# Automorphisms and derivations of quantum and classical Poisson algebras

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**A. Introduction**

**I. Characterizations by associative algebras**

1. Compact topological spaces, continuous functions (Gel'fand-Neumark theory):

$$C^0(T_1) \simeq C^0(T_2) \Rightarrow T_1 \simeq T_2$$

$$p^* = \{f \in C^0(T) : f(p) = 0\} \;\; (p \in T)$$

$\mathcal{I}(C^0(T))$: set of maximal ideals of $C^0(T)$

$$\begin{array}{ccc} p_1^* \in \mathcal{I}(C^0(T_1)) & \rightarrow & p_2^* \in \mathcal{I}(C^0(T_2)) \\ \uparrow & & \uparrow \\ p_1 \in T_1 & \rightarrow & p_2 \in T_2 \end{array}$$

2.$a$. Compact smooth manifolds, smooth functions: similar

2.$b$. Non-compact smooth manifolds, smooth functions: 1-codimensional ideals

3. Real-analytic and Stein manifolds, real-analytic and holomorphic functions: similar (second countable, paracompact)

## II. Characterizations by Lie algebras

1. Smooth manifolds, smooth compactly supported vector fields (Pursell and Shanks, 1954): maximal ideals

$$p^* = \{X \in \mathcal{X}_c(M) : X \text{ flat at } p\}$$

2. Analytic manifolds: maximal finite-codimensional subalgebras

$$p^0 = \{X \in \mathcal{X}_\bullet(M) : (Xf)(p) = 0, \forall f \in C^\bullet(M)\}$$

3. Lie algebras of vector fields that are modules over the algebra of functions (Skryabin, 1978): purely algebraic, module structure

4. Other Lie algebras of vector fields (preserving a symplectic or contact form, Hamiltonian vector fields, Poisson brackets of functions, ...): specific methods

## B. Lie algebraic characterizations by differential operators

### I. Goal

1. $\mathcal{D}(M), [.,.]$ : linear DO
2. $\mathcal{S}(M) = \text{Sec}(\mathcal{S}TM) = \text{Pol}(T^*M), \{.,.\}$ : symbols
3. $\mathcal{D}^1(M), [.,.]$ : 1st order linear DO

$$\mathcal{D}(M_1) \simeq \mathcal{D}(M_2) \Rightarrow M_1 \simeq M_2$$

### II. Quantum and classical Poisson algebras

1. Quantum Poisson algebra (model: $\mathcal{D}(M)$):

Associative *filtered* algebra $\mathcal{D}$,

$$\mathcal{D}^i \circ \mathcal{D}^j \subset \mathcal{D}^{i+j},$$

such that the *Lie bracket* verifies

$$[\mathcal{D}^i, \mathcal{D}^j] \subset \mathcal{D}^{i+j-1}$$

$\mathcal{A} = \mathcal{D}^0$ (model: $C^\infty(M)$) commutative subalgebra of $\mathcal{D}$

2. Classical Poisson algebra (model: $\mathcal{S}(M) = \mathrm{Sec}(\mathcal{S}TM) = \mathrm{Pol}(T^*M)$):

*Commutative* associative *graded* algebra $\mathcal{S}$,

$$\mathcal{S}_i \cdot \mathcal{S}_j \subset \mathcal{S}_{i+j},$$

equipped with a *Poisson bracket* such that

$$\{\mathcal{S}_i, \mathcal{S}_j\} \subset \mathcal{S}_{i+j-1}$$

$\mathcal{A} = \mathcal{S}_0$ (model: $C^\infty(M)$) associative and Lie-commutative subalgebra of $\mathcal{S}$

3. Quantum Poisson algebras induce classical Poisson algebras:

*Graded* vector space: $\mathcal{S}_i = \mathcal{D}^i/\mathcal{D}^{i-1}$

Symbol of order $i \geq \mathrm{deg}(D)$:

$$\sigma_i(D) = \mathrm{cl}_i(D) = \begin{cases} 0, \text{ if } i > \mathrm{deg}(D), \\ \sigma(D), \text{ if } i = \mathrm{deg}(D) \end{cases}$$

