Non-explosion of diffusion processes on manifolds with time-dependent metric

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Abstract

We study the problem of non-explosion of diffusion processes on a manifold with time-dependent Riemannian metric. In particular we obtain that Brownian motion cannot explode in finite time if the metric evolves under backwards Ricci flow. Our result makes it possible to remove the assumption of non-explosion in the pathwise contraction result established by Arnaudon, Coulibaly and Thalmaier (arXiv:0904.2762, to appear in Sém. Prob.).

As an important tool which is of independent interest we derive an Itô formula for the distance from a fixed reference point, generalizing a result of Kendall (Ann. Prob. 15 (1987), 1491–1500).

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1 Brownian motion with respect to time-changing Riemannian metrics

Let M be a d-dimensional differentiable manifold, $\pi : \mathcal{F}(M) \to M$ the frame bundle and $(g(t))_{t\in[0,T]}$ a family of Riemannian metrics on M depending smoothly on t such that $(M, g(t))$ is geodesically complete for all $t \in [0,T]$. Let $(e_i)_{i=1}^d$ be the standard basis of $\mathbb{R}^d$. For each $t \in [0,T]$ let $(H_i(t))_{i=1}^d$ be the associated $g(t)$-horizontal vector fields on $\mathcal{F}(M)$ (i.e. $H_i(t,u)$ is the $g(t)$-horizontal lift of ue_i), and let $(V_{\alpha,\beta})_{\alpha,\beta=1}^d$ be the canonical vertical vector fields. Let $(W_t)_{t\geq 0}$ be a standard $\mathbb{R}^d$-valued Brownian motion. In this situation Arnaudon, Coulibaly and Thalmaier [1, 5] defined horizontal Brownian motion on $\mathcal{F}(M)$ as the solution of the following Stratonovich SDE:

$$dU_t = \sum_{i=1}^d H_i(t,U_t) \circ dW_t^i - \frac{1}{2} \sum_{\alpha,\beta=1}^d \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) V_{\alpha\beta}(U_t) dt.$$

They showed that if $U_0 \in \mathcal{O}_{g(0)}(M)$, then $U_t \in \mathcal{O}_{g(t)}(M)$ for all $t \in [0,T]$. $g(t)$-Brownian motion on M is then defined as $X_t := \pi U_t$. We denote the law of $g(t)$-Brownian motion on M started at x by P^x, and expectation with respect to that measure by E^x.

2 Main result

The main result of this paper is the following theorem:

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Theorem 1. *If the family of metrics evolves under backwards super Ricci flow, i.e.*

$$\frac{\partial g}{\partial t} \leq \mathrm{Ric}, \tag{1}$$

then Brownian motion on M cannot explode up to time T. In particular this result holds for backwards Ricci flow $\frac{\partial g}{\partial t} = \mathrm{Ric}$.

By recent work (see section 3), it has turned out that backwards Ricci flow tends to compensate the effects of Ricci curvature on the behaviour of heat flow and Brownian motion. Thus our result is quite natural because a lower bound of Ricci curvature yields the non-explosion property in the fixed metric case.

Remark 1.

1. In Section 6 we will give an extension of Theorem 1 including the case of non-symmetric diffusion processes.
2. For the question of explosion or non-explosion of Brownian motion on a manifold equipped with a fixed Riemannian metric see e.g. [6], [8, Section 7.8] or [9, Section 4.2].

As an important tool we prove the following Itô formula for the radial process $\rho(t, X_t)$, where $\rho(t, x)$ denotes the distance with respect to $g(t)$ between x and a fixed reference point o:

Theorem 2. *There exists a nondecreasing continuous process L which increases only when $X_t \in \mathrm{Cut}_{g(t)}(o)$ such that*

$$\rho(t, X_t) = \rho(0, X_0) + \int_0^t \left[\frac{1}{2}\Delta_{g(s)}\rho + \frac{\partial \rho}{\partial s}\right](s, X_s)ds + \sum_{i=1}^{d} \int_0^t (U_s e_i)\rho(s, X_s)dW_s^i - L_t. \tag{2}$$

Remark 2.

1. The usual Itô formula fails to apply because the distance function is not smooth at the cut-locus. A priori it is even not clear that $\rho(t, X_t)$ is a semimartingale.
2. In the case of a fixed Riemannian metric Theorem 2 was proved by Kendall [11] (see also [8, Theorem 7.254] or [9, Theorem 3.5.1]). The idea of our proof is based on Kendall's original one.
3. For Theorem 2 we do not require any additional assumption on $g(t)$ such as (1).

3 Remarks concerning related work

McCann and Topping [13] (see also Topping [14] and Lott [12]) showed contraction in the Wasserstein metric for the heat equation under backwards Ricci flow on a compact manifold. More precisely, they showed that the following are equivalent:

1. g evolves under backwards super Ricci flow, i.e. $\frac{\partial g}{\partial t} \leq \mathrm{Ric}$.
2. Whenever u and v are two non-negative unit-mass solutions of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_{g(t)}u - \left(\frac{1}{2}\,\mathrm{tr}\,\frac{\partial g}{\partial t}\right)u$$

(the term $\left(\frac{1}{2}\operatorname{tr}\frac{\partial g}{\partial t}\right) u$ comes from the change in time of the volume element), the function $t \mapsto W_2(t, u(t,\cdot)\operatorname{vol}_{g(t)}, v(t,\cdot)\operatorname{vol}_{g(t)})$ is non-increasing. Here

$$W_2(t,\mu,\nu) := \left(\inf_{\pi}\int_{M\times M} d_{g(t)}(x,y)^2\pi(dx,dy)\right)^{1/2}$$

is the L^2-Wasserstein distance of two probability measures μ and ν on M. (The infimum is over all probability measures π on $M \times M$ whose marginals are μ and ν.)

