

SPECTRAL SYNTHESIS IN $L^2(G)$

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ABSTRACT. For locally compact, second countable, type I groups G , we characterize all closed (two-sided) translation invariant subspaces of $L^2(G)$. We establish a similar result for K -biinvariant L^2 -functions (K a fixed maximal compact subgroup) in the context of semisimple Lie groups.

1. INTRODUCTION

Questions of spectral analysis and spectral synthesis were at the origin of harmonic analysis. All started with the use and study of Fourier series. Very vaguely one may say that spectral analysis consists in determining the “elementary components” of a mathematical object and spectral synthesis means “reconstructing that object from its elementary components”. After all kinds of investigations on Fourier series and Fourier transform, these theories found their final formulation for the algebra $L^1(G)$, where G is a locally compact abelian group, in the first half of the 20th century: Let \widehat{G} be the dual group of G (set of unitary characters with the topology of uniform convergence on compacta). For any closed ideal I in $L^1(G)$ one defines the hull of I by

$$h(I) := \{\chi \in \widehat{G} \mid \widehat{f}(\chi) = 0, \forall f \in I\},$$

where $\widehat{f}(\chi) := \int_G f(x)\overline{\chi(x)}dx$ is the Fourier transform of f on G . For a closed subset E of \widehat{G} , one defines the kernel of E by

$$k(E) := \{f \in L^1(G) \mid \widehat{f}(\chi) = 0, \forall \chi \in E\}.$$

One says that E is a set of spectral synthesis if $h(k(E)) = E$. If E is a set of spectral synthesis, then $I := k(E)$ is the only closed ideal whose hull equals E , which means that I is uniquely determined by its hull E . This is generally not the case and positive results about this difficult question are rare; exceptions are for instance $E = \emptyset$, the empty set, or $E = \{\chi\}$, a singleton. The so-called Wiener theorem (named like this as it is linked to Wiener’s famous Tauberian theorem), says that the empty set is a set of spectral synthesis or, equivalently, that if I is an ideal of $L^1(G)$ such that $h(I) = \emptyset$, then I is dense in $L^1(G)$. For arbitrary locally compact abelian groups, this result was first proved independently by Godement ([5]) and Segal ([9]) in 1947. Moreover, singletons $\{\chi\}$, $\chi \in \widehat{G}$, are also sets of spectral synthesis (see, for instance, [6]). If the group G is not abelian, the questions become even more complicated: In the definition of the hull and the kernel, the characters have to be replaced by the classes of unitary irreducible representations and Wiener’s theorem is not necessary true anymore. For instance, it is true for connected, simply connected, nilpotent Lie groups, whereas in the class of connected, simply connected, exponential solvable Lie groups, there are some groups which satisfy Wiener’s theorem and others which don’t. Needless to say that more general questions of spectral synthesis are often out of reach.

These problems become much easier if we replace the group algebra $L^1(G)$ by the Hilbert space $L^2(G)$. In fact, it is well known that the Fourier series of a function in $L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, converges in the L^2 -norm to the function itself. This is a first result on spectral synthesis in the L^2 -context. On the other hand, the almost everywhere convergence of the Fourier series to the given function

2000 *Mathematics Subject Classification.* Primary 43A85; Secondary 22E30.

Key words and phrases. Spectral synthesis, translation invariant subspace of $L^2(G)$, Plancherel theorem.

f is a much harder problem which was solved by Carleson in 1966 ([1]). In order to formulate a general statement about spectral synthesis in the L^2 -context, let us make some remarks: As $L^2(G)$ is not an algebra for convolution (except if G is compact), ideals have to be replaced by closed translation invariant subspaces. For locally compact abelian groups, a closed subspace V of $L^2(G)$ is (two-sided) translation invariant if, for all $s \in G$ and all $f \in V$, ${}_s f, f_s \in V$ where ${}_s f(x) := f(s^{-1}x)$ and $f_s(x) := f(xs)$ for almost all $x \in G$. This is equivalent to the fact that $f * g, g * f \in V$, for all $f \in V$ and all $g \in L^1(G)$ (resp. $g \in \mathcal{C}_c(G)$, space of continuous functions with compact support, which is dense in $L^1(G)$). The general question of spectral synthesis in $L^2(G)$ may then be formulated as follows: Determine completely all closed, translation invariant subspaces of $L^2(G)$. May they be uniquely characterized by their ‘‘hull’’? For $G = \mathbb{R}$ or G locally compact abelian, we have the following result: The closed translation invariant subspaces of $L^2(G)$ coincide with the subsets of the form

$$V = \{f \in L^2(G) \mid \mathcal{F}\mathcal{P}f(\chi) = 0, \text{ for almost all } \chi \in E\},$$

where $\mathcal{F}\mathcal{P}$ denotes the Fourier-Plancherel transform and E is an arbitrary measurable subset of \widehat{G} . A nice proof of this result for $G = \mathbb{R}$ may be found in ([8]) and for arbitrary abelian locally compact groups one may for instance look in ([6]). But what about the non-abelian case? In a short notice ([11]), Sutherland proves an abstract result in the context of Von Neumann algebras, which may be considered as a result on L^2 -spectral synthesis, but which is not as explicit as the previous abelian result. The foundation of a result of this type certainly lies in the correspondence between $L^2(G)$ and $L^2(\widehat{G})$ given by Plancherel’s theorem, but it seems to be nevertheless worthwhile to work out the details explicitly and to strive for a formulation similar to the one in the abelian situation. Hence the aim of the present paper is to prove for locally compact non abelian groups, a result very similar to the one stated previously for G abelian, by methods as elementary as possible and relying mainly on the Plancherel theorem for second countable Type I locally compact groups.

