On the Cohomology of the Nijenhuis-Richardson Graded Lie Algebra of the Space of Functions of a Manifold*

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The term $\mathcal{E}^0$ of degree 0 of the Nijenhuis-Richardson graded Lie algebra $\mathcal{E}$, of course is a Lie algebra. The graded cohomology of weight -1 is the direct sum of the cohomology of some kernel and the Chevalley-Eilenberg cohomology of $\mathcal{E}^0$. We computed the first three spaces of this last cohomology in [7,8]. The aim of this paper is to determine the first three spaces of weight -1 or the adjoint cohomology of the Nijenhuis-Richardson algebra of the functions of a manifold, which are important in deformation theory (the first and second will be computed for an arbitrary weight $q \leq -1$).

INTRODUCTION

Let V be a vector space and denote by $(A(V), [.,.])$ the Nijenhuis-Richardson graded Lie algebra of V [5]. Recall the expression of the Nijenhuis-Richardson-bracket on $A^h(V) \times A^{-1}(V)$ and $A^0(V) \times A^b(V) (b \geq 0)$: for $B \in A^b(V)$ and $x \in A^{-1}(V) = V$,

$$[B, x] = i_x B = B(x, \cdot \cdots \cdot)$$

and for $A \in A^0(V), B \in A^b(V)$ and $x_0, \cdots, x_b \in V$,

$$[A, B](x_0, \cdots, x_b) = A(B(x_0, \cdots, x_b)) - \sum_k B(x_0, \cdots, A(X_k), \cdots, x_b).$$

If M is a smooth manifold and N the space of smooth functions on M, we set

$$\mathcal{E} := A(N)_{loc,\, n.c.'}$$

where the r.h.s. is the space of all multilinear, skew-symmetric $N-$valued mappings on N, which are local i.e. support preserving and vanish on $1 \in N$. This space $\mathcal{E}$ is a graded Lie subalgebra of the Nijenhuis-Richardson algebra of N. Let $H_{alt}(\mathcal{E})_{-1,\, loc}$ be the graded cohomology of $\mathcal{E}$ associated to the adjoint representation, all cochains being of weight -1 and support preserving. Observe that $\mathcal{E}^0 = A^0(N)_{loc,\, n.c.} = gl(N)_{loc,\, n.c.}$ is a Lie algebra, subalgebra of $gl(N)$. Denote by $H(\mathcal{E}^0, N)_{loc}$ the local Chevalley-Eilenberg cohomology of $\mathcal{E}^0$ valued in N. In [4], P.B.A. Lecomte, D. Melotte and C. Roger showed that the short sequence of differential spaces

$$0 \rightarrow \ker\theta \rightarrow \Lambda_{alt}(\mathcal{E})_{-1,\, loc} \xrightarrow{\theta} \Lambda(\mathcal{E}^0, N)_{loc} \rightarrow 0,$$

where θ is the restriction mapping of graded cochains to $\mathcal{E}^0 \times \cdots \times \mathcal{E}^0$, is exact and split. Finally,

$$H^p_{alt}(\mathcal{E})_{-1,\, loc} \approx H^p(\ker \theta) \oplus H^p(\mathcal{E}^0, N)_{loc}. \qquad (1)$$

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We developed in [7,8] and [9] two different methods to prove the following theorem :

THEOREM 1. *If $p \in \{1, 2, 3\}$ and M is a smooth connected second-countable Hausdorff manifold of dimension $m \geq p$, the space $H^p(\mathcal{E}^0, N)_{loc}$ is isomorphic to the corresponding space $H^p_{DR}(M)$ of the de Rham cohomology of M.*

We thus got one part of the first three spaces of the graded cohomology, important in deformation theory. This is a momentous motivation indeed, since the result of M. Kontsevich [3], concerning formal deformations of the algebra of functions of an arbitrary Poisson manifold, suggests the existence of some canonical universal graded cohomology classes, related to deformations and their classification [2].

The calculus of $H^p(\ker \theta)$ ($p \in \{2, 3\}$) involves the spaces $H^{p-1}_{alt}(\mathcal{E})_{-2,\, loc}$. We thus shall determine the first and the second spaces of the graded cohomology for an arbitrary weight $q \leq -1$. More precisely, we shall show that the first three spaces $H^p_{alt}(\mathcal{E})_{-1,\, loc} (p \in \{1, 2, 3\})$ of weights -1 are isomorphic to the corresponding spaces of the de Rham cohomology of M and that the first two spaces $H^p_{alt}(\mathcal{E})_{q,\, loc} (p \in \{1, 2\}, q \leq -2)$ of weight less than or equal to -2 vanish.

$$H^1_{alt}(\mathcal{E})_{q,\, loc}\ (q \leq -1), \quad H^2_{alt}(\mathcal{E})_{q,\, loc}\ (q \leq -1), \quad H^3_{alt}(\mathcal{E})_{-1,\, loc}. \qquad (2)$$

1. BASIC DEFINITIONS AND TOOLS

1.1. *Cohomology of the Nijenhuis-Richardson graded Lie algebra of the space of functions of a manifold*

Let M be a smooth manifold and set $N = C^\infty(M)$. Recall that the space $\mathcal{E} = A(N)_{loc,\, n.c.}$ is a graded Lie subalgebra of the Nijenhuis-Richardson graded Lie algebra $(A(N), [., .])$ [5]. Denote by $\Lambda^p_{alt}(\mathcal{E})_{q,\, loc}$, the space of p-linear $\mathcal{E}$-valued mappings T on $\mathcal{E}$, which are homogeneous of weight q, local and *alternate* i.e. such that

$$T(\ldots, A_{i+1}, A_i, \ldots) = (-1)^{a_i a_{i+1}+1}\, T(\ldots, A_i, A_{i+1}, \ldots)$$

(a_k is the degree of A_k for each k). The sum

$$\Lambda_{alt}(\mathcal{E})_{q,\, loc} = \oplus_p\, \Lambda^p_{alt}(\mathcal{E})_{q,\, loc}$$

is a differential space when equipped with the coboundary operator ∂ given by

$$\begin{aligned}(\partial T)(A_0,\ldots,A_p) &= \sum_{i=0}^{p}(-1)^{\alpha_i}\,[A_i, T(A_0,\ldots\hat{\imath}\ldots,A_p)]\\ &\quad+\sum_{i<j}(-1)^{\alpha_{ij}}\,T\left([A_i,A_j]\,,\ A_0,\ldots\hat{\imath}\ldots\hat{\jmath}\ldots,A_p\right),\end{aligned}$$

where $\alpha_i = i + a_i(a_0+\cdots+a_{i-1})$ and $\alpha_{ij} = \alpha_i+\alpha_j+a_ia_j$.

Remark 1. Observe that if T denotes a p-cochain ($p\geq 1$) of weight q, then $I_1T = T(\bullet,\ldots,1)$ ($1\in N=\mathcal{E}^{-1}$) is a $(p-1)$-cochain of weight $q-1$ and

$$I_1\partial T = \partial I_1 T.$$

Thus, $T\in\Lambda^p_{alt}(\mathcal{E})_{q,\,loc}\cap\ker\partial$ implies $I_1T\in\Lambda^{p-1}_{alt}(\mathcal{E})_{q-1,\,loc}\cap\ker\partial$.