This means that backwards super Ricci flow is characterized by the contractivity property for solutions of the heat equation. Moreover, in recent work by Topping [14] and Lott [12] (see also Brendle [3]) the heat equation and the theory of optimal transport are efficiently used to derive several monotonicity results including a new proof for the monotonicity of Perelman's reduced volume. These facts indicate that it would be effective for deeper understanding of Ricci flow to study the heat equation in conjunction with backwards Ricci flow and the theory of optimal transport.

The non-explosion property of the Brownian motion is one of the first problems we face when we begin to consider the heat equation on a noncompact manifold. Our result tells us that it is always satisfied as far as we consider the heat equation under backwards Ricci flow. It will be quite helpful for the study of Ricci flow on a noncompact manifold by means of the heat equation. In fact, our result enables us to remove the assumption on the non-explosion in recent work by Arnaudon, Coulibaly and Thalmaier [2, Section 4]. They extend McCann and Topping's implication 1 $\Rightarrow$ 2 in the case on a noncompact manifold. In addition, they sharpen the monotonicity of L^2-Wasserstein distance to a pathwise contraction in the following sense: There is a coupling $(\bar{X}_t^{(1)}, \bar{X}_t^{(2)})_{t\ge 0}$ of two Brownian motions starting from $x, y \in X$ respectively such that $t \mapsto d_{g(t)}(\bar{X}_t^{(1)}, \bar{X}_t^{(2)})$ is non-increasing almost surely. By taking expectation we can derive the monotonicity of the L^2-Wasserstein distance from it. The sharpness of their pathwise contraction looks useful for the study of the optimal transport associated with a more general cost function than the squared distance, e.g. $\mathcal{L}$-optimal transportation studied in the above mentioned papers [14, 12, 3]. As a consequence of our result, we can consider such a problem without assuming the compactness of the underlying space.

4 Proof of Theorem 2: Itô's formula for the radial process

Note that we only need to prove Theorem 2 until the exit time of X from an arbitrary large relatively compact open subset M_0 of M. Thus, by modifying M and $g(t)$ outside of a neighbourhood of M_0, we can reduce the proof to the case of compact M (recall Remark 2.3; as we will see, such a modification is harmless for proving Theorem 2). By the compactness of M, the injectivity radius

$$i_M := \inf\{d_{g(t)}(x,y) \mid t \in [0,T], y \in \operatorname{Cut}_{g(t)}(x)\}$$

is strictly positive and that we have a uniform bound for the sectional curvature $\operatorname{Sect}_{g(t)}$:

$$|\operatorname{Sect}_{g(t)}| \le K^2 \quad \text{for all } t \in [0,T].$$

We first state Itô's formula for smooth functions:

Lemma 1. *Let f be a smooth function on $[0,T]\times M$. Then*

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{1}{2}\Delta_{g(t)}f(t,X_t)dt + \sum_{i=1}^{d}(U_t e_i)f(t,X_t)dW_t^i.$$

Proof. Itô's formula applied to a smooth function $\tilde{f}$ on $[0,T]\times\mathcal{F}(M)$ gives

$$\begin{aligned} d\tilde{f}(t,U_t) &= \frac{\partial \tilde{f}}{\partial t}(t,U_t)dt + \sum_{i=1}^{d} H_i(t)\tilde{f}(t,U_t)dW_t^i + \frac{1}{2}\sum_{i=1}^{d} H_i(t)^2\tilde{f}(t,U_t)dt \\ &\quad - \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t,U_te_\alpha,U_te_\beta)V_{\alpha\beta}\tilde{f}(t,U_t)dt. \end{aligned} \tag{3}$$

Now let $\tilde{f}(t,u) := f(t,\pi u)$. By definition of $H_i(t)$, $H_i(t)\tilde{f}(t,u) = (ue_i)f(t,\pi u)$. Moreover, it is well known (see e.g. [9, Proposition 3.1.2]) that $\sum_{i=1}^{d} H_i(t)^2\tilde{f}(t,u) = \Delta_{g(t)}f(t,\pi u)$. Finally, since $\tilde{f}$ is constant in the vertical direction, the last term in (3) vanishes, so that the claim follows. □

Lemma 2. *Let $G(x,\tau,y,t)$ $(x,y\in M,\ 0\le t<\tau\le T)$ be the fundamental solution of the equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_{g(t)}u$ (see [7] for existence). Then for all $\tau\in(0,T]$ and all $x\in M$ the law of X_τ under P^x is absolutely continuous with respect to the volume measure (note that this property does not depend on the choice of the Riemannian metric), and its density with respect to the $g(0)$-volume measure is given by $y\mapsto G(x,\tau,y,0)$.*

