Bernoulli **16**(3), 2010, 798–824
DOI: 10.3150/09-BEJ230

Group representations and high-resolution central limit theorems for subordinated spherical random fields

DOMENICO MARINUCCI[1] and GIOVANNI PECCATI[2]

[1]*Department of Mathematics, University of Rome "Tor Vergata," Italy. E-mail: marinucc@mat.uniroma2.it*
[2]*Centre de Recherche Modal'X, Université Paris Ouest – Nanterre La Défense, 92000 Nanterre, France. E-mail: giovanni.peccati@gmail.com*

We study the weak convergence (in the high-frequency limit) of the frequency components associated with Gaussian-subordinated, spherical and isotropic random fields. In particular, we provide conditions for asymptotic Gaussianity and establish a new connection with random walks on the hypergroup $\widehat{SO(3)}$ (the dual of the group of rotations $SO(3)$), which mirrors analogous results previously established for fields defined on Abelian groups (see Marinucci and Peccati [*Stochastic Process. Appl.* **118** (2008) 585–613]). Our work is motivated by applications to cosmological data analysis.

Keywords: Clebsch–Gordan coefficients; cosmic microwave background; Gaussian subordination; group representations; high resolution asymptotics; spectral representation; spherical random fields

1. Introduction

This paper deals with weak limit theorems involving the high-frequency components (in the sense of the spherical harmonic decomposition) of random fields defined on the unit sphere $\mathbb{S}^2$. Our results are motivated by a number of mathematical issues arising in connection with the probabilistic and statistical analysis of cosmic microwave background radiation (see, e.g., [8]). We start by giving a description of our abstract mathematical framework, along with a sketch of the main results of the paper. The subsequent Section 1.2 focuses on the physical motivations and applications of our research. Here, and throughout the paper, all random elements are defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.1. General framework and outline of the main results

We shall consider real-valued random fields $\{\widetilde{T}(x) : x \in \mathbb{S}^2\}$ enjoying the following properties:

$$\mathbb{E}\widetilde{T}(x) = 0, \qquad \mathbb{E}\widetilde{T}^2(x) < +\infty \quad \text{and} \quad \widetilde{T}(gx) \stackrel{\text{law}}{=} \widetilde{T}(x), \tag{1}$$

for all $x \in \mathbb{S}^2$ and all $g \in SO(3)$, where $\stackrel{\text{law}}{=}$ denotes equality in law (in the sense of stochastic processes). A field verifying the last relation in (1) is usually called *isotropic* or *rotationally-invariant* (in law). It is a standard result that the following spectral representation holds in the mean square sense:

$$\widetilde{T}(x) = \sum_{l=0}^{\infty} \widetilde{T}_l(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x), \tag{2}$$

where $\{Y_{lm} : l \geq 0, m = -l, \dots, l\}$ is the collection of the spherical harmonics and the $\{a_{lm}\}$ are the associated (harmonic) Fourier coefficients. For $l \geq 0$, we also write $C_l \triangleq \mathbb{E}|a_{lm}|^2$ and we call the sequence $\{C_l : l \geq 0\}$ the *angular power spectrum* of the random field $\widetilde{T}$ (note that C_l does not depend on m; see, e.g., [2]). For every $l \geq 0$, the field $\widetilde{T}_l$ provides the projection of $\widetilde{T}$ onto the subspace of $L^2(\mathbb{S}^2, \mathrm{d}x)$ spanned by the class $\{Y_{lm} : m = -l, \dots, l\}$. The spherical harmonics form an orthonormal basis of $L^2(\mathbb{S}^2, \mathrm{d}x)$ which can be derived from the restriction to the sphere of harmonic polynomials. In particular, in spherical coordinates $x = (\theta, \varphi)$, they can be written explicitly as $Y_{00} \equiv 1/\sqrt{4\pi}$ and

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) \mathrm{e}^{\mathrm{i}m\varphi}, \qquad m \geq 0, \tag{3}$$

$$Y_{lm}(\theta, \varphi) = (-1)^m \overline{Y_{l,-m}}(\theta, \varphi), \qquad m < 0, 0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi, \tag{4}$$

where, for $l \geq 1$ and $m = 0, 1, 2, \dots, l$, $P_{lm}(\cdot)$ denotes the Legendre polynomial of index l, m, that is,

$$P_{lm}(x) = (-1)^m (1 - x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x), \qquad P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} (x^2 - 1)^l. \tag{5}$$

For a discussion of these and other properties of the spherical harmonics, see, for example, [31], [15], Chapter 9 or [34], Chapter 5. For $l \geq 0$, the real-valued field $\widetilde{T}_l$ is called the lth *frequency component* of $\widetilde{T}$. The expansion (2) can be achieved by many different routes, for instance by a Karhunen–Loéve argument or by means of the stochastic Peter–Weyl theorem (see, e.g., [1,3, 14] and [27]). The random harmonic coefficients $\{a_{lm}\}$ appearing in (2) form a triangular array of zero-mean random variables, which are complex-valued for $m \neq 0$ and such that $\mathbb{E} a_{lm} \overline{a_{l'm'}} = \delta_l^{l'} \delta_m^{m'} C_l$ (the bar denotes complex conjugation and δ is Kronecker's symbol; also, note that $a_{lm} = (-1)^m \overline{a_{l-m}}$). For a Gaussian random field $\widetilde{T}$ verifying (1), it is trivial that the set $\{a_{lm}\}$ is itself a complex-Gaussian array, with independent elements for $m \geq 0$. It is a simple but interesting fact that the converse also holds, that is, that under an isotropy assumption on $\widetilde{T}$, the independence of the a_{lm}s for $m \geq 0$ implies Gaussianity; see [2]. Apart from this result, the behaviour of the array $\{a_{lm}\}$ and of the projections $\{\widetilde{T}_l\}$ for non-Gaussian isotropic fields has thus far remained almost completely unexplored and open for research, although these objects are highly relevant for cosmological applications (see the next subsection). It should be stressed that the coefficients $\{a_{lm}\}$ depend on the choice of coordinates and are not intrinsic to the field, although their law is. In this sense, it is sometimes physically more sound to focus on the behaviour of the sequence of projections $\{\widetilde{T}_l\}$, which are indeed invariant with respect to the choice of coordinates.

In what follows, we focus on non-Gaussian fields $\widetilde{T}$ that are Gaussian-subordinated and we address the previous topic by studying the asymptotic behaviour of $\{a_{lm}\}$ and $\{\widetilde{T}_l\}$ as $l \to +\infty$. Recall that $\widetilde{T}$ is called Gaussian-subordinated whenever $\widetilde{T}(x) = F(T(x))$, where F is a suitable real-valued function and T is an isotropic spherical (real) Gaussian field. In particular, our purpose is to establish sufficient (and, sometimes, also necessary) conditions on F and on the law of T to ensure that the following two phenomena occur: (I) as $l \to +\infty$, for a fixed m and an appropriate sequence $\tau_1(l)$ ($l \geq |m|$), the sequence

$$\tau_1(l) \times a_{lm} = \tau_1(l) \int_{\mathbb{S}^2} F(T(z))\overline{Y_{lm}(z)}\,\mathrm{d}z, \qquad l \geq |m|$$

converges in law to a Gaussian random variable (real-valued for $m = 0$ and complex-valued for $m \neq 0$); (II) for a suitable real-valued sequence $\tau_2(l)$ ($l \geq 0$) and for l sufficiently large, the finite-dimensional distributions of the field

$$\tau_2(l) \times \widetilde{T}_l(\cdot) = \tau_2(l) \sum_{m=-l,\dots,l} a_{lm} Y_{lm}(\cdot)$$

are close (e.g., in the sense of the Prokhorov distance; see [26]) to those of a real spherical Gaussian field. Note that both results (I) and (II) can be interpreted as central limit theorems in the high-frequency (or high-resolution) sense since they involve Gaussian approximations and are established by letting the frequency index l diverge to infinity.

Our findings generalize previous results, obtained in [19], for fields defined on Abelian compact groups. One of our main tools is a result concerning the Gaussian approximation of multiple Wiener–Itô integrals established in [26] (see also [22,23,25,28] and [29]). These central limit theorems can be seen as a simplification of the combinatorial method of diagrams and cumulants (see, e.g., [32]). These techniques, combined with the use of group representation theory, lead to the main contribution of this paper: the derivation of sufficient (or necessary and sufficient) conditions for (I) and (II), expressed in terms of convolutions of Clebsch–Gordan coefficients (see, e.g., [34], Chapter 4), which are the elements of unitary matrices connecting specific reducible representations of the group of rotations in R^3 (labelled $SO(3)$, as usual).

1.2. Cosmological motivations

The cosmic microwave background radiation (hereafter CMB) can be viewed as a relic radiation of the Big Bang, providing maps of the primordial Universe before the formation of any of the current structures (approximately 3×10^5 years after the Big Bang); as such, it is acknowledged as a gold mine of information for fundamental physics. Many satellite experiments involving hundred of physicists throughout the world are devoted to the construction of spherical maps of the CMB radiation and for pioneering work in this area G. Smoot and J. Mather were awarded the Nobel Prize for Physics in 2006; see, for instance, http://map.gsfc.nasa.gov/ for more details.

The crucial point is that most cosmological models imply that the CMB radiation is the realization of a random field $\{\widetilde{T}(x) : x \in \mathbb{S}^2\}$, verifying the three conditions in (1); each $x \in \mathbb{S}^2$ corresponds to a direction in which the CMB radiation is measured. The isotropic property can be seen as a consequence of Einstein's cosmological principle, roughly stating that, on sufficiently large distance scales, the Universe looks identical everywhere in space (homogeneity) and appears the same in every direction (isotropy). A central issue in modern cosmology therefore relates to the distribution of the CMB random field $\widetilde{T}$, which is predicted to be (close to) Gaussian by some models for the dynamics at primordial epochs (e.g., by the so-called "inflationary scenario") and non-Gaussian by other models, where fluctuations are generated by topological defects arising in phase transitions of a thermodynamic nature; see, for instance, [8]. Many testing procedures have been proposed to tackle this issue; in some form, they all rely asymptotically on the behaviour of the field at the highest frequencies (see, e.g., [4,6,16] and the references therein). This is a sort of inescapable, foundational issue in cosmology. By definition, the latter is a science based on a single realization, that is, our Universe or the trace of its primordial structure in the form of the CMB radiation, which is observed at higher and higher resolutions. As such, an asymptotic theory for statistical tests is possible only in the sense of observations at higher and higher frequencies (smaller and smaller scales) becoming available as experiments become more sophisticated. In particular, any satellite experiment measuring the CMB radiation can reconstruct the spherical harmonic development appearing in (2) only up to a finite frequency $l_{\max}$, the quantity $\pi/l_{\max}$ representing approximately the angular resolution of the experiment (the pioneering satellite COBE (1993) could reach a frequency $l_{\max} \simeq 20$, WMAP (2003, 2006) improved this limit to $l_{\max} \simeq 600/800$ and Planck (launched in May 2009) is expected to reach $l_{\max} \simeq 2500/3000$). In order for such procedures to yield consistent outcomes, one should therefore determine the limiting behaviour of $\{\widetilde{T}_l\}$, for $l \gg 0$, under different distributional assumptions on $\widetilde{T}$. Some Monte Carlo evidence (see, e.g., [18] and the references therein) has suggested that this behaviour may be close to Gaussian, even in circumstances where the underlying field $\widetilde{T}$ clearly is not. The investigation of this issue is necessary for rigorous inference on CMB data and, in particular, for non-Gaussianity tests. The relevance of the asymptotic behaviour of the $\{\widetilde{T}_l\}$, however, goes far beyond the issue of such tests and indeed relates to the whole statistical analysis of CMB – which is largely dominated by likelihood approaches (see [9]).

