SOLVING EQUATIONS WITH NEGATIVES OR CROSSING THE FORMALIZING GAP

Joëlle Vlassis
Faculty of Psychology and of Sciences of Education - University of Liège - Belgium
Joelle.Vlassis@ulg.ac.be

Abstract
We designed an experiment in teaching the solving equations to 8th grades students. This experiment shows that students encounter many difficulties not really when passing through equations with the unknown in both members but rather when passing through equations with negatives. A clinical study was carried out with 13-14 year old pupils to investigate which obstacles students must overcome to solve equations with negatives. This article presents the principal results from experiment and from interview. The results show that the presence of negatives in the equations represents a cut with arithmetical knowledge and requires from students a formal reasoning in both semantic and formal aspects. The different levels of conceptualization of negative numbers proposed by Gallardo (1994) must be reached by the students in order they can give sense to the equations (with negatives) themselves and to the solving methods.

INTRODUCTION

In the research literature, we found authors such as Kieran (1981), Filloy and Rojano (1984), Sfard (1991), Colomb (1995), Linchevski and Herscovics (1996) who stressed cognitive obstacles met by students in their learning of first degree equations with one unknown : the algebraic sense of the equal sign, the presence of the unknown in both members (didactical cut), the complementarity of the procedural/structural conceptions of the expressions, the operations performed on the unknown, … In order to help the students to overcome these difficulties, we conceived a set of situations aimed to learn how to solve first degree equations with one unknown. These activities have been tested in two 8th-grade classes (aged 13-14). They are built on students' arithmetical knowledge and lead them to evolve towards algebraic methods, using among others the balance model. A particular attention was paid to the 'didactical cut' (Filloy and Rojano, 1984) – which occurs when the unknown appears on both sides of the equations – from a semantic as well as from a formal point of view. Our observations show that this demarcation point was passed through successfully for the resolution of equations resulting from the balance model, with such a structure : $ax + b = cx + d$. However, the presence of negative literal or numerical terms in the algebraic equations raised a lot of problems. In order to analyze more deeply the student reasoning in that context, we carried out several interviews with some pupils of the 8th and 9th grades.
This article reports on the results of the experiment as well as on interviews. It also presents reflections about the student cognitive processes when they are required to solve equations with negatives.

LEARNING SITUATIONS

Presentation of the situations

The whole of the activities proposed to the students of two 8th grade classes has been organized in two phases:

1. <u>The activities about equations with the unknown in one member</u> (arithmetical equations). The main objectives of these activities are to lead the students to use their arithmetical knowledge in solving equations (substitution, inversion of operations and cover-up) and to initiate the first notions related to the concepts of equation, solution and unknown. In this article, we will not discuss about the results of the first phase.

2. <u>The activities about equations with the unknown in both members</u> (algebraic equations). They are aimed to learn the formal method based on the equality properties (to perform the same operation in both members). Three situations have been designed :
   Situation 1 : Solving a problem : This activity consists in solving a problem of which modeling leads to an algebraic equation. At that stage, no new solving method is introduced. In the state of the students' knowledge, the solution can only be found by trials/errors. This situation is aimed to make students aware of arithmetic methods limits.
   Situation 2 : Balances : That situation introduces the formal method based on the equality properties with balance-based activities. Students are required to find the unknown value of a weight present on both pans.
   Situation 3 : Formalization : That step is aimed to systemize the formal process without the balance support. A set of 4 algebraic equations is presented without any context.

Results and comments

Data have been collected through observations in the classes and through students' productions analysis. They reveal two phenomenon's we would like to stress on :

1. <u>The interest of the balance model</u>
   Activities have been performed one after the other without raising any particular conceptual problems by the children until the formalization situation. When the balance situation has been got over, all the children were able to solve algebraic equations with additions, by performing the same operation in both members. Students productions analysis helped us to observe that the balance model offered a good mental picture of the required operations and the related concepts (sense and properties of equality). The recent interviews we propose hereafter confirm those results : after seven months of learning, students easily reactivate these techniques without needing any recall. A particular interest of using concrete models, like the balance model, is that students are able to reactivate at any moment that self-evident picture.

