Second order Poincaré inequalities and CLTs on Wiener space

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Abstract: We prove infinite-dimensional second order Poincaré inequalities on Wiener space, thus closing a circle of ideas linking limit theorems for functionals of Gaussian fields, Stein's method and Malliavin calculus. We provide two applications: (i) to a new "second order" characterization of CLTs on a fixed Wiener chaos, and (ii) to linear functionals of Gaussian-subordinated fields.

Key words: central limit theorems; isonormal Gaussian processes; linear functionals; multiple integrals; second order Poincaré inequalities; Stein's method; Wiener chaos

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1 Introduction

Let $N \sim \mathscr{N}(0,1)$ be a standard Gaussian random variable. In its most basic formulation, the *Gaussian Poincaré inequality* states that, for every differentiable function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbf{Var} f(N) \leqslant E f'(N)^2, \tag{1.1}$$

with equality if and only if f is affine. The estimate (1.1) is a fundamental tool of stochastic analysis: it implies that, if the random variable $f'(N)$ has a small $L^2(\Omega)$ norm, then $f(N)$ has necessarily small fluctuations. Relation (1.1) has been first proved by Nash in [14], and then rediscovered by Chernoff in [9] (both proofs use Hermite polynomials). The Gaussian Poincaré inequality admits extensions in several directions, encompassing both the case of smooth functionals of multi-dimensional (and possibly infinite-dimensional) Gaussian fields, and of non-Gaussian probability distributions – see e.g. Bakry *et al.* [1], Bobkov [2], Cacoullos *et al.*, Chen [5, 6, 7], Houdré and Perez-Abreu [10], and the references therein. In particular, the results proved in [10] (which make use of the Malliavin calculus) allow to recover the following infinite-dimensional version of (1.1). Let X be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$ (see Section 2), and let $F \in \mathbb{D}^{1,2}$ be a Malliavin-differentiable functional of X. Then, the Malliavin derivative of F, denoted by DF, is a random element with values in $\mathfrak{H}$, and it holds that

$$\mathbf{Var} F \leqslant E\|DF\|^2_{\mathfrak{H}}, \tag{1.2}$$

with equality if and only if F has the form of a constant plus an element of the first Wiener chaos of X. In Proposition 3.1 below we shall prove a more general version of (1.2), involving central moments of arbitrary even orders and based on the techniques developed in [16]. Note

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that (1.2) contains as a special case the well-known fact that, if $F = f(X_1, ..., X_d)$ is a smooth function of i.i.d. $\mathscr{N}(0,1)$ random variables $X_1, ..., X_d$, then

$$\mathbf{Var} F \leqslant E\|\nabla f(X_1, ..., X_d)\|^2_{\mathbb{R}^d}, \tag{1.3}$$

where ∇f is the gradient of f.

Now suppose that the random variable $F = f(X_1, ..., X_d)$ (where the $X_1, ..., X_d$ are again i.i.d. $\mathscr{N}(0,1)$) is such that f is twice differentiable. In the recent paper [4], Chatterjee has pointed out that if one focuses also on the $d \times d$ Hessian matrix $\mathrm{Hess}\, f$, and not only on ∇f, then one can state an inequality assessing the *total variation distance* (see Section 3.2, (3.21)) between the law of F and the law of a Gaussian random variable with matching mean and variance. The precise result goes as follows (see [4, Theorem 2.2]). Let $E(F) = \mu$, $\mathbf{Var} F = \sigma^2 > 0$, $Z \sim \mathscr{N}(\mu, \sigma)$, and denote by $d_{TV}(F, Z)$ the total variation distance between the laws of F and Z, see (3.21). Then

$$d_{TV}(F, Z) \leqslant \frac{2\sqrt{5}}{\sigma^2} E[\|\mathrm{Hess}\, f(X_1, ..., X_d)\|^4_{op}]^{\frac{1}{4}} \times E[\|\nabla f(X_1, ..., X_d)\|^4_{\mathbb{R}^d}]^{\frac{1}{4}}, \tag{1.4}$$

where $\|\mathrm{Hess}\, f(X_1, ..., X_d)\|_{op}$ is the operator norm of the (random) matrix $\mathrm{Hess} f(X_1, ..., X_d)$. A relation such as (1.4) is called a *second order Poincaré inequality*: it is proved in [4] by combining (1.3) with an adequate version of *Stein's method* (see e.g. [8, 24]).

In [16, Remark 3.6] the first two authors of the present paper pointed out that the finite-dimensional Stein-type inequalities leading to Relation (1.4) are special instances of much more general estimates, which can be obtained by combining Stein's method and Malliavin calculus on an infinite-dimensional Gaussian space. It is therefore natural to ask whether the results of [16] can be used in order to obtain a general version of (1.4), involving a "distance to Gaussian" for smooth functionals of arbitrary infinite-dimensional Gaussian fields. We shall show that the answer is positive. Indeed, one of the principal achievements of this paper is the proof of the following statement (d_W denotes the Wasserstein distance, see (3.22)):

Theorem 1.1 (Second order infinite-dimensional Poincaré inequality) *Let X be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$, and let $F \in \mathbb{D}^{2,4}$. Assume that $E(F) = \mu$ and $\mathrm{Var}(F) = \sigma^2 > 0$. Let $Z \sim \mathscr{N}(\mu, \sigma^2)$. Then*

$$d_W(F, Z) \leqslant \frac{\sqrt{10}}{2\sigma} E\left[\|D^2F\|^4_{op}\right]^{\frac{1}{4}} \times E\left[\|DF\|^4_{\mathfrak{H}}\right]^{\frac{1}{4}}. \tag{1.5}$$

If, in addition, the law of F is absolutely continuous with respect to the Lebesgue measure, then

$$d_{TV}(F, Z) \leqslant \frac{\sqrt{10}}{\sigma^2} E\left[\|D^2F\|^4_{op}\right]^{\frac{1}{4}} \times E\left[\|DF\|^4_{\mathfrak{H}}\right]^{\frac{1}{4}}. \tag{1.6}$$

The class $\mathbb{D}^{2,4}$ of twice Malliavin-differentiable functionals is formally defined in Section 2; note that D^2F is a random element with values in $\mathfrak{H}^{\odot 2}$ (the symmetric tensor product of $\mathfrak{H}$ with itself) and that we used $\left\|D^2F\right\|_{op}$ to indicate the operator norm (or, equivalently, the spectral radius) of the random Hilbert-Schmidt operator $f \mapsto \langle f, D^2F\rangle_{\mathfrak{H}}$. The proof of Theorem 1.1 is detailed in Section 4.1. As discussed in Section 4.2, a crucial point is that Theorem 1.1 leads to further (and very useful) inequalities, which we name *random contraction* *inequalities*. These estimates involve a "contracted version" of the second derivative D^2F, and will lead (see Section 5) to the proof of new necessary and sufficient conditions which ensure that a sequence of random variables belonging to fixed Wiener chaos converges in law to a standard Gaussian random variable. This result generalizes and unifies the findings contained in [16, 20, 21, 23], and virtually closes a very fruitful circle of recent ideas linking Malliavin calculus, Stein's method and central limit theorems (CLTs) on Wiener space (see also [15]). The role of contraction inequalities is further explored in Section 6, where we study CLTs for linear functionals of Gaussian subordinated fields.