We stress that the results we provide cover models that are quite relevant for cosmological applications, for instance, the so-called *Sachs–Wolfe model*, which represents the standard starting model for the inflationary scenario (see, e.g., [4,8]). In its simplest version, this model implies that the CMB is a straightforward quadratic transformation of an underlying Gaussian field, that is,

$$\widetilde{T}(x) = T(x) + f_{\mathrm{NL}}\{T(x)^2 - \mathbb{E}T(x)^2\}, \qquad x \in \mathbb{S}^2, \tag{6}$$

where f_{NL} is a nonlinearity parameter depending on constants from particle physics and T is Gaussian and isotropic. As a special case, our results allow for a complete characterization of the high-frequency behaviour of models such as (6) (see the last section of this work) and, in this sense, they are immediately applicable in the cosmological literature. In particular, in this paper, we show that the high frequency behaviour of $\widetilde{T}(x)$ is non-Gaussian for a polynomial decay of the angular power spectrum (as expected in the physics literature for this class of models), while this would not be the case for an exponential decay; we refer to Section 6 for more details.

1.3. Outline

In Section 2, we provide some background material on isotropic random fields on the sphere. Section 3 is devoted to a discussion of representation theory for the group of rotations $SO(3)$

and the so-called Clebsch–Gordan coefficients, which will play a crucial role in the analysis to follow. In Section 4, we state and prove a general central limit theorem result for the spherical harmonics' coefficients and the high-frequency components of a field arising from polynomial transformations of arbitrary order of a subordinating Gaussian process. In Section 5, we provide a more detailed analysis of necessary and sufficient condition for the central limit theorem to hold in the case of quadratic and cubic transformations; we also highlight the connections between our conditions and the theory of random walks on hypergroups. In Section 6, we turn our attention to more explicit conditions on the angular power spectrum and discuss an exponential/algebraic duality which, to some extent, parallels some earlier findings in the Abelian case.

2. Preliminaries on Gaussian and Gaussian-subordinated isotropic fields

As in the Introduction, we denote by $\mathbb{S}^2$ the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. For every rotation $g \in SO(3)$ and every $x \in \mathbb{S}^2$, the symbol gx indicates the canonical action of g on x (see [34], Chapter 1, as well as Section 3 below, for further details). We will systematically write $\mathrm{d}x$ for the Lebesgue measure on $\mathbb{S}^2$ and we denote by $L^2(\mathbb{S}^2, \mathrm{d}x)$ the class of complex-valued functions on $\mathbb{S}^2$ which are square-integrable with respect to $\mathrm{d}x$. We denote by $\{Y_{lm} : l \geq 0, m = -l, \ldots, l\}$ the basis of $L^2(\mathbb{S}^2, \mathrm{d}x)$ given by spherical harmonics, as defined via (3) and (4). From now on, we shall denote by $T = \{T(x) : x \in \mathbb{S}^2\}$ a centered, real-valued and Gaussian random field parametrized by $\mathbb{S}^2$. We also suppose that T is isotropic, that is, for every $g \in SO(3)$, one has that $T(x) \overset{\text{law}}{=} T(gx)$, where the equality holds in the sense of finite-dimensional distributions. To simplify the notation, we also assume that $\mathbb{E}T(x)^2 = 1$. Following, for example, [2] (but see also [3, 27] and [30]), one deduces from isotropy that T admits the spectral decomposition

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm;1} Y_{lm}(x) = \sum_{l=0}^{\infty} T_l(x), \qquad x \in \mathbb{S}^2, \tag{7}$$

where $a_{lm;1} \triangleq \int_{\mathbb{S}^2} T(x)\overline{Y_{lm}(x)}\,\mathrm{d}x$ (the role of the subscript "lm; 1" will be clarified in the following discussion), $T_l(x) \triangleq \sum_{m=-l}^{l} a_{lm;1} Y_{lm}(x)$ and the convergence takes place in $L^2(\mathbb{P})$ for every fixed x, as well as in $L^2(\mathbb{P} \otimes \mathrm{d}x)$. The next result gives a simple and very useful characterization of the joint law of the complex-valued array $\{a_{lm;1} : l \geq 0, m = -l, \ldots, l\}$. For every $z \in \mathbb{C}$, the symbols $\Re(z)$ and $\Im(z)$ indicate, respectively, the real and the imaginary part of z.

Proposition 1. *Let T be the centered, isotropic and Gaussian random field appearing in (7). Then:* (i) *for every $l \geq 0$, the random variable $a_{l0;1}$ is real-valued, centered and Gaussian*; (ii) *for every $l \geq 1$ and every $m = 1, \ldots, l$, the random variable $a_{lm;1}$ is Gaussian complex-valued and such that $a_{lm;1} = (-1)^m \overline{a_{l-m;1}}$, and, moreover, $\mathbb{E}(\Re(a_{lm;1})^2) = \mathbb{E}(\Im(a_{lm;1})^2) = \mathbb{E}(a_{l0;1}^2)/2 = C_l/2$ for some constant $C_l \in [0, +\infty)$ not depending on m and*

$$\mathbb{E}(\Re(a_{lm;1})\Im(a_{lm;1})) = 0; \tag{8}$$

(iii) *for every $l \geq 1$ and every $m = -l, \dots, l$, the random coefficient $a_{lm;1}$ is independent of $a_{l'm';1}$ for every $l' \geq 0$ such that $l' \neq l$ and every $m' = -l', \dots, l'$. By noting that $C_0 \triangleq \mathbb{E}(a_{00;1}^2)$, one also has the relation*

$$1 = \mathbb{E}[T(x)^2] = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l. \tag{9}$$

The reader is referred to [2] for a proof of Proposition 1, as well as for several converse statements. Here, we shall only stress that formula (9) is a consequence of the well-known relation (see, e.g., [31,34])

$$\sum_{m=-l}^{l} Y_{lm}(x)\overline{Y_{lm}(y)} = \frac{2l+1}{4\pi} P_l(\cos\langle x, y\rangle), \qquad x, y \in \mathbb{S}^2, \tag{10}$$

where $\langle x, y\rangle$ is the angle between x and y. Observe that property (8) implies that $\Re(a_{lm;1})$ and $\Im(a_{lm;1})$ are independent centered Gaussian random variables. Moreover, the combination of (8) and point (iii) in the statement of Proposition 1 yields that $\mathbb{E}(a_{lm;1}\overline{a_{l'm';1}}) = 0$, $\forall (l, m) \neq (l', m')$. Finally, it is also evident that points (i)–(iii) in the previous statement imply that the law of an isotropic Gaussian field such as T is completely characterized by its angular power spectrum $\{C_l : l \geq 0\}$. To avoid trivialities, we will always work under the following assumption.

Assumption. *The angular power spectrum $\{C_l : l \geq 0\}$ is such that $C_l > 0$ for every l.*

Note that the results of this paper could be extended without difficulty (but at the cost of heavier notation) to the case of a power spectrum such that $C_l \neq 0$ for infinitely many l's. In the subsequent sections, we shall obtain high-frequency central limit theorems for centered isotropic spherical fields that are subordinated to the Gaussian field T defined above.

Definition A (Subordinated fields). *Let $L_0^2(\mathbb{R}, \mathrm{e}^{-z^2/2}\,\mathrm{d}z)$ indicate the class of real-valued functions $F(z)$ on $\mathbb{R}$ which are square-integrable with respect to the measure $\mathrm{e}^{-z^2/2}\,\mathrm{d}z$ and such that $\int F(z)\mathrm{e}^{-z^2/2}\,\mathrm{d}z = 0$. A (centered) random field $\widetilde{T} = \{\widetilde{T}(x) : x \in \mathbb{S}^2\}$ is said to be subordinated to the Gaussian field T appearing in (2) if there exists $F \in L_0^2(\mathbb{R}, \mathrm{e}^{-z^2/2}\,\mathrm{d}z)$ such that $\widetilde{T}(x) = F[T](x)$, $\forall x \in \mathbb{S}^2$, where the symbol $F[T](x)$ stands for $F(T(x))$. Whenever $\widetilde{T}$ is subordinated, we will use the notation $F[T](x)$ instead of $\widetilde{T}(x)$, in order to emphasize the role of the function F. Of course, if $F(z) = z$, then $F[T](x) = \widetilde{T}(x) = T(x)$.*

It is immediate to check that, since T is isotropic, a subordinated field $F[T](\cdot)$, as in Definition A, is necessarily isotropic. As a consequence, again following [2] or [27], one deduces that $F[T]$ admits the spectral representation

$$F[T](x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(F) Y_{lm}(x) = \sum_{l=0}^{\infty} F[T]_l(x), \qquad x \in \mathbb{S}^2, \tag{11}$$

with convergence in $L^2(\mathbb{P})$ (for fixed x) and in $L^2(\Omega \times \mathbb{S}^2, \mathbb{P} \otimes \mathrm{d}x)$. Here,

$$a_{lm}(F) \triangleq \int_{\mathbb{S}^2} F[T](y)\overline{Y_{lm}(y)}\,\mathrm{d}y \quad \text{and} \tag{12}$$

$$F[T]_l(x) \triangleq \sum_{m=-l}^{l} a_{lm}(F)Y_{lm}(x). \tag{13}$$

The complex-valued array $\{a_{lm}(F) : l \geq 0, m = -l, \ldots, l\}$ always enjoys the following properties (a)–(c): (a) for every $l \geq 0$, the random variable $a_{l0}(F)$ is real-valued and centered; (b) for every $l \geq 1$ and every $m = 1, \ldots, l$, the random variable $a_{lm}(F)$ is complex-valued, centered and such that

$$a_{lm}(F) = (-1)^m \overline{a_{l-m}(F)}; \qquad \mathbb{E}(\Re(a_{lm}(F))\Im(a_{lm}(F))) = 0;$$
$$\mathbb{E}(\Re(a_{lm}(F))^2) = \mathbb{E}(\Im(a_{lm}(F))^2) = \mathbb{E}(a_{l0}(F)^2)/2 = C_l(F)/2,$$

where the finite constant $C_l(F) \geq 0$ depends solely on F and l; (c) $\mathbb{E}(a_{lm}(F) \times \overline{a_{l'm'}(F)}) = 0$, $\forall (l, m) \neq (l', m')$. Note that, in general, it is no longer true that $\Re(a_{lm}(F))$ and $\Im(a_{lm}(F))$ are independent random variables. Moreover, we state the following consequence of [2, Theorem 7]: *for every* $l \geq 1$, *the coefficients* $(a_{l0}(F), \ldots, a_{ll}(F))$ *are stochastically independent if and only if they are Gaussian*. Also, $\mathbb{E}(F[T](x)^2) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l(F)$.

In the subsequent sections, a crucial role will be played by the class of *Hermite polynomials*. Recall (see, e.g., [13], page 20) that the sequence $\{H_q : q \geq 0\}$ of Hermite polynomials is defined by the differential relation

$$H_q(z) = (-1)^q \mathrm{e}^{z^2/2} \frac{\mathrm{d}^q}{\mathrm{d}z^q} \mathrm{e}^{-z^2/2}, \qquad z \in \mathbb{R}, q \geq 0; \tag{14}$$

it is well known that the sequence $\{(q!)^{-1/2} H_q : q \geq 0\}$ defines an orthonormal basis of the space $L^2(\mathbb{R}, (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}\,\mathrm{d}z)$. When a subordinated field has the form (for $q \geq 2$) $H_q[T](x)$, $x \in \mathbb{S}^2$ (i.e., when $F = H_q$ in Definition A), we will use the following shorthand notation:

$$T^{(q)}(x) \triangleq H_q[T](x), \qquad x \in \mathbb{S}^2, \tag{15}$$

$$a_{lm;q} \triangleq a_{lm}(H_q), \tag{16}$$

$$T_l^{(q)}(x) \triangleq H_q[T]_l(x), \qquad l \geq 1, x \in \mathbb{S}^2, \tag{17}$$

$$\overline{T}_l^{(q)}(x) \triangleq \mathrm{Var}\big(T_l^{(q)}(x)\big)^{-1/2} T_l^{(q)}(x), \qquad l \geq 1, x \in \mathbb{S}^2, \tag{18}$$

$$\widetilde{C}_l^{(q)} \triangleq C_l(H_q) = \mathbb{E}|a_{lm;q}|^2, \qquad l \geq 1, m = -l, \ldots, l. \tag{19}$$

To justify our notation (15)–(19), we recall that for every fixed x, the random variable $H_q[T](x) = H_q(T(x))$ is just the qth Wick power of $T(x)$ (see, e.g., [13]). We conclude the section with an easy lemma which will be used in Section 4.

