Stochastic particle approximations for the Ricci flow on surfaces and the Yamabe flow

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Abstract

We present stochastic particle approximations for the normalized Ricci flow on surfaces and for the non-normalized Yamabe flow on manifolds of arbitrary dimension.

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1 Ricci and Yamabe flow

Thanks to Perelman's seminal work on the geometrization and hence the Poincaré conjecture [19, 20, 21] the Ricci flow has attracted worldwide attention and led to many new developments (see e.g. [17]). The importance of this subject has been underlined by the award of the Fields medal to Perelman. There is now a strong interest in understanding the microscopic structure of the Ricci flow.

The evolution of a Riemannian metric $g = g_t$ on a connected d-dimensional closed manifold M under the (normalized) Ricci flow is described by the partial differential equation

$$\frac{\partial g}{\partial t} = \frac{2}{d}\bar{R}g - 2\,\mathrm{Ric}\,. \tag{1}$$

Here Ric is the Ricci curvature and $\bar{R}$ the average scalar curvature of M, i.e. $\bar{R} := \frac{1}{\mathrm{vol}(M)}\int_M R$, where R is the scalar curvature (all quantities taken with respect to g_t).

In dimension $d = 2$ we have $\mathrm{Ric} = \frac{1}{2}Rg$, so that in this case (1) is equivalent to the Yamabe flow

$$\frac{\partial g}{\partial t} = (\bar{R} - R)g. \tag{2}$$

While in dimension $d \geq 3$ the Ricci flow does not usually admit global solutions (singularities can develop, see e.g. [17]), Hamilton [12] and Chow [5] (see also Theorem 5.1 in [6]) proved the following theorem concerning the 2-dimensional case:

Proposition 1. *Let g_0 be any Riemannian metric on M.*

1. *The Ricci flow equation has a unique solution $(g_t)_{t\geq 0}$ with initial data g_0.*
2. *As $t \to \infty$, g_t converges in any C^k-norm to a smooth metric g_∞ of constant curvature.*

For the Yamabe flow in dimension $d \geq 3$ Ye [26] proved global existence and uniqueness for arbitrary initial metrics. Convergence theorems under various assumptions were proved by Ye [26], Schwetlick and Struwe [23] and Brendle [2, 3].

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2 Stochastic particle approximation for the Ricci flow on surfaces

Since by (2) the conformal class of g is conserved under the two-dimensional Ricci flow it is possible to represent g_t in the form

$$g_t = u(t,\cdot)\tilde{g} \qquad \forall t \geq 0,$$

where $\tilde{g}$ is a fixed reference metric on M and u a smooth function on $\mathbb{R}_+ \times M$ (one can for instance take $\tilde{g} = g_0$, so that in this case $u_0 := u(0,\cdot) = 1$). This approach has the advantage that now the unknown is no longer a family of metrics, but just a function. In order to derive the partial differential equation satisfied by u we need the following lemma (for a proof see e.g. [6], Lemma 5.3).

Lemma 1. *Let $g = u\tilde{g}$. Then there is the following relation between the scalar curvatures with respect to g and $\tilde{g}$:*

$$R_g = \frac{1}{u}\left(-\Delta_{\tilde{g}}(\log u) + R_{\tilde{g}}\right).$$

Since moreover by the Gauss-Bonnet theorem $\bar{R}_g = 4\pi\chi(M)/\operatorname{vol}_g(M)$, it follows that

$$\frac{\partial u}{\partial t} = \Delta_{\tilde{g}}(\log u) - R_{\tilde{g}} + \frac{4\pi\chi(M)}{\operatorname{vol}_g(M)}u. \tag{3}$$

Remark 1. As a consequence of the convergence of g_t to g_∞ (Proposition 1) and the compactness of M, u is uniformly smooth, i.e. u and all its derivatives are uniformly bounded in $t \in \mathbb{R}_+$ and $x \in M$. Moreover, since $u(t,x) > 0$ for all $t \in \mathbb{R}_+$ and all $x \in M$ (because each g_t is a Riemannian metric), Proposition 1 implies that $u_{min} := \inf\{u(t,x) \mid t \geq 0,\ x \in M\}$ is strictly positive.

For the stochastic particle approximation it is essential that the metric $\tilde{g}$ has non-positive curvature.

Assumption 1. $R_{\tilde{g}} \leq 0$ everywhere on M.

For simplicity we assume:

Assumption 2. $\operatorname{vol}_g(M) = \operatorname{vol}_{\tilde{g}}(M) = 1$ ($\operatorname{vol}_g(M)$ is conserved under the Ricci flow, see e.g. [12], Section 2).

From now on all geometric and analytic quantities on M are taken with respect to the reference metric $\tilde{g}$.

The particle system lives on a set S_k of k points of M which is supposed to be approximately uniform in M in the sense that the discrete measure $\mu_k := \frac{1}{k}\sum_{x\in S_k}\delta_x$ converges weakly to the Riemannian volume measure on M as $k \to \infty$. By Theorem 11.3.3 in [8] this is equivalent to the property that

$$D_k := \sup_{f\in BL_1}\left|\frac{1}{k}\sum_{x\in S_k} f(x) - \int_M f(y)dy\right| \to 0 \qquad \text{as } k \to \infty, \tag{4}$$

where BL_1 is the set of all Lipschitz continuous functions on M with $\|f\|_{BL} := |f|_{lip} + \|f\|_\infty \leq 1$.

Remark 2. One possible choice is $S_k := \{\xi_1,\ldots,\xi_k\}$ with independent and uniformly (with respect to the Riemannian volume measure) distributed M-valued random variables $\xi_1,\ldots,\xi_k$. In this case the weak convergence of μ_k to the Riemannian volume measure and hence (4) holds almost surely (Theorem 11.4.1 in [8]).