Certain assumptions have to be made on the group G (G second countable, type I), in order to assure the existence of a Plancherel measure on \widehat{G} and the correctness of Plancherel’s theorem. We will even be able to work in the non unimodular case, thanks to the generalized Plancherel theorem due to Duflo and Moore ([3]). As the Fourier Plancherel transform of an L^2 -function is again an L^2 -function, and hence defined only almost everywhere, some care has to be taken to define the support of such a function, respectively the support $\text{supp}_{\widehat{G}}V$ of a subspace V of $L^2(G)$ in \widehat{G} . We then prove the following result, similar to the one in the abelian case: The closed translation invariant subspaces of $L^2(G)$ coincide with the subsets of $L^2(G)$ of the form

$$V_{[E]} := \{f \in L^2(G) \mid \mathcal{F}\mathcal{P}f(\pi) = 0, \text{ for almost all } \pi \in E\},$$

where E is an arbitrary measurable subset of \widehat{G} , where $[E]$ denotes the equivalence class of E if we identify sets that differ only by a set of measure zero and where $\mathcal{F}\mathcal{P}$ denotes again the Fourier Plancherel transform given by Plancherel’s theorem. Moreover, $\widehat{G} = \text{supp}_{\widehat{G}}V \cup E \cup N$, where N is a subset of measure zero and where the measurable sets $\text{supp}_{\widehat{G}}V$ and E may be chosen to be disjoint. The proof is elementary, except for the use of the Plancherel theorem and for the necessity of some result on Fourier algebras.

Finally, we prove a corresponding result for K -biinvariant L^2 -functions (K a fixed maximal compact subgroup) in the context of semisimple Lie groups. This proof is an easy adaptation of the proof of Rudin for $G = \mathbb{R}$ ([8]).

2. PRELIMINARIES

Let us first recall some results on convolution for non unimodular groups. Let Δ denote the modular function of the group G . We define the left translation of L^2 -functions by

$$(\lambda(x)\phi)(y) := \phi(x^{-1}y), \quad \phi \in L^2(G), \text{ for all } x \in G, \text{ for almost all } y \in G$$

and the right translation by

$$(\rho(x)\phi)(y) := \Delta(x)^{\frac{1}{2}}\phi(yx), \quad \phi \in L^2(G), \text{ for all } x \in G, \text{ for almost all } y \in G.$$

Due to the definitions,

$$\|\lambda(x)\phi\|_2 = \|\rho(x)\phi\|_2 = \|\phi\|_2.$$

The corresponding actions of $L^1(G)$ on $L^2(G)$ (left and right regular representations) are then given by

$$\lambda(f)\phi = f * \phi \text{ and } \rho(f)\phi = \phi \tilde{*} f, \quad f \in L^1(G), \phi \in L^2(G),$$

where $f * \phi$ is the usual convolution product obtained by

$$f * \phi(x) = \int_G f(y)\phi(y^{-1}x)dy$$

and $\phi \tilde{*} f$ is given by

$$\phi \tilde{*} f(x) = \int_G f(y)\Delta(y)^{\frac{1}{2}}\phi(xy)dy.$$

One checks that for $f, k \in L^1(G)$, $\phi \in L^2(G)$, $f * \phi, \phi \tilde{*} f \in L^2(G)$ and

$$\phi \tilde{*} (f * k) = (\phi \tilde{*} k) \tilde{*} f.$$

This reflects the fact that $\rho(f * k) = \rho(f) \circ \rho(k)$. One also has

$$f * (\phi \tilde{*} k) = (f * \phi) \tilde{*} k,$$

which means that $\lambda(f) \circ \rho(k) = \rho(k) \circ \lambda(f)$, i. e. right and left regular representations commute. The preceding formulas will in particular be used for $f, k \in \mathcal{C}_c(G)$ (space of continuous functions with compact support).

The group von Neumann algebra $VN_\lambda(G)$ of the left regular representation is defined as the double commutant of $\lambda(\mathcal{C}_c(G))$: $VN_\lambda(G) := (\lambda(\mathcal{C}_c(G)))''$. See ([2]) for more details on von Neumann algebras and for the definition of type I groups.

The most general version of Plancherel's theorem for locally compact, second countable, type I groups G was first given by Duflo and Moore ([3]). It says: There exists a positive measure μ on \widehat{G} , a measurable field of unitary irreducible representations of G

$$\xi \in \widehat{G} \mapsto (\pi_\xi, \mathcal{H}_\xi)$$

(where $\xi = [\pi_\xi]$, i. e. ξ is the equivalence class of π_ξ , and where \mathcal{H}_ξ denotes the representation space of π_ξ) and a measurable field of positive operators $\xi \in \widehat{G} \mapsto K_\xi$ on \mathcal{H}_ξ , such that K_ξ is semi-invariant with weight Δ for every ξ and such that if $f \in L^1(G) \cap L^2(G)$, the operator $\widehat{f}(\pi_\xi) = \mathcal{FP}f(\pi_\xi) := \pi_\xi(f)K_\xi^{\frac{1}{2}}$ ("L²-Fourier transform", or Fourier Plancherel transform, as opposed to the