Proof. Fix $\varphi\in\mathcal{C}^2(M)$, and let u be the solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta_{g(t)}u \\ u(0,\cdot) &= \varphi. \end{cases}$$

Then by Corollary 2.2 in [7],

$$u(\tau,x) = \int_M G(x,\tau,y,0)\varphi(y)dvol_{g(0)}(y).$$

Now apply Itô's formula to X and the function $(t,x)\mapsto u(\tau-t,x)$ to obtain

$$\begin{aligned} u(0,X_\tau) &= u(\tau,X_0) - \int_0^\tau \frac{\partial u}{\partial t}(\tau-t,X_t)dt + \frac{1}{2}\int_0^\tau \Delta_{g(t)}u(t-\tau,X_t)dt + \text{martingale} \\ &= u(\tau,X_0) + \text{martingale}, \end{aligned}$$

so that

$$E^x\left[\varphi(X_\tau)\right] = E^x\left[u(0,X_\tau)\right] = E^x\left[u(\tau,X_0)\right] = u(\tau,x) = \int_M G(x,\tau,y,0)\varphi(y)dvol_{g(0)}(y).$$

Since φ is arbitrary the claim is proved. □

Lemma 3. $\{t\in[0,T]\,|\,X_t\in\mathrm{Cut}_{g(t)}(o)\}$ *has Lebesgue measure zero almost surely.*

Proof. Since by Lemma 2 for each $t\in(0,T]$ and any starting point $x\in M$, the law of X_t under P^x is absolutely continuous with respect to the $g(t)$-Riemannian volume measure, and since moreover the cut-locus $\mathrm{Cut}_{g(t)}(o)$ has $g(t)$-volume zero (see e.g. [8, Theorem 7.253] or [4, Proposition 3.1]), we have

$$E^x\left[\int_0^T 1_{\{X_t\in\mathrm{Cut}_{g(t)}(o)\}}dt\right] = \int_0^T P^x\left[X_t\in\mathrm{Cut}_{g(t)}(o)\right]dt = 0,$$

so that almost surely $\int_0^T 1_{\{X_t\in\mathrm{Cut}_{g(t)}(o)\}}dt = 0$. □

We now apply Lemma 1 to the process $\rho(t, X_t)$ up to singularity. As long as X_t stays away from o and the $g(t)$-cut-locus of o,

$$d\rho(t, X_t) = d\beta_t + \frac{1}{2}\left[\Delta_{g(t)}\rho + 2\frac{\partial \rho}{\partial t}\right](t, X_t)dt, \tag{4}$$

where β_t is the martingale term given by

$$\beta_t := \sum_{i=1}^{d} \int_0^t H_i(s)\tilde{\rho}(s, U_s)dW_s^i.$$

As we will observe in Lemma 5, the singularity of $\rho(t, x)$ at o is negligible. The quadratic variation $\langle \beta \rangle_t$ of β_t is computed as follows:

$$\begin{aligned}
\langle \beta \rangle_t &= \sum_{i=1}^{d} \int_0^t \left[H_i(s)\tilde{\rho}(s, U_s)\right]^2 ds \\
&= \sum_{i=1}^{d} \int_0^t \left[(U_s e_i)\rho(s, X_s)\right]^2 ds \\
&= \int_0^t |\nabla_{g(s)}\rho(s, X_s)|^2 ds \\
&= t.
\end{aligned}$$

Thus β_t is a standard one-dimensional Brownian motion.

Lemma 4 (Lemma 5 and Remark 6 in [13]). *The function $(t, x) \mapsto \rho(t, x)$ is smooth whenever $x \notin \{o\} \cup \mathrm{Cut}_{g(t)}(o)$, and*

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{1}{2}\int_0^{\rho(t,x)} \frac{\partial g}{\partial t}(\dot{\gamma}(s), \dot{\gamma}(s))ds,$$

where $\gamma : [0, \rho(t, x)] \to M$ is the unique minimizing unit-speed $g(t)$-geodesic joining o to x.

Let $\mathrm{Cut}_{\mathrm{ST}}$ be the space-time cut-locus defined by

$$\mathrm{Cut}_{\mathrm{ST}} := \{(t, x, y) \in [0, T] \times M \times M \mid (x, y) \in \mathrm{Cut}_{g(t)}\}.$$

It is shown in [13] that $\mathrm{Cut}_{\mathrm{ST}}$ is a closed subset in $[0, T] \times M \times M$. Though they assumed M to be compact, extension to the noncompact case is straightforward. Since $[0, T] \times \{o\} \times \{o\}$ is a compact subset in $[0, T] \times M \times M$ and it is away from $\mathrm{Cut}_{\mathrm{ST}}$, we can take $r_1 > 0$ so that

$$d_{g(t)}(o, \mathrm{Cut}_{g(t)}(o)) > r_1 \tag{5}$$

holds for all $t \in [0, T]$. Thus we can use (4)) when X_t is in a small neighbourhood of o until X_t hits o. Since $g(t)$ is smooth, Lemma 4 and (4) together with the Laplacian comparison theorem imply the following by a standard argument:

Lemma 5. *With probability one, X_t never hits o.*

For $x, y \in M$, let

$$\overline{d}(x, y) := \sup_{t \in [0,T]} d_{g(t)}(x, y).$$

We consider $[0,T] \times M \times M$ equipped with the distance function $\hat{d}((s,x_1,x_2),(t,y_1,y_2)) := \max\{|t-s|, \overline{d}(x_1,y_1), \overline{d}(x_2,y_2)\}$.

