_{l_k.} \right\} \stackrel{d}{=} \left\{ a_{l_1.}, \dots, a_{l_k.} \right\}. \tag{4.24}$$

Proof. We provide the proof of (4.24) only when $k = 1$ and $l_1 = l \geq 1$. The general case is obtained analogously. By strong isotropy, we have that, for every $l \geq 1$, every $g \in SO(3)$ and every $x_1, \dots, x_n \in S^2$, the equality (2.1) takes place. Now, (2.1) can be rewritten as follows:

$$\begin{aligned} &\left\{ \sum_l \sum_m a_{lm} Y_{lm}(x_1), \dots, \sum_l \sum_m a_{lm} Y_{lm}(x_n) \right\} \stackrel{d}{=} \left\{ \sum_l \sum_m a_{lm} Y_{lm}(gx_1), \dots, \sum_l \sum_m a_{lm} Y_{lm}(gx_n) \right\} \\ &= \left\{ \sum_l \sum_m a_{lm} \sum_{m'} D^l_{m'm}(g) Y_{lm'}(x_1), \dots, \sum_l \sum_m a_{lm} \sum_{m'} D^l_{m'm}(g) Y_{lm'}(x_n) \right\} \\ &= \left\{ \sum_l \sum_{m'} \widetilde{a}_{lm'} Y_{lm'}(x_1), \dots, \sum_l \sum_{m'} \widetilde{a}_{lm'} Y_{lm'}(x_n) \right\}, \end{aligned} \tag{4.25}$$

Author's personal copy

where we write

$$\widetilde{a}_{lm'} \triangleq \sum_{m} a_{lm} D^{l}_{m'm}(g), \tag{4.26}$$

and we have used

$$\{Y_{lm}(gx_1), \ldots, Y_{lm}(gx_n)\} \equiv \left\{ \sum_{m'} D^{l}_{m'm}(g) Y_{lm'}(x_1), \ldots, \sum_{m'} D^{l}_{m'm}(g) Y_{lm'}(x_n) \right\} \tag{4.27}$$

which follows from the group representation property and the identity (3.9). To conclude, just observe that (4.25) implies that

$$\widetilde{a}_{lm'} = \int_{S^2} T\,(gx)\, \overline{Y_{lm'}(x)} \mathrm{d}x, \quad m' = -l, \ldots, l,$$

yielding that, due to strong isotropy and with obvious notation, $\widetilde{a}_{l.} \overset{d}{=} a_{l.}$. The conclusion follows from the fact that, thanks to (4.26),

$$\widetilde{a}_{l.} = D^{l}\,(g)\, a_{l.}. \quad \blacksquare$$

The next theorem connects the invariance properties of the vectors $\{a_{l.}\}$ to the representations of SO(3). We need first to establish some notation. For every $0 \le l_1, l_2, \ldots, l_n$, we shall write

$$\Delta_{l_1 \ldots l_n} \triangleq \int_{SO(3)} \left\{ D^{l_1}(g) \otimes D^{l_2}(g) \otimes \cdots \otimes D^{l_n}(g) \right\} \mathrm{d}g, \tag{4.28}$$

$$\Delta_{l_1 \ldots l_n}(g) \triangleq D^{l_1}(g) \otimes D^{l_2}(g) \otimes \cdots \otimes D^{l_n}(g), \quad g \in SO(3), \tag{4.29}$$

and use the symbol $S_{l_1 \ldots l_n}$ (whenever is well-defined), as given in formula (2.5). We stress that $\Delta_{l_1 \ldots l_n}$ and $\Delta_{l_1 \ldots l_n}(g)$ are square matrices with $(2l_1+1) \times \cdots \times (2l_n+1)$ rows and $S_{l_1 \ldots l_n}$ is a column vector with $(2l_1+1) \times \cdots \times (2l_n+1)$ elements. The following result applies to an arbitrary $n \ge 2$: see [16] for some related results in the case $n = 3, 4$.

Proposition 5. *Let T be a strongly isotropic field with moments of order $n \ge 2$. Then, for every $0 \le l_1, l_2, \ldots, l_n$ and every fixed $g^* \in SO(3)$*

$$\Delta_{l_1 \ldots l_n} S_{l_1 \ldots l_n} = S_{l_1 \ldots l_n} \tag{4.30}$$

$$\Delta_{l_1 \ldots l_n}\left(g^*\right) S_{l_1 \ldots l_n} = S_{l_1 \ldots l_n}. \tag{4.31}$$

On the other hand, fix $n \ge 2$ and assume that $T(x)$ is not necessarily an isotropic random field on the sphere s.t. $\sup_x \left(E\,|T(x)|^n\right) < \infty$. Then $T(.)$ is P-almost surely Lebesgue square integrable and the nth order spectral moments of T exist and are finite. If, moreover, (4.30) holds for every $0 \le l_1 \le \cdots \le l_n$, then one has that, for every $g \in SO(3)$,

$$E\left[D^{l_1}(g) a_{l_1.} \otimes \cdots \otimes D^{l_n}(g) a_{l_n.}\right] = E\left[a_{l_1.} \otimes \cdots \otimes a_{l_n.}\right], \tag{4.32}$$

and T is n-weakly isotropic.

Proof. By strong isotropy and Lemma 4, one has

$$E\left\{D^{l_1}(g) a_{l_1.} \otimes \cdots \otimes D^{l_n}(g) a_{l_n.}\right\} = E\left\{a_{l_1.} \otimes \cdots \otimes a_{l_n.}\right\} \quad \text{for all } g \in SO(3), l_1, \ldots, l_n \in \mathbb{N}^n.$$

Now assume that g is sampled randomly (and independently of the $\{a_{l.}\}$) according to some probability measure, say P_0, on SO(3). From the property (3.22) of tensor products and trivial manipulations, we obtain (with obvious notation and by independence)

$$\begin{aligned} E\left\{D^{l_1}(\cdot) a_{l_1.} \otimes \cdots \otimes D^{l_n}(\cdot) a_{l_n.}\right\} &= E\left\{\left[D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot)\right]\left[a_{l_1.} \otimes \cdots \otimes a_{l_n.}\right]\right\} \\ &= E_0\left\{D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot)\right\} E\left\{a_{l_1.} \otimes \cdots \otimes a_{l_n.}\right\}. \end{aligned}$$

Now, if one chooses P_0 to be equal to the Haar (uniform) measure on SO(3), one has that

$$E_0\left\{D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot)\right\} = \Delta_{l_1 \ldots l_n},$$

thus giving (4.30). On the other hand, if one chooses P_0 to be equal to the Dirac mass at some $g^* \in SO(3)$, one has that

$$E_0\left\{D^{l_1}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot)\right\} = \Delta_{l_1 \ldots l_n}\left(g^*\right),$$

which shows that (4.31) is satisfied.

Author's personal copy

Now let T satisfy the assumptions of the second part of the statement for some $n \geq 2$. We recall first that the representation (2.2) continues to hold, in a pathwise sense. To see that the nth order joint moments of the harmonic coefficients a_{lm} are finite it is enough to use Jensen's inequality, along with a standard version of the Fubini theorem, to obtain that

$$\begin{aligned} E\left|a_{lm}\right|^n &= E\left|\int_{S^2} T(x)\overline{Y_{lm}(x)}\mathrm{d}x\right|^n \leq E\int_{S^2}|T(x)|^n|Y_{lm}(x)|^n\mathrm{d}x \\ &\leq \left\{\sup_{x\in S^2}|Y_{lm}(x)|^n\right\}\left\{\sup_{x\in S^2}E|T(x)|^n\right\} \\ &\leq \left(\frac{2l+1}{4\pi}\right)^{n/2}\left\{\sup_{x\in S^2}E|T(x)|^n\right\} < \infty. \end{aligned}$$

It is then straightforward that, if $S_{l_1\ldots l_n}$ satisfies (4.30), one also has that for any fixed $\overline{g} \in SO(3)$

$$\begin{aligned} E\left\{\left[D^{l_1}(\overline{g})\otimes\cdots\otimes D^{l_n}(\overline{g})\right]\left[a_{l_1.}\otimes\cdots\otimes a_{l_n.}\right]\right\} &= \left[D^{l_1}(\overline{g})\otimes\cdots\otimes D^{l_n}(\overline{g})\right]E\left[a_{l_1.}\otimes\cdots\otimes a_{l_n.}\right] \\ &= \left[D^{l_1}(\overline{g})\otimes\cdots\otimes D^{l_n}(\overline{g})\right]\Delta_{l_1\ldots l_n}S_{l_1\ldots l_n} \\ &= \left\{\left[D^{l_1}(\overline{g})\otimes\cdots\otimes D^{l_n}(\overline{g})\right]\int_{SO(3)}\left\{D^{l_1}(g)\otimes\cdots\otimes D^{l_n}(g)\right\}\mathrm{d}g\right\}S_{l_1\ldots l_n} \\ &= \left\{\int_{SO(3)}\left\{D^{l_1}(\overline{g}g)\otimes D^{l_2}(\overline{g}g)\otimes\cdots\otimes D^{l_n}(\overline{g}g)\right\}\mathrm{d}g\right\}S_{l_1\ldots l_n} \\ &= \Delta_{l_1\ldots l_n}S_{l_1\ldots l_n} = E\left\{a_{l_1.}\otimes\cdots\otimes a_{l_n.}\right\}, \end{aligned}$$

which proves the n-th spectral moment is invariant to rotations. The fact that T is n-weakly isotropic is a consequence of the spectral representation (2.2). ■

Note that relation (4.30) can be rephrased by saying that, for a strongly isotropic field, the joint moment vector $E\left\{a_{l_1.}\otimes a_{l_2.}\otimes\cdots\otimes a_{l_n.}\right\}$ must be an eigenvector of the matrix (4.28) for every $n \geq 2$ and every $0 \leq l_1 \leq \cdots \leq l_n$. A similar characterization holds for cumulants polyspectra. Recall the notation $S^c_{l_1\ldots l_n}$ introduced in (2.6).

Proposition 6. *Let T be a strongly isotropic field with moments of order $n \geq 2$. Then, for every $0 \leq l_1, l_2, \ldots, l_n$ and every fixed $g^* \in SO(3)$,*

$$\Delta_{l_1\ldots l_n}S^c_{l_1\ldots l_n} = S^c_{l_1\ldots l_n} \tag{4.33}$$

$$\Delta_{l_1\ldots l_n}\left(g^*\right)S^c_{l_1\ldots l_n} = S^c_{l_1\ldots l_n}. \tag{4.34}$$

On the other hand, fix $n \geq 2$ and assume that $T(x)$ is not necessarily an isotropic random field on the sphere s.t. $\sup_x\left(E\left|T(x)\right|^n\right) < \infty$. Then $T(.)$ is P-almost surely Lebesgue square integrable and the nth order spectral moments of T exist and are finite. If, moreover, (4.33) holds for every $0 \leq l_1 \leq \cdots \leq l_n$, then one has that, for every $g \in SO(3)$, relation (4.32) holds, and T is n-weakly isotropic.