“ L^1 -Fourier transform” defined by $\mathcal{F}f(\xi) := \pi_\xi(f)$ extends to a Hilbert-Schmidt operator and

$$\begin{aligned}\|f\|_2^2 &= \int_{\widehat{G}}^\oplus \text{Tr}(K_\xi^{\frac{1}{2}} \pi_\xi(f)^* \pi_\xi(f) K_\xi^{\frac{1}{2}}) d\mu(\xi) \\ &= \int_{\widehat{G}}^\oplus \|\pi_\xi(f) K_\xi^{\frac{1}{2}}\|_{\text{HS}}^2 d\mu(\xi) \\ &= \int_{\widehat{G}}^\oplus \|\widehat{f}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi),\end{aligned}$$

as K_ξ is self-adjoint. Moreover, the map

$$\begin{aligned}L^1(G) \cap L^2(G) &\rightarrow L^2(\widehat{G}) := \int_{\widehat{G}}^\oplus \text{HS}(\mathcal{H}_\xi) d\mu(\xi) \\ f &\mapsto (\widehat{f}(\pi_\xi))_{\xi \in \widehat{G}}\end{aligned}$$

may be extended uniquely to an isometry between $L^2(G)$ and $L^2(\widehat{G})$. In this statement, semi-invariant with weight Δ means that

$$\pi_\xi(x) K_\xi \pi_\xi(x^{-1}) = \Delta(x) K_\xi, \quad \text{for all } x \in G.$$

Note that $K_\xi^{\frac{1}{2}}$ is then semi-invariant with weight $\Delta^{\frac{1}{2}}$ (thanks to the uniqueness of the positive square root). We will write $(\widehat{\phi}(\pi_\xi))_{\xi \in \widehat{G}}$ for the image of any L^2 -function ϕ under the previous isometry between $L^2(G)$ and $L^2(\widehat{G})$. Note that this operator field is only defined almost everywhere on \widehat{G} . Moreover, if

$$\langle \psi, \phi \rangle_2 := \int_G \overline{\psi(x)} \phi(x) dx$$

denotes the scalar product in $L^2(G)$, then

$$\langle \psi, \phi \rangle_2 = \int_{\widehat{G}}^\oplus \text{Tr}((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi)) d\mu(\xi).$$

If the group is type I and unimodular, the K_ξ 's are the identity operator and “ L^1 -” and “ L^2 -Fourier transforms” coincide. In that case, the Plancherel formula and the scalar product formula are

$$\|\phi\|_2^2 = \int_{\widehat{G}}^\oplus \|\pi_\xi(\phi)\|_{\text{HS}}^2 d\mu(\xi),$$

resp.

$$\langle \psi, \phi \rangle_2 = \int_{\widehat{G}}^\oplus \text{Tr}(\pi_\xi(\psi)^* \pi_\xi(\phi)) d\mu(\xi).$$

One checks easily the following results:

Proposition 2.1. *For $f \in \mathcal{C}_c(G)$ and $\phi \in L^2(G)$, we have:*

- (1) *For almost all ξ , $\pi_\xi(f * \phi) K_\xi^{\frac{1}{2}} = \pi_\xi(f) \pi_\xi(\phi) K_\xi^{\frac{1}{2}}$ if $\phi \in L^1(G) \cap L^2(G)$, hence $(f * \phi)\widehat{}(\pi_\xi) = \pi_\xi(f) \widehat{\phi}(\pi_\xi)$ if $\phi \in L^2(G)$ is arbitrary.*
- (2) *For almost all ξ , $\pi_\xi(\phi \check{*} f) K_\xi^{\frac{1}{2}} = (\pi_\xi(\phi) K_\xi^{\frac{1}{2}}) \pi_\xi(\check{f})$ if $\phi \in L^1(G) \cap L^2(G)$, hence $(\phi \check{*} f)\widehat{}(\pi_\xi) = \widehat{\phi}(\pi_\xi) \pi_\xi(\check{f})$ if $\phi \in L^2(G)$ is arbitrary. Here $\check{f}(x) := f(x^{-1})$.*
- (3) *$\|f * \phi\|_2 \leq \|f\|_1 \|\phi\|_2$ and $\|\phi \check{*} f\|_2 \leq \|\phi\|_2 \|f\|_1$*

Remark 2.2. Let $L^1(\widehat{G})$ be the Banach space of all measurable operator fields $\varphi = (\varphi(\xi))_{\xi \in \widehat{G}} \in (\mathcal{B}(\mathcal{H}_{\pi_\xi}))_{\xi \in \widehat{G}}$, such that

$$\|\varphi\|_1 := \int_{\widehat{G}} \|\varphi(\xi)\|_1 d\mu(\xi) < \infty.$$

In this formula, for a bounded operator a on a Hilbert space \mathcal{H} , $\|a\|_1 = \text{Tr}|a|$ denotes the trace class norm of a .

Proposition 2.3. *The Banach space $L^1(\widehat{G})$ and the Banach space $A(G)$ (Fourier algebra) of coefficients of the left regular representation are isomorphic.*

Proof. This proof is based on results of ([4]). As a matter of fact, every $\varphi \in L^1(\widehat{G})$ defines a bounded linear mapping ℓ_φ on $C^*(G)$:

$$\langle \ell_\varphi, a \rangle := \int_{\widehat{G}} \text{Tr}(\varphi(\xi)^* \circ \pi_\xi(a)) d\mu(\xi), \quad a \in C^*(G).$$

The operator norm of the functional ℓ_φ is then bounded by $\|\varphi\|_1$.

Let us show that ℓ_φ is contained in $A(G)$ and that the anti-linear mapping $L^1(\widehat{G}) \rightarrow A(G); \varphi \mapsto \ell_\varphi$ is isometric and surjective.