1.2. *Symbolic representation*

The formalism, allowing to compute the graded cohomology spaces (2), was developed in [7,8] and also used in [9].

Consider an open subset U of $\mathbb{R}^m$, two real finite-dimensional vector spaces V and W and

$$O\in\mathcal{L}(C^\infty(U,V),\ C^\infty(U,W))_{loc}.$$

This operator is fully defined by its values on the products fv ($f\in C^\infty(U)$, $v\in V$). It follows from a well-known theorem of J. Peetre [6], that

$$O(fv)=\sum_\mu O_\mu((D^\mu f)v),$$

where $D^\mu_x f = D^{\mu^1}_{x^1}\ldots D^{\mu^m}_{x^m} f$, where the sum is locally finite and where the coefficients $O_\mu\in C^\infty(U,\mathcal{L}(V,W))$ are well-determined. We symbolize the partial derivative $D^\mu f$ by the monomial $\zeta^\mu=\zeta_1^{\mu^1}\ldots\zeta_m^{\mu^m}$ in the components $\zeta_1,\ldots,\zeta_m$ of $\zeta\in(\mathbb{R}^m)^*$. The operator O may thus be symbolized by the polynomial

$$\mathcal{O}(\zeta;v)=\sum_\mu O_\mu(v)\,\zeta^\mu.$$

Observe that $\mathcal{O}\in C^\infty(U,\vee\mathbb{R}^m\otimes\mathcal{L}(V,W))$, where $\vee\mathbb{R}^m$ is the space of symmetric contravariant tensors on $\mathbb{R}^m$ i.e. the space of polynomials on $(\mathbb{R}^m)^*$.

If $M=U$, the preceding general rule, associating polynomials to local operators, gives the symbolic representation of – for instance – the arguments

$$A=\sum_{\alpha\neq 0}A_\alpha\,D^\alpha=\sum_{r=1}^{\infty}\sum_{|\alpha|=r}A_\alpha\,D^\alpha\in\mathcal{E}^0=\mathcal{L}(C^\infty(U),\ C^\infty(U))_{loc,\,n.c.}$$

($A_\alpha\in C^\infty(U),\ |\alpha|=\alpha^1+\cdots+\alpha^m$) and – in particular – of the elements

$$A=\sum_{|\alpha|=r}A_\alpha\,D^\alpha\in\mathrm{Diff}^r$$

($r \in \mathbb{N}^*$) of the space of homogeneous differential operators of degree r :

$$Au = \sum_{|\alpha|=r} A_\alpha \, D^\alpha u \approx \sum_{|\alpha|=r} A_\alpha \, \xi^\alpha = \mathcal{A}(\xi)$$

($u \in C^\infty(U),\ \xi \in (\mathbb{R}^m)^*$).

Remark 2. (i) Observe that we used the isomorphism $\ell \to \ell(1)$ to identify the spaces $\mathcal{L}(\mathbb{R}, \mathbb{R})$ and $\mathbb{R}$: A_α is in fact $A_\alpha(1)$ and $\mathcal{A}(\xi)$ is $\mathcal{A}(\xi; 1)$. It follows that, if $c \in \mathbb{R},\ A(cu) \approx \mathcal{A}(\xi; c) = c\mathcal{A}(\xi)$.

(ii) It's clear that the correspondence $A \to \mathcal{A}$ is an isomorphism between the spaces Diff^r and $C^\infty(U, \vee^r \mathbb{R}^m)$.

Consider now $T \in \Lambda^1_{alt}(\mathcal{E})_{-1,\,loc}$. Restricting T to $\mathcal{E}^0$ and even to Diff^r, we get

$$T \in \mathcal{L}(C^\infty(U, \vee^r \mathbb{R}^m),\ C^\infty(U))_{loc}$$

and, using the general association-rule, we obtain

$$T(fP_r) = \sum_\lambda T_\lambda((D^\lambda f) P_r) \approx \sum_\lambda T_\lambda(P_r)\eta^\lambda = T(\eta; P_r),$$

where $f \in C^\infty(U),\ P_r \in \vee^r \mathbb{R}^m$ and $T \in C^\infty(U, \vee \mathbb{R}^m \otimes (\vee^r \mathbb{R}^m)^*)$ i.e. $T = T(\eta; P_r)$ is a polynomial in $\eta \in (\mathbb{R}^m)^*$ with coefficients in $\mathcal{L}(\vee^r \mathbb{R}^m, \mathbb{R})$, depending smoothly on $x \in U$.

Remark 3. (i) It is important to remember that $T(fP_r) \approx T(\eta; P_r)$, where η represents the derivatives of f and where the $P_r = \sum_{|\alpha|=r} P_{r,\alpha}(\cdot)^\alpha$ on the r.h.s. symbolizes the $P_r = \sum_{|\alpha|=r} P_{r,\alpha}\, D^\alpha$ on the l.h.s.

(ii) Note also that T vanishes on $\mathcal{E}^0$, if $T(\eta; X^r) = 0$, for each $\eta \in (\mathbb{R}^m)^*,\ X \in \mathbb{R}^m$ and $r \geq 1$.

This construction extends to $T \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ restricted to $\mathcal{E}^{-q-1}$ ($q \leq -1$) and even to $T \in \Lambda^p_{alt}(\mathcal{E})_{q,\,loc}$ restricted to $\mathcal{E}^{b_1} \times \cdots \times \mathcal{E}^{b_p}$ ($p \geq 1,\ b_1, \ldots, b_p \geq 0,\ \sum b_i = -q-1$). Notice that these restrictions are, as above, valued in the space of functions. Remark that the arguments of T now may be multidifferential skew-symmetric operators. We then get a polynomial $T = T(\eta;\ \Lambda_{j=0}^{-q-1} X_j^{r_j})$ in η with coefficients in $\mathcal{L}(\Lambda_{j=0}^{-q-1}(\vee^{r_j}\mathbb{R}^m), \mathbb{R})$ (depending smoothly on x) respectively a polynomial in several variables $\eta^1, \ldots, \eta^p$ (each variable being associated to one argument and representing the derivatives acting on this argument) with coefficients in the space of multilinear forms of $\cdots \times \Lambda_{j=0}^{b_i}(\vee^{r_{i,j}}\mathbb{R}^m) \times \cdots$.

In the sequel, each object and its symbolic representation will be denoted by the same typographical sign.

2. FUNDAMENTAL EQUATION

The symbolic-representation-method will now be used to solve an equation, playing an important role in further computations.