By Lemma 4 and the compactness of M, there exists a constant $C_1 > 0$ such that

$$|d_{g(t)}(x,y) - d_{g(t')}(x,y)| \le C_1|t-t'| \tag{6}$$

holds for any $t,t' \in [0,T]$ and $x,y \in M$. We now define a set A by

$$A := \left\{ (t,x,y) \in [0,T] \times M \times M \;\middle|\; \begin{array}{l} d_{g(t)}(o,x) \ge 2i_M/3, d_{g(t)}(o,y) = i_M/3 \text{ and} \\ d_{g(t)}(x,y) = d_{g(t)}(o,x) - d_{g(t)}(o,y) \end{array} \right\}.$$

Note that A is closed and hence compact since $d_{g(t)}(x,y)$ is continuous as a function of (t,x,y). Note that, for $(t,x,y) \in A$, y is on a minimal $g(t)$-geodesic joining o and x. In particular, symmetry of the cutlocus implies that $A \cap \mathrm{Cut}_{ST} = \emptyset$. Thus we have

$$\delta_1 := \hat{d}(A, \mathrm{Cut}_{ST}) \wedge \frac{i_M}{3(C_1+1)} > 0.$$

We define the function $V : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$V(r) := \frac{d-1}{2} K \coth\left(K \cdot r \wedge \frac{i_M}{3}\right) + 2C_1.$$

The Laplacian comparison theorem implies that, for all $(t,x,y) \notin \mathrm{Cut}_{\mathrm{ST}}$, $|(\Delta_{g(t)} d_{g(t)}(y,\cdot))(x)| \le (d-1)K \coth(K d_{g(t)}(x,y))$ and hence Lemma 4 implies

$$\left| \frac{1}{2}(\Delta_{g(t)} d_{g(t)}(y,\cdot))(x) + \frac{\partial}{\partial t} d_{g(t)}(y,x) \right| \le V(d_{g(t)}(x,y)). \tag{7}$$

Lemma 6. *Let $(t_0,x_0) \in \mathrm{Cut}_{g(t_0)}(o)$ and $\delta \in (0,\delta_1)$. Let X be a $g(t)$-Brownian motion starting at x_0 at time t_0. Let $\tilde{T} := T \wedge (t_0+\delta) \wedge \inf\{t \ge t_0 \mid d_{g(t)}(x_0,X_t) = \delta\}$. Then*

$$E\left[\rho(t \wedge \tilde{T}, X_{t\wedge\tilde{T}}) - \rho(t_0,x_0) - \int_{t_0}^{t\wedge\tilde{T}} V(\rho(s,X_s))ds \right] \le 0.$$

Proof. We construct a point $\tilde{o} \in M$ as follows: we choose a minimizing unit-speed $g(t_0)$-geodesic γ from o to x_0 and define $\tilde{o} := \gamma(i_M/3)$. Then by construction $(t_0,x_0,\tilde{o}) \in A$. Moreover for all $t \in [t_0,\tilde{T}]$ we have $\hat{d}((t_0,x_0,\tilde{o}),(t,X_t,\tilde{o})) < \delta_1$ and therefore $X_t \notin \mathrm{Cut}_{g(t)}(\tilde{o})$. Let

$$\rho^+(t,x) := d_{g(t)}(o,\tilde{o}) + d_{g(t)}(\tilde{o},x).$$

Since $\tilde{o}$ lies on a minimizing $g(t_0)$-geodesic from o to x_0, we have $\rho^+(t_0,x_0) = \rho(t_0,x_0)$. Moreover, by the triangle inequality, $\rho^+(t,x) \ge \rho(t,x)$ for all (t,x). On $[t_0,\tilde{T}]$,

$$\begin{aligned} d_{g(t)}(\tilde{o},X_t) &\ge d_{g(t)}(\tilde{o},x_0) - d_{g(t)}(x_0,X_t) \\ &\ge d_{g(t_0)}(\tilde{o},x_0) - (1+C_1)\delta \\ &\ge \frac{i_M}{3} \end{aligned} \tag{8}$$

holds. By (7) and Lemma 4,

$$\left(\frac{1}{2}\Delta_{g(t)}\rho^+ + \frac{\partial \rho^+}{\partial t} \right)(t,x) \le V(\rho^+(t,x))$$

holds if $(t, x, \tilde{o}) \notin \mathrm{Cut}_{\mathrm{ST}}$. Note that $V(\rho^+(t, X_t)) = V(\rho(t, X_t))$ holds for all $t \in [t_0, \tilde{T}]$ since we can show $\rho(t, X_t) \geq i_M/3$ in a similar way as in (8). Therefore

$$\begin{aligned}
\rho(t \wedge \tilde{T}, X_{t\wedge\tilde{T}}) - \rho(t_0, X_{t_0}) - \int_{t_0}^{t\wedge\tilde{T}} V(\rho(s, X_s))ds& \\
&= \rho(t \wedge \tilde{T}, X_{t\wedge\tilde{T}}) - \rho^+(t_0, X_{t_0}) - \int_{t_0}^{t\wedge\tilde{T}} V(\rho^+(s, X_s))ds \\
&\leq \rho^+(t \wedge \tilde{T}, X_{t\wedge\tilde{T}}) - \rho^+(t_0, X_{t_0}) - \int_{t_0}^{t\wedge\tilde{T}} \left(\frac{1}{2}\Delta_{g(s)}\rho^+ + \frac{\partial \rho^+}{\partial s}\right)(s, X_s)ds.
\end{aligned}$$