Let us recall that $L^\infty(\widehat{G})$ is the Fourier transform of $VN_\lambda(G)$. This Fourier transform is the usual one defined by:

$$\mathcal{F}(u)(\xi) := \pi_\xi(u), \quad u \in VN_\lambda(G).$$

Let $\varphi \in L^1(\widehat{G})$. Let $\varphi = \theta|\varphi|$ be the polar decomposition of φ , with, in particular, $|\varphi| \in L^1(\widehat{G})$, $\theta \in L^\infty(\widehat{G})$ and $\theta^*\theta|\varphi| = |\varphi|$. Then $\theta^*\theta\sqrt{|\varphi|} = \sqrt{|\varphi|}$ and $\sqrt{|\varphi|}$ is contained in $L^2(\widehat{G})$. Let $f \in L^2(G)$ such that $\widehat{f} = \sqrt{|\varphi|}$ and v in $VN_\lambda(G)$ such that $\mathcal{F}(v) = \theta$. Then for $a \in C^*(G)$:

$$\begin{aligned} c_{f,v(f)}^\lambda(a) &:= \langle v(f), \lambda(a)f \rangle_2 \\ &= \langle f, v^*\lambda(a)f \rangle_2 \\ &= \int_{\widehat{G}} \text{Tr}(K_\xi^{1/2} \pi_\xi(f)^* \pi_\xi(v^*) \pi_\xi(a) \pi_\xi(f) K_\xi^{1/2}) d\mu(\xi) \\ &= \int_{\widehat{G}} \text{Tr}((\pi_\xi(f) K_\xi^{1/2})^* \pi_\xi(v)^* \pi_\xi(a) \pi_\xi(f) K_\xi^{1/2}) d\mu(\xi) \\ &= \int_{\widehat{G}} \text{Tr}(\pi_\xi(f) K_\xi^{1/2} (\pi_\xi(f) K_\xi^{1/2})^* \pi_\xi(v)^* \pi_\xi(a)) d\mu(\xi) \\ &= \int_{\widehat{G}} \text{Tr}(\sqrt{|\varphi(\xi)|} (\sqrt{|\varphi(\xi)|})^* \theta(\xi)^* \pi_\xi(a)) d\mu(\xi) \\ &= \int_{\widehat{G}} \text{Tr}(\varphi(\xi)^* \pi_\xi(a)) d\mu(\xi) \\ &= \langle \ell_\varphi, a \rangle. \end{aligned}$$

Hence ℓ_φ is equal to the coefficient function $c_{f,v(f)}^\lambda$, i. e. $\ell_\varphi \in A(G)$. Now the norm of the linear functional $c_{f,v(f)}^\lambda$ is equal to $\|f\|_2^2$, hence also to $\|\widehat{f}\|_2^2 = \|\varphi\|_1$. Indeed we have that $\theta^*\theta\sqrt{|\varphi|} = \sqrt{|\varphi|}$ which tells us that $v^*v(f) = f$. But then $\lim_i v^*\lambda(a_i)v(f) = f$ for an approximate identity $(a_i)_i$ of norm 1 in $C^*(G)$ and by Kaplansky's density theorem, we can replace v by a strongly converging

net $(\lambda(b_j))_j$ of elements b_j in $C^*(G)$ of norm ≤ 1 . Hence

$$\begin{aligned} \|f\|_2^2 &= |\lim_{i,j} \langle f, v^* \lambda(a_i b_j) f \rangle| \\ &= |\lim_{i,j} \langle v(f), \lambda(a_i b_j) f \rangle| \\ &= \lim_{i,j} |c_{f,v(f)}^\lambda(a_i b_j)| \\ &\leq \|c_{f,v(f)}^\lambda\| \\ &\leq \|f\|_2^2. \end{aligned}$$

So

$$\|l_\varphi\| = \|c_{f,v(f)}^\lambda\| = \|f\|_2^2 = \|\widehat{f}\|_2^2 = \|\varphi\|_1$$

and the anti-linear map $\varphi \mapsto l_\varphi$ from $L^1(\widehat{G})$ into $A(G)$ is isometric and hence injective.

Conversely, every coefficient function c of the left regular representation can be written as

$$c = c_{f,v(f)}^\lambda$$

where $v \in VN_\lambda(G)$ is a partial isometry satisfying $f = v^*v(f)$ and $\|c\| = \|f\|_2^2$ ([4]).

But the operator field defined by

$$\varphi(\xi)^* := \pi_\xi(f) K_\xi^{1/2} (\pi_\xi(f) K_\xi^{1/2})^* \pi_\xi(v)^*, \xi \in \widehat{G},$$

is contained in $L^1(\widehat{G})$, satisfies $l_\varphi = c$ and $\|c\| = \|\varphi\|_1 = \|l_\varphi\|$. This proves the surjectivity of the map $\varphi \mapsto l_\varphi$ from $L^1(\widehat{G})$ onto $A(G)$. □

Remark 2.4. Let $B(G)$ be the Fourier-Stieltjes algebra. As $A(G) \subset B(G) = (C^*(G))'$, dual of $C^*(G)$ (see [4]), an element φ of $L^1(\widehat{G})$ such that $\langle l_\varphi, a \rangle = \int_{\widehat{G}} \text{Tr}(\varphi(\xi)^* \pi_\xi(a)) d\mu(\xi) = 0$ for all $a \in C^*(G)$, has to be zero itself. This is the key to the proof of Theorem 3.3.

3. MAIN THEOREM

Let G be a locally compact, second countable, type I group. We shall characterize the closed translation invariant subspaces of $L^2(G)$. Since G is second countable, $L^2(G)$ is separable. That is, $L^2(G)$ admits a countable dense subset. The same is true for any closed subspace of $L^2(G)$.