The particle system consists of N particles moving as follows in discrete time steps of length $\tau > 0$ on S_k: Suppose that at time $n\tau$ (i.e. after n time steps) there are exactly $m \geq 1$ particles at a site $x \in S_k$. Then each of these particles, independently of the past and of the behaviour of all other particles, behaves as follows: With probability

$$\pi(x,y) := \frac{\tau}{k}\left[p_r(x,y)\varphi(km/N)/r - R(y)\right] \tag{5}$$

it jumps to a point $y \neq x$, otherwise it stays at x. Here $r > 0$ is a parameter and $p_r(x,y)$ is the heat kernel on M at time r, i.e. the fundamental solution of the heat equation $\frac{\partial p}{\partial t} = \Delta p$. Moreover

$$\varphi(u) := \begin{cases} \frac{\log u - \log u_{min}}{u} & \text{if } u \geq u_{min} \\ 0 & \text{if } u < u_{min}. \end{cases}$$

(Recall that $u_{min} := \inf\{u(t,x) \mid t \geq 0,\ x \in M\} > 0$, see Remark 1). Note that $\pi(x,y)$ is always nonnegative because of our assumption $R \leq 0$. In order to ensure that $\sum_{y \neq x} \pi(x,y) \leq 1$ (so that the jump rule (5) is well-defined), we will assume that

$$\tau \leq \left(\frac{1}{rkeu_{min}} \max_{x \in S_k} \sum_{y \in S_k \setminus \{x\}} p_r(x,y) - \frac{1}{k}\sum_{y \in S_k} R(y)\right)^{-1}, \tag{6}$$

where e is Euler's number (note that φ is bounded by $\frac{1}{eu_{min}}$). In order to give a formal definition of the particle system we introduce the following notation:

1. The particle configuration at time $n\tau$ (i.e. after n time steps) is denoted by

   $$X_n = (X_n^1, \ldots, X_n^N) \in S_k^N.$$

   This means that after n time steps the i-th particle is located at X_n^i.

2. For $\mathbf{x} = (x_1, \ldots, x_N) \in S_k^N$ and $y \in S_k$ we define

   $$\overline{\mathbf{x}}(y) := \frac{1}{N}\sum_{i=1}^N 1_{x_i}(y).$$

   In particular $\overline{X}_n(x)$ is the total mass at x after n time steps if we define the mass of each particle to be $1/N$.

Using this notation the stochastic dynamics can be exactly described as follows:

1. The particle system is a time-discrete Markov process $(X_n)_{n\in\mathbb{N}_0}$ with state space S_k^N.

2. The transition probabilities are as follows:

   (a) Independence of the jumps of different particles:

   $$P\left[X_{n+1} = \mathbf{y} | X_n = \mathbf{x}\right] = \prod_{i=1}^N P\left[X_{n+1}^i = y_i | X_n = \mathbf{x}\right]$$

   for $\mathbf{x} = (x_i)_{i=1}^N, \mathbf{y} = (y_i)_{i=1}^N \in S_k^N$.

   (b) Jump probability of each single particle:

   $$P\left[X_{n+1}^i = y | X_n = \mathbf{x}\right] := \begin{cases} \frac{\tau}{k}\left[p_r(x_i,y)\varphi(k\overline{\mathbf{x}}(x_i))/r - R(y)\right] & \text{if } y \neq x_i \\ 1 - \frac{\tau}{k}\sum_{z \in S_k \setminus \{x_i\}}\left[p_r(x_i,z)\varphi(k\overline{\mathbf{x}}(x_i))/r - R(z)\right] & \text{if } y = x_i \end{cases}$$

   for $y \in S_k$.

In order to study the macroscopic behaviour of the particle system we consider the empirical measure defined as

$$\overline{X}^N_{\tau,r,k}(t,x) := \overline{X}_{\lfloor t/\tau \rfloor}(x) + \left(\frac{t}{\tau} - \left\lfloor \frac{t}{\tau} \right\rfloor\right) \left[\overline{X}_{\lfloor t/\tau \rfloor + 1}(x) - \overline{X}^N_{\lfloor t/\tau \rfloor}(x)\right].$$

(Here we interpolate piecewise linearly in time).

Theorem 1. *There are constants $A_1, A_2, A_3 < \infty$ (depending on $(M, \tilde{g})$ and u_0) such that for all $T \geq 0$, all $N, k \in \mathbb{N}$, all $r \in (0,1]$ and all $\tau > 0$ satisfying (6):*

$$\begin{aligned} & E\left[\sup_{0\leq t\leq T}\sup_{f\in BL_1}\left|\int_M f(y)u(t,y)dy - \sum_{x\in S_k} f(x)\overline{X}^N_{\tau,r,k}(t,x)\right|\right] \\ \leq \;& E\left[\frac{1}{k}\sum_{x\in S_k}|u_0(x) - k\overline{X}_0(x)|\right] + T\left[A_1\tau + A_2 r + A_3 r^{-5/2} D_k + \frac{k}{2\tau\sqrt{N}}\right]. \end{aligned}$$

This means that the empirical measure of the particle system converges locally uniformly in time to the measure with density u as N and k tend to ∞ and τ and r tend to 0, provided that the initial configuration is chosen appropriately and the constraints $r^{-5/2}D_k \to 0$, $\frac{k}{\tau\sqrt{N}} \to 0$ and (6) are respected.

3 Remarks concerning related work

Particle approximations to nonlinear diffusion equations have been studied for a long time (see the books by Kipnis and Landim [15] and Spohn [24] for overviews). However, to the author's knowledge, no particle approximation for the Ricci flow has been proposed yet.

Written in the form (3), the Ricci flow equation on a surface is a logarithmic diffusion equation with source term and hence belongs to the class of filtration or generalized porous medium equations (see the book by Vázquez [25] for the analytical theory of this type of equations). For the porous medium equation particle approximations have been constructed by Ekhaus and Seppäläinen [9], Feng, Iscoe and Seppäläinen [10], Figalli and the author [11], Inoue [13], Jourdain [14], Oelschläger [18] and the author [22]. However, in all these papers the particle system and the limit equation are defined on a flat space ($\mathbb{R}^d$ or the flat torus $T^d := \mathbb{R}^d/\mathbb{Z}^d$). It seems that in the present paper for the first time a particle approximation for a nonlinear diffusion equation on a non-flat Riemannian manifold is given.

Another related work is the construction of a combinatorial (i.e. discrete) variant of the Ricci flow by Chow and Luo [7] (see also [16]). In their paper the surface is triangulated, and the metric is given as a function which assigns to each vertex v_i ($i = 1, \ldots, N$) of the triangulation a positive number r_i called radius. The distance between two adjacent vertices is defined as the sum of the two radii, and the triangles are realized geometrically as Euclidean (or even hyperbolic or spherical) triangles. In this case the curvature is concentrated at the vertices, and at the vertex v_i it is given as the angle defect $K_i := 2\pi - a_i$, where a_i is the sum of all angles adjacent to v_i. The (normalized) combinatorial Ricci flow is then given by the following system of ordinary differential equations:

$$\frac{dr_i}{dt} = -(K_i - K_{av})r_i, \tag{7}$$

where $K_{av} := 1/N \sum_{i=1}^N K_i$ is the average curvature. The main theorem in [7] is as follows: Under an approriate condition on the triangulation the solution of (7) exists for all time and converges exponentially fast to a metric of constant curvature.