*Commutative* multiplication:

$$\sigma(D_1) \cdot \sigma(D_2) = \sigma_{\mathrm{deg}(D_1)+\mathrm{deg}(D_2)}(D_1 \circ D_2)$$

*Poisson* bracket:

$$\{\sigma(D_1), \sigma(D_2)\} = \sigma_{\mathrm{deg}(D_1)+\mathrm{deg}(D_2)-1}([D_1, D_2])$$

4. Basic example:

$\mathcal{D}(M)$, $\mathcal{S}(M) = \mathrm{Sec}(\mathcal{S}TM) = \mathrm{Pol}(T^*M)$

## III. Algebraic approaches to differential operators

1. Differential operators associated to a unital associative commutative algebra $\mathcal{A}$ ($C^\infty(M)$)(Grothendieck, Vinogradov):

$$\mathcal{D}^0(\mathcal{A}) = \mathcal{A}$$

$$\mathcal{D}^{i+1}(\mathcal{A}) = \{D \in \mathrm{End}(\mathcal{A}) : [D, \mathcal{A}] \subset \mathcal{D}^i(\mathcal{A})\}$$

2. Quantum Poisson algebras

2.$a$. Algebraic characterization of filters:

$$\mathcal{D}^{i+1} = \{D \in \mathcal{D} : [D, \mathcal{A}] \subset \mathcal{D}^i\}?$$

2.$b$. Algebraic characterization of functions:

$$\begin{aligned} \mathcal{A} &= \mathrm{Nil}(\mathcal{D}) \\ &= \{D \in \mathcal{D} : \forall \Delta \in \mathcal{D}, \exists n \in \mathbb{N} : \\ & \qquad \overbrace{[D, [D, \ldots [D}^{n}, \Delta]]] = 0\}? \end{aligned}$$

2.$c$. Distinguishing Lie bracket:

$$\begin{aligned} \forall f \in \mathcal{A}, \exists n \in \mathbb{N} : \overbrace{[D, [D, \ldots [D}^{n}, f]]] &= 0 \\ &\Rightarrow D \in \mathcal{A} \end{aligned}$$

3. Examples

3.$a$. A non-distinguishing bracket:

$$\mathcal{A} = \mathsf{Pol}(\mathbb{R}), \mathcal{D} = \mathcal{D}(\mathcal{A})$$

3.$b$. A distinguishing bracket:

$$\mathcal{D}(M), \mathcal{S}(M) = \mathsf{Pol}(T^*M)$$

$$\forall f \in \mathcal{A}, \exists n \in \mathbb{N} : \overbrace{[D, [D, \ldots [D}^{n}, f]]] = 0$$
$$\Rightarrow D \in \mathcal{A}$$

## IV. Pursell-Shanks type results

**Theorem 1** *Let $\mathcal{D}_i$ $(i \in \{1,2\})$ be non-singular and distinguishing quantum Poisson algebras. Then every* **Lie algebra isomorphism** $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$ (*think:* $\mathcal{D}(M_1) \simeq \mathcal{D}(M_2)$) *respects the filtration and its restriction $\Phi|_{\mathcal{A}_1}$ to $\mathcal{A}_1$ has the form $\Phi|_{\mathcal{A}_1} = \kappa A$, where $\kappa \in \mathcal{A}_2$ is invertible and central in $\mathcal{D}_2$ and $A : \mathcal{A}_1 \to \mathcal{A}_2$ is an* **associative algebra isomorphism** (*think:* $C^\infty(M_1) \simeq C^\infty(M_2)$)

**Theorem 2** *The Lie algebras $\mathcal{D}(M_1)$ and $\mathcal{D}(M_2)$* (*resp.* $\mathcal{S}(M_1)$ *and* $\mathcal{S}(M_2)$, *or* $\mathcal{D}^1(M_1)$ *and* $\mathcal{D}^1(M_2)$) *of all differential operators* (*resp. all symmetric contravariant tensors, or all differential operators of order* 1) *on two* **smooth** (**resp. real-analytic, holomorphic**) *manifolds $M_1$ and $M_2$ are* **isomorphic** *if and only if the manifolds $M_1$ and $M_2$ are smoothly* (*resp. bianalytically, biholomorphically*) **diffeomorphic**.