Lemma 2. *Let $F[T](x)$, $x \in \mathbb{S}^2$, be an (isotropic) subordinated field, as in Definition A. Then, for every $l \geq 1$, one has the following*:

1. *the random field $x \mapsto F[T]_l(x)$ defined in (13) is real-valued and isotropic*;
2. *for every fixed $x \in \mathbb{S}^2$, $F[T]_l(x) \overset{\text{law}}{=} \sqrt{\frac{2l+1}{4\pi}} a_{l0}(F)$, where the coefficient $a_{l0}(F)$ is defined according to (12) and, consequently, $\mathbb{E}(F[T]_l(x)^2) = \frac{2l+1}{4\pi} C_l(F)$*;
3. *the normalized random field*

$$\overline{F[T]}_l(x) = \left[\frac{(2l+1)C_l(F)}{4\pi}\right]^{-1/2} F[T]_l(x) \tag{20}$$

has a covariance structure given as follows: for every $x, y \in \mathbb{S}^2$,

$$\mathbb{E}\big(\overline{F[T]}_l(x) \times \overline{F[T]}_l(y)\big) = P_l(\cos\langle x, y\rangle), \tag{21}$$

where $P_l(\cdot)$ is the lth Legendre polynomial defined in (5) and, as before, $\langle x, y\rangle$ is the angle between x and y.

Proof. Point 1 is straightforward. To prove Point 2, we define (in polar coordinates) $x_0 = (0, 0)$ and use the isotropy property stated in Point 1 to write

$$F[T]_l(x) \overset{\text{law}}{=} F[T]_l(x_0) = \sum_{m=-l}^{l} a_{lm}(F) Y_{lm}(x_0) = \sqrt{\frac{2l+1}{4\pi}} a_{l0}(F)$$

since (3) implies that $Y_{lm}(x_0) = \sqrt{(2l+1)/4\pi}\,\delta_m^0$. Finally, to prove relation (21), we use (10) to deduce that, for every $x, y \in \mathbb{S}^2$,

$$\mathbb{E}(F[T]_l(x) F[T]_l(y)) = C_l(F) \frac{2l+1}{4\pi} P_l(\cos\langle x, y\rangle),$$

thus giving the desired conclusion (recall that $P_l(1) = 1$). □

For instance, a first consequence of Lemma 2 is that, for every $q \geq 2$,

$$\mathbb{E}\big(T_l^{(q)}(x)^2\big) = (2l+1)\widetilde{C}_l^{(q)}/4\pi, \tag{22}$$

where we have used the notation introduced in (15)–(19) so that $\overline{T}_l^{(q)}(x) = [(2l+1)\widetilde{C}_l^{(q)}/4\pi]^{-1/2} T_l^{(q)}(x)$.

The main aim of the subsequent sections is to provide an accurate solution to the following problems:

(P-I) For a fixed $q \geq 2$, find conditions on the power spectrum $\{C_l : l \geq 0\}$ of T to ensure that the subordinated process $T^{(q)} = \{T^{(q)}(x) : x \in \mathbb{S}^2\}$ defined in (15) is such that, for every $x \in \mathbb{S}^2$,

$$\big\{(2l+1)\widetilde{C}_l^{(q)}/4\pi\big\}^{-1/2} \times T_l^{(q)}(x) \xrightarrow[l\to+\infty]{\text{law}} N, \tag{23}$$

where N is a centered standard Gaussian random variable.

(P-II) Under the conditions found at (P-I), study the asymptotic behaviour, as $l \to +\infty$, of the vector

$$\left\{(2l+1)\widetilde{C}_l^{(q)}/4\pi\right\}^{-1/2} \times \left(T_l^{(q)}(x_1), \ldots, T_l^{(q)}(x_k)\right) \tag{24}$$

for every $x_1, \ldots, x_k \in \mathbb{S}^2$.

(P-III) Combine (P-I) and (P-II) to study the asymptotic behaviour (in particular, the asymptotic Gaussianity), as $l \to +\infty$, of vectors of the type

$$\left\{(2l+1)\widetilde{C}_l^{(q)}/4\pi\right\}^{-1/2} \times (F[T]_l(x_1), \ldots, F[T]_l(x_k)) \tag{25}$$

for every $x_1, \ldots, x_k \in \mathbb{S}^2$ and every $F \in L_0^2(\mathbb{R}, \mathrm{e}^{-z^2/2}\,\mathrm{d}z)$.

Note that problems (P-I)–(P-III) are stated in increasing order of generality. We also observe the following fact: since (21) holds and since the limit of $P_l(\langle x, y\rangle)$ $(l \to +\infty)$ does not exist in general, it will not be possible to prove that the vectors in (24) and (25) converge in law to some Gaussian limit. However, by using the results developed in [26], we will be able to establish conditions under which the laws of such vectors are "asymptotically close" to a sequence of k-dimensional Gaussian distributions. As already mentioned, to study (P-I)–(P-III), we shall use estimates involving the so-called *Clebsch–Gordan coefficients*, which are elements of unitary matrices connecting some reducible representations of $SO(3)$. The definition and analysis of some crucial properties of Clebsch–Gordan coefficients are the subjects of the next section.

3. A primer on Clebsch–Gordan coefficients

In this section, we need to review some basic representation theory results for $SO(3)$, defined as the group of rotations in $\mathbb{R}^3$. We refer the reader to standard textbooks (e.g., [10,34] and [35]) for further details, as well as for any unexplained notion or definition. The recent paper [20], by the authors, contains several explicit examples that further illustrate the forthcoming definitions. It should be stressed that most of our arguments below could be extended to general compact groups with known representations; however, throughout the following, we shall stick to the group of rotations $SO(3)$, mainly for the sake of notational simplicity.

We start by reviewing some background material on the special group of rotations $SO(3)$, that is, the space of 3×3 real matrices A such that $A'A = I_3$ (the three-dimensional identity matrix) and $\det(A) = 1$. We first recall that each element $g \in SO(3)$ can be parametrized by the set (α, β, γ) of so-called Euler angles, where $0 \le \alpha < 2\pi, 0 \le \beta \le \pi$ and $0 \le \gamma < 2\pi$. More explicitly, each rotation in $\mathbb{R}^3$ can be realized sequentially as

$$A = A(g) = R(\alpha, \beta, \gamma) = R_z(\alpha) R_x(\beta) R_z(\gamma), \tag{26}$$

where $R_z(\alpha)$, $R_x(\beta)$, $R_z(\gamma) \in SO(3)$ can be expressed by means of the following general definitions, valid for every angle α:

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The parametrization (26) is unique, except for $\beta = 0$ or $\beta = \pi$, in which case only the sum $\alpha + \gamma$ is determined. In words, the rotation is realized by first rotating by γ around the axis z, then rotating around the initial x axis by β, then rotating by α around the initial z axis. It is clear that the last two rotations identify one point on the sphere, so the whole operation could also be interpreted as rotating by γ the tangent plane at the North Pole and then moving the latter to a location in $\mathbb{S}^2$.

Now, recall that an n-dimensional group representation is a homomorphism $M : SO(3) \to \mathcal{M}$ which maps the group into a space of matrices $\mathcal{M}$ and preserves the group structure, that is, $g_1 g_2 = g_3$ implies that $M(g_1)M(g_2) = M(g_1 g_2)$ for all $g_1, g_2 \in SO(3)$; see [7,10,35] for a much more detailed discussion. In the coordinates provided by Euler angles, a complete set of irreducible matrix representations for $SO(3)$ is provided by Wigner's so-called D matrices $D^l(\gamma, \beta, \alpha) = \{D^l_{nm}(\gamma, \beta, \alpha)\}_{m,n=-l,\dots,l}$, of dimensions $(2l+1) \times (2l+1)$ for $l = 0, 1, 2, \dots$; again, see [10,35]. An analytic expression for the elements of Wigner's D matrices is provided by

$$D^l_{nm}(\gamma, \beta, \varphi) = \mathrm{e}^{-\mathrm{i}n\gamma} d^l_{mn}(\beta) \mathrm{e}^{\mathrm{i}m\alpha}, \qquad m, n = -(2l+1), \dots, 2l+1,$$

where the indices n, m indicate, respectively, columns and rows,

$$\begin{aligned} d^l_{nm}(\beta) &= (-1)^{l-n}[(l+m)!(l-m)!(l+n)!(l-n)!]^{1/2} \\ &\quad \times \sum_k (-1)^k \frac{(\cos\beta/2)^{m+n+2k}(\sin\beta/2)^{2l-m-n-2k}}{k!(l-m-k)!(l-n-k)!(m+n+k)!} \end{aligned}$$

and the sum runs over all k such that the factorials are non-negative; see [34], Chapter 4, for a huge collection of alternative expressions. Here, we simply recall that the elements of $D^l(\gamma, \beta, \alpha)$ are related to the spherical harmonics via the relationship

$$D^l_{0m}(\alpha, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y_{l-m}(\beta, \alpha) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\beta, \alpha), \tag{27}$$

from which it is not difficult to show how the usual spectral representation for random fields on the spheres (e.g., (2) and (7)) is really just the stochastic Peter–Weyl theorem on $\mathbb{S}^2 = SO(3)/SO(2)$. Note that for $n = 0$, $D^l_{nm}(\cdot)$ does not depend on γ, whence the latter does not appear on the right-hand side of (27). The reader is referred to, for example, [10,35] and [33] for further discussions on the Peter–Weyl theorem and to [2,3,20] and [27] for several related probabilistic results.

It follows from standard representation theory that we can exploit the family $\{D^l\}_{l=0,1,2,\dots}$ to build alternative (reducible) representations, either by taking the tensor product family $\{D^{l_1} \otimes D^{l_2}\}_{l_1,l_2}$ or by considering direct sums $\{\bigoplus_{l=|l_2-l_1|}^{l_2+l_1} D^l\}_{l_1,l_2}$; these representations have dimensions $(2l_1+1)(2l_2+1) \times (2l_1+1)(2l_2+1)$ and are unitarily equivalent, that is, there exists a unitary matrix $C_{l_1 l_2}$ such that

$$\{D^{l_1} \otimes D^{l_2}\} = C_{l_1 l_2} \left\{ \bigoplus_{l=|l_2-l_1|}^{l_2+l_1} D^l \right\} C^*_{l_1 l_2}. \tag{28}$$

Here, $C_{l_1 l_2}$ is a $\{(2l_1+1)(2l_2+1)\times(2l_1+1)(2l_2+1)\}$ block matrix with blocks $C^{l}_{l_1(m_1)l_2}$ of dimensions $(2l_2+1)\times(2l+1)$, $m_1=-l_1,\dots,l_1$. The elements of such a block are indexed by m_2 (over rows) and m (over columns). More precisely,

$$C_{l_1 l_2} = \left[C^{l\cdot}_{l_1(m_1)l_2\cdot}\right]_{m_1=-l_1,\dots,l_1;\, l=|l_2-l_1|,\dots,l_2+l_1},$$

$$C^{l\cdot}_{l_1(m_1)l_2\cdot} = \{C^{lm}_{l_1 m_1 l_2 m_2}\}_{m_2=-l_2,\dots,l_2;\, m=-l,\dots,l}.$$

The Clebsch–Gordan coefficients for $SO(3)$ are then defined as $\{C^{lm}_{l_1 m_1 l_2 m_2}\}$, that is, as the elements of the unitary matrices $C_{l_1 l_2}$ (note that such matrices are real-valued, as are the $C^{lm}_{l_1 m_1 l_2 m_2}$). Explicit expressions for the Clebsch–Gordan coefficients of $SO(3)$ are known, but they are, in general, hardly manageable (see, e.g., [34], Section 8.2). The Clebsch–Gordan coefficients also enjoy a nice set of symmetry and orthogonality properties which will play a crucial role in our results to follow (see [16] and [17] for an account of such properties). Note, in particular, that the Clebsch–Gordan coefficients are different from zero only if $m_1+m_2=m$ and $|l_2-l_1|\le l\le l_1+l_2$ (the *triangle conditions*). Moreover, from unitary equivalence, we deduce that

$$\sum_{m_1,m_2} C^{lm}_{l_1 m_1 l_2 m_2} C^{l'm'}_{l_1 m_1 l_2 m_2} = \delta_l^{l'}\delta_m^{m'} \quad \text{and} \quad \sum_{l,m} C^{lm}_{l_1 m_1 l_2 m_2} C^{lm}_{l_1 m_1' l_2 m_2'} = \delta_{m_1}^{m_1'}\delta_{m_2}^{m_2'}. \tag{29}$$

We also recall that the Clebsch–Gordan coefficients are equivalent, up to a normalization factor, to *Wigner's 3j coefficients*, which are used in related works, such as [17].