The rest of the paper is organized as follows. In Section 2 we recall some preliminary results involving Malliavin operators. Section 3 concerns Poincaré type inequalities and bounds on distances between probabilities. Section 4 deals with the proof of Theorem 1.1, as well as with "random contraction inequalities". Section 5 and Section 6 focus, respectively, on CLTs on Wiener chaos and on CLTs for Gaussian subordinated fields. Finally, Section 7 is devoted to a version of (1.5) for random variables of the type $F = (F_1, \ldots, F_d)$.

2 Preliminaries

We shall now present the basic elements of Gaussian analysis and Malliavin calculus that are used in this paper. The reader is referred to the two monographs by Malliavin [12] and Nualart [19] for any unexplained definition or result.

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geqslant 1$ let $\mathfrak{H}^{\otimes q}$ be the qth tensor product of $\mathfrak{H}$ and denote by $\mathfrak{H}^{\odot q}$ the associated qth symmetric tensor product. We write $X = \{X(h), h \in \mathfrak{H}\}$ to indicate an isonormal Gaussian process over $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. This means that X is a centered Gaussian family, whose covariance is given in terms of the inner product of $\mathfrak{H}$ by $E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$. We also assume that $\mathcal{F}$ is generated by X.

For every $q \geqslant 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables of the type $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\big(e^{-\frac{x^2}{2}}\big)$. We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $q \geqslant 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ equipped with the modified norm $\sqrt{q!}\, \|\cdot\|_{\mathfrak{H}^{\otimes q}}$ and the qth Wiener chaos $\mathcal{H}_q$. For $q = 0$ we write $I_0(c) = c$, $c \in \mathbb{R}$.

It is well-known (Wiener chaos expansion) that $L^2(\Omega, \mathcal{F}, P)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. Therefore, any square integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ admits the following chaotic expansion

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.7}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geqslant 1$, are uniquely determined by F. For every $q \geqslant 0$ we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega, \mathcal{F}, P)$ is as in (2.7), then $J_q F = I_q(f_q)$ for every $q \geqslant 0$.

Let $\{e_k,\, k \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$

defined by

$$f \otimes_r g = \sum_{i_1,\dots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.8}$$

Notice that $f \otimes_r g$ is not necessarily symmetric: we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. In the particular case where $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and non-atomic measure, one has that $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener-Itô integral of order q of f with respect to X introduced by Itô in [11]. In this case, (2.8) can be written as

$$\begin{aligned}(f \otimes_r g)(t_1, \ldots, t_{p+q-2r}) \;&=\; \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r) \\ &\quad \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) d\mu(s_1) \ldots d\mu(s_r).\end{aligned}$$

It can then be also shown that the following *multiplication formula* holds: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g). \tag{2.9}$$

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X. Let $\mathcal{S}$ be the set of all cylindrical random variables of the form

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right), \tag{2.10}$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support and $\phi_i \in \mathfrak{H}$. The *Malliavin derivative* of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF \;=\; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \left(X(\phi_1), \ldots, X(\phi_n)\right) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the mth derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geqslant 2$. For $m \geqslant 1$ and $p \geqslant 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\| \cdot \|_{m,p}$, defined by the relation

$$\|F\|^p_{m,p} \;=\; E\left[|F|^p\right] + \sum_{i=1}^{m} E\left(\|D^i F\|^p_{\mathfrak{H}^{\otimes i}} \right).$$

The Malliavin derivative D verifies the following *chain rule*. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\,\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

Note also that a random variable F as in (2.7) is in $\mathbb{D}^{1,2}$ if and only if $\sum_{q=1}^{\infty} q\|J_qF\|^2_{L^2(\Omega)} < \infty$ and, in this case, $E\left(\|DF\|^2_{\mathfrak{H}}\right) = \sum_{q=1}^{\infty} q\|J_qF\|^2_{L^2(\Omega)}$. If $\mathfrak{H} = \mathrm{L}^2(A, \mathcal{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (2.7) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_xF = \sum_{q=1}^{\infty} qI_{q-1}\left(f_q(\cdot, x)\right), \quad x \in A. \tag{2.11}$$

We denote by δ the adjoint of the operator D, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, noted $\mathrm{Dom}\delta$, if and only if it verifies $|E\langle DF, u\rangle_{\mathfrak{H}}| \leqslant c_u \|F\|_{L^2(\Omega)}$ for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called *integration by parts formula*)

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathfrak{H}}, \tag{2.12}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The divergence operator δ is also called the *Skorohod integral* because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in [26].

The family $(T_t,\, t \geqslant 0)$ of operators is defined through the projection operators J_q as

$$T_t = \sum_{q=0}^{\infty} e^{-qt} J_q, \tag{2.13}$$

and is called the *Ornstein-Uhlenbeck semigroup*. Assume that the process X', which stands for an independent copy of X, is such that X and X' are defined on the product probability space $(\Omega\times\Omega', \mathscr{F}\otimes\mathscr{F}', P\times P')$. Given a random variable $Z \in \mathbb{D}^{1,2}$, we can regard its Malliavin derivative $DZ = DZ(X)$ as a measurable mapping from $\mathbb{R}^{\mathfrak{H}}$ to $\mathbb{R}$, determined $P \circ X^{-1}$-almost surely. Then, for any $t \geqslant 0$, we have the so-called *Mehler's formula* (see e.g. [12, Section 8.5, Ch. I] or [19, formula (1.54)]):

$$T_t(DZ) = E'\big(DZ(e^{-t}X + \sqrt{1-e^{-2t}}X')\big), \tag{2.14}$$

where E' denotes the mathematical expectation with respect to the probability P'.