4 Proof of Theorem 1

A crucial role in the proof is played by the *discrete Laplacian*, defined by

$$\Delta_{r,k} f(x) = \frac{1}{rk} \sum_{y \in S_k} p_r(x,y) \left[f(y) - f(x) \right].$$

The discrete Laplacian is the link between the Laplace-Beltrami operator appearing in equation (3) and the jump probabilities of the particles.

Lemma 2. *There is a constant $C(M) < \infty$ such that for all $f \in \mathcal{C}^4(M)$, all $x \in M$, all $r \in (0,1]$ and all $k \in \mathbb{N}$*

$$|\Delta_{r,k} f(x) - \Delta f(x)| \leq \frac{1}{2} \|\Delta\Delta f\|_\infty r + C(M) \|f\|_{\mathcal{C}^1(M)} r^{-(d+3)/2} D_k.$$

Proof. As an intermediate step we define the operator Δ_r by

$$\begin{aligned} \Delta_r f(x) &:= \frac{1}{r} \left[(P_r f)(x) - f(x) \right] \\ &= \frac{1}{r} \left[\int_M p_r(x,y) f(y) dy - f(x) \right] \end{aligned}$$

Here $(P_r)_{r \geq 0}$ is the heat semigroup on M. A Taylor expansion at $r = 0$ yields

$$(P_r f)(x) = f(x) + \Delta f(x) r + \frac{1}{2} \Delta\Delta (P_s f)(x) r^2$$

for a certain $s \in [0,r]$, hence

$$|(P_r f)(x) - f(x) - \Delta f(x) r| \leq \frac{1}{2} \|\Delta\Delta f\|_\infty r^2,$$

and therefore

$$|\Delta_r f(x) - \Delta f(x)| \leq \frac{1}{2} \|\Delta\Delta f\|_\infty r.$$

In order to estimate the difference between Δ_r and $\Delta_{r,k}$ we need the following heat kernel estimate (Theorems 4 and 6 in [4]): There is a constant $\tilde{C}(M) < \infty$ such that for all $r \in (0,1]$ and all $x \in M$

$$\|p_r(x,\cdot)\|_{\mathcal{C}^1(M)} \leq \tilde{C}(M) r^{-(d+1)/2}.$$

Using this estimate we obtain

$$\begin{aligned} |\Delta_{r,k} f(x) - \Delta_r f(x)| &= \frac{1}{r} \left| \frac{1}{k} \sum_{y \in S_k} p_r(x,y) \left[f(y) - f(x) \right] - \int_M p_r(x,y) [f(y) - f(x)] dy \right| \\ &\leq \frac{2}{r} \|p_r(x,\cdot)\|_{\mathcal{C}^1(M)} \|f\|_{\mathcal{C}^1(M)} D_k \\ &\leq 2\tilde{C}(M) r^{-(d+3)/2} \|f\|_{\mathcal{C}^1(M)} D_k, \end{aligned}$$

and the claim follows with $C(M) = 2\tilde{C}(M)$. □

We will use the following discrete version of the Euler characteristic of M (motivated by the Gauss-Bonnet theorem):

$$\chi_k(M) := \frac{1}{4\pi k} \sum_{x \in S_k} R(x).$$

Of course, $|\chi_k(M) - \chi(M)| = \frac{1}{4\pi} \left| \frac{1}{k} \sum_{x \in S_k} R(x) - \int_M R(y) dy \right| \leq \frac{1}{4\pi} \|R\|_{\mathcal{C}^1(M)} D_k$.

Proposition 2. *Let* $\Phi(u) := \varphi(u)u$. *For* $f : S_k \to \mathbb{R}$ *let*

$$K(f)(x) := f(x) + \tau \left[\Delta_{r,k}(\Phi(f))(x) - R(x) + 4\pi\chi_k(M)f(x)\right].$$

Let $t_n := n\tau$,

$$\varepsilon_n(x) := u(t_{n+1}, x) - K(u(t_n, \cdot))(x)$$

and

$$\delta_n(x) := k\overline{X}_{n+1}(x) - K(k\overline{X}_n)(x).$$

Then there are constants $C_1, C_2, C_3 < \infty$ *(depending on* $(M, \tilde{g})$, R *and* u_0*) such that for all* $n \in \mathbb{N}_0$

$$|\varepsilon_n(x)| \leq \tau \left[C_1\tau + C_2 r + C_3 r^{-5/2} D_k\right]$$

and

$$E\left[\delta_n(x)^2\right] \leq \frac{k^2}{4N}.$$

Proof. Clearly,

$$u(t_{n+1}, x) = u(t_n, x) + \int_{t_n}^{t_{n+1}} \left[\Delta(\log u)(s, x) - R(x) + 4\pi\chi(M)u(s, x)\right] ds,$$

so that

$$\begin{aligned} |\varepsilon_n(x)| \quad &\leq \quad \int_{t_n}^{t_{n+1}} |\Delta(\log u)(s, x) - \Delta_{r,k}(\log u)(t_n, x)| \, ds \\ &\quad + 4\pi \int_{t_n}^{t_{n+1}} |\chi(M)u(s, x) - \chi_k(M)u(t_n, x)| \, ds. \end{aligned}$$

For the first term we obtain using Lemma 2:

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} |\Delta(\log u)(s, x) - \Delta_{r,k}(\log u)(t_n, x)| \, ds \\ \leq \; & \int_{t_n}^{t_{n+1}} |\Delta(\log u)(s, x) - \Delta(\log u)(t_n, x)| \, ds + \tau \left|\Delta(\log u)(t_n, x) - \Delta_{r,k}(\log u)(t_n, x)\right| \\ \leq \; & \tilde{C}_1\tau^2 + \tau \left[C_2 r + \tilde{C}_3 r^{-5/2} D_k\right], \end{aligned}$$

where $\tilde{C}_1 := \frac{1}{2}\sup\left\{\frac{\partial}{\partial t}\Delta(\log u)(t, x) | t \in \mathbb{R}_+, x \in M\right\}$, $C_2 := \frac{1}{2}\sup\left\{\Delta\Delta(\log u)(t, x) | t \in \mathbb{R}_+, x \in M\right\}$ and $\tilde{C}_3 := C(M)\sup_{t\geq 0} \|\log u(t, \cdot)\|_{\mathcal{C}1(M)}$. Note that these constants are finite thanks to the uniform smoothness of u (see Remark 1). For the second term we obtain

$$\begin{aligned} & 4\pi \int_{t_n}^{t_{n+1}} |\chi(M)u(s, x) - \chi_k(M)u(t_n, x)| \, ds \\ \leq \; & 4\pi|\chi(M)| \int_{t_n}^{t_{n+1}} |u(s, x) - u(t_n, x)| \, ds + 4\pi\tau|\chi(M) - \chi_k(M)| u(t_n, x) \\ \leq \; & \tilde{C}_4\tau^2 + \tilde{C}_5\tau D_k, \end{aligned}$$

where $\tilde{C}_4 := 2\pi|\chi(M)|\sup\left\{\frac{\partial u}{\partial t}(t, x) | t \in \mathbb{R}_+, x \in M\right\}$ and $\tilde{C}_5 := \|R\|_{\mathcal{C}^1(M)}\|u\|_{L^\infty(\mathbb{R}_+\times M)}$. (These constants are finite as well by Remark 1). The claim concerning $\varepsilon_n(x)$ follows with $C_1 := \tilde{C}_1 + \tilde{C}_4$ and $C_3 := \tilde{C}_3 + \tilde{C}_5$.