First recall the following special cases of the definition of the Nijenhuis-Richardson-bracket. Consider $A \in \mathcal{E}^a,\ B \in \mathcal{E}^b$ and $u_0, \ldots, u_b \in N$. If $a = 0,\ b \geq 0$,

$$[A, B]\,(u_0, \ldots, u_b) = A(B(u_0, \ldots, u_b)) - \sum_{k=0}^{b} B(u_0, \ldots, A(u_k), \ldots, u_b)$$

and if $a \geq 0,\ b = -1$,

$$[A, B] = i_B\, A, \tag{3}$$

where the r.h.s. is the interior product of A by B.

LEMMA 1. *Let $T \in \Lambda^p_{alt}(\mathcal{E})_{q,\,loc}$ $(p \in \{1, 2\},\ q \leq -1)$ and $b_1, \ldots, b_p \in \mathbb{N}$, such that $\sum b_i = -q - 1$. If $m = \dim M \geq p + 1$ and if*

$$A(T(B_1, \ldots, B_p)) = \sum_{i=1}^{p} T(B_1, \ldots, [A, B_i], \ldots, B_p) \tag{4}$$

for all $A \in \mathrm{Diff}^1,\ B_1 \in \mathcal{E}^{b_1}, \ldots, B_p \in \mathcal{E}^{b_p}$, then

$$T(B_1, \ldots, B_p) = 0$$

for each $B_1 \in \mathcal{E}^{b_1}, \ldots, B_p \in \mathcal{E}^{b_p}$.

Proof. We of course may suppose that M is a connected open subset $U \subset \mathbb{R}^m$. Set

$$A = fW, \quad B_i = g_i\, \Lambda_{j=0}^{b_i}\, X_{i,j}^{r_{i,j}} \quad (= g_i\, \Lambda\, X_{i,j}^{r_{i,j}})$$

$(f, g_i \in C^\infty(U);\ W,\ X_{i,j} \in \mathbb{R}^m;\ r_{i,j} \in \mathbb{N}^*)$. The symbolization of equation (4) will lead to an algebraic equation, much easier to handle than the original. Indeed, let's represent the derivatives of the functions f, g_i by the linear forms ζ, η^i.

Since the operator

$$A = fW = \sum_{k=1}^{m} fW^k (\cdot)_k \approx \sum_{k=1}^{m} fW^k\, D_{x^k}$$

(see remark 2, (ii)) acts, on the l.h.s. of (4), on the coefficients of T and on $g_1, \ldots, g_p$ according to Leibniz's rule, we get

$$\begin{aligned} &(W.T)(\eta^1, \ldots, \eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}}, \ldots, \Lambda\, X_{p,j}^{r_{p,j}}) \\ &\quad + \sum_{i=1}^{p} \langle W, \eta^i \rangle\, T(\eta^1, \ldots, \eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}}, \ldots, \Lambda\, X_{p,j}^{r_{p,j}}), \end{aligned} \tag{5}$$

where $W.T$ denotes the action on the coefficients of T.

Consider now the r.h.s. of (4). We first symbolize $[A, B_i]$ (see remark 3, (i)). Omit index i and represent in

$$[A, B]\,(u_0, \ldots, u_b) = A(B(u_0, \ldots, u_b)) - \sum_{k=0}^{b} B(u_0, \ldots, A(u_k), \ldots, u_b),$$

the derivatives of each u_k by ξ^k. Then,

$$\begin{aligned} A(B(u_0,\dots,u_b)) &\approx fg\,\langle W, \eta+\sum \xi^k\ \rangle\ (\Lambda\ X_j^{r_j})(\xi^0,\dots,\xi^b), \\ A(u_k) &= \sum_{j=1}^{m} fW^j\ D_{x^j}u_k \approx f u_k\ \langle W,\xi^k\rangle \end{aligned}$$

and (see remark 2, (i))

$$\sum_{k=0}^{b} B(u_0,\dots,A(u_k),\dots,u_b) \approx fg \sum_{k=0}^{b} \langle W,\xi^k\rangle\ (\Lambda\ X_j^{r_j})(\xi^0,\dots,\zeta+\xi^k,\dots,\xi^b).$$

Finally,

$$\begin{aligned} &[A,B_i]\,(u_0,\dots,u_{b_i}) \\ &\quad\approx fg_i\left[\langle W,\eta^i\rangle\ (\Lambda\ X_{i,j}^{r_{i,j}})(\xi^0,\dots,\xi^{b_i}) - \sum_{k=0}^{b_i}\langle W,\xi^k\rangle\ (\tau_{\zeta,\xi^k}\ \Lambda\ X_{i,j}^{r_{i,j}})(\xi^0,\dots,\xi^{b_i})\right], \end{aligned} \tag{6}$$

where

$$(\tau_{\zeta,\xi^k}\ \Lambda\ X_{i,j}^{r_{i,j}})(\xi^0,\dots,\xi^{b_i}) = (\Lambda\ X_{i,j}^{r_{i,j}})(\xi^0,\dots,\zeta+\xi^k,\dots,\xi^{b_i}) - (\Lambda\ X_{i,j}^{r_{i,j}}(\xi^0,\dots,\xi^{b_i}).$$

The second term of the difference inside brackets may be transformed as follows. If we omit index i and if summation index ν describes the set of all permutations of $\{0,\dots,b\}$, we obtain

$$\begin{aligned} &\sum_{k=0}^{b}\sum_{\nu}\sum_{n=0}^{r_{\nu_k}-1}\binom{r_{\nu_k}}{n}\ \langle X_{\nu_k},\zeta\rangle^{r_{\nu_k}-n}\ \text{sign}\ \nu\ \langle X_{\nu_0},\xi^0\rangle^{r_{\nu_0}} \\ &\qquad\qquad \dots\ \langle W,\xi^k\rangle\ \langle X_{\nu_k},\xi^k\rangle^n\ \dots\ \langle X_{\nu_b},\xi^b\rangle^{r_{\nu_b}} \\ &= \sum_{k=0}^{b}\sum_{u=0}^{b}\sum_{\nu:\nu_k=u}\sum_{n=0}^{r_u-1}\binom{r_u}{n}\ \langle X_u,\zeta\rangle^{r_u-n}\ \text{sign}\ \nu\ \langle X_{\nu_0},\xi^0\rangle^{r_{\nu_0}} \\ &\qquad\qquad \cdots\frac{1}{n+1}(WD_{X_u})\ \langle X_u,\xi^k\rangle^{n+1}\ \dots\ \langle X_{\nu_b},\xi^b\rangle^{r_{\nu_b}} \\ &= \sum_{u=0}^{b}\sum_{n=0}^{r_u-1}\binom{r_u}{n}\ \langle X_u,\zeta\rangle^{r_u-n}\ \frac{1}{n+1}(WD_{X_u})\sum_{k=0}^{b}\sum_{\nu:\nu_k=u}\text{sign}\ \nu\ \langle X_{\nu_0},\xi^0\rangle^{r_{\nu_0}} \\ &\qquad\qquad \dots\ \langle X_u,\xi^k\rangle^{n+1}\ \dots\ \langle X_{\nu_b},\xi^b\rangle^{r_{\nu_b}}. \end{aligned}$$

This result will repeatedly be used in the sequel :