The operator L is defined as $L = \sum_{q=0}^{\infty} -qJ_q$, and it can be proven to be the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geqslant 0}$. The domain of L is

$$\mathrm{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_qF\|^2_{L^2(\Omega)} < \infty\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators D, δ and L (see e.g. [19, Proposition 1.4.3]). A random variable F belongs to $\mathbb{D}^{2,2}$ if and only if $F \in \mathrm{Dom}\,(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$), and in this case

$$\delta DF = -LF. \tag{2.15}$$

For any $F \in L^2(\Omega)$, we define $L^{-1}F = \sum_{q=0}^{\infty} -\frac{1}{q}J_q(F)$. The operator L^{-1} is called the *pseudo-inverse* of L. For any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \mathrm{Dom}L$, and

$$LL^{-1}F = F - E(F). \tag{2.16}$$

We end the preliminaries by noting that Shigekawa [25] has developed an alternative framework which avoids the inverse of the Ornstein-Uhlenbeck operator L. This framework could provide an alternative derivation of the integration by parts formula (2.30) in [16] which leads to Theorem 3.3.

3 Poincaré-type inequalities and bounds on distances

3.1 Poincaré inequalities

The following statement contains, among others, a general version (3.19) of the infinite-dimensional Poincaré inequality (1.2).

Proposition 3.1 *Fix $p \geqslant 2$ and let $F \in \mathbb{D}^{1,p}$ be such that $E(F) = 0$.*

1. *The following estimate holds:*

$$E \left\| DL^{-1}F \right\|_{\mathfrak{H}}^{p} \leqslant E \left\| DF \right\|_{\mathfrak{H}}^{p}. \tag{3.17}$$

2. *If in addition $F \in \mathbb{D}^{2,p}$, then*

$$E \left\| D^2 L^{-1}F \right\|_{op}^{p} \leqslant \frac{1}{2^p} E \left\| D^2 F \right\|_{op}^{p}, \tag{3.18}$$

where $\left\| D^2 F \right\|_{op}$ indicates the operator norm of the random Hilbert-Schmidt operator

$$\mathfrak{H} \to \mathfrak{H} : f \mapsto \left\langle f, D^2 F \right\rangle_{\mathfrak{H}}.$$

(and similarly for $\|D^2 L^{-1} F\|_{op}$).

3. *If p is an even integer, then*

$$E\big[F^p\big] \leqslant (p-1)^{p/2}\, E\big[\|DF\|_{\mathfrak{H}}^p\big]. \tag{3.19}$$

Proof. By virtue of standard arguments, we may assume throughout the proof that $\mathfrak{H} = \mathrm{L}^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and non-atomic measure.

1. In what follows, we will write X' to indicate an independent copy of X. Let $F \in L^2(\Omega)$ have the expansion (2.7). Then, from (2.11),

$$-D_x L^{-1} F = \sum_{q \geqslant 1} I_{q-1}\left(f_q\left(x, \cdot\right)\right),.$$

By combining this relation with Mehler's formula (2.14), one deduces that

$$\begin{aligned} -D_x L^{-1} F &= \int_0^\infty e^{-t} T_t D_x F(X) dt = \int_0^\infty e^{-t}\, E_{X'}\, D_x F\left(e^{-t} X + \sqrt{1-e^{-2t}} X'\right) dt \\ &= E_Y\, E_{X'}\, D_x F\left(e^{-Y} X + \sqrt{1-e^{-2Y}} X'\right) \end{aligned}$$

where $Y \sim \mathcal{E}(1)$ is an independent exponential random variable of mean 1, and $\{T_t : t \geqslant 0\}$ is the Ornstein-Uhlenbeck semigroup (2.13). Note that we regard every random variable D_xF as an application $\mathbb{R}^{\mathfrak{H}} \to \mathbb{R}$ and that (for a generic random variable G) we write E_G to indicate that we take the expectation with respect to G. It follows that

$$\begin{aligned} E\left\|DL^{-1}F\right\|_{\mathfrak{H}}^p &= E_X \left\|E_Y\, E_{X'}\, DF\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{\mathfrak{H}}^p \\ &\leqslant E_X\, E_Y\, E_{X'} \left\|DF\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{\mathfrak{H}}^p \\ &= E_Y\, E_X\, E_{X'} \left\|DF\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{\mathfrak{H}}^p \\ &= E_Y\, E_X \left\|DF(X)\right\|_{\mathfrak{H}}^p = E_X \left\|DF(X)\right\|_{\mathfrak{H}}^p = E\left\|DF\right\|_{\mathfrak{H}}^p \end{aligned}$$

where we used the fact that $e^{-t}X' + \sqrt{1-e^{-2t}}X \overset{law}{=} X$ for any $t \geqslant 0$.

2. From the relation

$$-D^2_{xy}L^{-1}F = \sum_{q\geqslant 2}(q-1)\, I_{q-2}\left(f_q(x,y,\cdot)\right)$$

one deduces analogously that

$$\begin{aligned} -D^2_{xy}L^{-1}F &= \int_0^\infty e^{-2t}T_t D^2_{xy}F\,dt \\ &= \int_0^\infty e^{-2t}\, E_{X'}\, D^2_{xy}F\left(e^{-t}X + \sqrt{1-e^{-2t}}X'\right)dt \\ &= \frac{1}{2} E_Y\, E_{X'}\, D^2_{xy}F\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right) \end{aligned}$$

where $Y \sim \mathcal{E}(2)$ is an independent exponential random variable of mean $\frac{1}{2}$. Thus

$$\begin{aligned} E\left\|D^2L^{-1}F\right\|_{op}^p &= \frac{1}{2^p} E_X \left\|E_Y\, E_{X'}\, D^2F\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{op}^p \\ &\leqslant \frac{1}{2^p} E_X\, E_Y\, E_{X'} \left\|D^2F\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{op}^p \\ &= \frac{1}{2^p} E_Y\, E_X\, E_{X'} \left\|D^2F\left(e^{-Y}X + \sqrt{1-e^{-2Y}}X'\right)\right\|_{op}^p \\ &= \frac{1}{2^p} E_Y\, E_X \left\|D^2F(X)\right\|_{op}^p = \frac{1}{2^p} E_X \left\|D^2F(X)\right\|_{op}^p = \frac{1}{2^p} E\left\|D^2F\right\|_{op}^p. \end{aligned}$$