Since ρ^+ is smooth at (t, X_t) for $t \in [t_0, \tilde{T}]$, the last term is a martingale. Hence the claim follows. □

For $\delta \in (0, \delta_1)$, we define a sequence of stopping times $(S_n^\delta)_{n\in\mathbb{N}}$ and $(T_n^\delta)_{n\in\mathbb{N}_0}$ by

$$\begin{aligned}
T_0^\delta &:= 0, \\
S_n^\delta &:= T \wedge \inf\{t \geq T_{n-1}^\delta \mid X_t \in \mathrm{Cut}_{g(t)}(o)\}, \\
T_n^\delta &:= T \wedge (S_n^\delta + \delta) \wedge \inf\{t \geq S_n^\delta \mid d_{g(t)}(X_{S_n^\delta}, X_t) = \delta\}.
\end{aligned}$$

Note that these are well-defined because $\mathrm{Cut}_{\mathrm{ST}}$ and $\{(t, x) \mid d_{g(t)}(y, x) = \delta\}$, where $y \in M$, are closed.

Proposition 1. *The process*

$$\rho(t, X_t) - \rho(0, X_0) - \int_0^t V(s, X_s)ds$$

is a supermartingale.

Corollary 1. *The process $\rho(t, X_t)$ is a semimartingale.*

Proof of Proposition 1. Thanks to the strong Markov property of Brownian motion it suffices to show that for all deterministic starting points $(t_0, x_0) \in [0, T] \times M$ and all $t \in [t_0, T]$

$$E\left[\rho(t, X_t) - \rho(t_0, X_{t_0}) - \int_{t_0}^t V(\rho(s, X_s))ds\right] \leq 0.$$

To show this we first observe that thanks to Lemma 1, (7) and Lemma 6 for all $n \in \mathbb{N}$

$$E\left[\rho(t \wedge S_n^\delta, X_{t\wedge S_n^\delta}) - \rho(t \wedge T_{n-1}^\delta, X_{t\wedge T_{n-1}^\delta}) - \int_{t\wedge T_{n-1}^\delta}^{t\wedge S_n^\delta} V(\rho(s, X_s))ds \,\middle|\, \mathcal{F}_{T_{n-1}^\delta}\right] \leq 0$$

and

$$E\left[\rho(t \wedge T_n^\delta, X_{t\wedge T_n^\delta}) - \rho(t \wedge S_n^\delta, X_{t\wedge S_n^\delta}) - \int_{t\wedge S_n^\delta}^{t\wedge T_n^\delta} V(\rho(s, X_s))ds \,\middle|\, \mathcal{F}_{S_n^\delta}\right] \leq 0.$$

It remains to show that $T_n \to T$ as $n \to \infty$. If $\lim_{n\to\infty} T_n =: T_\infty < T$ occurs, then $T_n^\delta - S_n^\delta$ converges to 0 as $n \to \infty$. In addition, $d_{g(t)}(X_{S_n^\delta}, X_{T_n^\delta}) = \delta$ must hold for infinitely many $n \in \mathbb{N}$. Take $N \in \mathbb{N}$ so large that $C_1(T_\infty - T_n) < \delta/2$ for all $n \geq N$. Then (6) yields $d_{g(T_\infty)}(X_{S_n^\delta}, X_{T_n^\delta}) \geq \delta/2$ for infinitely many $n \geq N$. But it contradicts with the fact that X_t is uniformly continuous on $[0, T]$. Hence $T_n \to T$ as $n \to \infty$. □

Lemma 7. $\lim_{\delta\to 0}\sum_{n=1}^{\infty}|T_n^\delta - S_n^\delta| = 0$ *almost surely.*

Proof. For $\delta > 0$, let us define a random subset E_δ and E in $[0,T]$ by

$$E_\delta := \{t \in [0,T] \,|\, \text{there exists } t' \in [0,T] \text{ satisfying } |t-t'| \le \delta \text{ and } (t', X_{t'}, o) \in \mathrm{Cut_{ST}}\},$$
$$E := \{t \in [0,T] \,|\, (t, X_t, o) \in \mathrm{Cut_{ST}}\}.$$