LEMMA 2. *If* $b \in \mathbb{N}$, $r_0,\dots,r_b \in \mathbb{N}^*$, $W, X_0,\dots,X_b \in \mathbb{R}^m$ *and* ζ,ξ^0,

$\ldots, \xi^b \in (\mathbb{R}^m)^*$, *we have*

$$\begin{aligned}&\sum_{k=0}^{b} \langle W, \xi^k\rangle (\tau_{\zeta,\xi^k}\ \Lambda_{j=0}^{b}\ X_j^{r_j})(\xi^0, \ldots, \xi^b)\\ &= \sum_{u=0}^{b}\sum_{n=0}^{r_u-1} \begin{pmatrix} r_u \\ n \end{pmatrix} \langle X_u, \zeta\rangle^{r_u-n}\ \frac{1}{n+1}\ (WD_{X_u})\\ &\qquad (X_0^{r_0}\ \Lambda \ldots \Lambda\ X_u^{n+1}\ \Lambda \ldots \Lambda\ X_b^{r_b})(\xi^0, \ldots, \xi^b).\end{aligned}$$

Proof of lemma 1 (continuation). Let's return to the symbolization of $[A, B_i](u_0, \ldots, u_{b_i})$. Applying lemma 2, we get

$$\begin{aligned}[A, B_i] \ \approx\ & fg_i \Bigg[\langle W, \eta^i\rangle\ \Lambda\ X_{i,j}^{r_{i,j}} - \sum_{u=0}^{b_i}\sum_{n=0}^{r_{i,u}-1} \begin{pmatrix} r_{i,u} \\ n \end{pmatrix} \langle X_{i,u}, \zeta\rangle^{r_{i,u}-n}\\ & \frac{1}{n+1}(WD_{X_{i,u}})(X_{i,0}^{r_{i,0}}\ \Lambda \ldots \Lambda\ X_{i,u}^{n+1}\ \Lambda \ldots \Lambda\ X_{i,b_i}^{r_{i,b_i}})\Bigg]. \end{aligned} \tag{7}$$

The symbolic script of the r.h.s. of (4) now is obvious. It is easy to verify that (4) reads (see (5), (6))

$$\begin{aligned}&(W.T)(\eta^1, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}})\\ &= \sum_{i=1}^{p} \langle W, \eta^i\rangle\ (\tau_{\zeta,\eta^i}\ T)(\eta^1, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}})\\ &- \sum_{i=1}^{p} T(\eta^1, \ldots, \zeta + \eta^i, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}},\\ &\qquad \ldots, \sum_{k=0}^{b_i} \langle W, \xi^k\rangle\ (\tau_{\zeta,\xi^k}\ \ \Lambda\ X_{i,j}^{r_{i,j}})(\xi^0, \ldots, \xi^{b_i}), \ldots, \Lambda\ X_{p,j}^{r_{p,j}}),\end{aligned} \tag{8}$$

or (see (5), (7))

$$\begin{aligned}&(W.T)(\eta^1, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}})\\ &= \sum_{i=1}^{p} \langle W, \eta^i\rangle\ (\tau_{\zeta,\eta^i}\ T)(\eta^1, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}})\\ &- \sum_{i=1}^{p}\sum_{u=0}^{b_i}\sum_{n=0}^{r_{i,u}-1} \begin{pmatrix} r_{i,u} \\ n \end{pmatrix} \langle X_{i,u}, \zeta\rangle^{r_{i,u}-n}\ \frac{1}{n+1}(WD_{X_{i,u}})\ T(\eta^1, \ldots, \zeta + \eta^i,\\ &\qquad \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, X_{i,0}^{r_{i,0}}\ \Lambda \ldots \Lambda\ X_{i,u}^{n+1}\ \Lambda \ldots \Lambda\ X_{i,b_i}^{r_{i,b_i}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}}),\end{aligned} \tag{9}$$

for all $\zeta, \eta^i \in (\mathbb{R}^m)^*$, all $W, X_{i,j} \in \mathbb{R}^m$ and all $r_{i,j} \in \mathbb{N}^*$.

Equation (8), (9) is polynomial in ζ (and in $\eta^1, \ldots, \eta^p$). Its term of degree 0 in ζ verifies

$$(W.T)(\eta^1, \ldots, \eta^p;\ \Lambda\ X_{1,j}^{r_{1,j}}, \ldots, \Lambda\ X_{p,j}^{r_{p,j}}) = 0.$$

Thus, the coefficients of $T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})$ are constant.

As easily seen, the first order terms in ζ (use Taylor's expansion) read

$$\sum_{i=1}^{p}\langle W,\eta^i\rangle\,(\zeta D_{\eta^i})\,T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})$$
$$-\sum_{i=1}^{p}T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},$$
$$\ldots,\sum_{k=0}^{b_i}\langle W,\xi^k\rangle\,(\zeta D_{\xi^k})(\Lambda\, X_{i,j}^{r_{i,j}})(\xi^0,\ldots,\xi^k,\ldots,\xi^{b_i}),\ldots,\Lambda\, X_{p,j}^{r_{p,j}})=0,$$

where ζD_{η^i} (resp. ζD_{ξ^k}) is the derivation with respect to η^i (resp. ξ^k) in the direction of ζ. This may be rewritten

$$\rho(W\otimes\zeta)\,T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})=0,$$

$\rho(W\otimes\zeta)$ denoting the natural action of $W\otimes\zeta\in gl(m,\mathbb{R})$. It then follows from a theorem of H. Weyl [11] that the polynomial $T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})$ in $X_{i,j}\in\mathbb{R}^m$ (it even is homogeneous of degree $r_{i,j}$ in these variables) and in $\eta^i\in(\mathbb{R}^m)^*$, is a polynomial in the contractions $\langle X_{i,j},\eta^k\rangle$ $(i,k\in\{1,\ldots,p\},\ j\in\{0,\ldots,b_i\})$:

$$T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})=P_{r_{i,j}}(\langle X_{i,j},\eta^k\rangle). \tag{10}$$

Seeking in (9) the terms of degree 2 in ζ, we get

$$\frac{1}{2}\sum_{i=1}^{p}\langle W,\eta^i\rangle\,(\zeta D_{\eta^i})^2\,T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})$$
$$-\sum_{i=1}^{p}\sum_{u=0}^{b_i}\langle X_{i,u},\zeta\rangle\,(WD_{X_{i,u}})(\zeta D_{\eta^i})\,T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})$$
$$-\sum_{i=1}^{p}\sum_{u=0}^{b_i}\frac{r_{i,u}}{2}\,\langle X_{i,u},\zeta\rangle^2\,(WD_{X_{i,u}})\,T(\eta^1,\ldots,\eta^p;\ \Lambda\, X_{1,j}^{r_{1,j}},$$
$$\ldots,X_{i,0}^{r_{i,0}}\,\Lambda\ldots\Lambda\, X_{i,u}^{r_{i,u}-1}\,\Lambda\ldots\Lambda\, X_{i,b_i}^{r_{i,b_i}},\ldots,\Lambda\, X_{p,j}^{r_{p,j}})=0 \tag{11}$$