Proof. For every $x_1, \ldots, x_n \in S^2$, write $\mathrm{Cum}\left\{T(x_1),\ldots,T(x_n)\right\}$ the joint cumulant of the random variables $T(x_1),\ldots,T(x_n)$. By using isotropy, one has that, for every $g \in SO(3)$,

$$\mathrm{Cum}\left\{T(x_1),\ldots,T(x_n)\right\} = \mathrm{Cum}\left\{T(gx_1),\ldots,T(gx_n)\right\}. \tag{4.35}$$

Hence, by using the well-known multilinearity properties of cumulants, one deduces that (with obvious notation)

$$\begin{aligned} \mathrm{Cum}\left\{T(x_1),\ldots,T(x_n)\right\} &= \sum_{l_1m_1,\ldots,l_nm_n}\mathrm{Cum}\left\{a_{l_1m_1},\ldots,a_{l_nm_n}\right\}Y_{l_1m_1}(x_1)\cdots Y_{l_nm_n}(x_n) \\ &= \sum_{l_1m_1,\ldots,l_nm_n}\mathrm{Cum}\left\{a_{l_1m_1},\ldots,a_{l_nm_n}\right\}Y_{l_1m_1}(gx_1)\cdots Y_{l_nm_n}(gx_n), \end{aligned} \tag{4.36}$$

and relations (4.33)–(4.34) are deduced by rewriting (4.36) by means of the identity

$$\left\{Y_{l_1m_1}(gx_1),\ldots,Y_{l_nm_n}(gx_n)\right\} \equiv \left\{\sum_{m'}D^{l_1}_{m'm}(g)Y_{l_1m'}(x_1),\ldots,\sum_{m'}D^{l_n}_{m'm}(g)Y_{l_nm'}(x_n)\right\}.$$

The second part of the statement is proved by arguments analogous to the ones used in the proof of Proposition 5. ■

Author's personal copy

We now present an alternative (and more involved) characterization of the cumulant polyspectra associated with an isotropic field. Given $n \geq 2$ and a partition $\pi = \{b_1, \ldots, b_k\} \in \Pi(n)$, we build a permutation $\upsilon^\pi = (\upsilon^\pi(1), \ldots, \upsilon^\pi(n)) \in \mathfrak{S}_n$ as follows: (i) write the partition

$$\pi = \{b_1, \ldots, b_k\} = \left\{ \left(i_1^1, \ldots, i_{|b_1|}^1\right), \ldots, \left(i_1^k, \ldots, i_{|b_k|}^k\right) \right\} \tag{4.37}$$

(where $|b_j| \geq 1$ stands for the size of b_j) by means of the convention outlined in Section 2 (that is, order the blocks and the elements within each block according to the lexicographic order); (ii) define $\upsilon^\pi = \mathfrak{S}_n$ by simply removing the brackets in (4.37), that is, set

$$\upsilon^\pi = (\upsilon^\pi(1), \ldots, \upsilon^\pi(n)) = \left(i_1^1, \ldots, i_{|b_1|}^1, i_1^2, \ldots, i_{|b_2|}^2, \ldots, i_1^k, \ldots, i_{|b_k|}^k\right).$$

For instance, if a partition π of $\{1, \ldots, 6\}$ is composed of the blocks $\{1, 3\}, \{6, 4\}$ and $\{2, 5\}$, one first writes π in the form $\pi = \{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$, and then defines $\upsilon^\pi = (\upsilon^\pi(1), \ldots, \upsilon^\pi(6)) = (1, 3, 2, 5, 4, 6)$. Given $n \geq 2$, $0 \leq l_1 \leq \cdots \leq l_n$, and $\pi \in \Pi(n)$, we define the matrix

$$\Delta^\pi_{l_1 \ldots l_n} \triangleq \int_{SO(3)} \left\{ D^{l_{\upsilon^\pi(1)}}(g) \otimes D^{l_{\upsilon^\pi(2)}}(g) \otimes \cdots \otimes D^{l_{\upsilon^\pi(n)}}(g) \right\} dg, \tag{4.38}$$

obtained from the matrix $\Delta_{l_1 \ldots l_n}$ in (4.28), by permuting the indexes l_i according to υ^π. Plainly, if υ^π is equal to the identity permutation, then $\Delta^\pi_{l_1 \ldots l_n} = \Delta_{l_1 \ldots l_n}$. We also set, for every fixed $g \in SO(3)$,

$$\Delta^\pi_{l_1 \ldots l_n}(g) \triangleq D^{l_{\upsilon^\pi(1)}}(g) \otimes D^{l_{\upsilon^\pi(2)}}(g) \otimes \cdots \otimes D^{l_{\upsilon^\pi(n)}}(g).$$

Proposition 7. *Let T be a strongly isotropic field with finite moments of order $n \geq 2$. For $0 \leq l_1, l_2, \ldots, l_n$, define $S^c_{l_1 \ldots l_n}$ according to (2.6). Then, for every $0 \leq l_1, l_2, \ldots, l_n$, and every $g \in SO(3)$*

$$S^c_{l_1,\ldots,l_n} = \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! \Delta^\pi_{l_1\ldots l_n} E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] \tag{4.39}$$

$$= \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! \Delta^\pi_{l_1\ldots l_n}(g) E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right]. \tag{4.40}$$

On the other hand, fix $n \geq 2$ and assume that $T(x)$ is a (not necessarily isotropic) random field on the sphere s.t. $\sup_x \left(E|T(x)|^n\right) < \infty$. Then, the nth order spectral moments and cumulants of T exist and are finite. If moreover (4.40) holds for every $0 \leq l_1, l_2, \ldots, l_n$ and every $g \in SO(3)$, then one has that T is n-weakly isotropic.

Proof. Fix $\pi = \{b_1, \ldots, b_k\} \in \Pi(n)$. By strong isotropy and Lemma 4, one has that, for a fixed $g^* \in SO(3)$, the quantity

$$E\left[\otimes_{i\in b_1} D^{l_i}(g) a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} D^{l_i}(g) a_{l_i\cdot}\right] = \Delta^\pi_{l_1\ldots l_n}\left(g^*\right) E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right]$$

does not depend on g^*, so that

$$\begin{aligned} E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] &= \Delta^\pi_{l_1\ldots l_n}\left(g^*\right) E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] \\ &= \int_{SO(3)} \Delta^\pi_{l_1\ldots l_n}(g) E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] dg \\ &= \Delta^\pi_{l_1\ldots l_n} E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right]. \end{aligned}$$

To prove the second part of the statement, suppose that $T(x)$ verifies $\sup_x \left(E|T(x)|^n\right) < \infty$, and that its associated harmonic coefficients verify (4.40). Then, for every fixed rotation $g^* \in SO(3)$,

$$\begin{aligned} &\sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! E\left[\otimes_{i\in b_1} D^{l_i}\left(g^*\right) a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} D^{l_i}\left(g^*\right) a_{l_i\cdot}\right] \\ &= \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! [D^{l_{\upsilon^\pi(1)}}\left(g^*\right) \otimes \cdots \otimes D^{l_{\upsilon^\pi(n)}}\left(g^*\right)] E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] \\ &= \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! \times \Delta^\pi_{l_1\ldots l_n}\left(g^*\right) E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right] \\ &= \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)! E\left[\otimes_{i\in b_1} a_{l_i\cdot}\right] \otimes \cdots \otimes E\left[\otimes_{i\in b_k} a_{l_i\cdot}\right]. \end{aligned}$$

By the definition of cumulants, this last equality gives that

$$E\left[D^{l_1}(g^*) a_{l_1\cdot} \otimes \cdots \otimes D^{l_n}(g^*) a_{l_n\cdot}\right] = E\left[a_{l_1\cdot} \otimes \cdots \otimes a_{l_n\cdot}\right].$$

Since g^* is arbitrary, the n-weak isotropy follows from (2.2). ■

Author's personal copy

Remark. By combining (4.33) and (4.39) we obtain for instance that the nth cumulant polyspectrum of an isotropic field verifies the identity

$$\begin{aligned} S^c_{l_1\ldots l_n} &= \Delta_{l_1\ldots l_n} S^c_{l_1\ldots l_n} \\ &= \sum_{\pi=\{b_1,\ldots,b_k\}\in\Pi(n)} (-1)^{k-1}(k-1)!\Delta^{\pi}_{l_1\ldots l_n} E\left[\otimes_{i\in b_1} a_{l_i.}\right]\otimes\cdots\otimes E\left[\otimes_{i\in b_k} a_{l_i.}\right]. \end{aligned}$$

5. Angular polyspectra and the structure of $\Delta_{l_1\ldots l_n}$

5.1. Spectra of strongly isotropic fields

Our aim in this section is to investigate more deeply the structure of the matrix $\Delta_{l_1\ldots l_n}$ appearing in (4.28), in order to derive an explicit characterization for the angular polyspectra. As a preliminary example, we deal with the case $n = 2$.

Proposition 8. *For integers $l_1, l_2 \geq 0$, one has that*

$$\Delta_{l_1 l_2} = \int_{SO(3)} \left\{D^{l_1}(g)\otimes D^{l_2}(g)\right\} dg = \delta_{l_1}^{l_2} C^{00}_{l_1.l_2.}(C^{00}_{l_1.l_2.})', \tag{5.41}$$

that is: if $l_1 \neq l_2$, then $\Delta_{l_1 l_2}$ is a $(2l_1+1)(2l_2+1)\times(2l_1+1)(2l_2+1)$ zero matrix; if $l_1 = l_2$, then $\Delta_{l_1 l_2} = \Delta_{l_1 l_1}$ is given by $C^{00}_{l_1.l_1.}(C^{00}_{l_1.l_1.})'$.

Proof. Using the equivalence of the two representations $D^{l_1}(g)\otimes D^{l_2}(g)$ and $\oplus_{\lambda=|l_2-l_1|}^{l_2+l_1} D^{\lambda}(g)$, as well as the definition of the Clebsch–Gordan matrices, we obtain that

$$\int_{SO(3)} \left\{D^{l_1}(g)\otimes D^{l_2}(g)\right\} dg = C_{l_1 l_2}\left[\int_{SO(3)} \left\{\bigoplus_{\lambda=|l_2-l_1|}^{l_2+l_1} D^{\lambda}(g)\right\} dg\right] C^{*}_{l_1 l_2}. \tag{5.42}$$

Now, if $l_1 \neq l_2$, then the RHS of (5.42) is equal to the zero matrix since, as a consequence of the Peter–Weyl theorem and for $\lambda \neq 0$, the entries of $D^{\lambda}(\cdot)$ are orthogonal to the constants. If $l_1 = l_2$, then the integrated matrix on the RHS of (5.42) becomes $\int_{SO(3)}\{\oplus_{\lambda=0}^{2l_1} D^{\lambda}(g)\}dg$, that is, a $(2l_1+1)^2\times(2l_1+1)^2$ matrix which is zero everywhere, except for the entry in the top-left corner, which is equal to one (since $\int_{SO(3)} dg = 1$). The proof is concluded by checking that

$$C_{l_1 l_1}\left[\int_{SO(3)} \left\{\bigoplus_{\lambda=0}^{2l_1} D^{\lambda}(g)\right\} dg\right] C^{*}_{l_1 l_1} = C^{00}_{l_1.l_1.}(C^{00}_{l_1.l_1.})'. \quad \blacksquare$$

Remark. Recall that $C^{00}_{l_1.l_2.}$ is a *column* vector of dimension $(2l_1+1)(2l_2+1)$, corresponding to the first column of the matrix $C_{l_1 l_2}$. Also, according e.g. to [12, formula 8.5.1.1], one has that

$$C^{00}_{l_1.l_2.} = \left\{\frac{(-1)^{m_1}}{2l_1+1}\delta_{l_1}^{l_2}\delta_{m_1}^{-m_2}\right\}_{m_1=-l_1,\ldots,l_1;m_2=-l_2,\ldots,l_2}.$$

Proposition 8 provides a characterization of the spectrum of a strongly isotropic field.

Corollary 9. *Let T be a strongly isotropic field with second moments, and let the vectors of the harmonic coefficients $\{a_{l.}\}$ be defined according to (2.2). Then, for any integers $l_1, l_2 \geq 0$, one has that*

$$E\left\{a_{l_1.}\otimes a_{l_2.}\right\} = \left\{\frac{(-1)^{m_1}}{2l_1+1}\delta_{l_1}^{l_2}\delta_{m_1}^{-m_2} C_{l_1}\right\} \tag{5.43}$$

for some $C_{l_1} \geq 0$ depending uniquely on l_1.