## C. Automorphisms and derivations

## I. Locality and filtration

$\mathcal{D}$ : one of our 3 (filtered) LA (think $\mathcal{D}(M)$)
$C$ : derivation of $\mathcal{D}$

$$\overbrace{\overbrace{[C\overbrace{D}^{\in\mathcal{D}^i}}^{\in\mathcal{D}^i}, f]}^{\in\mathcal{D}^{i-1}}$$

$C$ has a bounded weight, i.e. there is $d \in \mathbb{N}$ s.th. $C\mathcal{D}^i \subset \mathcal{D}^{i+d}, \forall i \in \mathbb{N}$

$$D = \sum_k [X_k, D_k], CD = \sum_k ([CX_k, D_k] + [X_k, CD_k])$$

$C$ is local, i.e. support preserving

$$0 = C[x^i, x^j] = [\overbrace{Cx^i}^{P_i(\xi)}, x^j] - [Cx^j, x^i]$$

$$\partial_{\xi_j} P_i = \partial_{\xi_i} P_j$$

$$\partial_{\xi_i} P = P_i$$

$$[P, x^i] = C x^i$$

$$C f - [P, f] \in \mathcal{A}$$

$$(C - \mathsf{ad}\, P)\, \mathcal{D}^i \subset \mathcal{D}^i, \forall i \in \mathbb{N} \; (P \in \mathcal{D}^{d+1})$$

**P1: $\mathrm{C} - \mathsf{ad}\,\mathrm{P}$ respects the filtration**

## II. Restriction to functions

$$\Phi \in \mathrm{Isom}_{LA}(\mathcal{D}(M_1), \mathcal{D}(M_2))$$

$$\Phi \in \mathrm{Isom}_{VS}(C^\infty(M_1), C^\infty(M_2))$$

$$[D, \Delta] \leftrightarrow f.g, f.D$$

$$[\overbrace{\mathbf{f}.\bullet}^{\ell_f}, \overbrace{[\mathbf{g}, \bullet]}^{\mathrm{ad}\,\mathcal{A}}] = 0$$

$$\ell_f, r_f \in \mathcal{C}_{\mathrm{ad}\,\mathcal{A}} \subset \mathcal{E} = \mathrm{End}\,\mathcal{D}$$

$$\mathrm{ad}_{\mathcal{E}}\,C \in \mathrm{Der}\,\mathcal{E}, (\mathrm{ad}_{\mathcal{E}}\,C)\,(\mathcal{C}_{\mathrm{ad}\,\mathcal{A}}) \subset \mathcal{C}_{\mathrm{ad}\,\mathcal{A}}$$

$$(C - C(1)\,\mathrm{id})\,|_{\mathcal{A}} \in \mathrm{Der}\,\mathcal{A} = \mathcal{X}(M)$$

$$(C - \mathrm{ad}\,Y)\,|_{\mathcal{A}} = \kappa\,\mathrm{id}$$

**P2: $\mathrm{C} - \mathrm{ad}\,\mathrm{Y}$ reduces on functions to a multiple of identity**

In the following we assume that the derivations are corrected and verify P1 and P2

**III. Derivations of** $(\mathcal{D}^1(\mathrm{M}), [.,.])$

$$\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{X}, C\,|_{\mathcal{X}} \in \mathcal{Z}^1(\mathcal{X}, \mathcal{A})$$

Finally,

$$C(f+X) = \overbrace{[Y, f+X]}^{\text{correction}} + \overbrace{\kappa f}^{C|_{\mathcal{A}}} + \overbrace{\lambda\ \operatorname{div} X + \omega(X)}^{C|_{\mathcal{X}}}$$

$$Y \in \mathcal{X}, \kappa, \lambda \in \mathbb{R}, \omega \in \Omega^1 \cap \ker d, \text{ unique}$$

$$H^1(\mathcal{D}^1, \mathcal{D}^1) = \mathbb{R}^2 \oplus H^1_{\mathrm{DR}}$$

## IV. Derivations of $(\mathcal{S}(\mathrm{M}) = \mathrm{Pol}(\mathrm{T}^*\mathrm{M}), \{.,.\})$

$$\mathcal{S}^1 \simeq \mathcal{D}^1, C(f+X) = \kappa f + \lambda \,\mathsf{div}\, X + \omega(X)$$