(the term (i,u) of the last sum is absent, if $r_{i,u}=1$). Observe that equation (11) is a polynomial in the contractions $\langle W,\zeta\rangle$, $\langle W,\eta^k\rangle$, $\langle X_{i,j},\zeta\rangle$, $\langle X_{i,j},\eta^k\rangle$ (of degree 1 in W and 2 in ζ), which may be viewed as independant variables, since $m\geq p+1$. Looking for the terms of (11) in $\langle W,\zeta\rangle\ \langle X_{\imath,\mu},\zeta\rangle(\imath\in\{1,\ldots,p\},\ \mu\in\{0,\ldots,b_\imath\})$, we find

$$-\langle W,\zeta\rangle\ \langle X_{\imath,\mu},\zeta\rangle\ D_{\langle X_{\imath,\mu},\eta^\imath\rangle}\ P_{r_{i,j}}(\langle X_{i,j},\eta^k\rangle)=0, \tag{12}$$

so that the polynomial $P_{r_{i,j}}(\langle X_{i,j},\eta^k\rangle)$ is independant of $\langle X_{\imath,\mu},\eta^\imath\rangle$, for each $\imath$ and μ.

If $p=1$, result (10) reads (we drop indexes varying in $\{1,\ldots,p\}$)

$$T(\eta;\ \Lambda\, X_j^{r_j})=P_{r_j}(\langle X_j,\eta\rangle)=c_{r_0,\ldots,r_b}\prod_{j=0}^{b}\langle X_j,\eta\rangle^{r_j}\quad(c_{r_0,\ldots,r_b}\in\mathbb{R}),$$

since $T(\eta;\ \Lambda\ X_j^{r_j})$ is homogeneous of degree r_j in X_j. It then follows from (12) that

$$T(\eta;\ \Lambda_{j=0}^{b}\ X_j^{r_j}) = 0.$$

If $p = 2$, we have

$$\begin{aligned} &T(\eta^1, \eta^2;\ \Lambda\ X_{1,j}^{r_{1,j}}, \Lambda\ X_{2,j}^{r_{2,j}}) \\ &= \ P_{r_{i,j}}(\langle X_{i,j}, \eta^k\rangle) = c_{\vec{r}_1,\vec{r}_2}\ \langle \vec{X}_1, \eta^2\rangle^{\vec{r}_1}\ \langle \vec{X}_2, \eta^1\rangle^{\vec{r}_2}, \quad (c_{\vec{r}_1,\vec{r}_2} \in \mathbb{R}) \end{aligned}$$

where

$$\langle \vec{X}_k, \eta^n\rangle^{\vec{r}_k} = \prod_{j=0}^{b_k} \langle X_{k,j}, \eta^n\rangle^{r_{k,j}}.$$

It then is easy to show that the term of (11) in

$$\langle W, \eta^1\rangle\ \langle X_{1,\mu}, \zeta\rangle\ \langle X_{2,\nu}, \zeta\rangle\ \langle \vec{X}_1, \eta^2\rangle^{\vec{r}_1 - \vec{e}_{\mu+1}}\ \langle \vec{X}_2, \eta^1\rangle^{\vec{r}_2 - \vec{e}_{\nu+1}}$$

$$(\mu \in \{0, \ldots, b_1\},\ \nu \in \{0, \ldots, b_2\},\ \vec{e}_i = (0, \ldots, \underset{(i)}{1}, \ldots, 0))$$

is

$$-r_{1,\mu}\ r_{2,\nu}\ c_{\vec{r}_1,\vec{r}_2}\ \langle W, \eta^1\rangle\ \langle X_{1,\mu}, \zeta\rangle\ \langle X_{2,\nu}, \zeta\rangle\ \langle \vec{X}_1, \eta^2\rangle^{\vec{r}_1 - \vec{e}_{\mu+1}}\ \langle \vec{X}_2, \eta^1\rangle^{\vec{r}_2 - \vec{e}_{\nu+1}} = 0,$$

so that

$$T(\eta^1, \eta^2;\ \Lambda_{j=0}^{b_1}\ X_{1,j}^{r_{1,j}}, \Lambda_{j=0}^{b_2}\ X_{2,j}^{r_{2,j}}) = 0. \qquad \blacksquare$$

3. CRITICAL CASE

LEMMA 3. *If M is a contractible open subset $U \subset \mathbb{R}^m$ with $m \geq 3$, each 2-cocycle*

$$T \in \Lambda^2_{alt}(\mathcal{E})_{q,\,loc} \cap \ker \partial$$

of weight $q \leq -2$ is cohomologous to some cocycle T', such that

$$T'(B, C) = 0, \quad \forall B \in \mathcal{E}^0,\ \forall C \in \mathcal{E}^{-q-1}.$$

Proof. Notations are the same as in the previous sections. Writing the equation $\partial T(A, B, C) = 0$ when $a = 0,\ b = 0$ and $c = -q-1$ (it is trivial when $a = -1, b = 0$ and $c = -q-1$), we get

$$A(T(B, C)) - B(T(A, C)) = T([A, B], C) + T(B, [A, C]) - T(A, [B, C]). \qquad (13)$$

Set $A = fW$, $B = gX$ and $C = h\ \Lambda_{j=0}^{-q-1}\ Y_j^{r_j}$ ($f, g, h \in C^\infty(U)$; $W, X, Y_j \in \mathbb{R}^m$; $r_j \in \mathbb{N}^*$) and symbolize the derivatives of f, g and h by ζ, η and θ respectively. Arguing in a similar way as in section 2, one sees that (13) reads

$$\begin{aligned}
&(W.T)(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) - (X.T)(\zeta, \theta;\ W, \Lambda\, Y_j^{r_j}) \\
&\quad = \langle W, \eta\rangle\ (\tau_{\zeta,\eta}T)(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) + \langle W, \theta\rangle\ (\tau_{\zeta,\theta}T)(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) \\
&\quad\quad -\langle X, \zeta\rangle\ (\tau_{\eta,\zeta}T)(\zeta, \theta;\ W, \Lambda\, Y_j^{r_j}) - \langle X, \theta\rangle\ (\tau_{\eta,\theta}T)(\zeta, \theta; W, \Lambda\, Y_j^{r_j}) \\
&\quad\quad -T\left(\eta, \zeta + \theta;\ X, \sum_{k=0}^{-q-1} \langle W, \xi^k\rangle\ (\tau_{\zeta,\xi^k}\ \Lambda\, Y_j^{r_j})(\xi^0, \ldots, \xi^{-q-1})\right) \\
&\quad\quad +T\left(\zeta, \eta + \theta;\ W, \sum_{k=0}^{-q-1} \langle X, \xi^k\rangle\ (\tau_{\eta,\xi^k}\ \Lambda\, Y_j^{r_j})(\xi^0, \ldots, \xi^{-q-1})\right). \qquad (14)
\end{aligned}$$