Proof. According to (4.30), one has that

$$E\left\{a_{l_1.}\otimes a_{l_2.}\right\} = \delta_{l_1}^{l_2} C^{00}_{l_1.l_2.}(C^{00}_{l_1.l_2.})' E\left\{a_{l_1.}\otimes a_{l_2.}\right\},$$

implying that $E\left\{a_{l_1.}\otimes a_{l_2.}\right\}$ is (a) equal to the zero vector for $l_1 \neq l_2$, and (b) of the form $C_{l_1}\times C^{00}_{l_1.l_2.}$, for some constant C_{l_1}, when $l_1 = l_2$. To see that C_{l_1} cannot be negative, just observe that $a_{l_1 0}$ is real-valued for every $l_1 \geq 0$, so that (5.43) yields that

$$C_{l_1} = (2l_1+1)\times E\left(a^2_{l_1 0}\right). \quad \blacksquare$$

Author's personal copy

In the subsequent two subsections, we shall obtain, for every $n \geq 3$, a characterization of $\Delta_{l_1 \ldots, l_n}$ and $E\{a_{l_1 .} \otimes \cdots \otimes a_{l_n .}\}$, respectively analogous to (5.41) and (5.43).

5.2. The structure of $\Delta_{l_1 \ldots l_n}$

We first need to establish some further notation.

Definition B. Fix $n \geq 3$. For integers $l_1, \ldots, l_n \geq 0$, we define $C_{l_1 \ldots l_n}$ to be the unitary matrix, of dimension

$$\prod_{j=1}^{n} (2l_j + 1) \times \prod_{j=1}^{n} (2l_j + 1),$$

connecting the following two equivalent representations of SO(3)

$$D^{l_1}(.) \otimes D^{l_2}(.) \otimes \cdots \otimes D^{l_n}(.) \tag{5.44}$$

and

$$\bigoplus_{\lambda_1 = |l_2 - l_1|}^{l_2 + l_1} \bigoplus_{\lambda_2 = |l_3 - \lambda_1|}^{l_3 + \lambda_1} \cdots \bigoplus_{\lambda_{n-1} = |l_n - \lambda_{n-2}|}^{l_n + \lambda_{n-2}} D^{\lambda_{n-1}}(.). \tag{5.45}$$

Remarks. (1) Fix $l_1, \ldots, l_n \geq 0$, as well as $g \in SO(3)$. Then, the matrix

$$\bigoplus_{\lambda_1 = |l_2 - l_1|}^{l_2 + l_1} \bigoplus_{\lambda_2 = |l_3 - \lambda_1|}^{l_3 + \lambda_1} \cdots \bigoplus_{\lambda_{n-1} = |l_n - \lambda_{n-2}|}^{l_n + \lambda_{n-2}} D^{\lambda_{n-1}}(g) \tag{5.46}$$

is a block-diagonal matrix, obtained as follows. (a) Consider vectors of integers $(\lambda_1, \ldots, \lambda_{n-1})$ satisfying the relations $|l_2 - l_1| \leq \lambda_1 \leq l_1 + l_2$, and $|l_{k+1} - \lambda_{k-1}| \leq \lambda_k \leq l_{k+1} + \lambda_{k-1}$, for $k = 2, \ldots, n-1$. (b) Introduce a (total) order $\prec_0$ on the collection of these vectors by saying that

$$(\lambda_1, \ldots, \lambda_{n-1}) \prec_0 \left(\lambda'_1, \ldots, \lambda'_{n-1}\right), \tag{5.47}$$

whenever either $\lambda_1 < \lambda'_1$, or there exists $k = 2, \ldots, n-2$ such that $\lambda_j = \lambda'_j$ for every $j = 1, \ldots, k$, and $\lambda_{k+1} < \lambda'_{k+1}$. (c) Associate to each vector $(\lambda_1, \ldots, \lambda_{n-1})$ the matrix $D^{\lambda_{n-1}}(g)$. (d) Construct a block-diagonal matrix by disposing the matrices $D^{\lambda_{n-1}}(g)$ from the top-left corner to the bottom-right corner, in increasing order with respect to $\prec_0$. As an example, consider the case where $n = 3$ and $l_1 = l_2 = l_3 = 1$. Here, the vectors (λ_1, λ_2) involved in the direct sum (5.45) are (in increasing order with respect to $\prec_0$)

$$(0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 2) \text{ and } (2, 3),$$

and the matrix (5.46) is therefore given by

$$\begin{pmatrix} D^1(g) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & D^1(g) & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & D^2(g) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & D^1(g) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & D^2(g) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & D^3(g) \end{pmatrix} \tag{5.48}$$

where the dots indicate *zero* entries, and we have used the fact that $D^0(g) \equiv 1$.

(2) The fact that the representation (5.45) has dimension $\prod_{j=1}^{n} \left(2l_j + 1\right)$ is a direct consequence of formula (3.14).

(3) The fact that the two representations (5.44) and (5.45) are equivalent can be proved by iteration. Indeed, by standard representation theory, (5.44) is equivalent to

$$\bigoplus_{\lambda_1 = |l_2 - l_1|}^{l_2 + l_1} D^{\lambda_1}(.) \otimes D^{l_3}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot),$$

which is in turn equivalent to

$$\bigoplus_{\lambda_1 = |l_2 - l_1|}^{l_2 + l_1} \bigoplus_{\lambda_2 = |l_3 - \lambda_1|}^{l_3 + \lambda_1} D^{\lambda_2}(.) \otimes D^{l_4}(\cdot) \otimes \cdots \otimes D^{l_n}(\cdot).$$

Author's personal copy

By iterating the same procedure until all tensor products have disappeared (that is, by successively replacing the tensor product $D^{\lambda_k}(.)\otimes D^{l_{k+2}}(\cdot)$ with $\oplus_{\lambda_{k+1}=|l_{k+2}-\lambda_k|}^{l_{k+2}+\lambda_k} D^{\lambda_2}(.)$ for $k=2,\dots,n-1$), one obtains the desired conclusion.

For every $n\geq 3$ and every $l_1,\dots,l_n\geq 0$, the elements of the matrix $C_{l_1\dots l_n}$, introduced in Definition B, can be written in the form $C_{l_1m_1\dots l_nm_n}^{\lambda_1\dots\lambda_{n-1},\mu_{n-1}}$. The indices $(m_1,\dots,m_n)$ are such that $-l_i\leq m_i\leq l_i$ $(i=1,\dots,n)$ and label rows; on the other hand, the indices $(\lambda_1\dots\lambda_{n-1},\mu_{n-1})$ label columns, and verify the relations $|l_2-l_1|\leq\lambda_1\leq l_1+l_2$, $|l_{k+1}-\lambda_{k-1}|\leq\lambda_k\leq l_{k+1}+\lambda_{k-1}$ $(k=2,\dots,n-1)$ and $-\lambda_{n-1}\leq\mu_{n-1}\leq\lambda_{n-1}$. It is well known (see e.g. [12]) that the quantity $C_{l_1m_1\dots l_nm_n}^{\lambda_1\dots\lambda_{n-1},\mu_{n-1}}$ can be represented as a *convolution* of the Clebsch–Gordan coefficients introduced in Section 3.2, namely:

$$\begin{aligned}
C_{l_1m_1\dots l_nm_n}^{\lambda_1,\dots,\lambda_{n-1},\mu_{n-1}} &= C_{l_1m_1\dots l_{n-1}m_{n-1}}^{\lambda_1,\dots,\lambda_{n-2},\cdot} C_{\lambda_{n-2}l_nm_n}^{\lambda_{n-1}\mu_{n-1}}\\
&= \sum_{\mu_{n-2}}\left\{\sum_{\mu_1\dots\mu_{n-3}} C_{l_1m_1l_2m_2}^{\lambda_1\mu_1}C_{\lambda_1\mu_1l_3m_3}^{\lambda_2\mu_2}\dots C_{\lambda_{n-3}\mu_{n-3}l_{n-1}m_{n-1}}^{\lambda_{n-2}\mu_{n-2}}\right\}C_{\lambda_{n-2}\mu_{n-2}l_nm_n}^{\lambda_{n-1}\mu_{n-1}}\\
&= \sum_{\mu_1\dots\mu_{n-2}} C_{l_1m_1l_2m_2}^{\lambda_1\mu_1}C_{\lambda_1\mu_1l_3m_3}^{\lambda_2\mu_2}\cdots C_{\lambda_{n-3}\mu_{n-3}l_{n-1}m_{n-1}}^{\lambda_{n-2}\mu_{n-2}}C_{\lambda_{n-2}\mu_{n-2}l_nm_n}^{\lambda_{n-1}\mu_{n-1}}.
\end{aligned}$$

Remark. Given an enumeration of the coefficients $C_{l_1m_1\dots l_nm_n}^{\lambda_1\dots\lambda_{n-1},\mu_{n-1}}$, the matrix $C_{l_1\dots l_n}$ can be built (analogously to the case of the Clebsch–Gordan matrices of Section 3.2) by disposing rows (from top to bottom) and columns (from left to right) increasingly according to two separate total orders. The order $\prec_r$ on the symbols $(m_1,\dots,m_n)$ is obtained by setting that $(m_1,\dots,m_n)\prec_r(m_1',\dots,m_n')$ whenever either $m_1<m_1'$, or there exists $k=2,\dots,n-1$ such that $m_j=m_j'$ for every $j=1,\dots,k$, and $m_{k+1}<m_{k+1}'$. The order $\prec_c$ on the symbols $(\lambda_1\dots\lambda_{n-1},\mu_{n-1})$ is obtained by setting that $(\lambda_1\dots\lambda_{n-1},\mu_{n-1})\prec_c(\lambda_1'\dots\lambda_{n-1}',\mu_{n-1}')$ whenever either $(\lambda_1,\dots,\lambda_{n-1})\prec_0(\lambda_1',\dots,\lambda_{n-1}')$, as defined in (5.47), or $\lambda_i=\lambda_i'$ for every $i=1,\dots,n-1$ and $\mu_{n-1}<\mu_{n-1}'$.

One has also the following (useful) alternative representation of generalized Clebsch–Gordan matrices.

Proposition 10. *For every $n\geq 3$ and every $l_1,\dots,l_n\geq 0$, one can represent the matrix $C_{l_1\dots l_n}$, as follows*

$$C_{l_1\dots l_n}=\left\{C_{l_1l_2l_3\dots l_{n-1}}\otimes I_{2l_n+1}\right\}\left\{\Big(\bigoplus_{\lambda_1=|l_2-l_1|}^{l_2+l_1}\cdots\bigoplus_{\lambda_{n-2}=|l_n-\lambda_{n-3}|}^{l_n+\lambda_{n-3}}C_{\lambda_{n-2}l_n}\Big)\right\},$$

where I_m indicates a $m\times m$ identity matrix. Also, one has that

$$\begin{aligned}
C_{l_1\dots l_n} &= (C_{l_1l_2}\otimes I_{2l_3+1}\otimes\cdots\otimes I_{2l_n+1})\times\left[\Big(\bigoplus_{\lambda=|l_2-l_1|}^{l_2+l_1}C_{\lambda l_3}\Big)\otimes\cdots\otimes I_{2l_n+1}\right]\\
&\quad\times\cdots\times\left[\Big(\bigoplus_{\lambda_1=|l_2-l_1|}^{l_2+l_1}\cdots\bigoplus_{\lambda_{n-2}=|l_n-\lambda_{n-3}|}^{l_n+\lambda_{n-3}}C_{\lambda_{n-2}l_n}\Big)\right],
\end{aligned}$$

where $\times$ stands for the usual product between matrices.