Since the map $t \mapsto (t, X_t, o)$ is continuous and $\mathrm{Cut_{ST}}$ is closed, E is closed and hence $E = \cap_{\delta>0}E_\delta$ holds. By the definition of S_n^δ and T_n^δ, we have

$$E \subset \bigcup_{n=1}^{\infty}[S_n^\delta, T_n^\delta] \subset E_\delta$$

and hence the monotone convergence theorem implies

$$\lim_{\delta\to 0}\sum_{n=1}^{\infty}|T_n^\delta - S_n^\delta| \le \lim_{\delta\to 0}\int_0^T 1_{E_\delta}(t)dt = \int_0^T 1_E(t)dt = 0$$

almost surely, where the last equality follows from Lemma 3. □

Lemma 8. *The martingale part of* $\rho(t, X_t)$ *is*

$$\sum_{i=1}^{d}\int_0^t (U_s e_i)\rho(s, X_s)dW_s^i.$$

Proof. By the martingale representation theorem there exists an $\mathbb{R}^d$-valued process η such that the martingale part of $\rho(t, X_t)$ equals $\int_0^t \eta_s dW_s$. Let

$$N_t := \int_0^t \eta_s dW_s - \sum_{i=1}^{d}\int_0^t (U_s e_i)\rho(s, X_s)dW_s^i.$$

Using the stopping times S_n^δ and T_n^δ, the quadratic variation $\langle N\rangle_T$ of N is expressed as follows;

$$\langle N\rangle_T = \sum_{i=1}^{d}\sum_{n=1}^{\infty}\left(\int_{T_{n-1}^\delta\wedge T}^{S_n^\delta\wedge T}|\eta_t^i - (U_t e_i)\rho(t, X_t)|^2 dt + \int_{S_n^\delta\wedge T}^{T_n^\delta\wedge T}|\eta_t^i - (U_t e_i)\rho(t, X_t)|^2 dt\right). \tag{9}$$

Since $X_t \notin \mathrm{Cut}_{g(t)}(o)$ if $t \in (T_{n-1}^\delta, S_n^\delta)$, Itô's formula (4) yields

$$\int_{T_{n-1}^\delta\wedge T}^{S_n^\delta\wedge T}|\eta_t^i - (U_t e_i)\rho(t, X_t)|^2 dt = 0$$

for $n \in \mathbb{N}$ and $i = 1, \cdots, d$. For the second term in the right-hand side of (9) we have

$$\sum_{n=1}^{\infty}\int_{S_n^\delta\wedge T}^{T_n^\delta\wedge T}|\eta_t^i - (U_t e_i)\rho(t, X_t)|^2 dt \le 2\int_{\bigcup_{n=1}^{\infty}[S_n^\delta, T_n^\delta]}\left(|\eta_t|^2 + 1\right)dt.$$

Since η_t is locally square-integrable on $[0,T]$ almost surely, Lemma 7 yields $\langle N\rangle_T = 0$ and the conclusion follows. □

We can now conclude the proof of Theorem 2: Set $I_\delta := \bigcup_{n=1}^{\infty} [S_n^\delta, T_n^\delta]$. Set L_t^δ by

$$L_t^\delta := -\rho(t, X_t) + \rho(0, X_0) + \sum_{i=1}^{d} \int_0^t (U_s e_i)\rho(s, X_s) dW_s^i \\ + \int_{[0,t]\setminus I_\delta} \left[\frac{1}{2}\Delta_{g(t)}\rho + \frac{\partial \rho}{\partial s} \right] (s, X_s) ds + \int_{[0,t]\cap I_\delta} V(\rho(s, X_s)) ds.$$

By Proposition 1, Lemma 8 and Itô's formula (4) on $[0, T] \setminus I_\delta$, L_t^δ is non-decreasing in t. In particular, L_t^δ can increase only when $t \in I_\delta$. Then we have

$$\rho(t, X_t) - \rho(0, X_0) - \sum_{i=1}^{d} \int_0^t (U_s e_i)\rho(s, X_s) dW_s^i - \int_0^t \left[\frac{1}{2}\Delta_{g(s)}\rho + \frac{\partial \rho}{\partial s} \right] (s, X_s) ds + L_t^\delta \\ = - \int_{[0,t]\cap I_\delta} \left[\frac{1}{2}\Delta_{g(t)}\rho + \frac{\partial \rho}{\partial s} \right] (s, X_s) ds - \int_{[0,t]\cap I_\delta} V(\rho(s, X_s)) ds. \tag{10}$$

Since (7) yields

$$\left| \int_{[0,t]\cap I_\delta} \left[\frac{1}{2}\Delta_{g(t)}\rho + \frac{\partial \rho}{\partial s} \right] (s, X_s) ds + \int_{[0,t]\cap I_\delta} V(\rho(s, X_s)) ds \right| \le 2 \int_{I_\delta} V(\rho(s, X_s)) ds$$

and $V(\rho(s, X_s))$ is bounded on I_δ, Lemma 7 yields that the right-hand side of (10) converges to 0 as $\delta \to 0$. Thus $L_t := \lim_{\delta \downarrow 0} L_t^\delta$ exists for all $t \in [0, T]$ almost surely and hence (2) holds. We can easily deduce the fact that L_t can increase only when $t \in \mathrm{Cut}_{g(t)}(o)$ from the corresponding property for L_t^δ. □

5 Proof of Theorem 1: Non-explosion of Brownian motion

We define $k_1 \ge 1$ and $\bar{F} : [0, \infty) \to \mathbb{R}$ by

$$k_1 := \inf \left\{ k \ge 1 \,|\, |\mathrm{Ric}_{g(t)}(x)| \le -(d-1)k^2 \text{ for } t \in [0, T] \text{ and } x \in M \text{ with } \overline{d}(o, x) \le r_1 \right\}, \\ \bar{F}(s) := (d-1)(k_1 \coth(k_1 \cdot s \wedge r_1) + k_1^2 \cdot s \wedge r_1),$$

where r_1 is defined in (5).