The Clebsch–Gordan coefficients play a crucial role in the evaluation of integrals involving products of spherical harmonics. In particular, the so-called *Gaunt integral* gives

$$\int_{\mathbb{S}^2} Y_{l_1 m_1}(x) Y_{l_2 m_2}(x) \overline{Y_{lm}(x)}\,\mathrm{d}x = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} C^{lm}_{l_1 m_1 l_2 m_2} C^{l0}_{l_1 0 l_2 0}. \tag{30}$$

Relation (30) can be established using (27), (28) and resorting to standard orthonormality properties of the elements of group representations; see [34], Expression 5.9.1.4. More generally, define

$$\mathcal{G}\{l_1,m_1;\dots;l_r,m_r\} \triangleq \int_{\mathbb{S}^2} Y_{l_1 m_1}(x)\cdots Y_{l_r m_r}(x)\,\mathrm{d}x \tag{31}$$

and call the quantity $\mathcal{G}\{l_1,m_1;\dots;l_r,m_r\}$ a *generalized Gaunt integral*. Then, iterating the previous argument, for $q\ge 3$, it can be shown that (by using, e.g., [34], Expression 5.6.2.12)

$$\begin{aligned}
&\mathcal{G}\{l_1,m_1;\dots;l_q,m_q;l,-m\} \\
&\quad= \sum_{L_1,\dots,L_{q-2}} \sum_{M_1,\dots,M_{q-2}} \left\{\prod_{i=1}^{q-3}\left(\sqrt{\frac{2l_{i+2}+1}{4\pi}} C^{L_{i+1}0}_{L_i 0 l_{i+2} 0} C^{L_{i+1}M_{i+1}}_{L_i M_i l_{i+2} m_{i+2}}\right)\right\} \\
&\qquad\times \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} C^{L_1 0}_{l_1 0 l_2 0} C^{L_1 M_1}_{l_1 m_1 l_2 m_2} \sqrt{\frac{2l_q+1}{4\pi}} C^{l0}_{L_{q-2} 0 l_q 0} C^{lm}_{L_{q-2} M_{q-2} l_q m_q},
\end{aligned} \tag{32}$$

where, for $q = 3$, we have used the convention $\Pi_{i=1}^{0} \equiv 0$. Note that expressions such as (32) imply that the generalized Gaunt integrals of the type (31) are indeed real-valued. To simplify the expression (32), let us introduce the coefficients

$$C_{l_1,m_1;\dots;l_p m_p}^{\lambda_1,\lambda_2,\dots,\lambda_{p-1};\mu} \triangleq \sum_{\mu_1=-\lambda_1}^{\lambda_1} \cdots \sum_{\mu_{p-2}=-\lambda_{p-2}}^{\lambda_{p-2}} C_{l_1,m_1,l_2,m_2}^{\lambda_1,\mu_1} C_{\lambda_1,\mu_1;l_3,m_3}^{\lambda_2,\mu_2} \cdots C_{\lambda_{p-2},\mu_{p-2};l_p,m_p}^{\lambda_{p-1},\mu}.$$

These coefficients are themselves the elements of unitary matrices connecting tensor product and direct sum representations of $SO(3)$ and, thus, it follows easily that the following orthonormality conditions hold:

$$\sum_{m_1,\dots,m_p} \{C_{l_1,m_1;\dots;l_p m_p}^{\lambda_1,\lambda_2,\dots,\lambda_{p-1};\mu}\}^2 = \sum_{\lambda_1} \cdots \sum_{\lambda_{p-1}} \sum_{\mu=-\lambda_{p-1}}^{\lambda_{p-1}} \{C_{l_1,m_1;\dots;l_p m_p}^{\lambda_1,\lambda_2,\dots,\lambda_{p-1};\mu}\}^2 = 1. \tag{33}$$

It is important to note that, due to the conditions $m_1 + m_2 = m_3$, the sums may actually vanish, for instance,

$$C_{l_1,0;\dots;l_p 0}^{\lambda_1,\lambda_2,\dots,\lambda_{p-1};0} = C_{l_1,0,l_2,0}^{\lambda_1,0} C_{\lambda_1,0;l_3,0}^{\lambda_2,0} \cdots C_{\lambda_{p-2},0;l_p,0}^{\lambda_{p-1},0}. \tag{34}$$

We have also that

$$\begin{aligned} &\mathcal{G}\{l_1, m_1; \dots; l_q, m_q; l, -m\} \\ &= \sqrt{\frac{4\pi}{2l+1}} \left\{ \prod_{i=1}^{q} \sqrt{\frac{2l_i+1}{4\pi}} \right\} \sum_{L_1,\dots,L_{q-2}} C_{l_1,0;\dots;l_q 0}^{L_1,L_2,\dots,L_{q-2},l;0} C_{l_1,m_1;\dots;l_q m_q}^{L_1,L_2,\dots,L_{q-2},l;m}. \end{aligned} \tag{35}$$

4. High-frequency central limit theorems: conditions in terms of Gaunt integrals

The aim of this section is to obtain conditions for high-frequency central limit theorems in terms of Gaunt integrals of the type (32). We start by focusing on the spherical field $T^{(q)}$ ($q \geq 2$) defined in (15), which is obtained by composing the Gaussian field T in (2) with the qth Hermite polynomial H_q (or, equivalently, by taking the qth Wick power of the random variable $T(x)$ for every x). Our first purpose is to characterize the asymptotic Gaussianity (when $l \to +\infty$) of the spherical harmonic coefficients $\{a_{lm;q}\}$ defined in (16). This aim is achieved by the following theorem. The general idea of the proof is to combine techniques recalled in the previous two sections; more explicitly, we use results from the theory of Gaussian-subordinated processes to characterize Gaussianity in terms of fourth order cumulants (see [25]) and we use group representation properties and Gaunt integrals to characterize fourth order cumulants in terms of convolutions of Clebsch–Gordan coefficients.

Theorem 3. *Fix $q \geq 2$.*

1. *For every $l \geq 1$, the positive constant $\widetilde{C}_l^{(q)}$ in (19) (which does not depend on m) equals the quantity*

$$q! \sum_{l_1,m_1} \cdots \sum_{l_q,m_q} C_{l_1} C_{l_2} \cdots C_{l_q} |\mathcal{G}\{l_1, m_1; \ldots; l_q, m_q; l, -m\}|^2 \tag{36}$$

$$= q! \sum_{l_1,\ldots,l_q=0}^{\infty} C_{l_1} \cdots C_{l_q} \frac{4\pi}{2l+1} \left\{ \prod_{i=1}^{q} \frac{2l_i+1}{4\pi} \right\} \sum_{L_1,\ldots,L_{q-2}} \{C_{l_1,0;\cdots;l_q 0}^{L_1,L_2,\ldots,L_{q-2},l;0}\}^2 \tag{37}$$

for every $m = -l, \ldots, l$, where the (generalized) Gaunt integral $\mathcal{G}\{\cdot\}$ is defined via (31).

2. *Fix $m \neq 0$. As $l \to +\infty$, the following two conditions* (A) *and* (B) *are equivalent:* (A)

$$\left(\widetilde{C}_l^{(q)}\right)^{-1/2} \times a_{lm;q} \overset{law}{\to} N + iN', \tag{38}$$

where $N, N' \sim \mathcal{N}(0, 1/2)$ are independent; (B) *for every $p = \frac{q-1}{2} + 1, \ldots, q-1$ if $q-1$ is even and every $p = q/2, \ldots, q-1$ if $q-1$ is odd,*

$$\begin{aligned}
&\left(\widetilde{C}_l^{(q)}\right)^{-2} \sum_{n_1,j_1} \cdots \sum_{n_{2(q-p)},j_{2(q-p)}} C_{j_1} \cdots C_{j_{2(q-p)}} \\
&\quad \times \Bigg| \sum_{l_1,m_1} \cdots \sum_{l_p,m_p} C_{l_1} \cdots C_{l_p} \\
&\qquad \times \mathcal{G}\{l_1, m_1; \ldots; l_p, m_p; j_1, n_1; \ldots; j_{q-p}, n_{q-p}; l, -m\} \\
&\qquad \times \mathcal{G}\big\{l_1, m_1; \ldots; l_p, m_p; j_{q-p+1}, n_{q-p+1}; \ldots; \\
&\qquad\qquad j_{2(q-p)}, n_{2(q-p)}; l, -m\big\}\Bigg|^2 \to 0.
\end{aligned} \tag{39}$$

3. *Let N be a centered Gaussian random variable with unitary variance. As $l \to +\infty$, the central limit theorem*

$$\left(\widetilde{C}_l^{(q)}\right)^{-1/2} \times a_{l0;q} \overset{law}{\to} N \tag{40}$$

takes place if and only if the asymptotic condition (39) holds for $m = 0$ and for every $p = \frac{q-1}{2} + 1, \ldots, q-1$ if $q-1$ is even and every $p = q/2, \ldots, q-1$ if $q-1$ is odd.

Proof. Consider a standard Brownian motion $W = \{W_t : t \in [0, 1]\}$ and denote by $L^2_{\mathbb{C}}([0, 1]) = L^2_{\mathbb{C}}([0, 1], \mathrm{d}\lambda)$ the class of complex-valued and square-integrable functions on $[0, 1]$, with respect to the Lebesgue measure $\mathrm{d}\lambda$. Now, select a complex-valued family $\{g_{lm} : l \geq 0, -l \leq$

$m \leq l\} \subseteq L^2_{\mathbb{C}}([0,1])$ with the following five properties: (1) g_{l0} is real for every $l \geq 0$; (2) $g_{lm} = (-1)^m \overline{g_{l-m}}$; (3) $\int g_{lm} \overline{g_{l'm'}} \, \mathrm{d}\lambda = 0, \forall (l,m) \neq (l',m')$; (4) $\int \Re(g_{lm}) \Im(g_{lm}) \, \mathrm{d}\lambda = 0$; (5) $\int \Re(g_{lm})^2 \, \mathrm{d}\lambda = \int \Im(g_{lm})^2 \, \mathrm{d}\lambda = \int g_{l0}^2 \, \mathrm{d}\lambda/2 = C_l/2$, where $\{C_l : l \geq 0\}$ is the power spectrum of the Gaussian field T. According to Proposition 1, the following identity in law holds:

$$\{a_{lm;1} : l \geq 0, -l \leq m \leq l\} \overset{\text{law}}{=} \{I_1(g_{lm}) : l \geq 0, -l \leq m \leq l\},$$

where $I_1(g_{lm}) = \int_0^1 g_{lm} \, \mathrm{d}W = \int_0^1 \Re(g_{lm}) \, \mathrm{d}W + \mathrm{i} \int_0^1 \Im(g_{lm}) \, \mathrm{d}W$ is the usual (complex-valued) Wiener–Itô integral of g_{lm} with respect to W. From this last relation, it also follows that, in the sense of stochastic processes, $T(x) \overset{\text{law}}{=} I_1(\sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm} Y_{lm}(x))$ (note that the function $z \mapsto \sum_{l,m} g_{lm}(z) Y_{lm}(x)$ is real-valued for every fixed $x \in \mathbb{S}^2$ and with norm equal to 1). Now, define $L^2_{s,\mathbb{C}}([0,1]^q)$ to be the class of complex-valued and symmetric functions on $[0,1]^q$ that are square-integrable with respect to Lebesgue measure. For every $f \in L^2_{s,\mathbb{C}}([0,1]^q)$, we define $I_q(f) = I_q(\Re(f)) + \mathrm{i} I_q(\Im(f))$ to be the multiple Wiener–Itô integral, of order q, of f with respect to the Brownian motion W (see, e.g., [24], Chapter 1 or [13]). From the previous discussion, it follows that, for every $q \geq 2$,

$$T^{(q)}(x) = H_q(T(x)) \overset{\text{law}}{=} I_q\left[\left\{\sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm} Y_{lm}(x)\right\}^{\otimes q}\right], \tag{41}$$

where the equality in law holds in the sense of finite-dimensional distributions and, for every $f \in L^2_{\mathbb{C}}([0,1])$, we use the notation $f^{\otimes q}(a_1, \ldots, a_q) = f(a_1) \times \cdots \times f(a_q)$. Note that to obtain the last equality in (41), we used the following well-known relation (see, e.g., [13]): for every real-valued $f \in L^2_{\mathbb{R}}([0,1])$ such that $\|f\|_{L^2_{\mathbb{R}}([0,1])} = 1$, it holds that $H_q[I_1(f)] = I_q(f^{\otimes q})$. Now, set $h^{(q)}_{l,m} = (-1)^m \sum_{l_1,m_1} \cdots \sum_{l_q,m_q} g_{l_1 m_1} \cdots g_{l_q m_q} \mathcal{G}\{l_1, m_1; \ldots; l_q, m_q; l, -m\}$, so that

$$a_{lm;q} \overset{\text{law}}{=} \int_{S^2} I_q\left[\left\{\sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_{lm} Y_{lm}(x)\right\}^{\otimes q}\right] \overline{Y_{lm}(x)} \, \mathrm{d}x = I_q\left[h^{(q)}_{l,m}\right] \tag{42}$$

and so (36) follows immediately from the well-known isometry relation

$$\mathbb{E}\left[\left|I_q\left[h^{(q)}_{l,m}\right]\right|^2\right] = q! \left\|h^{(q)}_{l,m}\right\|^2_{L^2([0,1]^q)}$$