In order to deal efficiently with sets defined modulo subsets of measure zero, let us introduce the following notations: We assume that \widehat{G} is endowed with the Plancherel measure. For any two measurable subsets E and F of \widehat{G} , we define their **symmetric difference** by

$$E \Delta F := (E \setminus F) \cup (F \setminus E).$$

Let $\mathcal{M}(\widehat{G})$ denote the set of all measurable subsets of \widehat{G} . We introduce an equivalence relation \sim on $\mathcal{M}(\widehat{G})$ by

$$E \sim F \Leftrightarrow E \Delta F \text{ has measure zero.}$$

For every measurable subset E of \widehat{G} , we write $[E]$ for the equivalence class of E and we denote by $\widetilde{\mathcal{M}}(\widehat{G}) := \mathcal{M}(\widehat{G}) / \sim$ the set of all such equivalence classes.

Definition 3.1. Let V be a closed subspace of $L^2(G)$ and let $\{\phi_i \mid i \in \mathbb{N}\}$ be a countable dense subset of V . We define the support of V in \widehat{G} as

$$\text{supp}_{\widehat{G}} V = \cup_{i \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\phi}_i(\pi_\xi) \neq 0\} \text{ modulo a set of measure zero.}$$

Then $[\text{supp}_{\widehat{G}}V] \subset \widetilde{\mathcal{M}}(\widehat{G})$ is well defined. In fact, let $\{\phi_i \mid i \in \mathbb{N}\}$ and $\{\eta_j \mid j \in \mathbb{N}\}$ be two countable dense subsets of V . Let $j \in \mathbb{N}$ be fixed such that $\{\xi \in \widehat{G} \mid \widehat{\eta}_j(\pi_\xi) \neq 0\}$ is a set of positive measure. Using $\eta_j = \lim_{k \rightarrow \infty} \phi_{j_k}$ in $L^2(G)$ and

$$\int_{\widehat{G}} \|\widehat{\eta}_j(\pi_\xi) - \widehat{\phi}_{j_k}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi) = \|\eta_j - \phi_{j_k}\|_2^2 \rightarrow 0, \text{ as } k \rightarrow \infty,$$

it follows that $\{\xi \mid \widehat{\phi}_{j_k}(\pi_\xi) \neq 0\}$ is a set of positive measure, for some $j_k \in \mathbb{N}$, and

$$\{\xi \in \widehat{G} \mid \widehat{\eta}_j(\pi_\xi) \neq 0\} \subseteq \cup_{i \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\phi}_i(\pi_\xi) \neq 0\} \text{ modulo a set of measure zero.}$$

Therefore,

$$\cup_{j \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\eta}_j(\pi_\xi) \neq 0\} \subseteq \cup_{i \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\phi}_i(\pi_\xi) \neq 0\} \text{ modulo a set of measure zero.}$$

Also similarly we can prove the converse. This shows that

$$\cup_{j \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\eta}_j(\pi_\xi) \neq 0\} = \cup_{i \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\phi}_i(\pi_\xi) \neq 0\} \text{ modulo a set of measure zero,}$$

and $[\text{supp}_{\widehat{G}}V]$ is well defined.

Definition 3.2. Let V be a closed subspace of $L^2(G)$. We say that V is (both sided) translation invariant if and only if $\lambda(x)\phi, \rho(x)\phi \in V$ for all $x \in G$, if $\phi \in V$. This is equivalent to say that $\lambda(f)\phi = f * \phi \in V$ and $\rho(f)\phi = \phi \tilde{*} f \in V$, for all $f \in \mathcal{C}_c(G)$, if $\phi \in V$.

We then have the following result:

Theorem 3.3. *Let G be a locally compact, second countable, type I group and let V be a closed (both sided) translation invariant subspace of $L^2(G)$. Then there exists a measurable subset E of \widehat{G} such that*

$$V = V_{[E]} := \{\phi \in L^2(G) \mid \widehat{\phi}(\pi_\xi) = \mathcal{FP}\phi(\pi_\xi) = 0, \text{ for almost every } \xi \in E\}.$$

Moreover, if \mathcal{V} denotes the collection of all translation invariant closed subspaces of $L^2(G)$, then the map

$$\widetilde{\mathcal{M}}(\widehat{G}) \rightarrow \mathcal{V} \tag{3.1}$$

$$[E] \mapsto V_{[E]} \tag{3.2}$$

is a bijection.

Proof. It is obvious that $V_{[E]}$ is a closed translation invariant subspace of $L^2(G)$.

To prove injectivity, let us assume that E and F are measurable subsets of \widehat{G} such that $[E] \neq [F]$. Then at least one of the two measurable sets $E \setminus F$ and $F \setminus E$ has non zero measure. Let us assume that $\mu(E \setminus F) \neq 0$. There there exists a $\vartheta \in L^2(\widehat{G})$ such that the operator field $\psi := 1_{E \setminus F} \vartheta$ is different from 0. By Plancherel's theorem, there exists $\zeta \in L^2(G)$ such that $\widehat{\zeta} = \psi$. By construction, $\zeta \notin V_{[E]}$ and $\zeta \in V_{[F]}$. This proves the injectivity of the map $[E] \mapsto V_{[E]}$.