Using lemma 2, we see that the two last terms of the r.h.s. of (14) equal

$$\begin{aligned}
- &\sum_{u=0}^{-q-1} \sum_{n=0}^{r_u-1} \binom{r_u}{n} \langle Y_u, \zeta\rangle^{r_u-n}\ \frac{1}{n+1}(W D_{Y_u})\ T(\eta, \zeta + \theta; \\
&\qquad\qquad X, Y_0^{r_0}\ \Lambda \ldots \Lambda\ Y_u^{n+1}\ \Lambda \ldots \Lambda\ Y_{-q-1}^{r_{-q-1}}) \\
+ &\sum_{u=0}^{-q-1} \sum_{n=0}^{r_u-1} \binom{r_u}{n} \langle Y_u, \eta\rangle^{r_u-n}\ \frac{1}{n+1}(X D_{Y_u})\ T(\zeta, \eta + \theta; \\
&\qquad\qquad W, Y_0^{r_0}\ \Lambda \ldots \Lambda\ Y_u^{n+1}\ \Lambda \ldots \Lambda\ Y_{-q-1}^{r_{-q-1}}).
\end{aligned}$$

(a) Now write the terms of (14) of degree 0 in ζ, then those of degree 0 in ζ and η. One obtains

$$\begin{aligned}
&(W.T)(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) - (X.T)(0, \theta;\ W, \Lambda\, Y_j^{r_j}) \\
&\quad = -\langle X, \theta\rangle\ (\tau_{\eta,\theta}\ T)(0, \theta;\ W, \Lambda\, Y_j^{r_j}) \\
&\quad\quad +T(0, \eta + \theta;\ W, \sum_{k=0}^{-q-1} \langle X, \xi^k\rangle\ (\tau_{\eta,\xi^k}\ \Lambda\, Y_j^{r_j})(\xi^0, \ldots, \xi^{-q-1})) \qquad (15)
\end{aligned}$$

respectively

$$(W.T)(0, \theta;\ X, \Lambda\, Y_j^{r_j}) = (X.T)(0, \theta;\ W, \Lambda\, Y_j^{r_j}). \qquad (16)$$

Since U is contractible, there is $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ (defined by 0 in $\mathcal{E}^a$ $(a \neq -q-1)$) such that

$$(X.S)(\theta;\ \Lambda\, Y_j^{r_j}) = T(0, \theta;\ X, \Lambda\, Y_j^{r_j}). \qquad (17)$$

Remark 4. If $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ vanishes on $\mathcal{E}^0$ and if we still set $B = gX$ and $C = h\ \Lambda_{j=0}^{-q-1}\ Y_j^{r_j}$, then

$$\partial S(B, C) = B(S(C)) - S([B, C])$$

i.e.

$$\partial S(\eta,\theta;\ X,\Lambda\, Y_j^{r_j})$$
$$= (X.S)(\theta;\ \Lambda\, Y_j^{r_j}) -\ \langle X,\theta\rangle\ (\tau_{\eta,\theta}\ S)(\theta;\ \Lambda\, Y_j^{r_j})$$
$$+ S(\eta+\theta; \sum_{k=0}^{-q-1}\ \langle X,\xi^k\rangle\ (\tau_{\eta,\xi^k}\ \Lambda\, Y_j^{r_j})(\xi^0,\ldots,\xi^{-q-1})).$$

If S denotes again the cochain with property (17), the new 2-cocycle $T' = T - \partial S$ verifies (we note T instead of T')

$$T(0,\theta;\ X,\Lambda\, Y_j^{r_j}) = 0 \tag{18}$$

and (15) becomes

$$(W.T)(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) = 0$$

i.e. $T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j})$ *has constant coefficients*. Equations (15) and (16) are now trivial.

(b) The first order terms of (14) in ζ resp. in ζ and η, give the conditions

$$\langle W,\eta\rangle\ (\zeta D_\eta)\ T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) + \langle W,\theta\rangle\ (\zeta D_\theta)\ T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j})$$
$$-\langle X,\zeta\ \rangle\ T(\eta,\theta;\ W,\Lambda\, Y_j^{r_j}) - \langle X,\theta\rangle\ (\tau_{\eta,\theta}\ T^{1,*})(\zeta,\theta;\ W,\Lambda\, Y_j^{r_j})$$
$$-T\left(\eta,\theta;\ X,\sum\ \langle W,\xi^k\rangle\ (\zeta D_{\xi^k})\ \Lambda\, Y_j^{r_j}\right) + T^{1,*}\left(\zeta,\eta+\theta;\ W,\sum\ \langle X,\xi^k\rangle\ \tau_{\eta,\xi^k}\ \Lambda\, Y_j^{r_j}\right) = 0 \tag{19}$$

and

$$\langle W,\eta\rangle\ (\zeta D_\eta)\ T^{1,*}(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) + \langle W,\theta\rangle\ (\zeta D_\theta)\ T^{1,*}(\eta,\theta;\ X,\Lambda\, Y_j^{r_j})$$
$$-\langle X,\zeta\rangle\ T^{1,*}(\eta,\theta;\ W,\Lambda\, Y_j^{r_j}) - \langle X,\theta\rangle\ (\eta D_\theta)\ T^{1,*}(\zeta,\theta;\ W,\Lambda\, Y_j^{r_j})$$
$$-T^{1,*}\left(\eta,\theta;\ X,\sum\ \langle W,\xi^k\rangle\ (\zeta D_{\xi^k})\ \Lambda\, Y_j^{r_j}\right) + T^{1,*}\left(\zeta,\theta;\ W,\sum\ \langle X,\xi^k\rangle\ (\eta D_{\xi^k})\ \Lambda\, Y_j^{r_j}\right) = 0, \tag{20}$$

where superscript $1,*$ means that we only consider the terms of degree 1 in the first linear form.

It is clear that $T^{1,*}$ is a bilinear mapping of $\mathbb{R}^m \times (\mathbb{R}^m)^*$ into the space $E_{\vec{r}}$ ($\vec{r} = (r_0,\ldots,r_{-q-1})$) of polynomials in $\theta \in (\mathbb{R}^m)^*$ with coefficients in $\mathcal{L}(\Lambda_{j=0}^{-q-1}(\vee^{r_j}\mathbb{R}^m),\mathbb{R})$ and can thus be viewed as a 1-cochain of the Chevalley-Eilenberg complex of $gl(m,\mathbb{R})$ associated to the natural representation space $E_{\vec{r}}$. Since

$$[W\otimes\zeta,\ X\otimes\eta] = \langle X,\zeta\rangle\ W\otimes\eta - \langle W,\eta\rangle\ X\otimes\zeta,$$

equation (20) reads

$$\left[(\partial_\rho T^{1,*})(W\otimes\zeta,\ X\otimes\eta)\right](\theta;\ \Lambda\, Y_j^{r_j}) = 0,$$

where ρ denotes the natural representation and ∂_ρ the corresponding coboundary operator. It then follows from the description of the cohomology of $gl(m,\mathbb{R})$, that there exist $S_{\vec{r}} \in E_{\vec{r}}$ and $I_{\vec{r}} \in E_{\vec{r},\,inv}$ ($E_{\vec{r},\,inv}$: space of all $gl(m,\mathbb{R})$-invariant polynomials of $E_{\vec{r}}$) such that

$$T^{1,*}(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) = \rho(X\otimes\eta)\ S_{\vec{r}}(\theta;\ \Lambda\, Y_j^{r_j}) + tr(X\otimes\eta)\ I_{\vec{r}}(\theta;\ \Lambda\, Y_j^{r_j}).$$