Definition C. For every $n\geq 3$ and every $l_1,\dots,l_n\geq 0$, we define $E_{l_1\dots l_n}$ to be the $\Pi_{j=1}^n(2l_j+1)\times\Pi_{j=1}^n(2l_j+1)$ square matrix

$$E_{l_1\dots l_n}:=\bigoplus_{\lambda_1=|l_2-l_1|}^{l_2+l_1}\cdots\bigoplus_{\lambda_{n-1}=|l_n-\lambda_{n-2}|}^{l_n+\lambda_{n-2}}\delta_{\lambda_{n-1}}^0 I_{2\lambda_{n-1}+1}. \tag{5.49}$$

In other words, $E_{l_1\dots l_n}$ is the diagonal matrix built from the matrix (5.46), by replacing every block of the type $D^{\lambda_{n-1}}(g)$, with $\lambda_{n-1}>0$, with a $(2\lambda_{n-1}+1)\times(2\lambda_{n-1}+1)$ zero matrix, and by letting the 1×1 blocks $D^0(g)=1$ unchanged. For instance, by setting $n=3$ and $l_1=l_2=l_3=1$ (and by using (5.48)) one obtains a 27×27 matrix E_{111} whose entries are all zero, except for the fourth element (starting from the top-left corner) of the main diagonal.

The following result states that the matrix $\Delta_{l_1\dots l_n}$ can be diagonalized in terms of $C_{l_1\dots l_n}$ and $E_{l_1\dots l_n}$.

Proposition 11. *The matrix $\Delta_{l_1\dots l_n}$ can be diagonalized as*

$$\Delta_{l_1\dots l_n}=C_{l_1\dots l_n}E_{l_1\dots l_n}C_{l_1\dots l_n}^*, \tag{5.50}$$

where $E_{l_1\dots l_n}$ is the matrix introduced in Definition C.

Author's personal copy

Proof. One has that

$$\begin{aligned}\Delta_{l_1\ldots l_n} &= \int_{SO(3)} D^{l_1}(g)\otimes D^{l_2}(g)\otimes\cdots\otimes D^{l_n}(g)\,dg\\ &= \int_{SO(3)}\left[C_{l_1\ldots l_n}\bigoplus_{\lambda_1=|l_2-l_1|}^{l_2+l_1}\bigoplus_{\lambda_2=|l_3-\lambda_1|}^{l_3+\lambda_1}\cdots\bigoplus_{\lambda_{n-1}=|l_n-\lambda_{n-2}|}^{l_n+\lambda_{n-2}} D^{\lambda_{n-1}}(g)C^{*}_{l_1\ldots l_n}\right]dg. \qquad (5.51)\end{aligned}$$

By linearity and by the definition of the integral of a matrix-valued function, one has that the last line of (5.51) equals

$$C_{l_1\ldots l_n}\left[\bigoplus_{\lambda_1=|l_2-l_1|}^{l_2+l_1}\bigoplus_{\lambda_2=|l_3-\lambda_1|}^{l_3+\lambda_1}\cdots\bigoplus_{\lambda_{n-1}=|l_n-\lambda_{n-2}|}^{l_n+\lambda_{n-2}}\int_{SO(3)} D^{\lambda_{n-1}}(g)dg\right]C^{*}_{l_1\ldots l_n}.$$

Now observe that, if $\lambda_{n-1} > 0$, then $\int_{SO(3)} D^{\lambda_{n-1}}(g)dg$ equals a $(2\lambda_{n-1}+1)\times(2\lambda_{n-1}+1)$ zero matrix, whereas $\int_{SO(3)} D^{0}(g)dg = \int_{SO(3)} 1dg = 1$. The conclusion is obtained by resorting to the definition of $E_{l_1\ldots l_n}$ given in (5.49). ■

5.3. Existence and characterization of reduced polyspectra of arbitrary orders

Combining the previous Proposition with 5, we obtain the main result of this paper.

Theorem 12. *If a random field is strongly isotropic with finite moments of order $n\geq 3$, then for every $l_1,\ldots,l_n$ there exists two arrays $P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3})$ and $P^{C}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3})$, with $|l_2-l_1|\leq\lambda_1\leq l_2+l_1$, $|l_3-\lambda_1|\leq\lambda_2\leq l_3+\lambda_1,\ldots,|l_{n-2}-\lambda_{n-4}|\leq\lambda_{n-3}\leq l_{n-2}+\lambda_{n-4}$, such that*

$$Ea_{l_1m_1}\ldots a_{l_nm_n} = (-1)^{m_n}\sum_{\lambda_1=l_2-l_1}^{l_2+l_1}\cdots\sum_{\lambda_{n-3}} C^{\lambda_1\ldots\lambda_{n-3}l_n-m_n}_{l_1m_1\ldots l_{n-1}m_{n-1}}P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}) \qquad (5.52)$$

$$\mathrm{Cum}\left\{a_{l_1m_1},\ldots,a_{l_nm_n}\right\} = (-1)^{m_n}\sum_{\lambda_1=l_2-l_1}^{l_2+l_1}\cdots\sum_{\lambda_{n-3}} C^{\lambda_1\ldots\lambda_{n-3}l_n-m_n}_{l_1m_1\ldots l_{n-1}m_{n-1}}P^{C}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}) \qquad (5.53)$$

$$C^{\lambda_1\ldots\lambda_{n-3};l_n-m_n}_{l_1m_1\ldots l_{n-1}m_{n-1}} = \sum_{\mu_1}\cdots\sum_{\mu_{n-3}} C^{\lambda_1\mu_1}_{l_1m_1l_2m_2}C^{\lambda_2\mu_2}_{\lambda_1\mu_1l_3m_3}\cdots C^{l_n,-m_n}_{\lambda_{n-3}\mu_{n-3}l_{n-1}m_{n-1}}. \qquad (5.54)$$

Remark. For a fixed $n\geq 2$, the real-valued arrays $\left\{P_{l_1\ldots l_n}(\cdot): l_1,\ldots,l_n\geq 0\right\}$ and $\left\{P^{C}_{l_1\ldots l_n}(\cdot): l_1,\ldots,l_n\geq 0\right\}$ are, respectively, the *reduced polyspectrum of order* $n-1$ and the *reduced cumulant polyspectrum of order* $n-1$ associated with the underlying strongly isotropic random field.

Proof of Theorem 12. We shall prove only (5.52), since the proof of (5.53) is entirely analogous. By Propositions 5 and 11, if the random field is isotropic, then

$$S_{l_1\ldots l_n} = C_{l_1\ldots l_n}E_{l_1\ldots l_n}C^{*}_{l_1\ldots l_n}S_{l_1\ldots l_n},$$

that is, because $C_{l_1\ldots l_n}$ is unitary

$$C^{*}_{l_1\ldots l_n}S_{l_1\ldots l_n} = E_{l_1\ldots l_n}C^{*}_{l_1\ldots l_n}S_{l_1\ldots l_n}.$$

It follows that $S_{l_1\ldots l_n}$ is a solution if and only if the column vector $C^{*}_{l_1\ldots l_n}S_{l_1\ldots l_n}$ has zeroes corresponding to the zeroes of $E_{l_1\ldots l_n}$, whereas the elements corresponding to unity can be arbitrary. In view of the orthonormality properties of $C^{*}_{l_1\ldots l_n}$, this condition is met if, and only if, $S_{l_1\ldots l_n}$ is a linear combination of the columns in the matrix $C^{*}_{l_1\ldots l_n}$ corresponding to non-zero elements of the diagonal $E_{l_1\ldots l_n}$. These linear combinations can be written explicitly as

$$\begin{aligned}&\sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}} C^{\lambda_1\ldots\lambda_{n-2}lm}_{l_1m_1\ldots l_nm_n}\widetilde{P}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3},\lambda_{n-2})\delta^{0}_{l}\\ &=\sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}}\left\{\sum_{\mu_1\ldots\mu_{n-2}} C^{\lambda_1\mu_1}_{l_1m_1l_2m_2}C^{\lambda_2\mu_2}_{\lambda_1\mu_1l_3m_3}\cdots C^{lm}_{\lambda_{n-2}\mu_{n-2}.l_nm_n}\delta^{0}_{l}\right\}\widetilde{P}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3},\lambda_{n-2})\\ &=\sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}}\left\{\sum_{\mu_1\ldots\mu_{n-2}} C^{\lambda_1\mu_1}_{l_1m_1l_2m_2}C^{\lambda_2\mu_2}_{\lambda_1\mu_1l_3m_3}\cdots C^{00}_{\lambda_{n-2}\mu_{n-2}.l_nm_n}\right\}\widetilde{P}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3},\lambda_{n-2}).\end{aligned}$$

Author's personal copy

Recalling again that

$$C^{0m}_{l_1m_1l_2m_2} = \frac{(-1)^{m_1}}{2l_1+1}\delta^{l_2}_{l_1}\delta^{-m_2}_{m_1}\delta^0_m,$$

(see [12], 8.5.1.1), we obtain that

$$= \sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}}\left\{\sum_{\mu_1\ldots\mu_{n-2}} C^{\lambda_1\mu_1}_{l_1m_1l_2m_2}C^{\lambda_2\mu_2}_{\lambda_1\mu_1l_3m_3}\cdots\frac{(-1)^{m_n}}{2l_n+1}\delta^{l_n}_{\lambda_{n-2}}\delta^{-m_n}_{\mu_{n-2}}\right\}\times\widetilde{P}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3},\lambda_{n-2})$$

$$= \sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}}\left\{\sum_{\mu_1\ldots\mu_{n-2}} C^{\lambda_1\mu_1}_{l_1m_1l_2m_2}C^{\lambda_2\mu_2}_{\lambda_1\mu_1l_3m_3}\cdots C^{l_n-m_n}_{\lambda_{n-3}\mu_{n-3}.l_{n-1}m_{n-1}}(-1)^{m_n}\right\}P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3})$$

$$= \sum_{\lambda_1=l_2-l_1}^{l_2-l_1}\sum_{\lambda_2=l_3-\lambda_1}^{l_3+\lambda_1}\cdots\sum_{\lambda_{n-1}=l_n-\lambda_{n-2}}^{l_n+\lambda_{n-2}} C^{\lambda_1\ldots\lambda_{n-3}l_n-m_n}_{l_1m_1\ldots l_{n-1}m_{n-1}}P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}),$$

where we have set

$$P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}) := \frac{1}{2l_n+1}\widetilde{P}_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3},l_n).$$

All there is left to show is that the coefficients of this linear combination are necessarily real. To see this, it is sufficient to specialize the previous discussion to the case where $m_1 = m_2 = \cdots = m_n = 0$, and to observe that, in this case

$$Ea_{l_10}\ldots a_{l_n0} = \sum_{\lambda_1}\cdots\sum_{\lambda_{n-3}} C^{\lambda_1\ldots\lambda_{n-3}l_n0}_{l_10\ldots l_{n-1}0}P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3})$$

is real by definition (note indeed that the columns of $C_{l_1\ldots l_n}$ are linearly independent). ■

Let us illustrate the previous results by some more examples.