Extension of $\kappa$ id:

$$\mathsf{Deg} : \mathcal{S}_i \ni P \to (i-1)P \in \mathcal{S}_i$$

$$-\kappa \;\mathsf{Deg}\; f = \kappa f$$

Extension of div: impossible

Extension of $\omega$:

$$\omega(X) \overset{\mathsf{loc}}{=} (df)(X) = \{X, f\}$$

$$\overline{\omega}(P) \overset{\mathsf{loc}}{=} \{P, f\} = \Lambda(dP, df)$$

$$\overline{\omega}(P) = (-\pi^*\omega)^\sharp(dP) = (dP)(\omega^v),\ \pi : T^*M \to M$$

Finally*,

$$C(P) = \overbrace{\{Q,P\}}^{\text{corrections}} \overbrace{-\kappa \;\mathsf{Deg}\; P}^{\kappa f} + \overbrace{0}^{\lambda\,\mathsf{div}\, X} + \overbrace{\omega^v(P)}^{\omega(X)}$$

$$Q \in \mathcal{S}, \kappa \in \mathbb{R}, \omega \in \Omega^1 \cap \ker d$$

$$\kappa \text{ unique}, (Q,\omega) : (Q+h, \omega + dh), h \in \mathcal{A}$$

$$H^1(\mathcal{S},\mathcal{S}) = \mathbb{R} \oplus H^1_{\mathsf{DR}}$$

$^*C\{X,P\} = \{X, CP\}$

## V. Derivations of $(\mathcal{D}(\mathrm{M}), [.,.])$

$$C(f+X)=$$

$$\overbrace{\kappa f}^{\substack{\mathcal{S}:\ -\kappa\,\mathsf{Deg}\\ \mathcal{D}:\ 0}} + \overbrace{\lambda\ \mathrm{div}\ X}^{\substack{\mathcal{S}:\ 0\\ \mathcal{D}:\ 0}} + \overbrace{\omega(X)}^{\substack{\mathcal{S}:\ \overline{\omega}=\{.,f\}\\ \mathcal{D}:\ \overline{\omega}=[.,f]}}$$

Finally†,

$$C(D) = \overbrace{[\Delta, D]}^{\text{corrections}} + \overbrace{\overline{\omega}(D)}^{\omega(X)}$$

$$\Delta \in \mathcal{D}, \omega \in \Omega^1 \cap \ker\ d$$

$$(\Delta, \omega) : (\Delta + h, \omega + dh), h \in \mathcal{A}$$

$$H^1(\mathcal{D}, \mathcal{D}) = H^1{}_{\mathrm{DR}}$$

†"Needle in haycock"

## VI. Canonical and equivariant quantizations

$$\mathcal{D}(\mathbb{R}^n) \overset{\sigma_{\text{aff}}}{\leftrightarrow} \text{Pol}(T^*\mathbb{R}^n)$$

$$\sum D^{i_1 \dots i_j}(x)\partial_{x^{i_1} \dots x^{i_j}} \leftrightarrow \sum D^{i_1 \dots i_j}(x)\xi_{i_1} \dots \xi_{i_j}$$

$$\mathcal{D}_\lambda(\mathbb{R}^n) \overset{\sigma_{\text{tot}}}{\leftrightarrow} \text{Pol}(T^*\mathbb{R}^n)$$

$$\sigma_{tot} \circ \mathcal{L}_X = L_X \circ \sigma_{tot}, \; \forall X \in \mathcal{X}(\mathbb{R}^n)$$

$$\sigma_{tot} D - \sigma D \in \text{Pol}^{\leq k-1}(T^*\mathbb{R}^n), \; \forall D \in \mathcal{D}^k(\mathbb{R}^n)$$

$$\mathcal{X}(\mathbb{R}^n) \leftrightarrow \text{sl}(n+1, \mathbb{R})$$

$$\mathcal{D}_\lambda(M) \overset{\sigma_{\text{tot}}}{\leftrightarrow} \text{Pol}(T^*M)$$