3. Writing $p = 2k$, we have

$$\begin{aligned} E\big[F^{2k}\big] &= E\big[LL^{-1}F \times F^{2k-1}\big] = -E\big[\delta DL^{-1}F \times F^{2k-1}\big] \\ &= (2k-1)E\big[\langle DF, -DL^{-1}F\rangle F^{2k-2}\big] \\ &\leqslant (2k-1)\left(E\big[\left|\langle DF, -DL^{-1}F\rangle\right|^k\big]\right)^{\frac{1}{k}} \left(E\big[F^{2k}\big]\right)^{1-\frac{1}{k}} \quad \text{by Hölder's inequality,} \end{aligned}$$

from which we infer that

$$\begin{aligned} E\big[F^{2k}\big] &\leqslant (2k-1)^k E\big[\left|\langle DF, -DL^{-1}F\rangle\right|^k\big] \leqslant (2k-1)^k E\big[\|DF\|_{\mathfrak{H}}^k \|DL^{-1}F\|_{\mathfrak{H}}^k\big] \\ &\leqslant (2k-1)^k \sqrt{E\big[\|DF\|_{\mathfrak{H}}^{2k}\big]}\sqrt{E\big[\|DL^{-1}F\|_{\mathfrak{H}}^{2k}\big]} \leqslant (2k-1)^k E\big[\|DF\|_{\mathfrak{H}}^{2k}\big]. \end{aligned}$$

■

We also state the following technical result which will be needed in Section 4. The proof is standard and omitted.

Lemma 3.2 *Let F and G be two elements of $\mathbb{D}^{2,4}$. Then, the two random elements $\langle D^2F, DG\rangle_{\mathfrak{H}}$ and $\langle DF, D^2G\rangle_{\mathfrak{H}}$ belong to $L^2(\Omega, \mathfrak{H})$. Moreover, $\langle DF, DG\rangle_{\mathfrak{H}} \in \mathbb{D}^{1,2}$ and*

$$D\langle DF, DG\rangle_{\mathfrak{H}} = \langle D^2F, DG\rangle_{\mathfrak{H}} + \langle DF, D^2G\rangle_{\mathfrak{H}}. \tag{3.20}$$

3.2 Bounds on the total variation and Wasserstein distances

Let U, Z be two generic real-valued random variables. We recall that the *total variation distance* between the law of U and the law of Z is defined as

$$d_{TV}(U, Z) = \sup_A |P(U \in A) - P(Z \in A)|, \tag{3.21}$$

where the supremum is taken over all Borel subsets A of $\mathbb{R}$. For two random vectors U and Z with values in $\mathbb{R}^d$, $d \geqslant 1$, the *Wasserstein distance* between the law of U and the law of Z is

$$d_W(U, Z) = \sup_{f:\|f\|_{Lip}\leqslant 1} |E[f(U)] - E[f(Z)]|, \tag{3.22}$$

where $\|\cdot\|_{Lip}$ stands for the usual Lipschitz seminorm. We stress that the topologies induced by d_{TV} and d_W, on the class of all probability measures on $\mathbb{R}$, are strictly stronger than the topology of weak convergence. The following statement has been proved in [16, Theorem 3.1] by means of Stein's method.

Theorem 3.3 *Suppose that $Z \sim \mathscr{N}(0,1)$. Let $F \in \mathbb{D}^{1,2}$ and $E(F) = 0$. Then,*

$$d_W(F, Z) \leqslant E|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leqslant E[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]^{1/2}. \tag{3.23}$$

If moreover F has an absolutely continuous distribution, then

$$d_{TV}(F, Z) \leqslant 2E|1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}| \leqslant 2E[(1 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})^2]^{1/2}. \tag{3.24}$$

4 Proof of Theorem 1.1 and contraction inequalities

4.1 Proof of Theorem 1.1

We can assume, without loss of generality, that $\mu = 0$ and $\sigma^2 = 1$. Set $W = \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}$. First, note that W has mean 1, as

$$E(W) = E[\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}] = -E[F \times \delta DL^{-1}F] = E[F \times LL^{-1}F] = E[F^2] = 1.$$

By Theorem 3.3 it follows that we only need to bound $\sqrt{\mathrm{Var}\,(W)}$. By (1.2), we have $\mathrm{Var}\,(W) \leqslant E\,\|DW\|^2_{\mathfrak{H}}$. So, our problem is now to evaluate $\|DW\|^2_{\mathfrak{H}}$. By using Lemma 3.2 in the special case $G = -L^{-1}F$, we deduce that

$$\begin{aligned}\|DW\|^2_{\mathfrak{H}} &= \left\|\langle D^2F, -DL^{-1}F\rangle_{\mathfrak{H}} + \langle DF, -D^2L^{-1}F\rangle_{\mathfrak{H}}\right\|^2_{\mathfrak{H}} \\ &\leqslant 2\left\|\langle D^2F, -DL^{-1}F\rangle_{\mathfrak{H}}\right\|^2_{\mathfrak{H}} + 2\left\|\langle DF, -D^2L^{-1}F\rangle_{\mathfrak{H}}\right\|^2_{\mathfrak{H}}.\end{aligned}$$

We evaluate the last two terms separately. We have

$$\left\|\langle D^2F, -DL^{-1}F\rangle_{\mathfrak{H}}\right\|_{\mathfrak{H}}^2 \leqslant \left\|D^2F\right\|_{op}^2 \left\|DL^{-1}F\right\|_{\mathfrak{H}}^2$$

and

$$\left\|\langle DF, -D^2L^{-1}F\rangle_{\mathfrak{H}}\right\|_{\mathfrak{H}}^2 \leqslant \|DF\|_{\mathfrak{H}}^2 \left\|D^2L^{-1}F\right\|_{op}^2.$$

It follows that

$$\begin{aligned} E\|DW\|_{\mathfrak{H}}^2 &\leqslant 2\,E\left[\left\|DL^{-1}F\right\|_{\mathfrak{H}}^2 \left\|D^2F\right\|_{op}^2 + \|DF\|_{\mathfrak{H}}^2 \left\|D^2L^{-1}F\right\|_{op}^2\right] \\ &\leqslant 2\left(E\left\|DL^{-1}F\right\|_{\mathfrak{H}}^4 \times E\left\|D^2F\right\|_{op}^4\right)^{1/2} + 2\left(E\|DF\|_{\mathfrak{H}}^4 \times E\left\|D^2L^{-1}F\right\|_{op}^4\right)^{1/2}. \end{aligned}$$

The desired conclusion follows by using, respectively, (3.17) and (3.18) with $p = 4$. ■

4.2 Random contraction inequalities

When the quantity $E\left\|D^2F\right\|_{op}^4$ appearing in (1.5)-(1.6) is analytically too hard to assess, one can resort to the following inequality, which we name *random contraction inequality*:

Proposition 4.1 (Random contraction inequality). *Let $F \in \mathbb{D}^{2,4}$. Then*

$$\left\|D^2F\right\|_{op}^4 \leqslant \left\|D^2F \otimes_1 D^2F\right\|_{\mathfrak{H}^{\otimes 2}}^2, \tag{4.25}$$

where $D^2F \otimes_1 D^2F$ is the random element of $\mathfrak{H}^{\odot 2}$ obtained as the contraction of the symmetric random tensor D^2F, see (2.8).