Examples. For $n = 3$, Theorem 12 implies that, under isotropy

$$Ea_{l_1m_1}a_{l_2m_2}a_{l_3m_3} = (-1)^{m_3}C^{l_3-m_3}_{l_1m_1l_2m_2}P_{l_1l_2l_3}.$$

From this last relation, we can recover the so-called reduced bispectrum, noted $b_{l_1l_2l_3}$, defined for instance in [16,18,19], which satisfies indeed the relationship

$$P_{l_1l_2l_3} = b_{l_1l_2l_3}C^{l_30}_{l_10l_20}\sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l_3+1)4\pi}}.$$

For $n = 4$ (i.e. the trispectrum, [16]) we obtain the expression

$$Ea_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4} = (-1)^{m_4}\sum_{\lambda=|l_2-l_1|}^{l_2+l_1} C^{\lambda l_4-m_4}_{l_1m_1l_2m_2l_3m_3}P_{l_1l_2l_3l_4}(\lambda)$$

$$= \sum_{\lambda=|l_2-l_1|}^{l_2+l_1}\sum_{\mu=-\lambda}^{\lambda} C^{\lambda\mu}_{l_1m_1l_2m_2}C^{l_4-m_4}_{\lambda\mu l_3m_3}P_{l_1l_2l_3l_4}(\lambda).$$

The next result gives a further probabilistic characterization of the reduced bispectrum.

Proposition 13. *Fix $n \geq 2$. A real-valued array $\left\{A_{l_1\ldots l_n}(\cdot) : l_1,\ldots,l_n \geq 0\right\}$ is the reduced polyspectrum of order $n-1$ (resp. the reduced cumulant polyspectrum of order $n-1$) of some strongly isotropic random field if, and only if, there exists a sequence $\{X_l : l \geq 0\}$ of zero-mean real-valued random variables such that*

$$\sum_{l\geq 0}(2l+1)\,E\left[X_l^2\right] < +\infty$$

and, for every $l_1,\ldots,l_n \geq 0$

$$E\left(X_{l_1}\cdots X_{l_n}\right) = \sum_{\lambda_1=l_2-l_1}^{l_2+l_1}\cdots\sum_{\lambda_{n-3}} C^{\lambda_1\ldots\lambda_{n-3}l_n0}_{l_10\ldots l_{n-1}0}A_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}) \tag{5.55}$$

Author's personal copy

resp.

$$\mathrm{Cum}\left\{X_{l_1},\ldots,X_{l_n}\right\}=\sum_{\lambda_1=l_2-l_1}^{l_2+l_1}\cdots\sum_{\lambda_{n-3}}C_{l_10\ldots l_{n-1}0}^{\lambda_1\ldots\lambda_{n-3}l_n0}A_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}). \tag{5.56}$$

Proof. We shall only prove (5.55). For the necessity, it is enough to take $X_l = a_{l0}$, where a_{l0} is the harmonic coefficient of index $(l, 0)$ associated with a strongly isotropic field with moments of all orders. For the sufficiency, we consider first the (anisotropic) random field

$$Z(x)=\sum_{l\geq 0}X_lY_{l0}(x).$$

Then, by taking $T(x) = Z(gx)$, where g is sampled randomly with the uniform Haar measure on SO(3), one obtains a random field with the desired characteristics. ■

There are two very important issues that are left open by Theorem 12. As a first issue, it seems natural to look for characterizations of the reduced polyspectra $P_{l_1\ldots l_n}$, at least under natural models of physical interest. As a second point, we note that the explicit expressions provided in Theorem 12 depend on the ordering $l_1,\ldots,l_n$ we chose for the decomposition of $\Delta_{l_1\ldots l_n}$. In the next two sections, we try to address these (and other) points.

6. Some explicit examples

In this section we provide explicit computations for the reduced polyspectra $P_{l_1\ldots l_n}$ $(n \geq 2)$, or $P^C_{l_1\ldots l_n}$, for some models of physical interest. Of course, the Gaussian isotropic fields can be easily dealt with. Indeed, in this case one has that $P^C_{l_1\ldots l_n} = 0$ for all $n \geq 3$. In what follows, we shall therefore be concerned with polyspectra of *Gaussian subordinated* isotropic fields, that is, random fields that can be written as a deterministic and non-linear function of some collection of Gaussian isotropic fields. In general, this class of random fields allow for a clear-cut mathematical treatment, whilst covering a great array of empirically relevant circumstances.

6.1. A simple physical model

The general Gaussian-subordinated model has the form

$$T=\sum_{j=1}^{q}f_jH_j\left(T_G/\sqrt{E\left(T_G^2\right)}\right)=f_1T_G+f_2(T_G^2/E\left(T_G^2\right)-1)+\cdots, \tag{6.57}$$

where f_j is a real constant, $H_j(.)$ denotes the jth Hermite polynomial (see e.g. [36]), and T_G is a Gaussian, zero-mean isotropic random field. Note that we have implicitly defined the sequence of Hermite polynomials in such a way that $H_1(x) = x$, $H_2(x) = x^2-1$, $H_3(x) = x^3-3x$, and so on. In this section, when no further specification is needed, the spectral decomposition of the underlying Gaussian field T_G is written

$$T_G=\sum_{lm}a_{lm}Y_{lm}.$$

We shall sometimes use the following notation

$$T=\sum_{lm}\widetilde{a}_{lm}Y_{lm}=\sum_{j=1}^{q}f_ja_{lm}(j)Y_{lm}, \tag{6.58}$$

$$a_{lm}(j)=\int_{S^2}H_j\left(T_G(x)/\sqrt{E\left(T_G^2\right)}\right)\overline{Y_{lm}}(x)\mathrm{d}x, \tag{6.59}$$

$$\widetilde{a}_{lm}=\sum_{j=1}^{q}a_{lm}(j). \tag{6.60}$$

For instance, models of Cosmic Microwave Background radiation are currently dominated by assumptions such as the Sachs–Wolfe model with the so-called *Bardeen's potential* (see e.g. [24] or [15]). The latter can be written down explicitly as

$$T=T_G+f_{NL}(T_G^2-ET_G^2), \tag{6.61}$$

where f_{NL} is a nonlinearity parameter which depends upon physical constants in the associated *"slow-roll" inflationary model* (see e.g. [24]). Note that (6.61) has can be written in the form (6.57), by setting $f_1 = 1$, $f_2 = f_{NL} \times E\left(T_G^2\right)$ and $f_j = 0$, for $j \geq 3$.

Author's personal copy

The value of the constant $f_{NL} \times E\left(T_G^2\right)$ is expected to be very small, namely of the order 10^{-4} [24]. To simplify the discussion, we now assume that $ET_G^2 = 1$. In this case, by using (6.58)–(6.60), one has that

$$\widetilde{a}_{lm} = a_{lm} + f_{NL} a_{lm}(2),$$
$$a_{lm}(2) = \int_{S^2} T^2 \overline{Y}_{lm} \mathrm{d}x = \int_{S^2} \sum_{\ell_1 \ell_2} \sum_{m_1 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} Y_{\ell_1 m_1} Y_{\ell_2 m_2} \overline{Y}_{lm} \mathrm{d}x$$
$$= \sum_{\ell_1 \ell_2} \sum_{m_1 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{(2l+1)4\pi}} C^{l0}_{\ell_1 0 \ell_2 0} C^{lm}_{\ell_1 m_1 \ell_2 m_2}.$$

It follows that

$$\widetilde{C}_l := E|\widetilde{a}_{lm}|^2 = C_l + 2f_{NL}^2 \sum_{l_1 l_2} C_{l_1} C_{l_2} \frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)} \left(C^{l0}_{l_1 0 l_2 0}\right)^2,$$

so that

$$\mathrm{Var}(T) = \sum_l \frac{2l+1}{4\pi} \widetilde{C}_l = \sum_l \frac{2l+1}{4\pi} C_l + 2f_{NL}^2 \sum_{l_1 l_2} C_{l_1} C_{l_2} \frac{(2l_1+1)(2l_2+1)}{(4\pi)^2} \sum_l \left(C^{l0}_{l_1 0 l_2 0}\right)^2$$
$$= \sum_l \frac{2l+1}{4\pi} C_l + 2f_{NL}^2 \left\{ \sum_{l_1} C_{l_1} \frac{(2l_1+1)}{4\pi} \right\}^2 = \mathrm{Var}(T_G) + f_{NL}^2 \mathrm{Var}(H_2(T_G)),$$

as expected, due to the orthogonality properties of Hermite polynomials. For the bispectrum, we obtain therefore

$$E\widetilde{a}_{l_1 m_1} \widetilde{a}_{l_2 m_2} \widetilde{a}_{l_3 m_3} = E\left\{ (a_{l_1 m_1} + f_2 a_{l_1 m_1}(2))(a_{l_2 m_2} + f_2 a_{l_2 m_2}(2))(a_{l_3 m_3} + f_2 a_{l_3 m_3}(2)) \right\}$$
$$= f_2 E a_{l_1 m_1}(2) a_{l_2 m_2} a_{l_3 m_3} + f_2 E a_{l_1 m_1} a_{l_2 m_2}(2) a_{l_3 m_3}$$
$$+ f_2 E a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}(2) + f_2^3 E a_{l_1 m_1}(2) a_{l_2 m_2}(2) a_{l_3 m_3}(2)$$
$$= (-1)^{m_3} C^{l_3 -m_3}_{l_1 m_1 l_2 m_2} P_{l_1 l_2 l_3},$$

where

$$P_{l_1 l_2 l_3} = 6f_2 \sqrt{\frac{(2l_1+1)(2l_2+1)}{(2l_3+1)4\pi}} C^{l_3 0}_{l_1 0 l_2 0} \left\{ C_{l_1} C_{l_2} + C_{l_1} C_{l_3} + C_{l_2} C_{l_3} \right\} \tag{6.62}$$

$$+ f_2^3 \sum_{\ell_1 \ell_2 \ell_3} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_2 0}_{\ell_1 0 \ell_3 0} C^{l_3 0}_{\ell_2 0 \ell_3 0} \frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{\sqrt{(4\pi)^3}} \frac{8(-1)^{l_3}}{\sqrt{2l_3+1}} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ l_3 & l_2 & l_1 \end{Bmatrix} \left\{ C_{\ell_1} C_{\ell_2} C_{\ell_3} \right\}. \tag{6.63}$$

The lack of symmetry with respect to the l_3 term is only apparent and can be easily dispensed with by permuting the multipoles in $C^{l_3 m_3}_{l_1 m_1 l_2 m_2}$ or using expression (3.18). Formula (6.62) is consistent with the cosmological literature, where (6.63) is considered a higher order term and hence neglected (see again [16]).

6.2. *The connection with higher order moments*

We now provide a simple result, connecting the reduced polyspectrum with the higher order moments of the associated spherical random field.