$$F *_\hbar G = Q_\hbar^{-1}(Q_\hbar F \circ Q_\hbar G)$$

$$(Q_\hbar P = \hbar^k \sigma_{tot}^{-1} P, P \in \text{Pol}^k(T^*M))$$

$$\overbrace{\mathcal{D}^k(\Omega^p(M), C^\infty(M))}^{=\mathcal{D}^k_p} \overset{\sigma_{tot}}{\leftrightarrow} \overbrace{\mathsf{Sec}(\mathcal{S}^{\leq k}TM \otimes \bigwedge{}^p TM)}^{=\mathcal{S}^k_p}$$

$$\begin{array}{ccc} \mathcal{D}^k_p & \overset{\mathcal{T}}{\to} & \mathcal{D}^\ell_q \\ \updownarrow \sigma_{tot} & & \updownarrow \sigma_{tot} \\ \mathcal{S}^k_p & \overset{T}{\to} & \mathcal{S}^\ell_q \end{array}$$

## VII. Integrability of derivations

Automorphisms-derivations:

$$\mathsf{Ad}(\exp(tg)) = e^{t\,\mathsf{ad}\,g}$$

Diffeomorphisms-automorphisms:

$$\phi \in \mathsf{Diff}(M) \text{ induces } \phi_* \in \mathsf{Aut}(\mathcal{D})$$

If $\mathcal{D} = \mathcal{D}^1(M)$ or $\mathcal{D} = \mathcal{D}(M)$,

$$(\phi_* D)f = D(f \circ \phi) \circ \phi^{-1}$$

($\phi_* X$ : push-forward of $X$)

If $\mathcal{D} = \mathcal{S}(M) = \mathsf{Sec}(STM) = \mathsf{Pol}(T^*M)$,

$$\phi_* P = P \circ (\phi^\sharp)^{-1}$$

($\phi_*$ : push-forward, $\phi^\sharp$ : phase lift)

## VII.a. First order differential operators

Automorphisms of $\mathcal{D}^1$:

$$\Phi_{\phi,K,\Lambda,\Omega}(f+X) =$$

$$\phi_*(X) + (K\,f + \Lambda\,\mathrm{div}\,X + \Omega(X)) \circ \phi^{-1}$$

$\phi \in \mathrm{Diff}(M), K \in \mathbb{R}^*, \Lambda \in \mathbb{R}, \Omega \in \Omega^1 \cap \ker d$, unique

Divergence:

$L_X\eta = (\mathrm{div}_\eta\, X)\ \eta$; $\mathrm{div}_{|\eta|}\, X, \mid \eta \mid$: odd volume[‡]

$\mathrm{div} : \mathcal{X} \to C^\infty, \mathrm{div}[X,Y] = [X, \mathrm{div}\,Y] - [Y, \mathrm{div}\,X]$

Group condition[§]:

$$\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t} \circ \Phi_{\phi_s,K_s,\Lambda_s,\Omega_s} = \Phi_{\phi_{t+s},K_{t+s},\Lambda_{t+s},\Omega_{t+s}}$$

$$\Lambda_t\ (\mathrm{div}\,\phi_{s*}X) \circ \phi_t^{-1}$$

[‡]Nowhere vanishing 1-density $\rho_0$; $L_X(f\rho_0^\lambda) - X(f)\rho_0^\lambda = \lambda f \gamma(X)\rho_0^\lambda$, $\gamma(X) = div_{\rho_0}X$; priviliged coho. class

[§]Smoothness with respect to the differential structure of $M$, i.e. $\mathbb{R} \times M \ni (t,x) \to (\Phi_t D)(f)(x) \in \mathbb{R}$ is smooth, for any $D$ and any $f$

$$\mathrm{div}_{|\eta|}\,\phi_* X = \left(\mathrm{div}_{\phi^*|\eta|}\,X\right)\circ\phi^{-1}$$

$$\phi^* \mid \eta \mid = \overbrace{J(\phi)}^{\in C^\infty(M,\mathbb{R}_+^*)} \mid \eta \mid$$

$$J(\phi)(x) \overset{loc}{=} \text{"} \mid det\,\partial_x\varphi \mid \text{"}$$