We now estimate $\delta_n(x)$:

$$\begin{aligned} E\left[\delta_n(x)^2\right] \quad &= \quad E\left[k\overline{X}_{n+1}(x) - K(k\overline{X}_n)(x)\right]^2 \\ &= \quad E\left[k^2\overline{X}_{n+1}(x)^2\right] - 2E\left[k\overline{X}_{n+1}(x)K(k\overline{X}_n)(x)\right] + E\left[K(k\overline{X}_n)(x)^2\right]. \end{aligned}$$

The claim concerning $\delta_n(x)$ now follows from the following lemma. □

Lemma 3. *We have*

$$E\left[k\overline{X}_{n+1}(x)K(k\overline{X}_n)(x)\right] = E\left[K(k\overline{X}_n)(x)^2\right]$$

and

$$\left|E\left[k^2\overline{X}_{n+1}(x)^2\right] - E\left[K(k\overline{X}_n)(x)^2\right]\right| \leq \frac{k^2}{4N}.$$

For the proof we need the following key lemma:

Lemma 4.

$$\frac{k}{N}\sum_{i=1}^{N} P\left[X_{n+1}^i = y | X_n = \mathbf{x}\right] = K(k\overline{\mathbf{x}})(y).$$

Proof. We first compute $P\left[X_{n+1}^i = y | X_n = \mathbf{x}\right]$. In the case $y = x_i$ we obtain

$$\begin{aligned} P\left[X_{n+1}^i = y|X_n = \mathbf{x}\right] &= 1 + \tau\Delta_{r,k}\left(\varphi\left(k\overline{\mathbf{x}}\right)1_{x_i}\right)(y) + \tau k^{-1}\sum_{z\in S_k} R(z) - \tau k^{-1}R(y) \\ &= 1 + \tau\Delta_{r,k}\left(\varphi\left(k\overline{\mathbf{x}}\right)1_{x_i}\right)(y) + \tau 4\pi\chi_k(M) - \tau k^{-1}R(y), \end{aligned}$$

and in the case $y \neq x_i$

$$P\left[X_{n+1}^i = y|X_n = \mathbf{x}\right] = \tau\Delta_{r,k}\left(\varphi\left(k\overline{\mathbf{x}}\right)1_{x_i}\right)(y) - \tau k^{-1}R(y),$$

so that in both cases we obtain

$$P\left[X_{n+1}^i = y|X_n = \mathbf{x}\right] = 1_{\{y=x_i\}} + \tau\left[\Delta_{r,k}\left(\varphi\left(k\overline{\mathbf{x}}\right)1_{x_i}\right)(y) + 4\pi\chi_k(M)1_{\{y=x_i\}} - k^{-1}R(y)\right].$$

The claim now follows by summing over i. □

Proof of Lemma 3. The first claim is proven as follows:

$$\begin{aligned} E\left[k\overline{X}_{n+1}(z)K(k\overline{X}_n)(z)\right] &= \sum_{\overline{\mathbf{x}},\overline{\mathbf{y}}\in S_k^N} k\overline{\mathbf{y}}(z)K(k\overline{\mathbf{x}})(z)P\left[X_{n+1} = \mathbf{y} \text{ and } X_n = \mathbf{x}\right] \\ &= \sum_{\overline{\mathbf{x}}\in S_k^N}\left\{\sum_{\overline{\mathbf{y}}\in S_k^N} k\overline{\mathbf{y}}(z)P\left[X_{n+1} = \mathbf{y}|X_n = \mathbf{x}\right]\right\}K(k\overline{\mathbf{x}})(z)P\left[X_n = \mathbf{x}\right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{\overline{\mathbf{y}}\in S_k^N} k\overline{\mathbf{y}}(z)P\left[X_{n+1} = \mathbf{y}|X_n = \mathbf{x}\right] &= \frac{k}{N}\sum_{i=1}^{N}\sum_{\overline{\mathbf{y}}\in S_k^N} 1_{\{y_i=z\}}P\left[X_{n+1} = \mathbf{y}|X_n = \mathbf{x}\right] \\ &= \frac{k}{N}\sum_{i=1}^{N}\sum_{\substack{\overline{\mathbf{y}}\in S_k^N \\ y_i=z}} P\left[X_{n+1} = \mathbf{y}|X_n = \mathbf{x}\right] \\ &= \frac{k}{N}\sum_{i=1}^{N} P\left[X_{n+1}^i = z|X_n = \mathbf{x}\right] \\ &= K(k\overline{\mathbf{x}})(z), \end{aligned}$$

and the first claim follows.

The second claim is proven as follows:

$$\begin{aligned}
E\left[k^2\overline{X}_{n+1}(y)^2\right] &= \frac{k^2}{N^2}\sum_{i,j=1}^{N} P\left[X^i_{n+1}=y \text{ and } X^j_{n+1}=y\right] \\
&= \frac{k^2}{N^2}\sum_{i,j=1}^{N}\sum_{\mathbf{x}\in S_k^N} P\left[X^i_{n+1}=y \text{ and } X^j_{n+1}=y|X_n=\mathbf{x}\right] P\left[X_n=\mathbf{x}\right] \\
&= \frac{k^2}{N^2}\sum_{i,j=1}^{N}\sum_{\mathbf{x}\in S_k^N} P\left[X^i_{n+1}=y|X_n=\mathbf{x}\right] P\left[X^j_{n+1}=y|X_n=\mathbf{x}\right] P\left[X_n=\mathbf{x}\right] \\
&\quad + \frac{k^2}{N^2}\sum_{i=1}^{N}\left[P\left[X^i_{n+1}=y\right]-P\left[X^i_{n+1}=y\right]^2\right] \\
&= \sum_{\mathbf{x}\in S_k^N}\left[\frac{k}{N}\sum_{i=1}^{N} P\left[X^i_{n+1}=y|X_n=\mathbf{x}\right]\right]^2 P\left[X_n=\mathbf{x}\right] \\
&\quad + \frac{k^2}{N^2}\sum_{i=1}^{N}\left[P\left[X^i_{n+1}=y\right]-P\left[X^i_{n+1}=y\right]^2\right] \\
&= \sum_{\mathbf{x}\in S_k^N}\left[K(k\overline{\mathbf{x}})(y)\right]^2 P\left[X_n=\mathbf{x}\right] + \frac{k^2}{N^2}\sum_{i=1}^{N}\left[P\left[X^i_{n+1}=y\right]-P\left[X^i_{n+1}=y\right]^2\right] \\
&= E\left[K(k\overline{X}_n)(y)^2\right] + \frac{k^2}{N^2}\sum_{i=1}^{N}\left[P\left[X^i_{n+1}=y\right]-P\left[X^i_{n+1}=y\right]^2\right].
\end{aligned}$$