Proposition 14. *The following identity holds for every isotropic field with finite moments of order p and with a reduced polyspectrum $\left\{P_{l_1 \dots l_p}(\cdot) : l_1, \dots, l_p \geq 0\right\}$: for every $x \in S^2$,*

$$ET(x)^p \equiv \sum_{l_1 \dots l_p} \sqrt{\frac{(2l_1+1)\cdots(2l_p+1)}{(4\pi)^p}} \sum_{\lambda_1 \dots \lambda_{p-3}} P_{l_1 \dots l_p}(\lambda_1, \dots, \lambda_{p-3}) C^{\lambda_1 \dots \lambda_{p-3} l_p 0}_{l_1 0 \dots l_{p-2} 0}.$$

Proof. We use the trivial fact that

$$T(x) \stackrel{d}{=} T(0) = \sum_l a_{l0} Y_{l0}(0) = \sum_l a_{l0} \sqrt{\frac{2l+1}{4\pi}},$$

Author's personal copy

where 0 is the North Pole and we used the fact that, for $m \neq 0$, $Y_{lm}(0) = 0$ and $Y_{l0}(0) = \sqrt{\frac{2l+1}{4\pi}}$ (see e.g. [12, Chapter 5]). Hence,

$$ET^p = \sum_{l_1 \dots l_p} \sqrt{\frac{(2l_1+1)\cdots(2l_p+1)}{(4\pi)^p}} E\left\{a_{l_1 0} \dots a_{l_p 0}\right\}$$

$$= \sum_{l_1 \dots l_p} \sqrt{\frac{(2l_1+1)\cdots(2l_p+1)}{(4\pi)^p}} \sum_{\lambda_1 \dots \lambda_{p-3}} P_{l_1 \dots l_p}(\lambda_1, \dots, \lambda_{p-3}) C^{\lambda_1 \dots \lambda_{p-3} l_p 0}_{l_1 0 \dots l_{p-2} 0}. \quad \blacksquare$$

Example. Take $T = H_q(T_G)$, where H_q is the qth Hermite polynomial. Then $ET^p = c_{pq}\left\{ET^2\right\}^{qp/2}$, where $c_{pq} \in \mathbb{N}$ denotes the number of Gaussian diagrams without flat edges with p rows and q columns (see [36]). Therefore, one has the identity

$$\sum_{l_1 \dots l_p} \sqrt{\frac{(2l_1+1)\dots(2l_p+1)}{(4\pi)^p}} \sum_{\lambda_1 \dots \lambda_{p-3}} P_{l_1 \dots l_p}(\lambda_1, \dots, \lambda_{p-3}) C^{\lambda_1 \dots \lambda_{p-3} l_p 0}_{l_1 0 \dots l_{p-2} 0} = c_{pq} \left\{ \sum_l \frac{(2l+1)}{4\pi} C_l \right\}^{pq/2}.$$

6.3. The χ^2_ν polyspectrum

Previously in (6.63), we have implicitly derived the "χ^2_1 bispectrum", that is, the bispectrum associated with a field of the type $T = H_2(T_G)$, where T_G is Gaussian, centered, isotropic and with unit variance. More precisely, with the notation (6.58)–(6.60)), one deduces from (6.63) that

$$Ea_{l_1 m_1}(2) a_{l_2 m_2}(2) a_{l_3 m_3}(2) = \sum_{\substack{\ell_1 \ell_2 \ell_3 \\ \ell_4 \ell_5 \ell_6}} \sum_{\mu_1 \dots \mu_6} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_1 m_1}_{\ell_1 \mu_1 \ell_2 \mu_2} C^{l_2 0}_{\ell_3 0 \ell_4 0} C^{l_2 m_2}_{\ell_3 \mu_3 \ell_4 \mu_4} C^{l_3 0}_{\ell_5 0 \ell_6 0} C^{l_3 m_3}_{\ell_5 \mu_5 \ell_6 \mu_6}$$

$$\times \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{(2l_1+1)4\pi} \frac{(2\ell_3+1)(2\ell_4+1)}{(2l_2+1)4\pi} \frac{(2\ell_5+1)(2\ell_6+1)}{(2l_3+1)4\pi}} E\left\{a_{\ell_1 \mu_1} a_{\ell_2 \mu_2} a_{\ell_3 \mu_3} a_{\ell_4 \mu_4} a_{\ell_5 \mu_5} a_{\ell_6 \mu_6}\right\}$$

$$= 8(-1)^{l_3 - m_3} \sum_{\ell_1 \ell_2 \ell_3} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_2 0}_{\ell_1 0 \ell_3 0} C^{l_3 0}_{\ell_2 0 \ell_3 0} \frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{\sqrt{(4\pi)^3}} \frac{C^{l_3 - m_3}_{l_1 m_1 l_2 m_2}}{\sqrt{2l_3+1}} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ l_3 & l_2 & l_1 \end{Bmatrix} \left\{C_{\ell_1} C_{\ell_2} C_{\ell_3}\right\}, \quad (6.64)$$

see [12, p. 260; p. 454]. We now wish to extend these results to polyspectra of order $p = 4, 5, 6$ for random fields of the type $T = H_2(T_G)$, where (as above) T_G is Gaussian, centered, isotropic and with unit variance. As anticipated, here we focus on cumulants instead of moments. We have the following result.

Proposition 15. *The cumulant $\chi\left(a_{l_1 m_1}(2), \dots, a_{l_p m_p}(2)\right)$ $(p = 4, 5, 6)$ associated with the harmonic coefficients of an isotropic random field of the type $H_2(T_G)$ (where T_G is Gaussian and isotropic, with angular power spectrum $\{C_l : l \geq 0\}$) given by*

$$\chi\left(a_{l_1 m_1}(2), \dots, a_{l_p m_p}(2)\right) = (-1)^{l_p - m_p} \sum_{\lambda_1 \dots \lambda_{p-3}} C^{\lambda_1 \dots \lambda_{p-3} l_p - m_p}_{l_1 m_1 \dots l_{p-1} m_{p-1}} \times P^{C;1}_{l_1 \dots l_p}\left(\lambda_1, \dots, \lambda_{p-3}\right),$$

where the reduced cumulant polyspectrum $\left\{P^C_{l_1 \dots l_p}(\cdot) : l_1, \dots, l_p \geq 0\right\}$ is given by

$$P^{C;1}_{l_1 l_2 l_3 l_4}(\lambda) = 48 \sqrt{\frac{(2\lambda+1)}{(4\pi)^4 (2l_4+1)}} \sum_{\ell_1 \dots \ell_4} C_{\ell_1} \dots C_{\ell_4} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_3 0}_{\ell_2 0 \ell_3 0} C^{l_4 0}_{\ell_3 0 \ell_4 0} C^{l_2 0}_{\ell_4 0 \ell_1 0}$$
$$\times (2\ell_1+1)\dots(2\ell_4+1)(-1)^{l_1 + l_2 + \ell_2 + \ell_4} \begin{Bmatrix} l_1 & l_2 & \lambda \\ \ell_4 & \ell_2 & \ell_1 \end{Bmatrix} \begin{Bmatrix} \lambda & l_3 & l_4 \\ \ell_3 & \ell_4 & \ell_2 \end{Bmatrix} \quad \text{for } p = 4,$$

$$P^{C;1}_{l_1 \dots l_5}(\lambda_1, \lambda_2) = 384 \sqrt{\frac{(2\lambda_1+1)(2\lambda_2+1)}{(4\pi)^5 (2l_5+1)}} \sum_{\ell_1 \dots \ell_5} C_{\ell_1} \dots C_{\ell_5} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_2 0}_{\ell_2 0 \ell_3 0} C^{l_4 0}_{\ell_3 0 \ell_4 0} C^{l_5 0}_{\ell_4 0 \ell_5 0} C^{l_3 0}_{\ell_5 0 \ell_1 0}$$
$$\times (2\ell_1+1)\dots(2\ell_5+1)(-1)^{\ell_1 + \ell_5 + l_3} \begin{Bmatrix} l_1 & l_2 & \lambda_1 \\ \ell_3 & \ell_1 & \ell_2 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & l_3 & \lambda_2 \\ \ell_5 & \ell_3 & \ell_1 \end{Bmatrix} \begin{Bmatrix} \lambda_2 & l_4 & l_5 \\ \ell_4 & \ell_5 & \ell_3 \end{Bmatrix} \quad \text{for } p = 5,$$

and

$$P^{C;1}_{l_1 \dots l_6}(\lambda_1, \lambda_2, \lambda_3) = 3840 \sqrt{\frac{(2\lambda_1+1)(2\lambda_2+1)(2\lambda_3+1)}{(4\pi)^6 (2l_5+1)}} \sum_{\ell_1 \dots \ell_5} C_{\ell_1} \dots C_{\ell_6} C^{l_1 0}_{\ell_1 0 \ell_2 0} C^{l_2 0}_{\ell_2 0 \ell_3 0} C^{l_3 0}_{\ell_3 0 \ell_4 0} C^{l_5 0}_{\ell_4 0 \ell_5 0} C^{l_6 0}_{\ell_5 0 \ell_6 0} C^{l_4 0}_{\ell_6 0 \ell_1 0}$$

Author's personal copy

$$\times (2\ell_1+1)\dots(2\ell_6+1)(-1)^{\lambda_1+\ell_3+\ell_6+l_4} \begin{Bmatrix} l_1 & l_2 & \lambda_1 \\ \ell_3 & \ell_1 & \ell_2 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & l_5 & \lambda_2 \\ \ell_5 & \ell_3 & \ell_1 \end{Bmatrix} \begin{Bmatrix} \lambda_2 & l_3 & l_4 \\ \ell_4 & \ell_5 & \ell_3 \end{Bmatrix}$$

for $p = 6$.

Proof. The result can be proved by means of the standard graphical techniques for convolutions of Clebsch–Gordan coefficients, as described in [12, Chapters 11 and 12]. Here, we only provide the complete proof for the case $p = 6$. Let $\{a_{\ell m}\}$ be the random harmonic coefficients associated with the underlying Gaussian field T_G. By definition, the field $H_2(T_G)$ admits the expansion

$$H_2(T_G) = \sum_{l\ge 0}\sum_{m=-l}^{l} a_{lm}(2)\, Y_{lm},$$

where

$$\begin{aligned} a_{lm}(2) &= \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \int_{S^2} Y_{\ell_1 m_1}(x) Y_{\ell_2 m_2}(x) \overline{Y_{lm}(x)}\mathrm{d}x \\ &= \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \begin{pmatrix} \ell_1 & \ell_2 & l \\ m_1 & m_2 & -m \end{pmatrix} \times (-1)^m \begin{pmatrix} \ell_1 & \ell_2 & l \\ 0 & 0 & 0 \end{pmatrix} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2l+1)}{4\pi}} \\ &= \sum_{\ell_1 m_1 \ell_2 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} C^{lm}_{\ell_1 m_1 \ell_2 m_2} C^{lm}_{\ell_1 0 \ell_2 0} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi\,(2l+1)}}. \end{aligned}$$

By using once again the multilinearity of cumulants, one obtains that

$$\begin{aligned} \mathrm{Cum}\left\{a_{l_1 m_1}(2),\dots,a_{l_6 m_6}(2)\right\} &= \sum_{\ell_{11} m_{11} \ell_{12} m_{12}} \cdots \sum_{\ell_{61} m_{61} \ell_{61} m_{61}} \mathrm{Cum}\left\{a_{\ell_{11} m_{11}} a_{\ell_{12} m_{12}},\dots,a_{\ell_{61} m_{61}} a_{\ell_{62} m_{62}}\right\} \\ &\quad \times \prod_{j=1}^{6} \left\{ C^{l_j m_j}_{\ell_{j1} m_{j1} \ell_{j2} m_{j2}} C^{l_j m_j}_{\ell_{j1} 0 \ell_{j2} 0} \sqrt{\frac{(2\ell_{j1}+1)(2\ell_{j2}+1)}{4\pi\,(2l_j+1)}} \right\}. \end{aligned}$$

For a given $\mathbf{lm} = (\ell_{11} m_{11}, \ell_{12} m_{12}; \dots; \ell_{61} m_{61}, \ell_{62} m_{62})$, the quantity $\mathrm{Cum}\left\{a_{\ell_{11} m_{11}} a_{\ell_{12} m_{12}},\dots,a_{\ell_{61} m_{61}} a_{\ell_{62} m_{62}}\right\}$ is computed as follows:

- Build the 6×2 matrix

$$\Lambda(\mathbf{lm}) = \begin{bmatrix} \ell_{11} m_{11} & \ell_{12} m_{12} \\ \ell_{21} m_{21} & \ell_{22} m_{22} \\ \ell_{31} m_{31} & \ell_{32} m_{32} \\ \ell_{41} m_{41} & \ell_{42} m_{42} \\ \ell_{51} m_{51} & \ell_{52} m_{52} \\ \ell_{61} m_{61} & \ell_{62} m_{62} \end{bmatrix}.$$