$$J(\phi\circ\psi) = \psi^*\left(J(\phi)\right).J(\psi)$$

$$G: \ \text{group}, \ M: \ \text{left } G-\text{module}$$

$$C : G \to M$$

$$g_1.C(g_2) - C(g_1\cdot g_2) + C(g_1) = 0$$

$$\psi^*\left((\ln\circ J)(\phi)\right) - (\ln\circ J)(\phi\circ\psi) + (\ln\circ J)(\psi) = 0$$

$$\mathrm{Div} = \ln\circ J \in \mathcal{Z}^1(\mathrm{Diff}(M), C^\infty(M))$$

$$\mathrm{Div}(\mathrm{Exp}(tX)) = \int_0^t \mathrm{div}\,X\circ\mathrm{Exp}(sX)ds$$

$$\Phi_{\phi,K,\Lambda,\Omega} = \phi_*(X) + (K\, f + \Lambda\, \mathrm{div}\, X + \Omega(X)) \circ \phi^{-1}$$

$$\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t} \circ \Phi_{\phi_s,K_s,\Lambda_s,\Omega_s} = \Phi_{\phi_{t+s},K_{t+s},\Lambda_{t+s},\Omega_{t+s}}$$

$$\left(\mathrm{div}_{J(\phi)|\eta|}\, X\right) \circ \phi^{-1}$$

$$= \left(\mathrm{div}_{|\eta|}\, X + X\left(\left(\ln \circ J\right)(\varphi)\right)\right) \circ \varphi^{-1}$$

$$= (\mathrm{div}\ X + d\,(\mathrm{Div}\,\varphi)\,(X)) \circ \varphi^{-1}$$

$$\phi_t \circ \phi_s = \phi_{t+s}, \phi_0 = \mathrm{id}; K_t K_s = K_{t+s}, K_0 = 1;$$

$$\Lambda_t + K_t \Lambda_s, \Lambda_0 = 0;$$

$$K_t \Omega_s + \phi_s^* \Omega_t + \Lambda_t\, d(\mathrm{Div}\, \phi_s) = \Omega_{t+s}, \Omega_0 = 0$$

$Y \in \mathcal{X}(M)$ *complete*

$\kappa, \lambda \in \mathbb{R}$

$\omega \in \Omega^1(M) \cap \ker d$

unique

Explicit form of $\Phi_{\phi_t,K_t,\Lambda_t,\Omega_t}$

**Theorem 3** A derivation

$$C_{Y,\kappa,\lambda,\omega}(X+f) =$$

$$[Y, X+f] + \kappa\, f + \lambda \ \mathsf{div}\, X + \omega(X)$$

of $\mathcal{D}^1(M)$ induces a one-parameter group $\Phi_t$ of automorphisms of $\mathcal{D}^1(M)$ if and only if the vector field $Y$ is complete. In this case the group is of the form

$$\Phi_t(X+f)$$

$$= \overbrace{(\mathsf{Exp}(tY))_*(X)}^{(\phi_t)_*(X)}$$

$$+ \overbrace{\left(e^{\kappa t}\, f + \lambda \frac{e^{\kappa t}-1}{\kappa}\, \mathsf{div}\, X\right) \circ \mathsf{Exp}(-tY)}^{(K_t\, f + \Lambda_t\ \mathsf{div}\, X)\circ\phi_t^{-1}}$$

$$+ \overbrace{\left[\int_0^t e^{\kappa(t-s)} \left[\lambda \int_0^s X(\mathsf{div}\, Y \circ \mathsf{Exp}(uY))du\right.\right.}^{\Omega_t(X)\circ\phi_t^{-1}}$$

$$\left.\left. + \left((\mathsf{Exp}(sY))^*\omega\right)(X)\right] ds\right] \circ \mathsf{Exp}(-tY)$$

## VII.b. Linear differential operators

Automorphisms of $\mathcal{D}(M)$:

$$\Phi = \phi_* \circ \mathcal{C}^a \circ e^{\overline{\Omega}}$$

$$\phi \in \mathrm{Diff}(M),$$

$$a \in \{0,1\}, \mathcal{C}^0 = \mathrm{id}, \mathcal{C}^1 = \mathcal{C} = -*,$$