Proof. We can associate with the symmetric random elements $D^2F \in \mathfrak{H}^{\odot 2}$ the random Hilbert-Schmidt operator $f \mapsto \left\langle f, D^2F\right\rangle_{\mathfrak{H}^{\otimes 2}}$. Denote by $\{\gamma_j\}_{j\geqslant 1}$ the sequence of its (random) eigenvalues. One has that

$$\left\|D^2F\right\|_{op}^4 = \max_{j\geqslant 1}|\gamma_j|^4 \leqslant \sum_{j\geqslant 1}|\gamma_j|^4 = \left\|D^2F \otimes_1 D^2F\right\|_{\mathfrak{H}^{\otimes 2}}^2,$$

and the conclusion follows. ■

The following result is an immediate corollary of Theorem 1.1 and Proposition 4.1.

Corollary 4.2 *Let $F \in \mathbb{D}^{2,4}$ with $E(F) = \mu$ and $\mathrm{Var}(F) = \sigma^2$. Assume that $Z \sim \mathscr{N}(\mu, \sigma^2)$. Then*

$$d_W(F, Z) \leqslant \frac{\sqrt{10}}{2\sigma} E\left[\|D^2F \otimes_1 D^2F\|_{\mathfrak{H}^{\otimes 2}}^2\right]^{\frac{1}{4}} \times E\left[\|DF\|_{\mathfrak{H}}^4\right]^{\frac{1}{4}}. \tag{4.26}$$

If, in addition, the law of F is absolutely continuous with respect to the Lebesgue measure, then

$$d_{TV}(F, Z) \leqslant \frac{\sqrt{10}}{\sigma^2} E\left[\|D^2F \otimes_1 D^2F\|_{\mathfrak{H}^{\otimes 2}}^2\right]^{\frac{1}{4}} \times E\left[\|DF\|_{\mathfrak{H}}^4\right]^{\frac{1}{4}}. \tag{4.27}$$

Remark 4.3 When used in the context of central limit theorems, inequality (4.27) does not give, in general, optimal rates. For instance, if $F_k = I_2(f_k)$ is a sequence of double integrals such that $E\left(F_k^2\right) \to 1$ and $F_k \xrightarrow{law} Z \sim \mathscr{N}(0,1)$ as $k \to \infty$, then (4.27) implies that

$$d_{TV}(F_k, Z) \leqslant cst \times \|f_k \otimes_1 f_k\|_{\mathfrak{H}^{\otimes 2}}^{1/2} \to 0,$$

and the rate $\|f_k \otimes_1 f_k\|_{\mathfrak{H}^{\otimes 2}}^{1/2}$ is suboptimal (by a power of 1/2), see Proposition 3.2 in [16].

5 Characterization of CLTs on a fixed Wiener chaos

The following statement collects results proved in [21] (for the equivalences between (i), (ii) and (iii)) and [20] (for the equivalence with (iv)).

Theorem 5.1 *Fix $q \geqslant 2$, and let $F_k = I_q(f_k)$, $k \geqslant 1$, be a sequence of multiple Wiener-Itô integrals such that $E\left(F_k^2\right) \to 1$. As $k \to \infty$, the following four conditions are equivalent:*

(i) $F_k \overset{law}{\longrightarrow} Z \sim \mathscr{N}(0,1)$;

(ii) $E(F_k^4) \longrightarrow E(Z^4) = 3$;

(iii) $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \longrightarrow 0$ *for all* $r = 1, \ldots, q-1$;

(iv) $\|DF\|_{\mathfrak{H}}^2 \overset{L^2(\Omega)}{\longrightarrow} q$.

See Section 9 in [22] for a discussion of the combinatorial aspects of the implication (ii) $\to$ (i) in the statement of Theorem 5.1. The next theorem, which is a consequence of the main results of this paper, provides two new necessary and sufficient conditions for CLTs on a fixed Wiener chaos.

Theorem 5.2 *Fix $q \geqslant 2$, and let $F_k = I_q(f_k)$ be a sequence of multiple Wiener-Itô integrals such that $E\left(F_k^2\right) \to 1$. Then, the following three conditions are equivalent as $k \to \infty$:*

(i) $F_k \overset{law}{\longrightarrow} Z \sim \mathscr{N}(0,1)$;

(ii) $\left\|D^2F_k \otimes_1 D^2F_k\right\|_{\mathfrak{H}^{\otimes 2}} \overset{L^2(\Omega)}{\longrightarrow} 0$;

(iii) $\left\|D^2F_k\right\|_{op} \overset{L^4(\Omega)}{\longrightarrow} 0$.