To prove the surjectivity, let V be a closed translation invariant subspace of $L^2(G)$. As $L^2(G)$ is separable, there exists a countable dense subset $\{\phi_k \mid k \in \mathbb{N}\}$ of V . Let

$$\text{supp}_{\widehat{G}}V = \cup_{k \in \mathbb{N}} \{\xi \in \widehat{G} \mid \widehat{\phi}_k(\pi_\xi) \neq 0\} =: \cup_{k \in \mathbb{N}} S_k,$$

defined modulo a set of measure zero. Let $F_k := \{\xi \in \widehat{G} \mid \widehat{\phi}_k(\pi_\xi) = 0\}$. We note that the sets S_k and F_k are measurable. We have, for all $k \in \mathbb{N}$, $\widehat{G} = S_k \cup F_k \cup N_k$, where N_k is a set of measure zero and $S_k \cap F_k = \emptyset$. Let $E := \cap_{k \in \mathbb{N}} F_k$. Then $\widehat{G} = \text{supp}_{\widehat{G}}V \cup E \cup N$, where N is a set of measure zero and where $\text{supp}_{\widehat{G}}V$ and E are chosen to be disjoint. Let

$$V_{[E]} := \{\phi \in L^2(G) \mid \widehat{\phi}(\pi_\xi) = 0, \text{ for almost all } \xi \in E\}.$$

By construction, $\phi_k \in V_{[E]}$ for all k and hence $V \subset V_{[E]}$ as $V_{[E]}$ is a closed subspace of $L^2(G)$. Let us assume that $V \subsetneq V_{[E]}$. Then there exists $0 \neq \psi \in V^\perp \cap V_{[E]}$. For any $\phi \in V$ we have

$$\begin{aligned} 0 &= \langle \psi, \phi \rangle_2 \\ &= \int_{\widehat{G}}^\oplus \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi)\right) d\mu(\xi). \end{aligned}$$

As V is right invariant, we may replace ϕ by $\phi \tilde{f}$, for $f \in \mathcal{C}_c(G)$ arbitrary. We obtain

$$\int_{\widehat{G}}^\oplus \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) \pi_\xi(f)\right) d\mu(\xi) = 0.$$

Let now $a \in C^*(G)$ be arbitrary. One has

$$\begin{aligned} \left| \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) (\pi_\xi(f) - a(\pi_\xi))\right) \right| &\leq \|(\widehat{\psi})(\pi_\xi)^*\|_{\text{HS}} \cdot \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}} \cdot \sup_{\xi \in \widehat{G}} \|\pi_\xi(f) - a(\pi_\xi)\|_{\text{op}} \\ &= \|\widehat{\psi}(\pi_\xi)\|_{\text{HS}} \cdot \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}} \cdot \|f - a\|_{C^*(G)} \end{aligned}$$

Hence

$$\begin{aligned} \left| \int_{\widehat{G}}^\oplus \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) a(\pi_\xi)\right) d\mu(\xi) \right| &= \left| \int_{\widehat{G}}^\oplus \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) (\pi_\xi(f) - a(\pi_\xi))\right) d\mu(\xi) \right| \\ &\leq \|f - a\|_{C^*(G)} \cdot \int_{\widehat{G}}^\oplus \|\widehat{\psi}(\pi_\xi)\|_{\text{HS}} \cdot \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}} d\mu(\xi) \\ &\leq \|f - a\|_{C^*(G)} \cdot \left(\int_{\widehat{G}}^\oplus \|\widehat{\psi}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_{\widehat{G}}^\oplus \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi) \right)^{\frac{1}{2}} \\ &= \|f - a\|_{C^*(G)} \cdot \|\psi\|_2 \cdot \|\phi\|_2 \end{aligned}$$

As $\mathcal{C}_c(G)$ is dense in $L^1(G)$ and hence in $C^*(G)$, this implies that

$$\int_{\widehat{G}}^\oplus \text{Tr}\left((\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) a(\pi_\xi)\right) d\mu(\xi) = 0,$$

for all $a \in C^*(G)$. On the other hand, as

$$\begin{aligned} \int_{\widehat{G}}^\oplus \text{Tr}\left|(\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi)\right| d\mu(\xi) &\leq \int_{\widehat{G}}^\oplus \|\widehat{\psi}(\pi_\xi)\|_{\text{HS}} \cdot \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}} d\mu(\xi) \\ &\leq \left(\int_{\widehat{G}}^\oplus \|\widehat{\psi}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi) \right)^{\frac{1}{2}} \cdot \left(\int_{\widehat{G}}^\oplus \|\widehat{\phi}(\pi_\xi)\|_{\text{HS}}^2 d\mu(\xi) \right)^{\frac{1}{2}} \\ &= \|\psi\|_2 \cdot \|\phi\|_2 \\ &< +\infty \end{aligned}$$

by formula (2.13) of ([7]), $(\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) \in L^1(\widehat{G})$. We may now use the fact that $L^1(\widehat{G})$ is isomorphic to the space $A(G)$ of coefficients of the representation λ (see Proposition 2.3 and Remark 2.4) to conclude that $(\widehat{\psi}(\pi_\xi))^* \widehat{\phi}(\pi_\xi) = 0$ for almost all $\xi \in \widehat{G}$.