The second claim now follows from the fact that $P(A)-P(A)^2 \le \frac{1}{4}$ for any event A. ☐

Proposition 3. *For all $n\in\mathbb{N}_0$,*

$$\begin{aligned}
\frac{1}{k}\sum_{x\in S_k}|u(t_n,x)-k\overline{X}_n(x)| &\le \frac{1}{k}\sum_{x\in S_k}|u_0(x)-k\overline{X}_0(x)| + n\tau\left[C_1\tau + C_2 r + C_3 r^{-5/2}D_k\right] \\
&\quad + \sum_{i=0}^{n-1}\frac{1}{k}\sum_{x\in S_k}|\delta_n(x)|.
\end{aligned}$$

Proof. We write

$$e_n(x) := u(t_n,x) - k\overline{X}_n(x)$$

and

$$a_n(x) := \frac{\Phi(u(t_n,x))-\Phi(k\overline{X}_n(x))}{u(t_n,x)-k\overline{X}_n(x)}$$

(with the convention $\frac{0}{0}:=0$), so that $\Phi(u(t_n,x))-\Phi(k\overline{X}_n(x)) = a_n(x)e_n(x)$. Note that $0\le$

$a_n(x) \leq 1/u_{min}$. Proposition 2 implies that

$$\begin{aligned}
& e_{n+1}(x) \\
= \; & u(t_{n+1}, x) - k\overline{X}_{n+1}(x) \\
= \; & K(u(t_n, \cdot))(x) - K\left(k\overline{X}_n\right)(x) + \varepsilon_n(x) - \delta_n(x) \\
= \; & e_n(x) + \tau\Delta_{r,k}\left(\Phi(u(t_n, \cdot)) - \Phi\left(k\overline{X}_n\right)\right)(x) + 4\pi\tau\chi_k(M)\left[u(t_n, x) - k\overline{X}_n(x)\right] + \varepsilon_n(x) - \delta_n(x) \\
= \; & (1 + 4\pi\tau\chi_k(M))e_n(x) + \frac{\tau}{rk}\sum_{y\in S_k} p_r(x, y)\left[\Phi(u(t_n, y)) - \Phi\left(k\overline{X}_n(y)\right) - (\Phi(u(t_n, x)) + \Phi\left(k\overline{X}_n(x)\right)\right] \\
& + \varepsilon_n(x) - \delta_n(x) \\
= \; & (1 + 4\pi\tau\chi_k(M))e_n(x) + \frac{\tau}{rk}\sum_{y\in S_k} p_r(x, y)\left[a_n(y)e_n(y) - a_n(x)e_n(x)\right] + \varepsilon_n(x) - \delta_n(x) \\
= \; & \left[1 + 4\pi\tau\chi_k(M) - \frac{\tau}{kr}\sum_{y\in S_k} p_r(x, y)a_n(x)\right] e_n(x) + \frac{\tau}{rk}\sum_{y\in S_k} p_r(x, y)a_n(y)e_n(y) + \varepsilon_n(x) - \delta_n(x).
\end{aligned}$$

Since by (6) $\tau[4\pi|\chi(M)| + \frac{1}{kr}\sum_{y\in S_k} p_r(x, y)a_n(x)] \leq 1$, it follows that

$$\begin{aligned}
\sum_{x\in S_k} |e_{n+1}(x)| \; \leq \; & \sum_{x\in S_k}\left[1 - \tau 4\pi|\chi(M)| - \tau\sum_{y\in S_k} p_r(x, y)a_n(x)\right]|e_n(x)| \\
& + \sum_{x\in S_k}\frac{\tau}{rk}\sum_{y\in S_k} p_r(x, y)a_n(y)|e_n(y)| + \sum_{x\in S_k}|\varepsilon_n(x) - \delta_n(x)| \\
= \; & (1 - \tau 4\pi|\chi(M)|)\sum_{x\in S_k}|e_n(x)| + \sum_{x\in S_k}|\varepsilon_n(x) - \delta_n(x)|.
\end{aligned}$$

The claim follows by induction over n. □

Proof of Theorem 1. Fix $T \geq 0$, $t \in [0, T]$ and $f \in BL_1$. Let $n := \lfloor t/\tau \rfloor$ and $\lambda := \lfloor t/\tau \rfloor + 1 - t/\tau$, so that

$$\overline{X}^N_{\tau,r,k}(t, x) = \lambda\overline{X}_n(x) + (1 - \lambda)\overline{X}_{n+1}(x).$$

We have

$$\begin{aligned}
& \left|\int_M f(y)u(t, y)dy - \sum_{x\in S_k} f(x)\overline{X}^N_{\tau,r,k}(t, x)\right| \\
\leq \; & \left|\int_M f(y)u(t, y)dy - \frac{1}{k}\sum_{x\in S_k} f(x)u(t, x)\right| + \frac{1}{k}\sum_{x\in S_k}\left|u(t, x) - k\overline{X}^N_{\tau,r,k}(t, x)\right|.
\end{aligned}$$