Theorem 1 follows immediately from the following estimate of the drift part of (2):

Proposition 2. *Suppose* (1). *Then, for all* $(t, x) \in [0, T] \times M$ *with* $(t, x, o) \notin \mathrm{Cut}_{ST}$,

$$\Delta_{g(t)}\rho(t, x) + 2\frac{\partial \rho}{\partial t}(t, x) \le \bar{F}(\rho(t, x)).$$

To prove this proposition it suffices to show the following lemma:

Lemma 9. *Suppose* (1). *Fix* $t \in [0, T]$ *and a minimizing unit-speed* $g(t)$*-geodesic* $\gamma : [0, b] \to M$ *with* $\gamma(0) = o$. *Then there exists a non-increasing function* $F : (0, b) \to \mathbb{R}$ *satisfying* $F(s) \le \bar{F}(s)$ *and*

$$\Delta_{g(t)}\rho(t, \gamma(s)) + 2\frac{\partial \rho}{\partial t}(t, \gamma(s)) \le F(s)$$

for all $s \in (0, b)$.

Proof. Let $(X_i)_{i=1}^d$ be orthonormal parallel fields along γ with $X_1 = \dot\gamma$. Fix $r \in (0, b)$, and let J_i be the Jacobi field along $\gamma|_{[0,r]}$ with $J_i(0) = 0$ and $J_i(r) = X_i(r)$. Then it is well known (see [4] for example) that

$$(\Delta_{g(t)} d_{g(t)}(\gamma(0), \cdot))(\gamma(r)) = \sum_{i=2}^{d} I(J_i, J_i),$$

where the index form I for smooth vector fields Y, Z along $\gamma|_{[0,r]}$ is defined by

$$I(Y, Z) := \int_0^r \Big(\langle \dot Y(s), \dot Z(s)\rangle_{g(t)} - \langle R_{g(t)}(Y(s), \dot\gamma(s))\dot\gamma(s), Z(s)\rangle_{g(t)} \Big)\, ds.$$

Let $G : [0, b] \to \mathbb{R}$ be the solution to the initial value problem

$$\begin{cases} G''(s) = -\dfrac{\mathrm{Ric}_{g(t)}(\dot\gamma(s), \dot\gamma(s))}{d-1} G(s), \\ G(0) = 0,\ G'(0) = 1. \end{cases}$$

Then we have

$$\begin{aligned}
\sum_{i=2}^{d} I(GX_i, GX_i) &= \sum_{i=2}^{d} \int_0^r \Big[\big|G'(s)X_i(s)\big|^2 - \langle R(G(s)X_i(s), \dot\gamma(s))\dot\gamma(s), G(s)X_i(s)\rangle \Big]\, ds \\
&= \int_0^r \big[(d-1)G'(s)^2 - G(s)^2 \,\mathrm{Ric}(\dot\gamma(s), \dot\gamma(s)) \big]\, ds \\
&= (d-1) \int_0^r \big[G'(s)^2 + G(s)G''(s) \big]\, ds \\
&= (d-1)G(r)G'(r). \qquad (11)
\end{aligned}$$

Since $\gamma(0)$ has no conjugate point along γ on $[0, r]$, the left-hand side of (11) must be strictly positive (see Theorem 2.10 in [4]). It follows that $G(r) > 0$ for all $r \in (0, b)$. Now let $Y_i(s) := \frac{G(s)}{G(r)} X_i(s)$. Note that Y_i has the same boundary values as J_i. Therefore, by the index lemma,

$$(\Delta_{g(t)} d_{g(t)}(\gamma(0), \cdot))(\gamma(r)) \le \sum_{i=2}^{d} I(Y_i, Y_i) = \frac{(d-1)G'(r)}{G(r)}.$$

Hence Lemma 4 and (1) yield

$$\begin{aligned}
(\Delta\rho + 2\frac{\partial\rho}{\partial t})(t, \gamma(r)) \quad &\le \quad \frac{(d-1)G'(r)}{G(r)} + \int_0^r \mathrm{Ric}(\dot\gamma(s), \dot\gamma(s)) ds \\
&=: \quad F(r).
\end{aligned}$$

By the definition of G we have

$$F'(r) = (d-1)\left[\frac{G(r)G''(r) - G'(r)^2}{G(r)^2} + \mathrm{Ric}(\dot\gamma(r), \dot\gamma(r)) \right] = -\frac{(d-1)G'(r)^2}{G(r)^2} < 0$$

and hence F is decreasing. In particular, we have $F(r) \le F(r \wedge r_1)$. A usual comparison argument implies that $G'(r \wedge r_1)/G(r \wedge r_1) \le k_1 \coth(k_1 \cdot r \wedge r_1)$, and hence the conclusion follows from the definitions of F and k_1. □

6 Generalization to non-symmetric diffusion

We generalize the previous results on more general setting including the case for nonsymmetric diffusions. Let X_t be time-dependent diffusion whose generator is $\Delta_{g(t)}/2 + Z(t)$, where $Z(t)$ is a time-dependent vector field on M which is smooth on $[0, T] \times M$.