As V is left invariant, we may now replace ϕ by $f * \phi$, for $f \in L^1(G)$ arbitrary. We obtain $(\widehat{\psi}(\pi_\xi))^* \pi_\xi(f) \widehat{\phi}(\pi_\xi) = 0$ for $\xi \in \widehat{G} \setminus N(f, \phi)$, where $N(f, \phi)$ is a set of measure zero. We know that $\widehat{G} = \text{supp}_{\widehat{G}} V \cup E \cup N$, where N is of measure zero and where $\text{supp}_{\widehat{G}} V$ and E are chosen to be disjoint. Moreover, $\widehat{\psi}(\pi_\xi) = 0$, if $\xi \in E$ (up to a set of measure zero). Let now $\{f_j \mid j \in \mathbb{N}\} \subset \mathcal{C}_c(G)$ be a countable dense subset of $L^1(G)$ (such a dense subset exists, because G is second countable). Let $N(\phi) := \cup_{j \in \mathbb{N}} N(f_j, \phi)$, which is again a set of measure zero. We have $(\widehat{\psi}(\pi_\xi))^* \pi_\xi(f_j) \widehat{\phi}(\pi_\xi) = 0$,

for all j , for all $\xi \in \widehat{G} \setminus N(\phi)$. In particular, let us consider the countable dense subset $\{\phi_k \mid k \in \mathbb{N}\}$ of V and let us put $N' := \cup_k N(\phi_k)$, which is again of measure zero. We get $(\widehat{\psi}(\pi_\xi))^* \pi_\xi(f_j) \widehat{\phi}_k(\pi_\xi) = 0$ for all j, k , if $\xi \in \widehat{G} \setminus N'$. Recall that $\widehat{\psi}(\pi_\xi) = 0$, if $\xi \in E$. Let now $\xi \in \text{supp}_{\widehat{G}} V \setminus (\text{supp}_{\widehat{G}} V \cap N')$. Take any $\phi \in V$ such that $\widehat{\phi}(\pi_\xi) \neq 0$ (for instance $\phi = \phi_k$ for some k) and let $v_\xi \in \mathcal{H}_\xi$ such that $\widehat{\phi}(\pi_\xi)v_\xi \neq 0$. Then, by irreducibility of π_ξ , the set $\{\pi_\xi(f_j)\widehat{\phi}(\pi_\xi)v_\xi \mid j \in \mathbb{N}\}$ generates a dense subspace of \mathcal{H}_ξ . This implies that $\widehat{\psi}(\pi_\xi) = 0$. As $\widehat{G} = \text{supp}_{\widehat{G}} V \cup E \cup N$ where N is a set of measure zero, we have proven that $\widehat{\psi}(\pi_\xi) = 0$ almost everywhere. So $\psi = 0$ in $L^2(G)$, by Plancherel's theorem, which is a contradiction. This proves that $V = V_{[E]}$ and that the map $[E] \mapsto V_{[E]}$ is a surjection. \square

Corollary 3.4. *Let G be a locally compact, second countable, type I group. Let V be a closed (both sided) translation invariant subspace of $L^2(G)$ such that $\text{supp}_{\widehat{G}} V = \widehat{G}$. Then $V = L^2(G)$.*

4. EASY PROOF FOR SEMISIMPLE LIE GROUPS

Let G be a connected, noncompact, semisimple Lie group with finite centre and let K be a fixed maximal compact subgroup of G . Let \mathfrak{g} be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition of \mathfrak{g} . The Lie algebra of K is \mathfrak{k} . We fix a maximal abelian subspace \mathfrak{a} of \mathfrak{s} . Let $\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha)$ be the root space decomposition of \mathfrak{g} . Here $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ where \mathfrak{m} is the centralizer of \mathfrak{a} in \mathfrak{k} and $\Sigma \subseteq \mathfrak{a}^*$ is the root system of $(\mathfrak{g}, \mathfrak{a})$. Let W be the Weyl group associated to Σ . We choose a set Σ^+ of positive roots. Let $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Then \mathfrak{n} is a nilpotent subalgebra of \mathfrak{g} . The element $\rho \in \mathfrak{a}^*$ is defined by

$$\rho(H) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H)$$

where $m_\alpha = \dim(\mathfrak{g}_\alpha)$. Let A be the analytic subgroup of G with Lie algebra \mathfrak{a} . Then A is a closed subgroup of G and the exponential map is an isomorphism from \mathfrak{a} onto A . Let N be the analytic subgroup of G with Lie algebra \mathfrak{n} , and let M be the centralizer of A in K . Any element $x \in G$ has a unique (Iwasawa) decomposition $x = k(x)e^{H(x)}n(x)$ where $k(x) \in K, H(x) \in \mathfrak{a}, n(x) \in N$.

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the elementary spherical function ϕ_λ on G is given by

$$\phi_\lambda(x) = \int_K e^{-(i\lambda + \rho)H(x^{-1}k)} dk.$$

It satisfies the following properties:

- (1) It is K -biinvariant i.e., $\phi_\lambda(k_1 x k_2) = \phi_\lambda(x)$ for all $k_1, k_2 \in K$ and $x \in G$. It is also W -invariant in $\lambda \in \mathfrak{a}_\mathbb{C}^*$ i.e., $\phi_{w \cdot \lambda}(x) = \phi_\lambda(x)$ for all $w \in W$ and $x \in G$.
- (2) The function $\phi_\lambda(x)$ is C^∞ in x and holomorphic in λ .
- (3) It is a joint eigenfunction for all G -invariant differential operators on G/K . In particular for the Laplacian Δ on G/K ,

$$\Delta \phi_\lambda = -(\langle \lambda, \lambda \rangle + \|\rho\|^2) \phi_\lambda.$$

- (4) $\int_K \phi_\lambda(xky) dk = \phi_\lambda(x)\phi_\lambda(y)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $x, y \in G$.

For a K -biinvariant function f on G its spherical Fourier transform is defined by

$$\mathcal{F}f(\lambda) = \widehat{f}(\lambda) = \int_G f(x)\phi_\lambda(x) dx$$

for suitable $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then we have the following inversion formula:

$$f(x) = \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2} d\lambda$$

where $|c(\lambda)|^{-2}$ is the Harish-Chandra c -function.