Proof. Since $E\|DF_k\|_{\mathfrak{H}}^2 = qE(F_k^2) \to q$, and since the random variables $\|DF_k\|_{\mathfrak{H}}^2$ live inside a finite sum of Wiener chaoses (where all the $L^p(\Omega)$ norms are equivalent), we deduce that the sequence $E\|DF_k\|_{\mathfrak{H}}^4$, $k \geqslant 1$, is bounded. In view of (1.5) and (4.25), it is therefore enough to prove the implication (i) $\to$ (ii). Without loss of generality, we can assume that $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$ where $(A, \mathscr{A})$ is a measurable space and μ is a σ-finite measure with no atoms. Now observe that

$$D_{a,b}^2 F_k = q(q-1)I_{q-2}\big(f_k(\cdot, a, b)\big), \quad a, b \in A.$$

Hence, using the multiplication formula (2.9),

$$
\begin{aligned}
& D^2F_k \otimes_1 D^2F_k(a,b) \\
= \;& q^2(q-1)^2 \int_A I_{q-2}\big(f_k(\cdot,a,u)\big) I_{q-2}\big(f_k(\cdot,b,u)\big)\mu(du) \\
= \;& q^2(q-1)^2 \sum_{r=0}^{q-2} r! \binom{q-2}{r}^2 I_{2q-4-2r}\left(\int_A f_k(\cdot,a,u)\widetilde{\otimes}_r f_k(\cdot,b,u)\mu(du)\right) \\
= \;& q^2(q-1)^2 \sum_{r=0}^{q-2} r! \binom{q-2}{r}^2 I_{2q-4-2r}\left(f_k(\cdot,a)\widetilde{\otimes}_{r+1} f_k(\cdot,b)\right) \\
= \;& q^2(q-1)^2 \sum_{r=1}^{q-1} (r-1)! \binom{q-2}{r-1}^2 I_{2q-2-2r}\left(f_k(\cdot,a)\widetilde{\otimes}_{r} f_k(\cdot,b)\right).
\end{aligned}
$$

Using the orthogonality and isometry properties of the integrals I_q, we get

$$
E\left\|D^2F_k \otimes_1 D^2F_k\right\|^2_{\mathfrak{H}^{\otimes 2}} \leqslant q^4(q-1)^4 \sum_{r=1}^{q-1} (r-1)!^2 \binom{q-2}{r-1}^4 (2q-2-2r)! \|f_k \otimes_r f_k\|^2_{\mathfrak{H}^{\otimes(2q-2r)}}.
$$

The desired conclusion now follows since, according to Theorem 5.1, if (i) is verified then, necessarily, $\|f_k \otimes_r f_k\|_{\mathfrak{H}^{\otimes(2q-2r)}} \to 0$ for every $r = 1, ..., q-1$. ■

6 CLTs for linear functionals of Gaussian subordinated fields

We now provide an explicit application of the inequality (4.26). Let B denote a centered Gaussian process with stationary increments and such that $\int_{\mathbb{R}} |\rho(x)|dx < \infty$, where $\rho(u-v) := E\big[(B_{u+1}-B_u)(B_{v+1}-B_v)\big]$. Also, in order to avoid trivialities, assume that ρ is not identically zero.

The Gaussian space generated by B can be identified with an isonormal Gaussian process of the type $X = \{X(h),\, h \in \mathfrak{H}\}$, for $\mathfrak{H}$ defined as follows: (i) denote by $\mathscr{E}$ the set of all step functions on $\mathbb{R}$, (ii) define $\mathfrak{H}$ as the Hilbert space obtained by closing $\mathscr{E}$ with respect to the inner product $\langle \mathbf{1}_{[s,t]}, \mathbf{1}_{[u,v]}\rangle_{\mathfrak{H}} = \mathrm{Cov}(B_t - B_s, B_v - B_u)$. In particular, with such a notation, one has that $B_t - B_s = X(\mathbf{1}_{[s,t]})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function of class $\mathscr{C}^2$, and $Z \sim \mathscr{N}(0,1)$. We assume that f is not constant, that $E|f(Z)| < \infty$ and that $E|f''(Z)|^4 < \infty$. As a consequence of the generalized Poincaré inequality (3.19), we see that we also automatically have $E|f'(Z)|^4 < \infty$ and $E|f(Z)|^4 < \infty$.

Fix $a < b$ in $\mathbb{R}$ and, for any $T > 0$, consider

$$
F_T = \frac{1}{\sqrt{T}} \int_{aT}^{bT} \big(f(B_{u+1} - B_u) - E[f(Z)]\big)du.
$$

Theorem 6.1 *As $T \to \infty$,*

$$
d_W\left(\frac{F_T}{\sqrt{\mathrm{Var} F_T}}, Z\right) = O(T^{-1/4}). \tag{6.28}
$$

Remark 6.2 We believe that the rate in (6.28) is not optimal (it should be $O(T^{-1/2})$ instead), see also Remark 4.3.

Proof of Theorem 6.1. We have

$$DF_T = \frac{1}{\sqrt{T}} \int_{aT}^{bT} f'(B_{u+1} - B_u) \mathbf{1}_{[u,u+1]} du$$

and

$$D^2 F_T = \frac{1}{\sqrt{T}} \int_{aT}^{bT} f''(B_{u+1} - B_u) \mathbf{1}_{[u,u+1]}^{\otimes 2} du.$$

Hence

$$\|DF_T\|_{\mathfrak{H}}^2 = \frac{1}{T} \int_{[aT,bT]^2} f'(B_{u+1} - B_u) f'(B_{v+1} - B_v)\, \rho(u-v) dudv$$

so that

$$\begin{aligned}\|DF_T\|_{\mathfrak{H}}^4 &= \frac{1}{T^2} \int_{[aT,bT]^4} f'(B_{u+1} - B_u) f'(B_{v+1} - B_v) f'(B_{w+1} - B_w) \\ &\qquad \times f'(B_{z+1} - B_z) \rho(w-z) \rho(u-v) dudvdwdz.\end{aligned}$$

By applying Cauchy-Schwarz inequality twice, and by using the fact that $B_{u+1} - B_u \overset{law}{=} Z$, we get

$$\left| E\big(f'(B_{u+1} - B_u) f'(B_{v+1} - B_v) f'(B_{w+1} - B_w) f'(B_{z+1} - B_z)\big) \right| \leqslant E|f'(Z)|^4$$

so that

$$\begin{aligned}E\|DF_T\|_{\mathfrak{H}}^4 &\leqslant E|f'(Z)|^4 \left(\frac{1}{T} \int_{[aT,bT]^2} |\rho(u-v)| dudv \right)^2 \\ &\leqslant E|f'(Z)|^4 \left(\frac{1}{T} \int_{aT}^{bT} du \int_{\mathbb{R}} |\rho(x)| dx \right)^2 = O(1). \qquad (6.29)\end{aligned}$$

On the other hand, we have

$$D^2 F_T \otimes_1 D^2 F_T = \frac{1}{T} \int_{[aT,bT]^2} f''(B_{u+1} - B_u) f''(B_{v+1} - B_v) \rho(u-v) \mathbf{1}_{[u,u+1]} \otimes \mathbf{1}_{[v,v+1]} dudv.$$