2. <u>The algebraic equations with negatives : beyond the balances</u>
   On the other hand, the next step of formalization (situation 3) was very difficult for a lot of students. As long as both members were composed of an addition and that the unknown value was a positive whole number, students solved it quite easily. They mentally used the balance model. But when the equation was composed of

subtractions, a lot of difficulties appeared. We identified two different error origins:

a) *'The detachment from minus sign preceding a numeral or literal term'*. That type of difficulty - identified by Herscovics and Linchevski (1991) - produced the following errors :

Some students simplified like this the given equation members:

$-3x + 6 = 2x + 16$	$2 - 3x + 6 = 2x + 18$
$-2x\downarrow \quad \downarrow -2x$	$-2x \downarrow \quad \downarrow -2x$
$1x + 6 = 16$	$2 - 1x + 6 = 18$

We can see that apparently, the students do not take into account the minus sign before $3x$, no matter it is presented as a number attribute (first example) or as an operation sign (second example).

b) *Subtracting in order to neutralize a negative expression*

In order to cancel a negative numerical (or literal) term, some students use subtraction. An equation such as $3x - 4 = x + 9$ was solved like that :

$3x - 4 = x + 9$

$-4\downarrow \quad \downarrow -4$

$3x = x - 5$

We can imagine two hypothesis to explain that error. The first one may come from an abusive generalization of the balance model. Stimulated by their previous success with the balance process, the children use the subtraction in order to cancel an expression, just like they did in order to withdraw some weight from the pans. The second hypothesis refers to the first type of error mentioned here above. It could be related to the inability of some students to consider the sign before the expression. For them, it is not –4 that is needed to be cancelled, but 4, the sign '–' before 4 is not taken into consideration.

Authors such as Filloy and Rojano (1984) and Linchevski and Herscovics (1996), who experimented learning situations about the same theme, also say that similar processes appear with their subjects when these are facing equations with negatives. Filloy and Rojano (1984) think that difficulty is related to the fact the students do not succeed in generalizing their knowing stemmed from their experiences with the model. These authors bring into question the use of those concrete models since they do not improve significantly the students competencies. On the basis of our analysis, we assert, on the contrary, that the balance model is a useful instrument to help students to understand notions such as the properties of equality and the related techniques.

The introduction of negatives puts the equation solving at an abstract and formal level. In that case, we can no longer consider the equation terms just as weights needing to be withdrawn from the pans (what would it mean indeed to withdraw some weights $-3x$ or -4?). New obstacles have indeed to be overcome. Balances are not designed to overcome that type of obstacle. We have to consider transition activities in order to help students to leave the model while keeping the principles it introduces.

CLINICAL INTERVIEW

Presentation of the interview

We interviewed 7 students: five amongst them were in the beginning of the 9th grade and had learnt during the previous school year how to solve equations in the frame of our experiments. Two other students were at the beginning of the 8th grade and had not yet learnt equation solving. They were all of average ability.
The topic of the interview was to go deeper in the analysis of the data we collected during the previous experiments. The questions concerned mainly:

- Equations with the unknown in one member : i) $a - x = b$ (with $x < a$); ii) $a - x = b$ (with $x > a$), iii) $-a - x = c$ and iv) $-x = a$. Numbers were whole numbers.
- Equations with the unknown in both members.
- Numerical operations, with for instance, ‘ 237 + 89 – 89 + 67 – 92 + 92 = ? ’ (Herscovics & Linchevski, 1991).
- Expressions reductions, with for instance, ‘ $19n + 67 - 11n - 48$ = ? ’ (Linchevski and Herscovics,1996).

Results

The most significant results obtained with this interview are the following :
<u>in arithmetic equations with negatives</u> :
Here is an example of each type of equation presented in the interview :
i) $12 - x = 7$; ii) $4 - x = 5$; iii) $-4 - x = 10$; iv) $-x = 7$.