The first term is bounded by $\|u(t, \cdot)\|_{\mathcal{C}^1(M)}D_k$. For the second term let us note that

$$\begin{aligned}
\left|u(t, x) - k\overline{X}^N_{\tau,r,k}(t, x)\right| \; \leq \; & |u(t, x) - \lambda u(n\tau, x) - (1 - \lambda)u((n + 1)\tau, x)| \\
& + \lambda\left|u(t_n, x) - k\overline{X}_n(x)\right| + (1 - \lambda)\left|u(t_{n+1}, x) - k\overline{X}_{n+1}(x)\right|
\end{aligned}$$

and that $|u(t, x) - \lambda u(n\tau, x) - (1 - \lambda)u((n + 1)\tau, x)| \leq C_5\tau^2$, where $C_5 := \frac{1}{8}\sup\{\frac{\partial^2 u}{\partial t^2}(t, x)|t \geq 0, x \in M\}$. Using Proposition 3 it follows that

$$\begin{aligned}
\frac{1}{k}\sum_{x\in S_k}\left|u(t, x) - k\overline{X}^N_{\tau,r,k}(t, x)\right| \; \leq \; & C_5\tau^2 + \frac{1}{k}\sum_{x\in S_k}|u_0(x) - k\overline{X}_0(x)| \\
& + (n + 1)\tau\left[C_1\tau + C_2 r + C_3 r^{-5/2}D_k\right] + \sum_{i=0}^{n}\frac{1}{k}\sum_{x\in S_k}|\delta_n(x)|.
\end{aligned}$$

Altogether we have

$$
\begin{aligned}
&\sup_{f\in BL_1}\sup_{0\le t\le T}\left|\int_M f(y)u(t,y)dy-\sum_{x\in S_k}f(x)\overline{X}^N_{\tau,r,k}(t,x)\right|\\
\le\ &\sup_{0\le t\le T}\|u(t,\cdot)\|_{\mathcal{C}^1(M)}D_k+C_5\tau^2+\frac{1}{k}\sum_{x\in S_k}|u_0(x)-k\overline{X}_0(x)|\\
&+T\Big[C_1\tau+C_2r+C_3r^{-5/2}D_k\Big]+\sum_{i=0}^{\lceil T/\tau\rceil}\frac{1}{k}\sum_{x\in S_k}|\delta_n(x)|,
\end{aligned}
$$

and the claim follows from Proposition 2. □

5 Particle approximation for Yamabe flow

Our method can also be applied to the non-normalized Yamabe flow

$$\frac{\partial g}{\partial t}=-Rg$$

on a closed manifold M of arbitrary dimension. The conformal class of g is still conserved so that for all $t\ge 0$ g_t can be written in the form $g_t=u(t,\cdot)^{4/(d+2)}\tilde{g}$ with a fixed reference metric $\tilde{g}$. In dimension $d\ge 3$ the analogue of Lemma 1 is

$$R_g=-\frac{4(d-1)}{d-2}\frac{1}{u}\Delta_{\tilde{g}}(u^{\frac{d-2}{d+2}})+R_{\tilde{g}}u^{-\frac{4}{d+2}}$$

(see e.g. [1], Corollary 1.161). This implies that

$$\frac{\partial u}{\partial t}=\begin{cases}\frac{(d-1)(d+2)}{d-2}\Delta_{\tilde{g}}(u^{\frac{d-2}{d+2}})-\frac{d+2}{4}R_{\tilde{g}}u^{\frac{d-2}{d+2}} & \text{if } d\ge 3\\ \Delta_{\tilde{g}}(\log u)-R_{\tilde{g}} & \text{if } d=2.\end{cases}\tag{8}$$

In contrast to (3) the total mass of solutions of (8) is not conserved. Therefore this equation cannot be approximated by a particle system with a fixed number of particles, but one has to allow for creation or destruction of particles. In order to prevent the manifold from shrinking in a finite time, we restrict ourselves to the case $R_{\tilde{g}}\le 0$. In this case, thanks to the maximum principle, u is bounded away from 0. We can therefore write (8) in the form

$$\frac{\partial u}{\partial t}=\Delta_{\tilde{g}}(\varphi(u)u)-R_{\tilde{g}}\psi(u).$$

with a smooth function φ and a smooth, non-decreasing and concave function ψ.

We now consider a system consisting of a finite number of particles of mass $\frac{1}{N}$ defined on a set S_k of k points of M satisfying (4). The particle dynamics takes place in discrete time steps of length $\tau>0$ and consists of two effects: particles can jump from one site to another, and independently of these jumps, new particles can be created. The jumps and the creation of particles happen in the following way: suppose that at time $n\tau$ (i.e. after n time steps) there are exactly $m\ge 1$ particles at a site $x\in S_k$. Then each of these particles, independently of the past and of the behaviour of all other particles, behaves as follows: With probability

$$\pi(x,y):=\frac{\tau}{k}p_r(x,y)\varphi(km/N)/r\tag{9}$$

it jumps to a point $y \neq x$, otherwise it stays at x. Moreover, independently of that and the past, new particles are created at x; their number is Poisson-distributed with expectation

$$\tau \frac{N}{k} |R(x)| \psi(\frac{km}{N}).$$

As in Section 2 $p_r(x, y)$ is the heat kernel on M at time r. In order to ensure that $\sum_{y \neq x} \pi(x, y) \leq 1$ (so that the jump rule (9) is well-defined), and for some technical reasons we will assume that

$$\tau \leq \left(\frac{1}{rk} \max(\|\varphi\|_\infty, |\Phi|_{lip}) \max_{x \in S_k} \sum_{y \in S_k \setminus \{x\}} p_r(x, y) \right)^{-1}, \tag{10}$$

where $\Phi(u) := \varphi(u)u$.

We denote the number of particles at site $x \in S_k$ at time $n\tau$ by $X_n(x)$, and we define $\overline{X}_n(x) := \frac{1}{N} X_n(x)$.

In order to study the macroscopic behaviour of the particle system we consider the empirical measure defined as

$$\overline{X}^N_{\tau,r,k}(t, x) := \overline{X}_{\lfloor t/\tau \rfloor}(x) + \left(\frac{t}{\tau} - \left\lfloor \frac{t}{\tau} \right\rfloor \right) \left[\overline{X}_{\lfloor t/\tau \rfloor + 1}(x) - \overline{X}^N_{\lfloor t/\tau \rfloor}(x) \right].$$

(Here we interpolate piecewise linearly in time). Then we have the following result:

Theorem 2. *For all $T \geq 0$ there are constants $A_1, A_2, A_3, A_4, A_5 < \infty$ (depending on $(M, \tilde{g})$, u_0 and T) such that for all $N, k \in \mathbb{N}$, all $r \in (0, 1]$ and all $\tau > 0$ satisfying (10):*

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq T} \sup_{f \in BL_1} \left| \int_M f(y) u(t, y) dy - \sum_{x \in S_k} f(x) \overline{X}^N_{\tau,r,k}(t, x) \right| \right] \\ \leq \;& A_1 E \left[\frac{1}{k} \sum_{x \in S_k} |u_0(x) - k\overline{X}_0(x)| \right] + A_2 \tau + A_3 r + A_4 r^{-(d+3)/2} D_k + A_5 \frac{k}{\tau \sqrt{N}}. \end{aligned}$$

This means that the empirical measure of the particle system converges locally uniformly in time to the measure with density u as N and k tend to ∞ and τ and r tend to 0, provided that the initial configuration is chosen appropriately and the constraints $r^{-(d+3)/2} D_k \to 0$, $\frac{k}{\tau\sqrt{N}} \to 0$ and (10) are respected.