(to obtain (42), we interchanged stochastic and deterministic integration, by means of a standard stochastic Fubini argument). To prove that (37) is equal to (36), first observe first that (33) yields

$$\sum_{m_1=-l_1}^{l_1} \cdots \sum_{m_q=-l_q}^{l_q} C^{L_1, L_2, \ldots, L_{q-2}, l; m}_{l_1, m_1; \ldots; l_q m_q} C^{L'_1, L'_2, \ldots, L'_{q-2}, l; m}_{l_1, m_1; \ldots; l_q m_q} = \delta^{L'_1}_{L_1} \cdots \delta^{L'_{q-2}}_{L_{q-2}}$$

(the right-hand side of the previous expression does not depend on m). Then, use (35) to deduce that

$$\begin{aligned}&\sum_{m_1=-l_1}^{l_1}\cdots\sum_{m_q=-l_q}^{l_q}\mathcal{G}\{l_1,m_1;\ldots;l_q,m_q;l,-m\}^2\\&\quad=\frac{4\pi}{2l+1}\left\{\prod_{i=1}^{q}\frac{2l_i+1}{4\pi}\right\}\sum_{L_1,\ldots,L_{q-2}}\{C_{l_1,0;\ldots;l_q0}^{L_1,L_2,\ldots,L_{q-2},l;0}\}^2.\end{aligned}$$

This proves Point 1 in the statement. To prove Point 2, recall that, according to [19], Proposition 6, relation (38) holds if and only if

$$\big(\widetilde{C}_l^{(q)}\big)^{-2}\|h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}\|^2_{L^2([0,1]^{2(q-p)})}\to 0$$

for every $p=1,\ldots,q-1$, where the complex-valued (and not necessarily symmetric) function $h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}$ (which is an element of $L^2([0,1]^{2(q-p)})$) is defined as the *contraction*

$$\begin{aligned}&h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}\big(a_1,\ldots,a_{2(q-p)}\big)\\&\quad=\int_{[0,1]^p}h_{l,m}^{(q)}(\mathbf{x}_p,a_1,\ldots,a_{q-p})\overline{h_{l,m}^{(q)}\big(\mathbf{x}_p,a_{q-p+1},\ldots,a_{2(q-p)}\big)}\,\mathrm{d}\mathbf{x}_p\end{aligned}\tag{43}$$

for every $(a_1,\ldots,a_{2(q-p)})\in[0,1]^{2(q-p)}$, where $d\mathbf{x}_p$ is the Lebesgue measure on $[0,1]^p$. Since, trivially, $\|h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}\|^2=\|h_{l,m}^{(q)}\otimes_{q-p}\overline{h_{l,m}^{(q)}}\|^2$ (we stress that, in the last equality, the first norm is taken in $L^2([0,1]^{2(q-p)})$, whereas the second is in $L^2([0,1]^{2p})$), one deduces that it is sufficient to check that the norm of $h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}$ is asymptotically negligible for every $p=\frac{q-1}{2}+1,\ldots,q-1$ if $q-1$ is even and every $p=q/2,\ldots,q-1$ if $q-1$ is odd. It follows that the result is proved once it is shown that, for every p in such range, the norm $\|h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}\|^2$ equals the multiple sum appearing in (39). To see this, use (43) to deduce that (recall that Gaunt integrals are real-valued)

$$\begin{aligned}&h_{l,m}^{(q)}\otimes_p\overline{h_{l,m}^{(q)}}\big(a_1,\ldots,a_{2(q-p)}\big)\\&\quad=\sum_{n_1,j_1}\cdots\sum_{n_{2(q-p)},j_{2(q-p)}}g_{j_1n_1}\cdots g_{j_{q-p}n_{q-p}}\overline{g_{j_{q-p+1}n_{q-p+1}}\cdots g_{j_{2(q-p)}n_{2(q-p)}}}\\&\qquad\times\sum_{l_1,m_1}\ldots\sum_{l_p,m_p}C_{l_1}\cdots C_{l_p}\mathcal{G}\{l_1,m_1;\ldots;l_p,m_p;j_1,n_1;\ldots;j_{q-p},n_{q-p};l,-m\}\\&\qquad\qquad\times\mathcal{G}\{l_1,m_1;\ldots;l_p,m_p;j_{q-p+1},n_{q-p+1};\ldots;j_{2(q-p)},n_{2(q-p)};l,-m\}\end{aligned}$$

and the result is obtained by using the orthogonality properties of the g_{jn}'s. Point 3 in the statement is proved in exactly the same way, by first observing that $a_{l0;q}$ is a real-valued random variable and then by applying Theorem 1 in [25]. $\square$

Remarks. (1) One has the relation $\mathbb{E}[T^{(q)}(x)^2] = q![\mathbb{E}\{T(x)^2\}]^q$. This equality can be proven in two ways: (i) by exploiting the representation of $T^{(q)}(x)$ as a multiple Wiener–Itô integral; or (ii) by using the equality $\mathbb{E}[T^{(q)}(x)^2] = \sum_l \frac{2l+1}{4\pi}\widetilde{C}_l^{(q)}$ and then by expanding $\widetilde{C}_l^{(q)}$ according to Theorem 3 so that one can apply the orthogonality relations (33).

(2) By using the results proven by Nourdin and Peccati in [22], one can prove that the convergence in (40) also takes place in the sense of total variation. This means that as $l \to +\infty$,

$$\sup_B \bigl|\mathbb{P}\bigl[\bigl(\widetilde{C}_l^{(q)}\bigr)^{-1/2} \times a_{l0;q} \in B\bigr] - \mathbb{P}[N \in B]\bigr| \to 0,$$

where the supremum is taken over all Borel sets B.

Now, recall that, according to part 2 of Lemma 2, $T_l^{(q)}(x) \stackrel{\text{law}}{=} \sqrt{\frac{2l+1}{4\pi}} a_{l0;q}$ so that relation (22) holds. This immediately gives a first (exhaustive) solution to Problem (P-I), as stated in Section 2.

Corollary 4. *For every $q \ge 2$, the following conditions are equivalent*:

1. *the central limit theorem* (23) *holds for every $x \in \mathbb{S}^2$*;
2. *the asymptotic relation* (39) *takes place for $m = 0$ and for every $p = \frac{q-1}{2} + 1, \ldots, q-1$ if $q-1$ is even and every $p = q/2, \ldots, q-1$ if $q-1$ is odd.*

To deal with Problem (P-II) of Section 2, we recall the notation $\overline{T}_l^{(q)}$ (indicating the lth normalized frequency component of $T^{(q)}$) introduced in (18). We also introduce (for every $l \ge 1$) the normalized lth frequency component of the Gaussian field T, which is defined as

$$\overline{T}_l(x) = \frac{T_l(x)}{\operatorname{Var}(T_l(x))^{1/2}} = \frac{T_l(x)}{((2l+1)/(4\pi)C_l)^{1/2}}, \qquad x \in \mathbb{S}^2. \tag{44}$$

According to Lemma 2 (in the special case $F(z) = z$), $\overline{T}_l$ is a real-valued, isotropic, centered and Gaussian field. Moreover, one has that $\mathbb{E}[\overline{T}_l(x)\overline{T}_l(y)] = \mathbb{E}[\overline{T}_l^{(q)}(x)\overline{T}_l^{(q)}(y)] = P_l(\langle x, y\rangle)$ for every $q \ge 2$ and every $l \ge 1$. The next result – which gives an exhaustive solution to Problem (P-II) – states that whenever Condition 1 (or, equivalently, Condition 2) in the statement of Corollary 4 is verified (and without any additional assumption), the "distance" between the finite-dimensional distributions of the normalized field $\overline{T}_l^{(q)}$ and those of $\overline{T}_l$ converge to zero. For every $k \ge 1$, we denote by $\mathbf{P}(\mathbb{R}^k)$ the class of all probability measures on $\mathbb{R}^k$. We say that a metric $\gamma(\cdot,\cdot)$ metricizes the weak convergence on $\mathbf{P}(\mathbb{R}^k)$ whenever the following double implication holds for every $Q \in \mathbf{P}(\mathbb{R}^k)$ and every $\{Q_l : l \ge 1\} \subset \mathbf{P}(\mathbb{R}^k)$ (as $l \to +\infty$): $\gamma(Q_l, Q) \to 0$ if and only if Q_l converges weakly to Q. The quantity $\gamma(P, Q)$ is sometimes called the *γ-distance* between P and Q.

Theorem 5. *Let $q \ge 2$ be fixed and suppose that Condition* 1 (*or* 2) *of Corollary* 4 *is satisfied.*

1. *For every* $k \geq 1$, *every* $x_1, \ldots, x_k \in \mathbb{S}^2$ *and every compact subset* $M \subset \mathbb{R}^k$,

$$\sup_{(\lambda_1,\ldots,\lambda_k)\in M} \left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^k \lambda_j \overline{T}_l^{(q)}(x_j)}\right] - \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{j=1}^k \lambda_j \overline{T}_l(x_j)}\right]\right| \underset{l\to+\infty}{\longrightarrow} 0. \tag{45}$$

2. *Fix* $x_1, \ldots, x_k$ *and denote by* $\mathcal{L}(\overline{T}_l^{(q)}; x_1, \ldots, x_k)$ *and* $\mathcal{L}(\overline{T}_l; x_1, \ldots, x_k)$ ($l \geq 1$), *respectively, the law of* $(\overline{T}_l^{(q)}(x_1), \ldots, \overline{T}_l^{(q)}(x_k))$ *and the law of* $(\overline{T}_l(x_1), \ldots, \overline{T}_l(x_k))$. *For every metric* $\gamma(\cdot,\cdot)$ *on* $\mathbf{P}(\mathbb{R}^k)$ *such that* $\gamma(\cdot,\cdot)$ *metricizes the weak convergence, it holds that*

$$\lim_{l\to+\infty} \gamma\left(\mathcal{L}\left(\overline{T}_l^{(q)}; x_1, \ldots, x_k\right), \mathcal{L}(\overline{T}_l; x_1, \ldots, x_k)\right) = 0.$$

Proof. The crucial point is that the spherical field $x \mapsto \overline{T}_l^{(q)}(x)$ lives in the qth Wiener chaos associated with the Gaussian space generated by T. By using this fact and arguing as in the proof of Theorem 3, one can show that the vector $(\overline{T}_l^{(q)}(x_1), \ldots, \overline{T}_l^{(q)}(x_k))$ is indeed equal in law to a vector of multiple Wiener–Itô integrals, of order q, with respect to a Brownian motion. Since each element of this vector converges in law to a standard Gaussian random variable, one can directly apply Theorem 1 and Proposition 2 in [26] to achieve the desired conclusion (see also [26], Proposition 5). □

Remark. Consider now Problem (P-III), as stated at the end of Section 2, where F is a general real-valued function belonging to the class $L_0^2(\mathbb{R}, \mathrm{e}^{-x^2/2}\,\mathrm{d}x)$. The function F admits a unique representation of the form

$$F(z) = \sum_{q=1}^{\infty} \frac{c_q(F)}{q!} H_q(z), \qquad z \in \mathbb{R}, \sum_{q=1}^{\infty} \frac{c_q(F)^2}{q!} < +\infty \tag{46}$$

and, for every $l \geq 0$, the frequency component $F[T]_l(x)$ defined in (13) admits the expansion

$$F[T]_l(x) = \sum_{q=1}^{\infty} \frac{c_q(F)}{q!} T_l^{(q)}(x), \qquad x \in \mathbb{S}^2, \tag{47}$$

where the series converges in $L^2(\mathbb{P})$ for every fixed x. Formula (47) combined with Lemma 2 also yields that

$$\mathbb{E}(F[T]_l(x)F[T]_l(y)) = \frac{2l+1}{4\pi} P_l(\cos\langle x, y\rangle) \sum_{q=1}^{\infty} \left(\frac{c_q(F)}{q!}\right)^2 \widetilde{C}_l^{(q)},$$

where $\widetilde{C}_l^{(q)}$ is given by (19) or, equivalently, by (37). The asymptotic Gaussianity of F-subordinated spherical random fields can then be simply characterized along the same lines as before, as a direct application of results in [12], Theorem 4. Exact conditions and proofs are standard and hence omitted for the sake of brevity.