- Define the class $M(\Lambda(\mathbf{lm}))$ of connected, Gaussian non-flat diagrams over Λ, that is, every $\gamma \in M(\Lambda(\mathbf{lm}))$ is a partition of the entries of $\Lambda(\mathbf{lm})$, into pairs belonging to different rows; moreover, such a partition has to be *connected*, in the sense that γ cannot be divided into two separate diagrams. For instance, an element of $M(\Lambda(\mathbf{lm}))$ is

$$\gamma = \{\{\ell_{11} m_{11}, \ell_{21} m_{21}\}\{\ell_{22} m_{22}, \ell_{32} m_{32}\}\{\ell_{31} m_{31}, \ell_{41} m_{41}\}\{\ell_{42} m_{42}, \ell_{52} m_{52}\}\{\ell_{51} m_{61}, \ell_{61} m_{61}\}\{\ell_{62} m_{62}, \ell_{12} m_{12}\}\}$$

- For every $\gamma \in M(\Lambda(\mathbf{lm}))$, write

$$\delta(\gamma) = \prod_{\{\ell_{ab} m_{ab}, \ell_{cd} m_{cd}\}\in\gamma} \delta^{\ell_{ab}}_{\ell_{cd}} \delta^{-m_{cd}}_{m_{ab}} (-1)^{m_{ab}} C_{\ell_{ab}}$$

  (where δ^b_a is the usual Kronecker symbol).
- Use the standard diagram formula (see again [36]), to obtain that

$$\mathrm{Cum}\left\{a_{\ell_{11} m_{11}} a_{\ell_{12} m_{12}},\dots,a_{\ell_{61} m_{61}} a_{\ell_{62} m_{62}}\right\} = \sum_{\gamma\in M(\Lambda(\mathbf{lm}))} \delta(\gamma).$$

It follows that

$$\mathrm{Cum}\left\{a_{l_1 m_1}(2),\dots,a_{l_6 m_6}(2)\right\} = \sum_{\mathbf{lm}} \sum_{\gamma\in M(\Lambda(\mathbf{lm}))} \delta(\gamma) \prod_{j=1}^{6} \left\{ C^{l_j m_j}_{\ell_{j1} m_{j1} \ell_{j2} m_{j2}} C^{l_j m_j}_{\ell_{j1} 0 \ell_{j2} 0} \sqrt{\frac{(2\ell_{j1}+1)(2\ell_{j2}+1)}{4\pi\,(2l_j+1)}} \right\},$$

Author's personal copy

where the first sum runs over all vectors of the type $\mathbf{lm} = (\ell_{11}m_{11}, \ell_{12}m_{12}; \ldots; \ell_{61}m_{61}, \ell_{62}m_{62})$. The proof now follows directly from graphical techniques. In particular, the previous term can be associated with a hexagon, having in each vertex an outward line corresponding to a "free" (i.e. not summed up) index $l_i m_i$, $i = 1, \ldots, 6$. An expression for convolutions of Clebsch–Gordan coefficients corresponding to such a configuration can be found in [12, p. 461], Eq. 12.1.6.30. From this, standard combinatorial arguments and a convenient relabelling of the indexes, we obtain that

$$P^{C;1}_{l_1\ldots l_6}(\lambda_1,\lambda_2,\lambda_3) = 3840\sqrt{\frac{\left\{\prod_{j=1}^{3}(2\lambda_j+1)\right\}}{(4\pi)^6(2l_p+1)}} \times (-1)^{\lambda_1+\ell_3+\ell_6+l_4}$$
$$\times \sum_{\ell_1\ldots\ell_6}(2\ell_1+1)\cdots(2\ell_6+1)C_{\ell_1}\ldots C_{\ell_6}C^{l_1 0}_{\ell_1 0\ell_2 0}C^{l_2 0}_{\ell_2 0\ell_3 0}C^{l_3 0}_{\ell_3 0\ell_4 0}C^{l_5 0}_{\ell_4 0\ell_5 0}C^{l_6 0}_{\ell_5 0\ell_6 0}C^{l_4 0}_{\ell_6 0\ell_1 0}$$
$$\times \begin{Bmatrix} l_1 & l_2 & \lambda_1 \\ \ell_3 & \ell_1 & \ell_2 \end{Bmatrix}\begin{Bmatrix} \lambda_1 & \lambda_2 & l_3 \\ \ell_4 & \ell_3 & \ell_1 \end{Bmatrix}\begin{Bmatrix} \lambda_2 & l_4 & \lambda_3 \\ \ell_6 & \ell_4 & \ell_1 \end{Bmatrix}\begin{Bmatrix} \lambda_3 & l_5 & l_6 \\ \ell_5 & \ell_6 & \ell_4 \end{Bmatrix}.$$

Note that $3840 = 2^{p-1}(p-1)! = 2^5 5!$ is the number of automorphisms between graphs belonging to $M(\Lambda(\mathbf{lm}))$. ■

We recall that the Clebsch–Gordan coefficients $\left\{C^{c0}_{a0b0}\right\}$ are identically zero unless $a+b+c$ is even; it is hence easy to see that the previous polyspectra are non-zero only if the sum $\left\{l_1+\cdots+l_p\right\}$ is even as well.

From the previous Proposition, we can derive the corresponding expressions for the cumulant polyspectra for χ^2_ν random field.

Definition B. We say the random field $T_{\chi^2_\nu}$ has a chi-square law with $\nu \geq 1$ degrees of freedom if there exist ν independent and identically distributed Gaussian random fields T_i such that

$$T_{\chi^2_\nu} \stackrel{law}{=} T_1^2 + \cdots + T_\nu^2.$$

It is trivial to show that $T_{\chi^2_\nu}$ is mean-square continuous and isotropic if T_i is. We have the following

Proposition 16. *The cumulant polyspectra of* $T_{\chi^2_\nu}$ *(for* $p \geq 2$*) are given by*

$$P^{C;\nu}_{l_1\ldots l_p}(\lambda_1,\ldots,\lambda_{p-3}) = \nu P^{C;1}_{l_1\ldots l_p}(\lambda_1,\ldots,\lambda_{p-3}).$$

Proof. Note that the cumulant polyspectra of order $p \geq 2$ of $T_{\chi^2_\nu}$ coincide with those of the centered field $T_{\chi^2_\nu} - ET_{\chi^2_\nu}$ (due to the translation-invariance properties of cumulants). Then, the proof is an immediate consequence of Proposition 15 and the of the standard multinearity properties of cumulants. ■

7. Further issues and applications

The purpose of this final Section is to introduce what we view as promising directions for further research, where the ideas of this paper may perhaps yield further insights. We shall delay to future work a more thorough investigation of the issues which are left open below.

7.1. Representations of the symmetric group

As a further link between representation theory and higher order angular power spectra, we mention the following. It is to be stressed that the decomposition of $\Delta_{l_1\ldots l_n}$ that we achieved in the previous Proposition 11 is by no means unique. In particular, what we did was to choose a particular sequence of "couplings", i.e. we partitioned tensor products of the Wigner's matrices D^l in a specific order before decomposing them into direct sums. Alternative partitions yield different eigenvectors and, therefore, different expressions for the polyspectra/joint moments. Alternatively, we could maintain the same coupling scheme (for instance, "start always from the first pair on the left", as we did earlier) but acting on $(l_1,\ldots,l_n)$ by the symmetric group S_n. However, not all coupling schemes can be achieved by simply permuting the elements of $(l_1,l_2,\ldots,l_n)$. This is the well-known *problem of parentheses*in Mathematical Physics (see for instance [35]).

We suggest here that one can establish a link between alternate expressions for the angular polyspectra and representations of the symmetric group. More precisely, the alternate expressions that we find for the polyspectra $P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3})$ of a strongly isotropic field (with n-moments) must be such that for every permutation $\pi \in \mathfrak{S}_n$,

$$\sum_{\lambda_1}\cdots\sum_{\lambda_{n-3}} C^{\lambda_1\ldots\lambda_{n-3};l_n-m_n}_{l_1m_1\ldots l_{n-1}m_{n-1}} P_{l_1\ldots l_n}(\lambda_1,\ldots,\lambda_{n-3}) = \sum_{\lambda'_1}\cdots\sum_{\lambda'_{n-3}} C^{\lambda'_1\ldots\lambda'_{n-3};l_n-m_n}_{\pi(l_1)m_1\ldots\pi(l_{n-1})m_{n-1}} P_{\pi(l_1)\ldots\pi(l_n)}(\lambda'_1,\ldots,\lambda'_{n-3}).$$

Author's personal copy

Now let us multiply both sides by $C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1'' \dots \lambda_{n-3}'' ; l_n m_n'}$, where $(\lambda_1'', \dots, \lambda_{n-3}'')$ is fixed, and sum over $(m_1, \dots m_n)$. In view of the unitary properties of Clebsch–Gordan coefficients we obtain for the left-hand side

$$\sum_{m_1 \dots m_n} C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1'' \dots \lambda_{n-3}'' ; l_n - m_n} \left\{ \sum_{\lambda_1} \dots \sum_{\lambda_{n-3}} C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1 \dots \lambda_{n-3} ; l_n - m_n} P_{l_1 \dots l_n}(\lambda_1, \dots, \lambda_{n-3}) \right\}$$
$$= \sum_{\lambda_1} \dots \sum_{\lambda_{n-3}} \left\{ \sum_{m_1 \dots m_n} C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1'' \dots \lambda_{n-3}'' ; l_n - m_n} C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1 \dots \lambda_{n-3} ; l_n - m_n} P_{l_1 \dots l_n}(\lambda_1, \dots, \lambda_{n-3}) \right\}$$
$$= \sum_{\lambda_1} \dots \sum_{\lambda_{n-3}} \left\{ \delta_{\lambda_1}^{\lambda_1''} \dots \delta_{\lambda_{n-3}}^{\lambda_{n-3}''} P_{l_1 \dots l_n}(\lambda_1, \dots, \lambda_{n-3}) \right\} = P_{l_1 \dots l_n}(\lambda_1'', \dots, \lambda_{n-3}''); \tag{7.65}$$

on the right-hand side we get

$$\sum_{m_1 \dots m_n} C_{l_1 m_1 \dots l_{n-1} m_{n-1}}^{\lambda_1'' \dots \lambda_{n-3}'' ; l_n m_n} \left\{ \sum_{\lambda_1'} \dots \sum_{\lambda_{n-3}'} C_{\pi(l_1) m_1 \dots \pi(l_{n-1}) m_{n-1}}^{\lambda_1' \dots \lambda_{n-3}' ; l_n m_n} P_{\pi(l_1) \dots \pi(l_n)}(\lambda_1', \dots, \lambda_{n-3}') \right\}$$
$$= \sum_{\lambda_1'} \dots \sum_{\lambda_{n-3}'} \sum_{m_1 \dots m_n} C_{\pi(l_1) m_1 \dots \pi(l_{n-1}) m_{n-1}}^{\lambda_1'' \dots \lambda_{n-3}'' ; l_n m_n} C_{\pi(l_1) m_1 \dots \pi(l_{n-1}) m_{n-1}}^{\lambda_1' \dots \lambda_{n-3}' ; l_n m_n} P_{\pi(l_1) \dots \pi(l_n)}(\lambda_1', \dots, \lambda_{n-3}'). \tag{7.66}$$

Similarly as in the previous section, the sum of products of Clebsch–Gordan coefficients on the right hand side can be expressed in terms of higher order Wigner's coefficients. Since this section is just informal, for brevity's sake we do not give explicit expressions (see e.g. [12, Chapter 10]). The two expressions (7.65) and (7.66) imply that, for every fixed $(l_1, \dots, l_n)$ and every permutation π, there exists a square matrix $A((l_1, \dots, l_n); \pi)$ such that

$$P_{l_1 \dots l_n} = A\{(l_1, \dots, l_n); \pi\} P_{\pi(l_1) \dots \pi(l_n)},$$

where $P_{l_1 \dots l_n}$ is the vector with entries $P_{l_1 \dots l_n}(\lambda_1, \dots, \lambda_n)$. We conjecture that in this way one can build a representation of the symmetric group $\mathfrak{S}_n$ on the vector space generated by admissible polyspectra $P_{l_1 \dots l_n}$. If this is indeed the case, some important questions are left open: for instance, whether or not the representation is *faithful* (see [1]), and whether these ideas can lead to algorithms for the numerical simulation of representation matrices, along the lines of what we shall pursue in the next subsection.