The proof of Theorem 2 is quite similar to the one of Theorem 1. We start with an analogue of Lemma 2:

Proposition 4. *Let $\Phi(u) := \varphi(u)u$. For $f : S_k \to \mathbb{R}$ let*

$$K(f)(x) := f(x) + \tau \left[\Delta_{r,k}(\Phi(f))(x) - R(x)\psi(f(x)) \right].$$

Let $t_n := n\tau$,

$$\varepsilon_n(x) := u(t_{n+1}, x) - K(u(t_n, \cdot))(x)$$

and

$$\delta_n(x) := k\overline{X}_{n+1}(x) - K(k\overline{X}_n)(x).$$

Then for all $T \geq 0$ there are constants $C_1, C_2, C_3, C_4 < \infty$ (depending on $(M, \tilde{g})$, R, u_0 and T) such that for all $n \leq T/\tau$:

$$|\varepsilon_n(x)| \leq \tau \left[C_1 \tau + C_2 r + C_3 r^{-(d+3)/2} D_k \right]$$

and

$$E \left[\delta_n(x)^2 \right] \leq C_4 \frac{k^2}{N}.$$

Proof. The claim concerning $\varepsilon_n(x)$ is proven as the corresponding claim in Proposition 2.

In order to estimate $\delta_n(x)$ let us observe that the particles which are at x after $n+1$ time steps belong to three different classes:

1. those which were at another site $z \neq x$ at time $n\tau$,
2. those which were at x at time $n\tau$, and
3. those which were created in the last time step.

We denote their respective numbers by $X^{1,z}_{n+1}(x)$, $X^2_{n+1}(x)$ and $X^3_{n+1}(x)$. By definition of the particle system, conditioned on $X_n = X$, these quantities are independent and distributed as follows:

$$\begin{aligned}
X^{1,z}_{n+1}(x) &\sim B\left(X(z), \tau \frac{p_r(z,x)}{kr} \varphi(k\overline{X}(z))\right), \\
X^{2}_{n+1}(x) &\sim B\left(X(x), 1 - \sum_{z\neq x} \tau \frac{p_r(x,z)}{kr} \varphi(k\overline{X}(x))\right), \\
X^{3}_{n+1}(x) &\sim \pi\left(-\tau \frac{N}{k} R(x)\psi(k\overline{X}(x))\right).
\end{aligned}$$

Here $B(n,p)$ and $\pi(\lambda)$ denote the binomial and the Poisson distribution, respectively.

It follows that

$$\begin{aligned}
& E\left[k\overline{X}_{n+1}(x) | X_n = X\right] \\
= & \frac{k}{N}\left[\sum_{z\neq x} E\left[X^{1,z}_{n+1}(x)|X_n = X\right] + E\left[X^2_{n+1}(x)|X_n = X\right] + E\left[X^3_{n+1}(x)|X_n = X\right]\right] \\
= & \frac{k}{N}\left[\sum_{z\neq x} X(z)\tau\frac{p_r(z,x)}{kr}\varphi(k\overline{X}(z)) + X(x)\left[1 - \sum_{z\neq x}\tau\frac{p_r(x,z)}{kr}\varphi(k\overline{X}(x))\right] - \tau\frac{N}{k}R(x)\psi(k\overline{X}(x))\right] \\
= & \frac{k}{N}\left[X(x) + \tau\sum_{z\neq x}\frac{p_r(x,z)}{kr}\left[X(z)\varphi(k\overline{X}(z)) - X(x)\varphi(k\overline{X}(x))\right] - \tau\frac{N}{k}R(x)\psi(k\overline{X}(x))\right] \\
= & k\overline{X}(x) + \tau\sum_{z\neq x}\frac{p_r(x,z)}{kr}\left[k\overline{X}(z)\varphi(k\overline{X}(z)) - k\overline{X}(x)\varphi(k\overline{X}(x))\right] - \tau R(x)\psi(k\overline{X}(x)) \\
= & k\overline{X}(x) + \tau\left[\Delta_{r,k}(\Phi(k\overline{X}))(x) - R(x)\psi(k\overline{X}(x))\right] \\
= & K(k\overline{X})(x).
\end{aligned}$$

Moreover, using Bienaymé's equality and the fact that $p(1-p) \leq 1/4$ for every $p \in [0,1]$,

$$\begin{aligned}
& \operatorname{Var}\left(k\overline{X}_{n+1}(x)|X_n = X\right) \\
= \ & \frac{k^2}{N^2}\operatorname{Var}\left(X_{n+1}(x)|X_n = X\right) \\
= \ & \frac{k^2}{N^2}\left[\sum_{z\neq x}\operatorname{Var}\left[X^{1,z}_{n+1}(x)|X_n = X\right] + \operatorname{Var}\left[X^2_{n+1}(x)|X_n = X\right] + \operatorname{Var}\left[X^3_{n+1}(x)|X_n = X\right]\right] \\
\leq \ & \frac{k^2}{N^2}\left[\sum_{z\neq x} X(z)\frac{1}{4} + X(x)\frac{1}{4} - \tau\frac{N}{k}R(x)\psi(k\overline{X}(x))\right] \\
= \ & \frac{k^2}{4N}\sum_{z\in S_k}\overline{X}(z) - \tau\frac{k}{N}R(x)\psi(k\overline{X}(x)).
\end{aligned}$$

It follows that

$$\begin{aligned}
& E\left[\delta_n(x)^2\right] \\
= \ & E\left[\left(k\overline{X}_{n+1}(x) - K(k\overline{X}_n(x))\right)^2\right] \\
= \ & \sum_{X\in\mathbb{N}_0^{S_k}} E\left[\left(k\overline{X}_{n+1}(x) - K(k\overline{X}_n(x))\right)^2 |X_n = X\right] P\left[X_n = X\right] \\
= \ & \sum_{X\in\mathbb{N}_0^{S_k}} \operatorname{Var}\left(k\overline{X}_{n+1}(x)|X_n = X\right) P\left[X_n = X\right] \\
\leq \ & \sum_{X\in\mathbb{N}_0^{S_k}} \left[\frac{k^2}{4N}\sum_{z\in S_k}\overline{X}(z) - \tau\frac{k}{N}R(x)\psi(k\overline{X}(x))\right] P\left[X_n = X\right] \\
= \ & E\left[\frac{k^2}{4N}\sum_{z\in S_k}\overline{X}_n(z) - \tau\frac{k}{N}R(x)\psi(k\overline{X}_n(x))\right].
\end{aligned}$$