The set of all square integrable K -biinvariant functions is denoted by $L^2(G//K)$. For a function $f \in L^2(G//K)$, the left translation of f by an element $x \in G$, is defined by $\tau_x f(y) = f(x^{-1}y)$ for all $y \in G$. The function $\tau_x f$ is not a K -biinvariant function. Define the projection of $\tau_x f$ on $L^2(G//K)$ by

$$(\tau_x f)_K(y) = \int_K \tau_x f(ky) dk = \int_K f(x^{-1}ky) dk.$$

Then for $f \in L^2(G//K)$, the left translated projection $(\tau_x f)_K \in L^2(G//K)$. Let M be a subspace of $L^2(G//K)$. We say that M is (left) translation invariant in $L^2(G//K)$ if $(\tau_x M)_K \subseteq M$ for all $x \in G$. We characterize such closed translation invariant subspaces of $L^2(G//K)$ (cf. [8, 9.16]).

Lemma 4.1. *For $f \in C_c^\infty(G//K)$, we have $\widehat{(\tau_x f)_K}(\lambda) = \phi_\lambda(x)\widehat{f}(\lambda)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.*

Proof. We have

$$\begin{aligned} \widehat{(\tau_x f)_K}(\lambda) &= \int_G (\tau_x f)_K(y) \phi_\lambda(y) dy \\ &= \int_G \int_K f(x^{-1}ky) \phi_\lambda(y^{-1}) dk dy \\ &= \int_G \int_K f(z) \phi_\lambda(k^{-1}xz) dk dz. \end{aligned}$$

Since $\phi_\lambda(k_1 x k_2) = \phi_\lambda(x)$ for all $x \in G$ and $k_1, k_2 \in K$, we have

$$\begin{aligned} \widehat{(\tau_x f)_K}(\lambda) &= \int_G f(z) \phi_\lambda(xz) dz \\ &= \int_G \int_K f(kz) \phi_\lambda(xz) dz dk \\ &= \int_G \int_K f(z) \phi_\lambda(xk^{-1}z) dz dk \\ &= \phi_\lambda(x)\widehat{f}(\lambda) \end{aligned}$$

□

Theorem 4.2. *Let M be a closed (left) translation invariant subspace of $L^2(G//K)$. Then there exists a measurable set $E \subseteq \mathfrak{a}^*$ such that*

$$M = \left\{ f \in L^2(G//K) \mid \widehat{f}(\lambda) = 0 \text{ almost every where on } E \right\}.$$

Proof. By Lemma 4.1, $\{f \in L^2(G//K) \mid \widehat{f}(\lambda) = 0 \text{ almost everywhere on } E\}$ is a closed, left translation invariant subspace of $L^2(G//K)$.

Let now M be an arbitrary closed, left translation invariant subspace of $L^2(G//K)$. The image of $L^2(G//K)$ under the spherical transform is $L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} d\lambda)^W$. Let \widehat{M} be the image of M , under the spherical transform. Also let

$$P : L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} d\lambda)^W \rightarrow \widehat{M}$$

be the orthogonal projection. Then for all $f, g \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} d\lambda)^W$ we have $f - Pf \perp Pg$. Since M is a translation invariant subspace of $L^2(G//K)$, it follows from Lemma 4.1 that $f - Pf \perp \phi_\lambda(x)Pg$ for all $x \in G$. That is,

$$\int_{\mathbb{R}} (f - Pf)(\lambda) \overline{Pg(\lambda)} \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda = 0.$$

So $\mathcal{F}^{-1}((f - Pf)\overline{Pg})(x) = 0$ for all $x \in G$. By Plancherel's theorem we have

$$\|(f - Pf)\overline{Pg}\|_{L^2(\mathfrak{a}^*, |c(\lambda)|^{-2} d\lambda)} = \|\mathcal{F}^{-1}((f - Pf)\overline{Pg})\|_{L^2(G//K)}.$$

This implies that $(f - Pf)(\lambda)\overline{Pg}(\lambda) = 0$ a.e. on \mathfrak{a}^* . That is, $f \cdot \overline{Pg} = Pf \cdot \overline{Pg}$ a.e.. Interchanging f and g we get $g \cdot \overline{Pf} = Pg \cdot \overline{Pf}$ a.e.. Therefore, $f \cdot \overline{Pg} = \overline{g} \cdot Pf$ a.e. on \mathfrak{a}^* . Let $g(\lambda) = e^{-\lambda^2}$ on \mathfrak{a}^* and $\phi(\lambda) = \frac{\overline{Pg}(\lambda)}{\overline{g}(\lambda)}$. For almost every λ , we have $Pf(\lambda) = \phi(\lambda)f(\lambda)$. Also for $f \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2}d\lambda)^W$, we have $\phi^2(\lambda)f(\lambda) = \phi(\lambda)Pf(\lambda) = P^2f(\lambda) = Pf(\lambda) = \phi(\lambda)f(\lambda)$. This shows that, $\phi^2(\lambda) = \phi(\lambda)$. That is, $\phi(\lambda) = 1$ or 0 almost everywhere. Let $E = \{\lambda \in \mathfrak{a}^* \mid \phi(\lambda) = 0\}$. Then

$$\begin{aligned}\widehat{M} &= \{f \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2}d\lambda)^W \mid f(\lambda) = Pf(\lambda) = \phi(\lambda)f(\lambda)\} \\ &= \{f \in L^2(\mathfrak{a}^*, |c(\lambda)|^{-2}d\lambda)^W \mid f(\lambda) = 0 \text{ almost everywhere on } E\}.\end{aligned}$$

Hence

$$M = \{\alpha \in L^2(G//K) \mid \widehat{\alpha}(\lambda) = 0 \text{ almost everywhere on } E\}.$$

□

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