Remark. Another interesting issue is whether the representation associated with the matrices $A\{(l_1, \dots, l_n); \pi\}$ is *irreducible*. Generally speaking, it is well-known that the collection of the (equivalence classes of the) irreducible representations of $\mathfrak{S}_n$ can be indexed by the family of *partitions* of the integer n. In particular, recall that: (i) A partition of n is a vector $\lambda = (\lambda_1, \dots, \lambda_k)$ of weakly decreasing positive integers such that $\sum_{j=1}^k \lambda_j = n$. (ii) The partitions of n can be represented by so-called *Young diagrams,* a finite collection of boxes, or cells, arranged in left-justified rows, with the row sizes weakly decreasing (each row has the same or shorter length than its predecessor). (iii) A *Young tableau* is obtained by filling the diagram with letters from some alphabet (for instance the integers $1, \dots, n$). (iv) A *Young symmetrizer* (also called a *Specht module*) of parameter λ is an element of the group algebra of the symmetric group, constructed in such a way that the image of the element corresponds to an irreducible representation of the symmetric group over the complex numbers. (v) Every irreducible representation of $\mathfrak{S}_n$ is equivalent to a Young symmetrizer of parameter λ, for some partition λ of n. Young symmetrizers can also be related to the problem of parentheses we mentioned above, and from a broader point of view to representation theory for the general linear group $GL(n)$. It is then natural to ask whether the formalism of Young diagrams and tableaux could help to shed further lights on the results of this paper, and in particular to explore further the connection with representations of the symmetric group. We plan to investigate these connections further in future work; the reader is referred e.g. to [1], [37, pp. 61–62] and [38] for more discussion on these issues.

7.2. Random data compression

In this subsection we shall show how we can exploit the previous results to develop a probabilistic algorithm to compress information on Clebsch–Gordan coefficients. Note first that

$$\#\left\{ C_{l_1 m_1 l_2 m_2}^{l_3 m_3} : l_1, l_2, l_3 \le L, \left| C_{l_1 m_1 l_2 m_2}^{l_3 m_3} \right| \ne 0 \right\} \approx O(L^6);$$

it is therefore clear how for most applications the storage of Clebsch–Gordan coefficients for future usage is simply unfeasible, whatever the supercomputing facilities (for instance, for CMB data analysis, $L \approx 3 \times 10^3$ is currently required, so that the number of Clebsch–Gordan coefficients to be saved would exceed 10^{20}). Let us consider again a chi-square field as defined before, i.e.

$$T_{\chi^2}(x) = H_2(T_G(x)) = \sum_{lm} a_{lm}(2) Y_{lm}(x);$$

Author's personal copy

we have proved earlier in (6.64) that

$$Ea_{l_1m_1}(2)a_{l_2m_2}(2)a_{l_3m_3}(2) = (-1)^{m_3}C^{l_3m_3}_{l_1m_1l_2m_2}h_{l_1l_2l_3}$$

where

$$h_{l_1l_2l_3} := 8\sum_{\ell_1\ell_2\ell_3} C^{l_10}_{\ell_10\ell_20}C^{l_20}_{\ell_10\ell_30}C^{l_30}_{\ell_20\ell_30}\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{\sqrt{(4\pi)^3}}\frac{1}{\sqrt{2l_3+1}}\begin{Bmatrix}\ell_1 & \ell_2 & \ell_3\\ l_1 & l_2 & l_3\end{Bmatrix}\left\{C_{\ell_1}C_{\ell_2}C_{\ell_3}\right\},$$

which can be calculated analytically and stored, with storage dimension

$$\#\left\{h_{l_1l_2l_3} : l_1, l_2, l_3 \le L, \left|C^{l_30}_{l_10l_20}\right| \neq 0\right\} \approx O(L^3).$$

Let us assume we simulate B times $T_{\chi^2}(x)$, which is trivially done by simply squaring a Gaussian field: the latter is obtained by sampling independent complex Gaussian variables with variance C_l. We store the triangular arrays $\left\{a^i_{lm}\right\}_{l=1,\dots,L;m=-l,\dots,l}$, $i = 1,\dots,B$; here the dimension is of order $B\times L^2$. We can then recover any value $C^{l_3m_3}_{l_1m_1l_2m_2}$ by means of the Monte Carlo estimate

$$\widehat{C}^{l_3m_3}_{l_1m_1l_2m_2} = h^{-1}_{l_1l_2l_3}\sum_{i=1}^{B}\frac{a^{(i)}_{l_1m_1}a^{(i)}_{l_2m_2}a^{(i)}_{l_3m_3}}{B},$$

which requires B steps and $B\times L^2 + L^3$ storage capacity, as opposed to L^6 storage capacity by the direct method. We leave for further research a more thorough investigation on the convergence properties of this algorithm; we stress, however, that the procedure we advocate is completely general, i.e. it does not depend on peculiar features of the group SO(3) we are currently considering. We believe, hence, that similar ideas can be implemented for the numerical estimation of Clebsch–Gordan coefficients for other compact groups of interest for theoretical physicists. We leave this and the previous issues in this Section as topics for further research.

References

[1] P. Diaconis, Group Representations in Probability and Statistics, in: IMS Lecture Notes – Monograph Series, vol. 11, Hayward, California, 1988.
[2] D. Bump, Lie Groups, Springer-Verlag, Berlin, Heidelberg, New York, 2004.
[3] P. Diaconis, E. Mayer-Wolf, O. Zeitouni, M. Zerner, The Poisson–Dirichlet law is the unique invariant distribution for uniform split-merge transformations, The Annals of Probability 32 (1B) (2004) 915–938.
[4] J. Fulman, Convergence rates of random walk on irreducible representations of finite groups, Journal of Theoretical Probability 21 (2008) 193–211.
[5] Y. Guivarc'h, M. Keane, B. Roynette, Marches Aléatoires sul les Groupes de Lie, in: Lecture Notes in Mathematics, vol. 624, Springer-Verlag, 1977.
[6] J.-R. Pycke, A decomposition for invariant tests of uniformity on the sphere, Proceedings of the American Mathematical Society (2007) (in press).
[7] O. Raimond, Flots browniens isotropes sur la sph ère, Annales de l'institut Henri Poincaré (B) Probabilité s et Statistiques 35 (3) (1999) 313–354.
[8] M.I. Yadrenko, Spectral Theory of Random Fields, Optimization Software, Inc., New York, 1983.
[9] G. Peccati, J.-R. Pycke, Decompositions of stochastic processes based on irreducible group representations, Theory of Probability and its Applications (2005) (in press).
[10] D. Marinucci, G. Peccati, High-frequency asymptotics for subordinated stationary fields on an Abelian compact group, Stochastic Processes and their Applications 118 (2008) 585–613.
[11] D. Marinucci, G. Peccati, Group representations and high-resolution central limit theorems for subordinated spherical random fields, 2008, Preprint. arXiv:0706.2851.
[12] D.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, Quantum Theory of Angular Momentum, World Scientific Press, 1988.
[13] N.Ja. Vilenkin, A.U. Klimyk, Representation of Lie Groups and Special Functions, Kluwer, Dordrecht, 1991.
[14] P. Cabella, D. Marinucci, Statistical Challenges in the Analysis of Cosmic Microwave Background, The Annals of Applied Statistics 3 (1) (2009).
[15] S. Dodelson, Modern Cosmology, Academic Press, 2003.
[16] W. Hu, The angular trispectrum of the CMB, Physical Review D 64 (2001) id. 083005.
[17] E. Komatsu, D.N. Spergel, Acoustic signatures in the primary microwave background bispectrum, Physical Review D 63 (2001) id. 063002.
[18] D. Marinucci, High-resolution asymptotics for the angular bispectrum of spherical random fields, The Annals of Statistics 34 (2006) 1–41.
[19] D. Marinucci, A central limit theorem and higher order results for the angular bispectrum, Probability Theory and Related Fields 141 (2007) 389–409.
[20] P. Cabella, F.K. Hansen, M. Liguori, D. Marinucci, S. Matarrese, L. Moscardini, N. Vittorio, The integrated bispectrum as a test of cosmic microwave background non-Gaussianity: Detection power and limits on f_{NL} with WMAP data, Monthly Notices of the Royal Astronomical Society 369 (2006) 819–824.
[21] A.P.S. Yadav, B.D. Wandelt, Detection of primordial non-Gaussianity (f_{NL}) in the WMAP 3-year data at above 99.5% confidence, 2007. arxiv:0712.1148.
[22] A.P.S. Yadav, E. Komatsu, B.D. Wandelt, M. Liguori, F.K. Hansen, S. Matarrese, Fast estimator of primordial Non-Gaussianity from temperature and polarization anisotropies in the cosmic microwave background II: Partial sky coverage and inhomogeneous noise, 2007, Preprint. arXiv:0711.4933.
[23] D. Babich, P. Creminelli, M. Zaldarriaga, The Shape of non-Gaussianities, Journal of Cosmology and Astroparticle Physics 8 (2004) 009.
[24] N. Bartolo, E. Komatsu, S. Matarrese, A. Riotto, Non-Gaussianity from inflation: Theory and observations, Physical Reports 402 (2004) 103–266.
[25] J. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, Journal of High Energy Physics 5 (2003) 0–13.
[26] M.K. Chung, K.M. Dalton, A.C. Evans, R.J. Davidson, Tensor-based cortical surface morphometry via weighted spherical harmonics representation, IEEE Transactions on Medical Imaging (2007).
[27] F.J. Simons, F.A. Dahlen, M.A. Wieczorek, Spatiospectral concentration on a sphere, SIAM Review 48 (3) (2006) 504–536.
[28] M.A. Wieczorek, F.J. Simons, Minimum-variance multitaper spectral estimation on the sphere, The Journal of Fourier Analysis and Applications 13 (6) (2007) 665–692.
[29] R.J. Adler, J.E. Taylor, Random Fields and Geometry, Springer, 2007.
[30] P. Baldi, D. Marinucci, Some characterizations of the spherical harmonics coefficients for isotropic random fields, Statistics and Probability Letters 77 (5) (2007) 490–496.
[31] P. Baldi, D. Marinucci, V.S. Varadarajan, On the characterization of isotropic random fields on homogeneous spaces of compact groups, Electronic Communications in Probability 12 (2007) 291–302.
[32] D. Geller, D. Marinucci, Spin wavelets on the sphere, 2008, Preprint. arXiv:0811.2935.
[33] R.L. Liboff, Introductory Quantum Mechanics, Addison-Wesley, 1999.

Author's personal copy

[34] R.P. Stanley, Enumerative Combinatorics, vol. I, Cambridge University Press, 1997.
[35] L.C. Biedenharn, J.D. Louck, The Racah–Wigner Algebra in Quantum Theory, in: Encyclopaedia of Mathematics and its Applications, vol. 10, Addison-Wesley, 1981.
[36] D. Surgailis, CLTs for polynomials of linear sequences: Diagram formula with illustrations, in: Theory and Applications of Long Range Dependence, Birkhäuser, 2003, pp. 111–128.
[37] B.E. Sagan, The Symmetric Group. Representations, Combinatorial Algorithms and Symmetric Functions, 2nd ed., Springer-Verlag, 2001.
[38] S. Sternberg, Group Theory and Physics, Cambridge University Press, 1999.