Hence

$$\begin{aligned}& E\|D^2 F_T \otimes_1 D^2 F_T\|_{\mathfrak{H}^{\otimes 2}}^2 \\ &= \frac{1}{T^2} \int_{[aT,bT]^4} E\left(f''(B_{u+1} - B_u) f''(B_{v+1} - B_v) f''(B_{w+1} - B_w) f''(B_{z+1} - B_z) \right) \\ &\qquad \times \rho(u-v) \rho(w-z) \rho(u-w) \rho(z-v) dudvdwdz \\ &\leqslant E|f''(Z)|^4 \frac{1}{T^2} \int_{[aT,bT]^4} |\rho(u-v)||\rho(w-z)||\rho(u-w)||\rho(z-v)| dudvdwdz \\ &\leqslant E|f''(Z)|^4 \frac{b-a}{T} \int_{\mathbb{R}^3} |\rho(x)||\rho(y)||\rho(t)||\rho(x-y-t)| dxdydt = O(T^{-1}).\end{aligned}$$

By combining all these facts and (4.26), the desired conclusion follows. ■

Theorem 6.1 does not guarantee that $\lim_{T\to\infty} \mathrm{Var} F_T$ exists. The following proposition shows that the limit does indeed exist, at least when f is symmetric.

Proposition 6.3 *Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a symmetric real function of class $\mathscr{C}^2$. Then $\sigma^2 := \lim_{T\to\infty}\mathrm{Var}F_T$ exists in $(0,\infty)$. Moreover, as $T\to\infty$,*

$$F_T \overset{law}{\longrightarrow} Z \sim \mathscr{N}(0,\sigma^2). \tag{6.30}$$

Proof of Proposition 6.3. We expand f in terms of Hermite polynomials. Since f is symmetric, we can write

$$f(x) = E[f(Z)] + \sum_{q=1}^{\infty} c_{2q} H_{2q}(x), \quad x \in \mathbb{R},$$

where the real numbers c_{2q} are given by $(2q)!c_{2q} = E[f(Z)H_{2q}(Z)]$. Thus

$$\begin{aligned}
\mathrm{Var}F_T &= \frac{1}{T}\int_{[aT,bT]^2} \mathrm{Cov}\big(f(B_{u+1}-B_u), f(B_{v+1}-B_v)\big)dudv \\
&= \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \frac{1}{T}\int_{[aT,bT]^2} \rho^{2q}(v-u)dudv \\
&= \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \frac{1}{T}\int_{aT}^{bT} du \int_{aT-u}^{bT-u} dx \rho^{2q}(x) \\
&= \sum_{q=1}^{\infty} c_{2q}^2 (2q)! \int_a^b du \int_{-T(u-a)}^{T(b-u)} dx \rho^{2q}(x) \\
&\underset{T\nearrow\infty}{\longrightarrow} (b-a)\sum_{q=1}^{\infty} c_{2q}^2 (2q)! \int_{\mathbb{R}} \rho^{2q}(x)dx =: \sigma^2, \quad \text{by monotone convergence.}
\end{aligned}$$

Since f is not constant, there exists $q \geqslant 1$ such that $c_{2q} \neq 0$ so that $\sigma^2 > 0$ (recall that we assumed $\rho \not\equiv 0$). Moreover, we also have

$$\mathrm{Var}F_T \leqslant E\big[\|DF_T\|_{\mathfrak{H}}^2\big] \leqslant \sqrt{E\big[\|DF_T\|_{\mathfrak{H}}^4\big]} = O(1), \quad \text{see (6.29)},$$

so that $\sigma^2 < \infty$. The assertion now follows from Theorem 6.1. ∎

When B is a fractional Brownian motion with Hurst index $H < 1/2$, Theorem 6.1 applies because, in this case, it is easily checked that $\int_{\mathbb{R}} |\rho(x)|dx < \infty$. On the other hand, using the scaling property of B, observe that $F_{1/h} \overset{law}{=} \frac{1}{\sqrt{h}}\int_a^b \left[f\left(\frac{B_{x+h}-B_x}{h^H}\right) - E\big(f(Z)\big)\right]dx$ for all fixed $h>0$. Hence, since $E|B_t - B_s|^2 = \sigma^2(|t-s|)$ with $\sigma^2(r) = r^{2H}$ a concave function, the general Theorem 1.1 in [13] also applies, and this gives another proof of (6.30). We believe however that, even in this particular case, our proof is simpler (since not based on the rather technical method of moments). Moreover, note that [13] is not concerned with bounds on distance between the laws of $F_{1/h}/\sqrt{\mathrm{Var}F_{1/h}}$ and $Z \sim \mathscr{N}(0,1)$.

7 A multidimensional extension

Let V, Y be two random vectors with values in $\mathbb{R}^d$, $d \geqslant 2$. Recall that the Wasserstein distance between the laws of V and Y is defined in (3.22). The following statement, whose proof is based on the results obtained in [18], provides a multidimensional version of (1.5).

Theorem 7.1 *Fix $d \geqslant 2$, and let $C = \{C(i,j) : i,j = 1, \ldots, d\}$ be a $d \times d$ positive definite matrix. Suppose that $F = (F_1, \ldots, F_d)$ is a $\mathbb{R}^d$-valued random vector such that $E[F_i] = 0$ and $F_i \in \mathbb{D}^{2,4}$ for every $i = 1, \ldots, d$. Assume moreover that F has covariance matrix C. Then*

$$d_W\big(F, \mathscr{N}_d(0,C)\big) \leqslant \frac{3\sqrt{2}}{2} \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i=1}^{d} \big(E\|D^2 F_i\|_{op}^4\big)^{1/4} \times \sum_{j=1}^{d} \big(E\|DF_j\|_{\mathfrak{H}}^4\big)^{1/4},$$

where $\mathscr{N}_d(0,C)$ indicates a d-dimensional centered Gaussian vector, with covariance matrix equal to C.