1) Giving some sense to equations such ii), iii) et iv)

In order to solve those equations, all the students tried to give them some sense. No student of the 9th grade did solve spontaneously those equations through a formal method. We can reasonably assume that the numbers simplicity as well as the resolution of the first equation (i) $12 - x = 7$, incited the students to use arithmetic methods based on a concrete meaning.
‘ Inhibitory mechanisms ’ (Gallardo & Rojano, 1994) then appeared. On the one hand, most students were amazed before an equation such as (ii) : ‘It is impossible to subtract a number to 4 to obtain a bigger result!’. On the other hand, that difficulty to give some sense to those equations led to the inability to generalize the inversion of operations, even if it was widely used for (i). Either children did not think to use that method for (ii) et (iii), or they thought they were using it when adding 4 and 5 in $4 - x = 5$ (ii) and adding –4 and 10 in $-4 - x = 10$ (iii).
After having hesitating before ‘the subtraction which makes bigger’, the students who explicitly asked the question like that : ‘By which number have I to substitute x in order to obtain a bigger result?’ succeeded in finding the solution, remembering the rule ‘minus by minus gives plus’.
Only one student solved $12 - x = 7$ with the substitution method (‘which is the number which, when subtracted to 12, gives 7?’). The same student solved spontaneously and easily items (ii) and (iii) with the same method.

This observation supports our hypothesis (Fagnant & Vlassis, in press), according which the substitution process involves thinking structures that help students to conceptualize more easily the notions involved in the equations. Kieran (1988 and 1990) also supports that position.

2) Solving '$-x = 7$'

That equation was the most difficult. No student could solve it spontaneously. Five of them began by giving 7 as a solution. Some of them understood it was not possible since they had then $-7 = 7$, but they could not find the solution.

This difficulty has of course its origin in the absence of sense given by the students to an equation like $-x = a$. But why is that particular equation more difficult to solve for the students than equations such as (ii) and (iii)? We think the main reason is that students consider $-x$ statically, just as –4, for instance. That representation leads to confusion in the minus signs. For them, the '–' sign before the x is the same that the '–' sign in –4. In numeric cases, the numbers written behind '–' are 'naturals', so the students consider that x can only be a natural.

Some students who proposed the correct solution $x = -7$ explained that the solution could not be –7 because if it was, it should be written --7. When we proposed to other students the solution –7, they answered that the '–' was already written, they refused our argument and suggested $x = 7$ again. Cortès (1993) also indicates that in that type of equation, the unknown considered by the student is not x but $-x$.

In order to give some sense to an equation such as $-x = 7$, it is required the students can make the distinction between the minus sign and x. Two different thinking procedures can be considered:

- Considering the expression $-x = 7$, in a procedural manner, like $-1 \cdot x = 7$. This raises the following question 'Which is the number that, when multiplied by –1, gives 7'. During the interview, we tried to involve the students in that way of thinking. But none of them seemed to understand where we wanted them to go.
- The equation $-x = 7$ can be considered with the idea of opposite numbers. That perspective leads to express the equation like this : '$-x$ means it is needed to take the opposite of the number x ; thus, which is the number of which the opposite is 7'. That perspective makes possible the idea that x could be a negative. Since it is required to take the opposite of a number, that number can be a negative. That second way of reasoning seems to be easier for the students. It helps them to keep their static conception of the expression while leading them to the correct solution. That method was used to find the solution by the two students who answered correctly.

<u>in algebraic equations</u> (for the 5 students of the 9th grade)

We were surprised to observe that algebraic equations with negatives raised fewer difficulties than the arithmetical ones. The students indeed did not try to give some sense to the algebraic equations to solve them. The presence of the unknown in both members seemed to act as a starter of the solving method consisting in performing the same operation in both members. The students did not meet any particular difficulty in the application of that method. But the presence of negatives nevertheless produced the following errors:

4*) ' The detachment from minus sign '*

One student made that error. But no student did subtract a negative term to cancel it. That observation is surprising when we consider the results of our previous experiments. We'll need to interview more pupils to analyze more deeply the problem.