Even in this case, Theorem 2 still holds by replacing $\Delta_{g(t)}/2$ with $\Delta_{g(t)}/2 + Z(t)$. In what follows, we briefly mention the proof. Except for Lemma 2 and Lemma 3, an extension of each assertion is straightforward. For Lemma 3, some difficulties come from the fact that the result corresponding to Lemma 2, especially the existence of a fundamental solution, is not yet known at this moment for non-symmetric diffusions. But, for our purpose, it suffices to show the following:

Lemma 10. *Suppose that M is compact. Then $P^x[X_t \in \mathrm{Cut}_{g(t)}(o)] = 0$.*

Proof. Let $\hat{Z}(t)$ be a differential 1-form corresponding to $Z(t)$ by duality with respect to $g(t)$. Let M_t^Z be the martingale part of the stochastic line integral of $\hat{Z}(t)$ along X_t. Note that there is a constant $c > 0$ such that $\langle M^Z \rangle_t \le ct$ holds since M is compact. Let us define a probability measure $\tilde{P}^x$ on the same probability space as P^x by $\tilde{P}^x[A] := E^x[\exp(-M^Z - \langle M^Z \rangle_t/2) 1_A]$. By the Girsanov formula, the law of X_t under $\tilde{P^x}$ coincides with that of $g(t)$-Brownian motion at time t. The Schwarz inequality yields

$$\begin{aligned} P^x[X_t \in \mathrm{Cut}_{g(t)}(o)] &\le E^x\left[\mathrm{e}^{-M_t^Z - \langle M^Z \rangle_t/2} 1_{\{X_t \in \mathrm{Cut}_{g(t)}(o)\}}\right]^{1/2} E^x\left[\mathrm{e}^{M_t^Z + \langle M^Z \rangle_t/2}\right]^{1/2} \\ &\le \tilde{P}^x\left[X_t \in \mathrm{Cut}_{g(t)}(o)\right]^{1/2} \mathrm{e}^{ct/2}. \end{aligned}$$

Hence the conclusion follows from Lemma 2. □

To state an extension of Theorem 1, define a tensor field $(\nabla Z(t))^\flat$ by

$$(\nabla Z(t))^\flat(X, Y) := \frac{1}{2}\left(\langle \nabla_X Z(t), Y \rangle_{g(t)} + \langle \nabla_Y Z(t), X \rangle_{g(t)}\right),$$

where ∇ is the Levi-Civita connection with respect to $g(t)$.

Assumption 1. There exists a locally bounded measurable function b on $[0, \infty)$ so that

1. $(\nabla Z(t, x))^\flat + \partial_t g(t, x) \le \mathrm{Ric}_{g(t)}(x) + b(\rho(t, x)) g(t, x)$ for all $t \in (0, T)$ and all $x \in M$.
2. The 1-dimensional diffusion process y_t given by $dy_t = d\beta_t + \left(\bar{F}(y_t) + \int_0^{y_t} b(s) ds\right) dt$ does not explode. (This is the case if and only if

$$\int_1^\infty \exp\left[-2 \int_1^y \hat{b}(z) dz\right] \left\{\int_1^y \exp\left[2 \int_1^z \hat{b}(\xi) d\xi\right] dz\right\} dy = \infty,$$

where $\hat{b}(y) := \bar{F}(y) + \int_0^y b(s) ds$, see e.g. [8, Theorem 6.50] or [10, Theorem VI.3.2].)

Once we obtain the following, non-explosion of X_t follows in the same way as above by the comparison argument.

Lemma 11. *Suppose that Assumption 1 holds. Fix $t \in [0, T]$ and a minimizing unit-speed $g(t)$-geodesic $\gamma : [0, b] \to M$ with $\gamma(0) = 0$. Then there exists a constant $C_Z > 0$ depending only on $\{Z(t)\}_{t \in [0,T]}$ and $\gamma(0)$ such that*

$$((\Delta_{g(t)} + Z(t)) d_{g(t)}(\gamma(0), \cdot))(\gamma(s)) + 2\frac{\partial}{\partial t} d_{g(t)}(\gamma(0), \gamma(s)) \le C_Z + F(s) + \int_0^s b(d_{g(t)}(\gamma(0), \gamma(u))) du$$

for all $s \in (0, b)$, where F is the same function as in Lemma 9.

Proof. By a direct calculation, $(\nabla Z(t))^\flat(\dot\gamma(s),\dot\gamma(s)) = \partial_s\langle Z(t),\dot\gamma(s)\rangle_{g(t)}(\gamma(s))$. Hence we obtain

$$\begin{aligned}(Z(t)d_{g(t)}(\gamma(0),\cdot))(\gamma(r)) &= \langle Z(t),\dot\gamma(r)\rangle_{g(t)}(\gamma(r))\\ &= \langle Z(t),\dot\gamma(0)\rangle_{g(t)}(\gamma(0)) + \int_0^t (\nabla Z(t))^\flat(\dot\gamma(s),\dot\gamma(s))ds.\end{aligned}$$

Then, by setting $C_Z := \sup_{s\in[0,T]}\sup\{\langle Z(s),X\rangle_{g(s)}(\gamma(0)) \mid X \in T_{\gamma(0)}M, \|X\|_{g(s)} = 1\}$, the conclusion follows in a similar way as we did in the proof of Lemma 9. □

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