5. Explicit sufficient conditions: convolutions and random walks

The purpose of this section is to provide more explicit conditions for the central limit theorems proven in Section 4 for the (Hermite) frequency components $T_l^{(q)}$, $l \geq 0$. In particular, we shall establish sufficient conditions that are more directly linked to primitive assumptions on the behaviour of the angular power spectrum $\{C_l : l \geq 0\}$. The idea we shall pursue is the following: in the (much simpler) case of random fields defined on the circle [19], Section 3, the conditions for the central limit theorem could be expressed in terms of convolutions of the angular power spectra, leading, on the one hand, to more explicit conditions and, on the other, to their possible interpretation in terms of a random walk on the representations of the associated group (the torus). In this section, we shall exploit this analogy and group representation properties of Clebsch–Gordan coefficients to write our conditions for the central limit theorem in terms of a random walk on the representations of $SO(3)$. This will allow us to achieve two aims, that is, we shall obtain more explicit conditions in terms of the asymptotic behaviour of angular power spectra (which shall be further developed in Section 6) and we shall provide a unifying framework which may point to a more general theory. In particular, the results of Section 5.2 cover the cases $q = 2$ and $q = 3$. Section 5.3 contains some partial findings for the case of a general q, as well as several conjectures. These results will be used in Section 6 to deduce explicit conditions on the rate of decay of the angular power spectrum $\{C_l : l \geq 0\}$.

Our analysis is inspired by the following result, which is a particular case of the statements contained in [19], Section 3, concerning fields on Abelian groups. Indeed, consider a centered real-valued Gaussian field $V = \{V(\theta) : \theta \in \mathbb{T}\}$ defined on the torus $\mathbb{T} = [0, 2\pi)$ (which we regard as an Abelian compact group with group operation given by $xy = (x + y) \operatorname{mod}(2\pi)$). We suppose that the law of V is isotropic, that is, that $V(\theta) \stackrel{\text{law}}{=} V(x\theta)$ (in the sense of stochastic processes) for every $x \in \mathbb{T}$, and also that $\mathbb{E}V(\theta)^2 = 1$. We denote by $V(\theta) = \sum_{l \in \mathbb{Z}} a_l \mathrm{e}^{\mathrm{i}l\theta}$ the Fourier decomposition of V and we write $\Gamma_l^V = \mathbb{E}|a_l|^2$ (note that $\Gamma_l^V = \Gamma_{-l}^V$). Fix $q \geq 2$ and consider the Hermite-subordinated field $H_q[V](\theta) = H_q(V(\theta))$, where q is the qth Hermite polynomial. The Fourier decomposition of $H_q[V]$ is $H_q[V](\theta) = \sum_{l \in \mathbb{Z}} a_l^{(q)} \mathrm{e}^{\mathrm{i}l\theta}$. We write N, N' to indicate a pair of independent centered Gaussian random variables with common variance equal to $1/2$: in [19] it is proved that to have the high-frequency central limit theorem

$$\frac{a_l^{(q)}}{\operatorname{Var}(a_l^{(q)})^{1/2}} = \frac{\int_{\mathbb{T}} H_q[V](\theta)\mathrm{e}^{-\mathrm{i}l\theta}\,\mathrm{d}\theta}{\operatorname{Var}(a_l^{(q)})^{1/2}} \xrightarrow[l \to \infty]{\text{law}} N + \mathrm{i}N', \tag{48}$$

it is necessary and sufficient that, for every $p = 1, \ldots, q-1$,

$$\lim_{l \to +\infty} \sup_{j \in \mathbb{Z}} \mathbb{P}[U_p = j \mid U_q = l] = 0, \tag{49}$$

where $\{U_n : n \geq 0\}$ is the random walk on $\mathbb{Z}$ whose law is given by $U_0 = 0$ and $\mathbb{P}[U_{n+1} = j | U_n = k] = \Gamma_{j-k}^V$. Note that the law of the random variable U_n has, trivially, the form of a convolution of the coefficients Γ_l^V (see also the discussion below). The correspondence between (48) and the "random walk bridge" (49) has been used in [19] to establish explicit conditions on the power spectrum $\{\Gamma_l^V\}$ to ensure that (48) holds.

Note that the random walk $\{U_n : n \geq 0\}$ can be viewed as being defined on the space of representations of the torus $\mathbb{T}$; indeed, as the latter is an Abelian group, its representations are provided by the one-dimensional matrices $\{\exp(ik\cdot)\}$, $k \in \mathbb{Z}$. In what follows, we shall unveil (and apply) an analogous connection between the central limit theorems proved in Section 4 and some specific convolutions and random walks on the space of representations of the group of rotations, which, as usual, we label $\widehat{SO(3)}$.

5.1. Convolutions on $\widehat{SO(3)}$

In the light of Part 3 of Theorem 3 and by Corollary 4, we will focus on the sequence $\{a_{l0;q} : l \geq 0\}$ (see (16)), whose behaviour as $l \to +\infty$ yields an asymptotic characterization of the fields $T_l^{(q)}(\cdot)$ defined in (17). A crucial point is the simple fact that the numerator of (39), for $m = 0$, can be developed as a multiple sum involving products of four generalized Gaunt integrals so that, by (32), the asymptotic expressions appearing in Theorem 3 can be studied by means of the properties of linear combinations of products of Clebsch–Gordan coefficients. As anticipated, a very efficient tool for our analysis will be the use of convolutions on $\mathbb{N}$ that we endow with an hypergroup structure isomorphic to $\widehat{SO(3)}$, that is, the dual of $SO(3)$. This will be the object of the subsequent discussion.

From now on, and for the rest of the section, we shall fix a sequence $\{C_l : l \geq 0\}$, representing the angular power spectrum of an isotropic, centered, normalized Gaussian field T over $\mathbb{S}^2$, as in Section 2. Whenever convenient, we shall write

$$\Gamma_l \triangleq (2l+1)C_l, \qquad l \geq 0, \tag{50}$$

so that, for $l \geq 1$ and up to the constant $1/4\pi$, the parameter Γ_l represents the variance of the projection of the Gaussian field T in (2) on the frequency l; indeed, according to Lemma 2, $\mathrm{Var}(T_l) = \Gamma_l/4\pi$. As motivated earlier, to exploit the analogy with the Abelian case and derive more explicit conditions, we define the following convolutions of the coefficients Γ_l (in the following expressions, the sums over indices $l_i, L_i, \ldots$ range implicitly from 0 to $+\infty$):

$$\widehat{\Gamma}_{2,l} = \sum_{l_1,l_2} \Gamma_{l_1}\Gamma_{l_2}(C^{l0}_{l_1 0 l_2 0})^2, \tag{51}$$

$$\widehat{\Gamma}_{3,l} = \sum_{L_1,l_3} \widehat{\Gamma}_{2,L_1}\Gamma_{l_3}(C^{l0}_{L_1 0 l_3 0})^2 = \sum_{l_1,l_2,l_3} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{l_3} \sum_{L_1} (C^{L_1 l;0}_{l_1 0 l_2 0 l_3 0})^2, \ldots, \tag{52}$$

$$\widehat{\Gamma}_{q,l} = \sum_{L_1,l_q} \widehat{\Gamma}_{q-1,L_{q-1}}\Gamma_{l_q}(C^{l0}_{L_{q-1} 0 l_q 0})^2 = \sum_{l_1,\ldots,l_q} \Gamma_{l_1}\cdots\Gamma_{l_q} \sum_{L_1,\ldots,L_{q-2}} (C^{L_1,\ldots,L_{q-2} l;0}_{l_1 0,\ldots,l_q 0})^2. \tag{53}$$

We stress that the equalities in formulae (52) and (53) are consequences of (34). It will be also convenient to define a *-convolution of order $p \geq 2$ as

$$\widehat{\Gamma}^*_{p,l;l_1} = \sum_{l_2} \cdots \sum_{l_p} \Gamma_{l_2}\cdots\Gamma_{l_p} \sum_{L_1,\ldots,L_{p-2}} \{C^{L_1,\ldots,l;0}_{l_1 0 l_2 0,\ldots,l_p 0}\}^2. \tag{54}$$

Note that the number of sums following the equalities in formula (54) is $p-1$; however, we choose to keep the symbol p to denote *-convolutions since it is consistent with the probabilistic representations given in formulae (58) and (59) below. The above *-convolution has the following property: for every $p = 2, \ldots, q$,

$$\sum_{l_1} \widehat{\Gamma}_{q+1-p,l_1} \widehat{\Gamma}^*_{p,l;l_1} = \widehat{\Gamma}_{q,l} \quad \text{and, in particular,} \quad \sum_{l_1} \Gamma_{l_1} \widehat{\Gamma}^*_{q,l;l_1} = \widehat{\Gamma}_{q,l}.$$

The *-convolution of order 2 can be written more explicitly as

$$\widehat{\Gamma}^*_{2,l;l_1} = \sum_{l_2} \Gamma_{l_2} (C^{l0}_{l_1 0 l_2 0})^2. \tag{55}$$

Remarks. (1) (*Probabilistic interpretation of the convolutions*). First, write $\Gamma_* \triangleq \sum_l \Gamma_l$ (plainly, in our framework, $\Gamma_* = 4\pi$, but the following discussion applies to coefficients $\{\Gamma_l\}$ such that $\Gamma_* > 0$ is arbitrary) so that $l \longmapsto \Gamma_l / \Gamma_*$ defines a probability on $\mathbb{N}$. The second orthonormality relation in (29) implies that, for fixed l_1, l_2, the application $l \longmapsto (C^{l0}_{l_1 0 l_2 0})^2$ is a probability on $\mathbb{N}$. Now, define the law of a (homogeneous) Markov chain $\{Z_n : n \geq 1\}$ as follows:

$$\mathbb{P}\{Z_1 = l\} = \Gamma_l / \Gamma_*, \tag{56}$$

$$\mathbb{P}\{Z_{n+1} = l \mid Z_n = L\} = \sum_{l_0} \frac{\Gamma_{l_0}}{\Gamma_*} (C^{l0}_{l_0 0 L 0})^2. \tag{57}$$

It is clear that $\mathbb{P}\{Z_q = l\} = \widehat{\Gamma}_{q,l} / (\Gamma_*)^q$ and also, for $p \geq 2$,

$$\frac{\widehat{\Gamma}^*_{p,l:l_1}}{(\Gamma_*)^{p-1}} = \mathbb{P}\{Z_p = l | Z_1 = l_1\}, \tag{58}$$

$$\frac{\widehat{\Gamma}^*_{p,l:l_1} \widehat{\Gamma}_{q+1-p,l_1}}{(\Gamma_*)^q} = \mathbb{P}\{(Z_q = l) \cap (Z_{q+1-p} = l_1)\} \qquad (q > p-1). \tag{59}$$

The following quantity will be crucial in the subsequent sections:

$$\frac{\widehat{\Gamma}^*_{q+1-p,l;\lambda} \widehat{\Gamma}_{p,\lambda}}{\sum_L \widehat{\Gamma}_{p,L} \widehat{\Gamma}^*_{q+1-p,l;L}} = \frac{\widehat{\Gamma}^*_{q+1-p,l;\lambda} \widehat{\Gamma}_{p,\lambda}}{\widehat{\Gamma}_{q,l}} = \mathbb{P}\{Z_p = \lambda | Z_q = l\} \qquad (q > p); \tag{60}$$

observe that the last relation in (60) derives from

$$\widehat{\Gamma}^*_{q+1-p,l;\lambda} / (\Gamma_*)^{q-p} = \mathbb{P}\{(Z_{q+1-p} = l) | (Z_1 = \lambda)\} = \mathbb{P}\{(Z_q = \lambda) | (Z_p = l)\},$$

where the last equality is a consequence of the homogeneity of Z. Also, note that we can identify each natural number $l \geq 0$ with an irreducible representation of $SO(3)$. It follows that the formal addition $l_1 + l_2 \triangleq \sum_l l (C^{l0}_{l_1 0 l_2 0})^2$ may be used to endow $\widehat{SO(3)}$ with a hypergroup structure (see, e.g., [5] for some general results on hypergroups). In this sense, we can interpret the chain $\{Z_n : n \geq 1\}$ as a random walk on the hypergroup $\widehat{SO(3)}$, in a spirit similar to [11].