5) Going from –ax=b to x=-b/a (like in –5x = 10)

It is the most important difficulty we observed in the algebraic equations. We noticed two main obstacles for the students who apply completely the method consisting in performing the same operation in both members:

- Finding the right operation which links –5 with x. Whereas going from $6x = 7$ to $x = 7/6$ seems easy (the students explicitly explain they divide by 6 in each member), it is surprising to observe they do not know which operation they have to perform in order to transform $-5x$ into x. Some of them think they have to make +5 in each side. Others do not know at all what to do. It seems that the multiplication sign between -5 and x is not evident when the coefficient of the unknown is a negative number. The students are not able to decode easily the expression $-5x$ in terms of $-5 . x$.
- Dividing by a negative : Once the students succeeded in decoding correctly the expression, another obstacle appeared : they had some difficulty to accept the idea of dividing by –5.

<u>in operations with numerical or literal terms</u>

6) Detaching from the minus sign

- For two students, the expression $6 + n - 2 + 5$ is simplified in $-1 + n$. They justify it by explaining that if $2 + 5 = 7$ then $6 - 7 = -1$.
- A student makes $237 + 89 - 89 + 67 - 92 + 92$ like that : he crosses out +89 et –89 and makes : $237 + 67 - (92 + 92) = 304 - 184 = 120$

7) Considering the sign which follows the numerical or literal terms

- For two students, the reduced expression of $6 + n - 2 + 5$ is '$13 + n$' because $2 + 5 = 7$ so $6 + 7 = 13$. They consider the sign '+' which follows 6.
- In $19n + 67 - 11n - 48 = ?$, we find the following errors, the same ones already identified by Linchevski et Herscovics (1996) (3 students) :

  « *Jumping off with the posterior operation* »: in order to reduce the expression $19n + 67 - 11n - 48 = ?$, some students grouped $19n$ and $11n$ by making an addition rather than a subtraction. The considered sign is the '+' sign which follows 19n.

  « *Inability to select the appropriate operation for the partial sum* »:

  For some students, $19n + 67 - 11n - 48 = 8n - 19$, because 67 is followed by the '–' sign.

  We attribute these three different types of errors to the same trend as the detachment from the minus and which consists, over all in that particular case, in paying more attention to the sign which follows the operation rather than to the one which precedes it.

According to Linchevski and Herscovics (1991), the difficulties related to the detachment from the minus sign, could possibly come from confusion in the priority rules of the operations. Our interviews do not confirm that hypothesis. The students did not propose any explanation of that kind. We tend to explain those difficulties by the arithmetical practices of the students. According to Vergnaud (1989), it seems that there is confusion between the numbers without signs (measures of size or quantities) and the numbers with signs (quantification of transformations and relations). In the primary school, over all the components of an operation, that is the numbers without signs (we make the operations with meters, francs, etc.) are stressed on, but not so often are the relations intervening in the operations. In the same way, we can also mention the ambiguity of the minus, which can be considered in a procedural way, as a sign of the operation to be performed, or in a static way, as the attribute of a number. It seems that, in this particular case, the students do not consider the second possibility.

8) Confusing the algebraic rules

A student wrote $19n + 67 - 11n - 48 = 8n - 19$ with a quite distinct justification from the other students'one. To our question 'Why –19?', he answered '+67 – 48 = –19 because plus by minus gives minus '.

DISCUSSION

The operations with the negative whole numbers present a lot of difficulties for the students. With the introduction of negatives, a formal reasoning, which goes most often into contradiction with the arithmetical knowledge, becomes necessary. In case of solving equations, that difficulty turns out to be still more important because that context makes it necessary for the students to have perfectly integrated the distinct levels of conceptualization of the negatives stressed by Gallardo (1994) : subtraction, signed number (plus or minus sign is associated with the number), relative (or directed) number (idea of opposite and symmetry) and isolated number (result of an operation or solution of an equation). The consideration of the various negative dimensions is needed to give some sense to the equations with negatives themselves, as well as to the formal solving procedures. For instance, the conception of 'relative number' enables to give sense to an equation such as $-x = a$, or to the procedure of neutralization of a term (canceling a numerical or a literal term); the idea of 'signed number' is essential to avoid errors of 'detachment of the minus sign', ...