$$\Omega \in \Omega^1(M) \cap \ker d$$

$$\int_M D(f).g \mid \eta \mid = \int_M f.D^*(g) \mid \eta \mid$$

$$(\eta \text{ volume of } M,\ f, g \in C_c^\infty(M))$$

$$(D \circ \Delta)^* = \Delta^* \circ D^*$$

$$\mathcal{C}(D \circ \Delta) = -\,\mathcal{C}(\Delta) \circ \mathcal{C}(D)$$

$$\mathcal{C}(D \circ \Delta - \Delta \circ D) = \mathcal{C}(D) \circ \mathcal{C}(\Delta) -\, \mathcal{C}(\Delta) \circ \mathcal{C}(D)$$

$$\Phi_{\phi,\Omega} = \phi_* \circ e^{\overline{\Omega}}$$

$$C_{\Delta,\omega} = \mathrm{ad}\,\Delta + \overline{\omega}$$

**Theorem 4** A derivation

$$C_{\Delta,\omega}(D) = [\Delta, D] + \overline{\omega}(D)$$

of the Lie algebra $\mathcal{D}(M)$ of all differential operators is integrable if and only if $\Delta \in \mathcal{X}(M)$ and $\Delta$ is complete. In this case the one-parameter group of automorphisms $\Phi_t$ generated by $C_{\Delta,\omega}$ reads

$$\Phi_t = (\mathrm{Exp}(t\Delta))_* \circ e^{\overline{\int_0^t (\mathrm{Exp}(s\Delta))^* \omega \, ds}}$$

## VII.c. Symmetric contravariant tensor fields

Automorphisms of $\mathcal{S}(M)$:

$$\Phi = \phi_* \circ \mathcal{U}_K \circ e^{\overline{\Omega}}$$

$$\phi \in \mathsf{Diff}(M), K \in \mathbb{R}^*, \Omega \in \Omega^1(M) \cap \ker d$$

$$\mathcal{U}_K : \mathcal{S}_i(M) \ni P \to K^{i-1} P \in \mathcal{S}_i(M)$$

Derivations of $\mathcal{S}(M)$:

$$C = \mathrm{ad}\, S + \kappa\, \mathrm{Deg} + \overline{\omega}$$

$$S \in \mathcal{S}(M), \kappa \in \mathbb{R}, \omega \in \Omega^1(M) \cap \ker d$$

$$\kappa\, \mathrm{Deg} : \mathcal{S}_i(M) \ni P \to (i-1)\kappa\, P \in \mathcal{S}_i(M)$$

Automorphisms-derivations:

$$S \in \mathcal{X}(M),\ S \text{ complete}$$

**D. Differential operators over a real line bundle**

$\pi : L \to M$: real line bundle

$\mathcal{D}(L)$: differential operators mapping Sec($L$) into Sec($L$) (definition " à la Vinogradov ")

$\varsigma \in$ Sec($L_U$): nowhere vanishing section of $L$ over $U$

$$\iota_\varsigma : \mathrm{Sec}(L_U) \ni \varphi\varsigma \to \varphi \in C^\infty(U)$$

$$\mathcal{I}_\varsigma : \mathcal{D}^k(L_U) \ni \Delta \to \iota_\varsigma \circ \Delta \circ \iota_\varsigma^{-1} \in \mathcal{D}^k(U)$$

$$\mathcal{I}_{\varsigma'}(\Delta)(f) = \psi^{-1}.\left(\mathcal{I}_\varsigma(\Delta)(\psi.f)\right), \;\; \varsigma' = \psi\varsigma$$

$$\sigma_k(\mathcal{I}_{\varsigma'}(\Delta)) = \sigma_k(\mathcal{I}_\varsigma(\Delta))$$

$$\Phi : \mathcal{S}(\mathcal{D}(L)) \overset{\text{cPa isom}}{\to} \mathcal{S}(\mathcal{D}(M))$$

$$|L_0| = L_0/\mathbb{Z}_2$$

$$|\varsigma|, \{\varsigma_\alpha, -\varsigma_\alpha\} \overset{U_\alpha \cap U_\beta}{=} \{\varsigma_\beta, -\varsigma_\beta\}, \tilde{\varsigma}_\alpha \overset{U_\alpha \cap U_\beta}{=} \pm\tilde{\varsigma}_\beta$$

$$\Psi : \mathcal{D}(L) \overset{\text{qPa isom}}{\to} \mathcal{D}(M)$$