(2) (*A comparison with the Abelian case*). In [19], where we dealt with similar problems in the case of homogenous spaces of Abelian groups, we extensively used convolutions over $\mathbb{Z}$. These kinds of convolutions, that we denote ${}_A\widehat{\Gamma}_{q,l}$ ($q \geq 2$, $l \in \mathbb{Z}$), are obtained as in (51)–(55), by taking sums over $\mathbb{Z}$ (instead of over $\mathbb{N}$) and by replacing the Clebsch–Gordan symbols $(C^{l0}_{l_1 0 l_2 0})^2$ with the indicator $\mathbf{1}_{l_1+l_2=l}$. Note that these indicator functions do indeed provide the Clebsch–Gordan coefficients associated with the irreducible representations of the 1-dimensional torus $\mathbb{T} = [0, 2\pi)$, regarded as a compact Abelian group with group operation $xy = (x + y)(\operatorname{mod}(2\pi))$ (this is equivalent to the trivial relation $\mathrm{e}^{\mathrm{i}l_1 x}\mathrm{e}^{\mathrm{i}l_2 x} = \sum_l \mathbf{1}_{l_1+l_2=l}\mathrm{e}^{\mathrm{i}lx} = \mathrm{e}^{\mathrm{i}(l_1+l_2)x}$). Note, also, that in the Abelian case, one has ${}_A\widehat{\Gamma}^*_{p,l;l_1} = {}_A\widehat{\Gamma}_{p,l-l_1}$. Also, if $\Gamma_l = \Gamma^V_l$, where $\{\Gamma^V_l\}$ is the power spectrum of the Gaussian field V on $\mathbb{T}$ appearing in (48), one has that ${}_A\widehat{\Gamma^V_{q,l}} = \mathbb{P}[U_q = l]$, where $\{U_n\}$ is the random walk given in (49).

5.2. The cases $q = 2$ and $q = 3$

In this subsection, we provide a sufficient condition on the spectrum $\{C_l : l \geq 0\}$ (or, equivalently, on $\{\Gamma_l : l \geq 0\}$, as defined in (50)) to have the central limit theorem (40) in the quadratic and cubic cases $q = 2, 3$. The proofs are very technical and require very careful manipulations of Clebsch–Gordan coefficients. For the sake of brevity, they are not reported here; full details are given in the arXiv preprint [21].

Proposition 6. *For $q = 2$, a sufficient condition for the central limit theorem* (40) *is the asymptotic relation*

$$\lim_{l\to+\infty} \sup_{l_1} \frac{\sum_{l_1} \Gamma_{l_1}\Gamma_{l_2}\{C^{l0}_{l_1 0 l_2 0}\}^2}{\sum_{l_1,l_2} \Gamma_{l_1}\Gamma_{l_2}(C^{l0}_{l_1 0 l_2 0})^2} = \lim_{l\to+\infty} \sup_{l_1} \mathbb{P}\{Z_1 = l_1 \mid Z_2 = l_2\} = 0, \tag{61}$$

where the $\{\Gamma_l\}$ are given by (50) *and $\{Z_l\}$ is the Markov chain defined in formulae* (56) *and* (57).

Remark. Note that, using (53) and (55), condition (61) becomes

$$\lim_{l\to\infty} \sup_{\lambda} \frac{\Gamma_\lambda \widehat{\Gamma}^*_{2,l;\lambda}}{\sum_{l_1} \Gamma_{l_1}\widehat{\Gamma}_{2,l;l_1}} = 0. \tag{62}$$

Also, note that if, in the convolutions (53), one replaces each squared Clebsch–Gordan coefficient $(C^{l0}_{l_1 0 l_2 0})^2$ by the indicator $\mathbf{1}_{l_1+l_2=l}$ and extends the sums over $\mathbb{Z}$, one obtains the relation

$$\lim_{l\to\infty} \sup_{l_1} \frac{\Gamma_{l_1}\Gamma_{l-l_1}}{\sum_{l_1} \Gamma_{l_1}\Gamma_{l-l_1}} = 0. \tag{63}$$

In particular, when $\{\Gamma_l\} = \{\Gamma^V_l\}$ (the power spectrum of the field V on $\mathbb{T}$ given in (48)), it is not difficult to show that formula (63) gives exactly the asymptotic (necessary and sufficient) condition (49).

Proposition 7. *Sufficient conditions for the central limit theorem* (40) *when* $q = 3$ *are*

$$\lim_{l\to\infty} \sup_{L_1} \frac{\sum_{l_1 l_2 j_1} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{j_1}\{C^{L_1 l 0}_{l_1 0 l_2 0 j_1 0}\}^2}{\sum_{L_1}\sum_{l_1,l_2,l_3} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{l_3}\{C^{L_1 l 0}_{l_1 0 l_2 0 l_3 0}\}^2} = 0 \quad \textit{and} \tag{64}$$

$$\lim_{l\to\infty} \sup_{j_1} \frac{\sum_{l_1 l_2 L_1} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{j_1}\{C^{L_1 l 0}_{l_1 0 l_2 0 j_1 0}\}^2}{\sum_{L_1}\sum_{l_1,l_2,l_3} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{l_3}\{C^{L_1 l 0}_{l_1 0 l_2 0 l_3 0}\}^2} = 0. \tag{65}$$

Remark. In the light of (53)–(55) and of the definition of the random walk Z given in (56) and (57), it is not difficult to see that (64) can be rewritten as

$$\begin{aligned}\lim_{l\to\infty} \sup_{\lambda} \frac{\widehat{\Gamma}_{2,\lambda}\sum_{j_1}\Gamma_{j_1}\{C^{l0}_{\lambda j_1 0}\}^2}{\widehat{\Gamma}_{3,l}} &= \lim_{l\to\infty} \sup_{\lambda} \frac{\widehat{\Gamma}_{2,\lambda}\widehat{\Gamma}^*_{2,l;\lambda}}{\sum_{L_1}[\widehat{\Gamma}_{2,L_1}\widehat{\Gamma}^*_{1,l;L_1}]} \\ &= \lim_{l\to\infty} \sup_{\lambda} \mathbb{P}[Z_2 = \lambda \mid Z_3 = l] = 0.\end{aligned} \tag{66}$$

Likewise, one obtains that (65) is equivalent to

$$\begin{aligned}&\lim_{l\to\infty} \sup_{j_1} \frac{\Gamma_{j_1}\widehat{\Gamma}^*_{3,l;j_1}}{\sum_{L_1}\sum_{l_1,l_2,l_3} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{l_3}\{C^{L_1 l 0}_{l_1 0 l_2 0 l_3 0}\}^2} \\ &\quad = \lim_{l\to\infty} \sup_{j_1} \mathbb{P}[Z_1 = j_1 \mid Z_3 = l] = 0.\end{aligned} \tag{67}$$

It should be noted that the conditions (66) and (67) can be written compactly as

$$\lim_{l\to\infty} \max_{q=1,2} \sup_{j_1} \frac{\widehat{\Gamma}_{q,j_1}\widehat{\Gamma}^*_{3-q,l;j_1}}{\sum_{L_1}\sum_{l_1,l_2,l_3} \Gamma_{l_1}\Gamma_{l_2}\Gamma_{l_3}\{C^{L_1 l 0}_{l_1 0 l_2 0 l_3 0}\}^2} = 0. \tag{68}$$

Relation (68) once again parallels analogous conditions established for stationary fields on a torus; see [19].

5.3. A conjecture for general q

The relation (39) (which implies (40)), in the general case where $q \geq 4$, is still being investigated as it requires a difficult analysis of higher order Clebsch–Gordan coefficients by means of graphical techniques (see, e.g., [34], Chapter 11). In view of the results of the previous subsection and some preliminary computations for the case $p = q - 1$ (see ([21])), it is natural at this stage to propose the following conjecture. Recall that we focus on the central limit theorem (40) because of the equality in law $T_l^{(q)}(x) = \sqrt{\frac{2l+1}{4\pi}} a_{l0;q}$ and Corollary 4.

Conjecture. *A sufficient condition for the central limit theorem* (40) *is*

$$\lim_{l\to\infty}\max_{1\le p\le q-1}\sup_{\lambda}\frac{\widehat{\Gamma}_{p,\lambda}\widehat{\Gamma}^{*}_{q+1-p,l;\lambda}}{\sum_{L}\widehat{\Gamma}_{p,L_{q-2}}\widehat{\Gamma}^{*}_{q+1-p,l;L}}=\lim_{l\to\infty}\max_{1\le p\le q-1}\sup_{\lambda}\mathbb{P}\{Z_p=\lambda\mid Z_q=l\}=0. \quad (69)$$

It is worth emphasizing how condition (69) is the exact analog of the necessary and sufficient condition (49), established in [19] for the high-frequency central limit theorem on the torus $\mathbb{T}=[0,2\pi)$. This remarkable circumstance suggests that an analogous result may exist for homogeneous spaces of general compact groups. We leave the previous conjecture and such an extension as open issues for future research.

Remark (On "no privileged path" conditions). In terms of Z, condition (69) can be further interpreted as follows: for every l, define a "bridge" of length q, by conditioning Z to equal l at time q. Then (69) is verified if and only if the probability that the bridge hits λ at time q converges to zero, uniformly on λ, as $l\to+\infty$. It is also evident that when (69) is verified for every $p=1,\dots,q-1$, one also has that

$$\lim_{l\to+\infty}\sup_{\lambda_1,\dots,\lambda_{q-1}\in\mathbb{N}}\mathbb{P}[Z_1=\lambda_1,\dots,Z_{m-1}=\lambda_{q-1}|Z_q=l]=0, \quad (70)$$

meaning that, asymptotically, the law of Z does not charge any "privileged path" of length q leading to l. The interpretation of condition (70) in terms of bridges can be reinforced by putting, by convention, $Z_0=0$ so that the probability in (70) is that of the particular path $0\to\lambda_1\to\cdots\to\lambda_{q-1}\to l$ associated with a random bridge linking 0 and l.

6. Application: algebraic/exponential dualities

In this section, we discuss explicit conditions on the angular power spectrum $\{C_l : l\ge 0\}$ of the Gaussian field T introduced in Section 2, ensuring that the central limit theorem (40) will hold. Our results show that if the power spectrum decreases exponentially, then a high-frequency central limit theorem holds, whereas the opposite implication holds if the spectrum decreases as a negative power. This duality mirrors analogous conditions previously established in the Abelian case; see [19]. For simplicity, we stick to the case $q=2$. Note that the results below allow one to deal with the asymptotic (high-frequency) behaviour of the Sachs–Wolfe model (6).