Moreover, with the presence of negatives, algebraic equations resolution can no longer be considered concretely : for example, it becomes impossible to maintain the 'subtraction' idea (withdrawing some weights from the balance pans) to neutralize a term. The letter has to be given a mathematical interpretation and no longer an intuitive one, as an object (a weight) in the meaning of Kuchemann (1981). Students have to be taught to abstract, from the concrete manipulations, the general mathematical method needed to solve all types of equations with one unknown.

Solving equations with negatives, means thus crossing the formalizing gap. This transition cannot be left in the students'hands. It needs a teaching performing explicitly the transition towards the abstract concepts involved by the formal solving methods. The data collected through the studies presented here above will help us to modify our learning sequence. Further experiments will be carried out.

References

Colomb, J. (1995). *Calcul littéral. Savoir des élèves de collège*. Document et travaux de recherche en éducation. Paris : Institut National de Recherche Pédagogique.

Cortès, A. (1993). Analysis of errors and a cognitive model in the resolving of equations, *Proceedings of the XVII Annual Meeting of PME,* Tsukuba, Japan : ed. University of Tsukuba, vol 1, 146-153.

Fagnant, A. & Vlassis, J. (in press). Utiliser les connaissances antérieures des élèves pour construire les mathématiques : des calculs relationnels au primaire à la mise en équation au secondaire, *Actes du premier colloque des chercheurs en éducation : Le point sur la recherche en Éducation en Communauté française de Belgique*. Université libre de Bruxelles.

Filloy, E. & Rojano, T. (1984). From an arithmetical to an algebraic thought. *Proceedings of the Sixth Annual Meeting of PME-NA*, J.M. Moser (Ed.), Madison: University of Wisconsin, vol.3, 51-56.

Gallardo, A. (1994). Negative numbers in algebra, the use of a teaching model, J. P. da Ponte & J.F. Matos (Eds), *Proceedings of the XVIII Annual Meeting of PME,* vol II, 376-383.

Gallardo, A. & Rojano, T. (1994). School algebra, syntactic difficulties in the operativity with negative numbers, *Proceedings of the XVI Meeting of PME-NA*, D. Kirshner (Ed.), USA : Louisiana State University, vol 1, 159-165.

Herscovics, N. & Linchevski, L. (1991). Pre-algebraic thinking : range of equations and informal solution processes used by seventh graders prior to any instruction, *Proceedings of the XV Annual Meeting of PME*, F. Furinghetti (ed), Assissi, Italy, vol.3, 173-180.

Kieran, C. (1981). Concepts Associated With Equality Symbol. *Educational Studies in Mathematics,* 12, 317-326.

Kieran, C. (1988). Two different approaches among algebra learner, in A. F. Coxford (Ed.), *The ideas of algebra*, K-12 (1988 Yearbook), Reston, VA: National Council of Teachers of Mathematics, 91-96.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nescher et J. Kilpatrick (Eds), *Mathematics and Cognition*, 96-102, Cambridge University Press.

Kuchemann, D.E. (1981). Chapitre 8 : Algebra in *Children's understanding of mathematics, 11-16*. London : The CSMS Mathematics Team.

Linchevski, L & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra : operation on the unknown in the context of equations. *Educational Studies in Mathematics*, 30, 39-65.

Pirie, S. & Kieren, T. (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics,* 14, 1, 39-43.

Sfard, A. (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics, 22*, 1-36.

Vergnaud, G. (1989). L'obstacle des nombres négatifs et l'introduction à l'algèbre. Construction des savoirs, *Colloque International Obstacle Epistémologique et Conflit socio-cognitif*, CIRADE, Montréal, 76-83.