The $S_{\vec{r}}$ define $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\, loc}$ (vanishing on $\mathcal{E}^a$, if $a \neq -q-1$). The 2-cocycle $T' = T + \partial S$ inherits all properties of T. Indeed, since the coefficients of S are constant, those of T' are constant too and it follows from remark 4 that

$$T'(0, \theta;\ X, \Lambda\, Y_j^{r_j}) = 0.$$

Moreover, applying once more Weyl's theorem, we see that

$$\begin{aligned} T'^{1,*}(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) &= T^{1,*}(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) - \rho(X \otimes \eta)\ S(\theta;\ \Lambda\, Y_j^{r_j}) \\ &= tr(X \otimes \eta)\ I_{\vec{r}}(\theta;\ \Lambda\, Y_j^{r_j}) = c_{\vec{r}}\ \langle X, \eta \rangle\ \langle \vec{Y}, \theta \rangle^{\vec{r}} \qquad (21) \end{aligned}$$

($c_{\vec{r}} \in \mathbb{R}$).

In the sequel, we note again T instead of T'. We of course may regard T as 0-cochain of $gl(m, \mathbb{R})$ valued in the space $F_{\vec{r}}$ of all polynomials in $\eta, \theta \in (\mathbb{R}^m)^*$ with coefficients in $\mathcal{L}_2(\mathbb{R}^m \times \Lambda_{j=0}^{-q-1}(\vee^{r_j}\mathbb{R}^m), \mathbb{R})$. Equation (19) may now be written

$$\begin{aligned} &[(\partial_\rho T)(W \otimes \zeta)]\,(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) - \langle X, \theta \rangle\ (\tau_{\eta,\theta}\, T^{1,*})(\zeta, \theta;\ W, \Lambda\, Y_j^{r_j}) \\ &\quad + T^{1,*}\left(\zeta, \eta + \theta;\ W, \sum \langle X, \xi^k \rangle\ \tau_{\eta, \xi^k}\ \Lambda\, Y_j^{r_j}\right) = 0 \qquad (22) \end{aligned}$$

(∂_ρ : Chevalley-Eilenberg coboundary operator). The explicit form of the terms in $T^{1,*}$ of (22) is necessary for later computations. It is easy to verify that lemma 2 and conclusion (21) yield

$$\begin{aligned} &[(\partial_\rho T)(W \otimes \zeta)]\,(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) \\ &\quad + tr(W \otimes \zeta) \Big[-c_{\vec{r}}\ \langle X, \theta \rangle\ \{ \langle \vec{Y}, \eta + \theta \rangle^{\vec{r}} - \langle \vec{Y}, \theta \rangle^{\vec{r}} \} \\ &\quad + \langle X, \eta + \theta \rangle \sum_{u=0}^{-q-1} \langle \vec{Y}, \eta + \theta \rangle^{\vec{r} - r_u \vec{e}_{u+1}} \sum_{n=0}^{r_u + 1} \frac{1}{(r_u - n)!}\ c_{(r_0, \ldots, \underset{(u)}{n+1}, \ldots, r_{-q-1})} \\ &\qquad (\eta D_\theta)^{r_u - n}\ \langle Y_u, \eta + \theta \rangle^{r_u} \Big] = 0, \qquad (23) \end{aligned}$$

where $(\vec{e}_i)$ is the canonical basis of $\mathbb{R}^{-q}$. We claim that the l.h.s. of (23) is the value at $W \otimes \zeta$ and $(\eta, \theta;\ X, \Lambda\, Y_j^{r_j})$ of an element of

$$B^1(gl(m, \mathbb{R}), F_{\vec{r}}) \oplus \Lambda^1_{inv}(gl(m, \mathbb{R}), \mathbb{R}) \otimes F_{\vec{r},\, inv},$$

where $B^1(gl(m, \mathbb{R}), F_{\vec{r}})$ denotes the space of the 1-coboundaries, $\Lambda^1_{inv}(gl(m, \mathbb{R}), \mathbb{R})$ the space of the invariant scalar 1-cochains and $F_{\vec{r},\, inv}$ the space of the invariant elements of $F_{\vec{r}}$. This is quite evident. Indeed, since the coefficients $c_{\vec{r}}$ obviously are skew-symmetric in the components of $\vec{r}$, the (second) expression in brackets is multilinear and skew-symmetric in the $Y_j^{r_j}$. That's one of the methods allowing to see that the bracketed aggregation is linear in $\Lambda\, Y_j^{r_j}$. Thus we have the announced result and both terms of (23) vanish. The vanishing of the coboundary means that $T(\eta, \theta;\ X, \Lambda\, Y_j^{r_j})$ *is invariant under the natural action of* $gl(m, \mathbb{R})$. The vanishing of the second term, a polynomial in the evaluations of $W, X, Y_0, \ldots, Y_{-q-1}$ at ζ, η, θ, which may be looked upon as independant variables, implies that its coefficients vanish. Determine now its term in $\langle W,\zeta\rangle\ \langle X,\eta\rangle\ \langle Y_{-q-1},\eta\rangle\ \langle \vec{Y},\theta\rangle^{\vec{r}-\vec{e}_{-q}}$. Noting that the degree in $\langle Y_u,\eta\rangle$ of each term of the double sum over u and n is at least 1, one obtains

$$r_{-q-1}\ c_{\vec{r}}\ \langle W,\zeta\rangle\ \langle X,\eta\rangle\ \langle Y_{-q-1},\eta\rangle\ \langle \vec{Y},\theta\rangle^{\vec{r}-\vec{e}_{-q}}.$$

Hence the result

$$T^{1,*}(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) = 0. \tag{24}$$

The properties of T (constancy of the coefficients, invariance, (18), (24)) may be resumed as follows

$$\begin{aligned} &T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) \\ &= \langle X,\eta\rangle\ P_{r_j}(\langle Y_j,\eta\rangle,\ \langle Y_j,\theta\rangle) + \langle X,\theta\rangle\ Q_{r_j}(\langle Y_j,\eta\rangle,\ \langle Y_j,\theta\rangle) \\ &= \langle X,\eta\rangle \sum_{\substack{0\le u_j\le r_j\\ |\vec{u}|\ge 1}} a^{\vec{r}}_{\vec{u}}\ \langle \vec{Y},\eta\rangle^{\vec{u}}\ \langle \vec{Y},\theta\rangle^{\vec{r}-\vec{u}} + \langle X,\theta\rangle \sum_{\substack{0\le v_j\le r_j\\ |\vec{v}|\ge 2}} b^{\vec{r}}_{\vec{v}}\ \langle \vec{Y},\eta\rangle^{\vec{v}}\ \langle \vec{Y},\theta\rangle^{\vec{r}-\vec{v}}, \end{aligned} \tag{25}$$

where $a^{\vec{r}}_{\vec{u}},\ b^{\vec{r}}_{\vec{v}} \in \mathbb{R}$ and where $|\cdot|$ denotes the sum of the components. Observe also that the coefficients $a^{\vec{r}}_{\vec{u}},\ b^{\vec{r}}_{\vec{v}}$ are skew-symmetric in the columns of indexes.