Since by the following lemma $E\left[\sum_{x\in S_k}\overline{X}_n(x)\right] \leq C(T)$ the claim follows. □

Lemma 5. *Let*

$$\overline{\psi}(u) := \int_0^u \frac{1}{\psi(x)}dx.$$

Then

$$E\left[\sum_{x\in S_k}\overline{X}_n(x)\right] \leq \overline{\psi}^{-1}\left(\overline{\psi}\left(E\left[\sum_{x\in S_k}\overline{X}_0(x)\right]\right) + \|R\|_\infty n\tau\right).$$

Proof.

$$
\begin{aligned}
E\left[\sum_{x\in S_k}\overline{X}_{n+1}(x)\right] &= \frac{1}{k}\sum_{x\in S_k}E\left[k\overline{X}_{n+1}(x)\right] \\
&= \frac{1}{k}\sum_{x\in S_k}E\left[K(k\overline{X}_n)(x)\right] \\
&= \frac{1}{k}\sum_{x\in S_k}E\left[k\overline{X}_n(x)+\tau\Delta_{r,k}(\Phi(k\overline{X}_n))(x)-\tau R(x)\psi(k\overline{X}_n(x))\right] \\
&= E\left[\sum_{x\in S_k}\overline{X}_n(x)\right]+\frac{\tau}{k}\sum_{x\in S_k}|R(x)|E\left[\psi(k\overline{X}_n(x))\right].
\end{aligned}
$$

Since ψ is concave it follows that

$$
\begin{aligned}
E\left[\sum_{x\in S_k}\overline{X}_{n+1}(x)\right]-E\left[\sum_{x\in S_k}\overline{X}_n(x)\right] &\leq \|R\|_\infty\tau\frac{1}{k}\sum_{x\in S_k}E\left[\psi(k\overline{X}_n(x))\right] \\
&\leq \|R\|_\infty\tau\psi\left(\frac{1}{k}\sum_{x\in S_k}E\left[k\overline{X}_n(x)\right]\right) \\
&= \|R\|_\infty\tau\psi\left(\sum_{x\in S_k}E\left[\overline{X}_n(x)\right]\right).
\end{aligned}
$$

Moreover, since $\overline{\psi}'=\frac{1}{\psi}$ and because ψ is nondecreasing,

$$
\begin{aligned}
&\overline{\psi}\left(E\left[\sum_{x\in S_k}\overline{X}_n(x)\right]\right)-\overline{\psi}\left(E\left[\sum_{x\in S_k}\overline{X}_{n-1}(x)\right]\right) \\
\leq\ & \frac{1}{\psi\left(E\left[\sum_{x\in S_k}\overline{X}_{n-1}(x)\right]\right)}\left[E\left[\sum_{x\in S_k}\overline{X}_n(x)\right]-E\left[\sum_{x\in S_k}\overline{X}_{n-1}(x)\right]\right] \\
\leq\ & \frac{1}{\psi\left(E\left[\sum_{x\in S_k}\overline{X}_{n-1}(x)\right]\right)}\|R\|_\infty\tau\psi\left(\sum_{x\in S_k}E\left[\overline{X}_{n-1}(x)\right]\right) \\
=\ & \|R\|_\infty\tau.
\end{aligned}
$$

The claim now follows by induction. □

Proposition 5. *For all $n\leq T/\tau$:*

$$
\begin{aligned}
\frac{1}{k}\sum_{x\in S_k}|u(t_n,x)-k\overline{X}_n(x)| \leq\ & (1+\tau\|R\|_\infty|\psi|_{lip})^n\frac{1}{k}\sum_{x\in S_k}|u_0(x)-k\overline{X}_0(x)| \\
&+\sum_{i=0}^{n-1}(1+\tau\|R\|_\infty|\psi|_{lip})^{n-1-i}\frac{1}{k}\sum_{x\in S_k}|\varepsilon_i(x)-\delta_i(x)|.
\end{aligned}
$$

Proof. As in the proof of Proposition 3 we write

$$
e_n(x):=u(t_n,x)-k\overline{X}_n(x),
$$

$$a_n(x) := \frac{\Phi(u(t_n,x)) - \Phi(k\overline{X}_n(x))}{u(t_n,x) - k\overline{X}_n(x)}$$

and

$$b_n(x) := \frac{\psi(u(t_n,x)) - \psi(k\overline{X}_n(x))}{u(t_n,x) - k\overline{X}_n(x)}$$

(with the convention $\frac{0}{0} := 0$) and obtain

$$\begin{aligned} e_{n+1}(x) &= \left[1 - \tau R(x) b_n(x) - \frac{\tau}{kr}\sum_{y\in S_k} p_r(x,y)a_n(x)\right] e_n(x) \\ &\quad + \frac{\tau}{rk}\sum_{y\in S_k} p_r(x,y)a_n(y)e_n(y) + \varepsilon_n(x) - \delta_n(x). \end{aligned}$$

Since by (10) $\frac{\tau}{kr}\sum_{y\in S_k} p_r(x,y)a_n(x) \leq 1$, it follows that

$$\begin{aligned} \sum_{x\in S_k} |e_{n+1}(x)| &\leq \sum_{x\in S_k}\left[1 - \tau R(x)b_n(x) - \frac{\tau}{rk}\sum_{y\in S_k} p_r(x,y)a_n(x)\right] |e_n(x)| \\ &\quad + \sum_{x\in S_k}\frac{\tau}{rk}\sum_{y\in S_k} p_r(x,y)a_n(y)|e_n(y)| + \sum_{x\in S_k}|\varepsilon_n(x) - \delta_n(x)| \\ &= \sum_{x\in S_k}[1 - \tau R(x)b_n(x)]\,|e_n(x)| + \sum_{x\in S_k}|\varepsilon_n(x) - \delta_n(x)| \\ &\leq [1 + \tau\|R\|_\infty|\psi|_{lip}]\sum_{x\in S_k}|e_n(x)| + \sum_{x\in S_k}|\varepsilon_n(x) - \delta_n(x)| \end{aligned}$$

The claim follows by induction over n. □

The proof of Theorem 2 can now be concluded in a similar way as the proof of Theorem 1.

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