Proof. In [18, Theorem 3.5] it is shown that

$$d_W\big(F, \mathscr{N}_d(0,C)\big) \leqslant \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sqrt{\sum_{i,j=1}^{d} E\big[(C(i,j) - \langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}})^2\big]}.$$

Since, using successively (2.12), (2.15) and (2.16), we have

$$E\big[\langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}}\big] = -E\big[F_i \times \delta DL^{-1}F_j\big] = E\big[F_i \times LL^{-1}F_j\big] = E[F_iF_j] = C(i,j),$$

we deduce, applying successively (1.2), (3.20), Cauchy-Schwarz inequality and Proposition 3.1,

$$\begin{aligned}
& d_W\big(F, \mathscr{N}_d(0,C)\big) \\
\leqslant\ & \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{d} \sqrt{\mathrm{Var}\big[\langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}}\big]} \\
\leqslant\ & \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{d} \sqrt{E\big[\|D\langle DF_i, -DL^{-1}F_j\rangle_{\mathfrak{H}}\|_{\mathfrak{H}}^2\big]} \\
\leqslant\ & \sqrt{2}\|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{d} \left(\sqrt{E\big[\|\langle D^2F_i, -DL^{-1}F_j\rangle_{\mathfrak{H}}\|_{\mathfrak{H}}^2\big]} + \sqrt{E\big[\|\langle DF_i, -D^2L^{-1}F_j\rangle_{\mathfrak{H}}\|_{\mathfrak{H}}^2\big]} \right) \\
\leqslant\ & \sqrt{2}\|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{d} \Big[\big(E\big[\|D^2F_i\|_{op}^4\big]\big)^{1/4} \big(E\big[\|DL^{-1}F_j\|_{\mathfrak{H}}^4\big]\big)^{1/4} \\
& \qquad + \big(E\big[\|DF_i\|_{\mathfrak{H}}^4\big]\big)^{1/4} \big(E\big[\|D^2L^{-1}F_j\|_{op}^4\big]\big)^{1/4} \Big] \\
\leqslant\ & \sqrt{2}\|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i,j=1}^{d} \Big[\big(E\big[\|D^2F_i\|_{op}^4\big]\big)^{1/4} \big(E\big[\|DF_j\|_{\mathfrak{H}}^4\big]\big)^{1/4} \\
& \qquad + \frac{1}{2} \big(E\big[\|DF_i\|_{\mathfrak{H}}^4\big]\big)^{1/4} \big(E\big[\|D^2F_j\|_{op}^4\big]\big)^{1/4} \Big] \\
=\ & \frac{3\sqrt{2}}{2} \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \sum_{i=1}^{d} \big(E\big[\|D^2F_i\|_{op}^4\big]\big)^{1/4} \times \sum_{j=1}^{d} \big(E\big[\|DF_j\|_{\mathfrak{H}}^4\big]\big)^{1/4}.
\end{aligned}$$

■

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References

[1] D. Bakry, F. Barthe, P. Cattiaux and A. Guillin (2008). A simple proof of the Poincaré inequality for a large class of probability measures including the log-concave case. *Elect. Comm. in Probab.* **13**, 60–66 (Electronic)

[2] S.G. Bobkov (1999). Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Probab.* **27**(4), 1903–1921.

[3] T. Cacoullos, V. Papahanasiou and S.A. Utev (1994). Variational inequalities with examples and an application to the central limit theorem. *Ann. Probab.* **22**(3), 1607–1618.

[4] S. Chatterjee (2009). Fluctuation of eigenvalues and second order Poincaré inequalities. *Probab. Theory Related Fields* **143**, 1–40.

[5] L.H.Y. Chen (1985). Poincaré-type inequalities via stochastic integrals. *Z. Wahrsch. verw. Gebiete* **69**, 251–277.

[6] L.H.Y. Chen (1987). Characterization of probability distributions by Poincaré-type inequalities. *Ann. Inst. H. Poincaré Probab. Statist.* **23**(1), 91–110.

[7] L.H.Y. Chen (1988). The central limit theorem and Poincaré-type inequalities. *Ann. Probab.* **16**(1), 300–304.

[8] L. Chen and Q.-M. Shao (2005). Stein's method for normal approximation. In: *An introduction to Stein's method*, 1–59. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **4**, Singapore Univ. Press, Singapore, 2005.

[9] H. Chernoff (1981). A note on an inequality involving the normal distribution. *Ann. Probab.* **9**(3), 533–535.

[10] C. Houdré and V. Pérez-Abreu (1995). Covariance identities and inequalities for functionals on Wiener and Poisson spaces. *Ann. Probab.* **23**, 400–419.

[11] K. Itô (1951). Multiple Wiener integral. *J. Math. Soc. Japan* **3**, 157–169.

[12] P. Malliavin (1997). *Stochastic Analysis*. Springer-Verlag, Berlin, Heidelberg, New York.

[13] M. Marcus and J. Rosen (2008). CLT for L^p moduli of continuity of Gaussian processes. *Stoch. Proc. Appl.* **118**, 1107–1135.

[14] J. Nash (1956). Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* **80**, 931–954.

[15] I. Nourdin and G. Peccati (2008). Non-central convergence of multiple integrals. *Ann. Probab.*, to appear.

[16] I. Nourdin and G. Peccati (2008). Stein's method on Wiener chaos. *Probab. Theory Rel. Fields*, to appear.

[17] I. Nourdin and G. Peccati (2008). Stein's method and exact Berry-Esséen bounds for functionals of Gaussian fields. Preprint.

[18] I. Nourdin, G. Peccati and A. Réveillac (2008). Multivariate normal approximation using Stein's method and Malliavin calculus. *Ann. Inst. H. Poincaré Probab. Statist.*, to appear.

[19] D. Nualart (2006). *The Malliavin calculus and related topics.* Springer-Verlag, Berlin, 2nd edition.

[20] D. Nualart and S. Ortiz-Latorre (2008). Central limit theorem for multiple stochastic integrals and Malliavin calculus. *Stoch. Proc. Appl.* **118** (4), 614–628.

[21] D. Nualart and G. Peccati (2005). Central limit theorems for sequence of multiple stochastic integrals. *Ann. Probab.* **33** (1), 177–193.

[22] G. Peccati and M.S. Taqqu (2008). Moments, cumulants and diagram formulae for non-linear functionals of random measures. Preprint available at `http://fr.arxiv.org/abs/0811.1726`.

[23] G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: *Séminaire de Probabilités XXXVIII* , 247–262. Lecture Notes in Math. **1857**, Springer-Verlag, Berlin.

[24] G. Reinert (2005). Three general approaches to Stein's method. In: *An introduction to Stein's method*, 183–221. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. **4**, Singapore Univ. Press, Singapore.

[25] I. Shigekawa (1986). De Rham-Hodge-Kodaira's decomposition on an abstract Wiener space. *J. Math. Kyoto Univ.* **26** (2), 191–202.

[26] A. V. Skorohod (1975). On a generalization of a stochastic integral. *Theory Probab. Appl.* **20**, 219–233.