6.1. The exponential case

Assume that

$$C_l\approx(l+1)^{\alpha}\exp(-l),\qquad \alpha\ge 0. \quad (71)$$

To prove that, in this case, (40) is verified for $q=2$, we will prove that (61) holds (recall the definition of Γ_l given in (50)). For the denominator of the previous expression, we obtain the lower bound

$$\sum_{l_1,l_2=1}^{\infty} \Gamma_{l_1}\Gamma_{l_2}(C^{l0}_{l_1 0 l_2 0})^2 \geq \sum_{l_1=[l/3]}^{[2l/3]} \Gamma_{l_1}\Gamma_{l-l_1}(C^{l0}_{l_1 0 l-l_1 0})^2 \approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} (C^{l0}_{l_1 0 l-l_1 0})^2 \tag{72}$$

and, in view of [34], equation (8.5.2.33) and Stirling's formula,

$$\begin{aligned}
(72) &\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \left(\frac{l!}{l_1!(l-l_1)!}\right)^2 \left(\frac{(2l_1)!(2l-2l_1)!}{(2l)!}\right) \\
&\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \frac{l^{2l+1}}{l_1^{2l_1+1}(l-l_1)^{2l-2l_1+1}} \left(\frac{(2l_1)^{2l_1+1/2}(2l-2l_1)^{2l-2l_1+1/2}}{(2l)^{2l+1/2}}\right) \\
&\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \frac{l^{1/2}}{l_1^{1/2}(l-l_1)^{1/2}} \approx \exp(-l) l^{2(\alpha+1)} l^{1/2}.
\end{aligned}$$

On the other hand, recall that by the triangle conditions (Section 3), $\{C^{l0}_{l_1 0 l_2 0}\}^2 \equiv 0$ unless $l_1 + l_2 \geq l$. Hence,

$$\begin{aligned}
\sup_{l_1} \sum_{l_2} \Gamma_{l_1}\Gamma_{l_2}\{C^{l0}_{l_1 0 l_2 0}\}^2 &\leq K \sup_{l_1} \exp(-l) l_1^{\alpha+1} \\
&\quad \times \left\{ |l-l_1|^{\alpha+1} + \sum_{u=1}^{\infty} \exp(-u)|l_1+u|^{\alpha+1} \right\} \approx \exp(-l) l^{2(\alpha+1)}.
\end{aligned}$$

It is then immediate to see that (61) is satisfied.

6.2. Regularly varying functions

For $q = 2$, we show below that the central limit theorem fails for all sequences C_l such that: (a) C_l is quasi-monotonic, that is, $C_{l+1} \leq C_l(1 + K/l)$; and (b) C_l is such that $\liminf_{l\to\infty} C_l/C_{l/2} > 0$. In particular, a necessary condition for the central limit theorem (40) to hold is that $C_l/C_{l/2} \to 0$. This is exactly the same necessary condition as was derived by [19] in the Abelian case. For the general case $q \geq 2$, we expect the central limit theorem to fail for all regularly varying angular power spectra, that is, for all C_l such that $\liminf_{\ell\to\infty} C_l/C_{\alpha l} > 0$ for all $\alpha > 0$. Note that we are thus covering all polynomial forms for C_l^{-1}.

Since (61) only provides a sufficient condition for the central limit theorem, we need to directly analyze the more primitive condition (39) for $m = 0$ (however, the case $m \neq 0$ just entails a more complicated notation). We first consider an upper bound for the square root of the denominator of (39), which is given by $\widetilde{C}_l^{(2)}$.

We have

$$\begin{aligned}
\widetilde{C}_l^{(2)} &= \sum_{j_1,j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{4\pi(2l+1)} (C_{j_1 0 j_2 0}^{l0})^2 \\
&\leq 2 \sum_{j_1,j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{4\pi(2l+1)} (C_{j_1 0 j_2 0}^{l0})^2 \\
&= \frac{1}{2\pi} \sum_{j_1} C_{j_1}(2j_1+1) \sum_{j_2=j_1}^{\infty} C_{j_2} (C_{j_1 0 l 0}^{j_2 0})^2 \\
&\leq \frac{1}{2\pi} \sum_{j_1} C_{j_1}(2j_1+1) \Big\{ \sup_{j_2 \geq j_1, j_1+j_2>l} C_{j_2} \Big\} \sum_{j_2=0}^{\infty} (C_{j_1 0 l 0}^{j_2 0})^2 \leq K C_{l/2},
\end{aligned}$$

where we have used the relation $\frac{2j_2+1}{2l+1}(C_{j_1 0 j_2 0}^{l0})^2 = (C_{j_1 0 l 0}^{j_2 0})^2$, as well as

$$\sup_{j_2 \geq j_1, j_1+j_2>l} C_{j_2} \leq K C_{l/2}, \quad \text{and} \quad \sum_{l=|l_2-l_1|}^{l_2+l_1} (C_{l_1 0 l_2 0}^{l0})^2 \equiv 1.$$

For the numerator of (39), one has that it is greater than

$$\begin{aligned}
&\sum_{j_1,j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{(4\pi(2l+1))^2} \left| \sum_{l_1} C_{l_1}(2l_1+1) C_{l_1 0 j_1 0}^{l0} C_{l_1 0 j_1 0}^{l0} C_{l_1 0 j_2 0}^{l0} C_{l_1 0 j_2 0}^{l0} \right|^2 \\
&\geq \sum_{j_1,j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{(4\pi(2l+1))^2} |5C_2 \{C_{20 j_1 0}^{l0} C_{20 j_2 0}^{l0}\}^2|^2 \\
&\geq C_l^2 \frac{1}{(4\pi)^2} |5C_2 \{C_{20 l 0}^{l0}\}^2|^2 \geq K C_l^2.
\end{aligned}$$

The left-hand side of condition (39) is then bounded below by $\lim_{l\to\infty}(K_1 C_l^2)/(K_2 C_{l/2}^2) \neq 0$ so that the central limit theorem (40) cannot hold.

Remark. In cosmology, for the simplest version of the so-called Sachs–Wolfe model, (6) holds and the decay of the angular power spectra of the underlying Gaussian field T is polynomial. Our results show that under these circumstances, the Fourier components $\widetilde{T}_l/[\mathrm{Var}\{\widetilde{T}_l\}]^{1/2}$ are also non-Gaussian asymptotically, so statistical procedures searching for non-Gaussianities at high-frequencies can be justified. Note that

$$\widetilde{T}_l(x) = T_l(x) + f_{\mathrm{NL}} T_l^{(2)}(x)$$

and it can be shown that for polynomially decaying angular power spectra, there exist positive constants $c_1, c_2 > 0$ such that $c_1 \leq \operatorname{Var}\{T_l(x)\}/\operatorname{Var}\{T_l^{(2)}(x)\} \leq c_2$ for all $l \geq 2$. Hence, the Gaussian and non-Gaussian parts have the same stochastic order of magnitude.

Acknowledgements

We are grateful to an Associate Editor and two referees for suggestions that greatly improved the presentation of the paper.

References

[1] Adler, R.J. and Taylor, J. (2007). *Random Fields and Geometry*. New York: Springer. MR2319516
[2] Baldi, P. and Marinucci, D. (2007). Some characterizations of the spherical harmonics coefficients for isotropic random fields. *Statist. Probab. Lett.* **77** 490–496. MR2344633
[3] Baldi, P., Marinucci, D. and Varadarajan, V.S. (2007). On the characterization of isotropic random fields on homogeneous spaces of compact groups. *Electron. Comm. Probab.* **12** 291–302. MR2342708
[4] Bartolo, N., Komatsu, E., Matarrese, S. and Riotto, A. (2004). Non-Gaussianity from inflation: Theory and observations. *Phys. Rep.* **402** 103–266. MR2106384
[5] Bloom, W.R. and Heyer, H. (1995). *The Harmonic Analysis of Probability Measures on Hypergroups*. Berlin: de Gruyter. MR1312826
[6] Cabella, P. and Marinucci, D. (2008). Statistical challenges in the analysis of Cosmic Microwave Background radiation. *Ann. Appl. Statist.* **3** 61–95.
[7] Diaconis, P. (1988). *Group Representations in Probability and Statistics*. *IMS Lecture Notes – Monograph Series* **11**. Hayward, CA: IMS. MR0964069
[8] Dodelson, S. (2003). *Modern Cosmology*. Academic Press.
[9] Efstathiou, G. (2004). Myths and truths concerning estimation of power spectra: The case for a hybrid estimator. *Monthly Notices of the Royal Astronomical Society* **349** 603–626.
[10] Faraut, J. (2006). *Analyse sur le groupes de Lie*. Calvage et Mounet.
[11] Guivarc'h, Y., Keane, M. and Roynette, B. (1977). *Marches Aléatoires sur les Groupes de Lie*. *Lecture Notes in Math.* **624**. Berlin: Springer. MR0517359
[12] Hu, Y. and Nualart, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.* **33** 948–983. MR2135309
[13] Jansson, S. (1997). *Gaussian Hilbert Spaces.* Cambridge Univ. Press. MR1474726
[14] Leonenko, N. (1999). *Limit Theorems for Random Fields with Singular Spectrum*. Dordrecht: Kluwer. MR1687092
[15] Liboff, R.L. (1999). *Introductory Quantum Mechanics*. Addison-Wesley.
[16] Marinucci, D. (2006). High-resolution asymptotics for the angular bispectrum of spherical random fields. *Ann. Statist.* **34** 1–41. MR2275233
[17] Marinucci, D. (2008). A central limit theorem and higher order results for the angular bispectrum. *Probab. Theory Related Fields* **141** 389–409. MR2391159
[18] Marinucci, D. and Piccioni, M. (2004). The empirical process on Gaussian spherical harmonics. *Ann. Statist.* **32** 1261–1288. MR2065205
[19] Marinucci, D. and Peccati, G. (2008). High-frequency asymptotics for subordinated stationary fields on an Abelian compact group. *Stochastic Process. Appl.* **118** 585–613. MR2394764
[20] Marinucci, D. and Peccati, G. (2009). Group representations and angular polyspectra. *J. Multivariate Anal.* To appear.

[21] Marinucci, D. and Peccati, G. (2007). Group representations and high-resolution central limit theorems for subordinated spherical random fields. Preprint. Available at arXiv0706.2851v3.

[22] Nourdin, I. and Peccati, G. (2009). Stein's method on Wiener chaos. *Probab. Theory Related Fields* **145** 75–118. MR2520122

[23] Nourdin, I., Peccati, G. and Réveillac, A. (2009). Multidimensional normal approximation via Stein's method and Malliavin calculus. *Ann. Inst. H. Poincaré Probab. Statist.* To appear.

[24] Nualart, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. Berlin: Springer. MR2200233

[25] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863

[26] Peccati, G. (2007). Gaussian approximations of multiple integrals. *Electron. Comm. Probab.* **12** 350–364. MR2350573

[27] Peccati, G. and Pycke, J.-R. (2005). Decompositions of stochastic processes based on irreducible group representations. *Theory Probab. Appl.* To appear.

[28] Peccati, G. and Taqqu, M.S. (2008). Stable convergence of multiple Wiener–Itô integrals. *J. Theoret. Probab.* **21** 527–570. MR2425357

[29] Peccati, G. and Tudor, C.A. (2005). Gaussian limits for vector-valued multiple stochastic integrals. In *Séminaire de Probabilités XXXVIII* 247–262. Berlin: Springer. MR2126978

[30] Pycke, J.-R. (2007). A decomposition for invariant tests of uniformity on the sphere. *Proc. Amer. Math. Soc.* To appear. MR2317977

[31] Stein, E.M. and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces* **32**. Princeton, NJ: Princeton Univ. Press. MR0304972

[32] Surgailis, D. (2003). CLTs for polynomials of linear sequences: Diagram formula with illustrations. In *Theory and Applications of Long Range Dependence* 111–128. Boston, MA: Birkhäuser. MR1956046

[33] Varadarajan, V.S. (1999). *An Introduction to Harmonic Analysis on Semisimple Lie Groups*. Cambridge Univ. Press. Corrected reprint of the 1989 original. MR1725738

[34] Varshalovich, D.A., Moskalev, A.N. and Khersonskii, V.K. (1988). *Quantum Theory of Angular Momentum*. Teaneck, NJ: World Scientific Press. MR1022665

[35] Vilenkin, N.J. and Klimyk, A.U. (1991). *Representation of Lie Groups and Special Functions*. Dordrecht: Kluwer. MR1143783

Received September 2008 and revised July 2009