(c) The terms of (14) of degree 2 in ζ read

$$\begin{aligned}
&\frac{1}{2}\ \langle W,\eta\rangle\ (\zeta D_\eta)^2\ T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) && \text{(L1)}\\
&+\frac{1}{2}\ \langle W,\theta\rangle\ (\zeta D_\theta)^2\ T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) && \text{(L2)}\\
&-\langle X,\zeta\rangle\ (\zeta D_\eta)\ \ T(\eta,\theta;\ W,\Lambda\, Y_j^{r_j}) && \text{(L3)}\\
&-\langle X,\theta\rangle\ (\tau_{\eta,\theta}\ T^{2,*})(\zeta,\theta;\ W,\Lambda\, Y_j^{r_j}) && \text{(L4)}\\
&-\sum_{u=0}^{-q-1}\ \langle Y_u,\zeta\rangle\ (W D_{Y_u})(\zeta D_\theta)\ T(\eta,\theta;\ X,\Lambda\, Y_j^{r_j}) && \text{(L5)}\\
&-\sum_{u=0}^{-q-1}\ \frac{r_u}{2}\ \langle Y_u,\zeta\rangle^2\ (W D_{Y_u})\ T(\eta,\theta;\ X, Y_0^{r_0}\ \Lambda\ldots\Lambda\ Y_u^{r_u-1}\ \Lambda\ldots\Lambda\ Y_{-q-1}^{r_{-q-1}}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if } r_u\ge 2) && \text{(L6)}\\
&+\sum_{u=0}^{-q-1}\sum_{n=0}^{r_u-1} \binom{r_u}{n} \langle Y_u,\eta\rangle^{r_u-n}\ \frac{1}{n+1} \\
&\qquad\qquad (X D_{Y_u})\ T^{2,*}(\zeta,\eta+\theta;\ W, Y_0^{r_0}\ \Lambda\ Y_u^{n+1}\ \Lambda\ldots\Lambda\ Y_{-q-1}^{r_{-q-1}}) && \text{(L7)}\\
&= 0.
\end{aligned} \tag{26}$$

It is useful for later argumentations to notice that (25) yields

$$\begin{aligned}
&T^{2,*}(\zeta,\eta+\theta;\ W,\Lambda\, Y_j^{r_j})\\
&= \langle W,\zeta\rangle \sum_{i=1}^{-q} a^{\vec r}_{\vec e_i}\ \langle Y_{i-1},\zeta\rangle\ \langle \vec Y,\eta+\theta\rangle^{\vec r-\vec e_i}\\
&\quad +\langle W,\eta+\theta\rangle \sum_{i\le k} b^{\vec r}_{\vec e_i+\vec e_k}\ \langle Y_{i-1},\zeta\rangle\ \langle Y_{k-1},\zeta\rangle\ \langle \vec Y,\eta+\theta\rangle^{\vec r-\vec e_i-\vec e_k}
\end{aligned} \tag{27}$$

(if $r_j=1$, the term $i=k=j+1$ is missing).

By seeking in (26) (its l.h.s. is a polynomial in the evaluations) the terms in $\langle W,\zeta\rangle\,\langle X,\zeta\rangle$ (note that (26) is homogeneous of degree 1 in W and X and of degree 2 in ζ), one sees that

$$\begin{aligned}
&P_{r_j}(\langle Y_j,\eta\rangle\,,\langle Y_j,\theta\rangle)\\
&= \sum_{u=0}^{-q-1}\sum_{n=0}^{r_u-1}\binom{r_u}{n}\frac{1}{n+1}\, a^{\vec r+(n+1-r_u)\vec e_{u+1}}_{\vec e_{u+1}}\ \langle Y_u,\eta\rangle^{r_u-n}\\
&\qquad\qquad \left\langle \vec Y,\eta+\theta\right\rangle^{\vec r+(n-r_u)\vec e_{u+1}}.
\end{aligned} \tag{28}$$

Set now

$$S(\eta;\ \Lambda\, Y_j^{r_j}) = c_{\vec r}\left\langle \vec Y,\eta\right\rangle^{\vec r},$$

where $c_{\vec r}\in\mathbb{R}$ is antisymmetric in the components of $\vec r$. This defines $S\in\Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ vanishing on $\mathcal{E}^a$, if $a\neq -q-1$. It's possible to choose the coefficients $c_{\vec r}$ so that the cohomologous cocycle $T'=T-\partial S$ verifies (25) with $P_{r_j}=0$ (and thus has all the properties of T). In fact, applying (28), remark 4 and lemma 2, we obtain

$$\begin{aligned}
&T'(\eta,\theta;\ X,\Lambda\, Y_j^{r_j})\\
&= \langle X,\eta\rangle \sum_{u=0}^{-q-1}\sum_{n=0}^{r_u-1}\binom{r_u}{n}\left(\frac{1}{n+1}\, a^{\vec r+(n+1-r_u)\vec e_{u+1}}_{\vec e_{u+1}} - c_{\vec r+(n+1-r_u)\vec e_{u+1}}\right)\\
&\qquad\qquad \langle Y_u,\eta\rangle^{r_u-n}\left\langle \vec Y,\eta+\theta\right\rangle^{\vec r+(n-r_u)\vec e_{u+1}}\\
&\quad + \langle X,\theta\rangle\left[\sum_{\substack{0\le v_j\le r_j\\ |\vec v|\ge 2}} b^{\vec r}_{\vec v}\left\langle \vec Y,\eta\right\rangle^{\vec v}\left\langle \vec Y,\theta\right\rangle^{\vec r-\vec v} + c_{\vec r}(\left\langle \vec Y,\eta+\theta\right\rangle^{\vec r}-\left\langle \vec Y,\theta\right\rangle^{\vec r})\right.\\
&\quad \left. -\sum_{u=0}^{-q-1}\sum_{n=0}^{r_u-1}\binom{r_u}{n} c_{\vec r+(n+1-r_u)\vec e_{u+1}}\ \langle Y_u,\eta\rangle^{r_u-n}\left\langle \vec Y,\eta+\theta\right\rangle^{\vec r+(n-r_u)\vec e_{u+1}}\right].
\end{aligned} \tag{29}$$

It's natural to set

$$c_{\vec r+(n+1-r_u)\vec e_{u+1}} = \frac{1}{n+1}\, a^{\vec r+(n+1-r_u)\vec e_{u+1}}_{\vec e_{u+1}}, \tag{30}$$

for each $\vec r\in(\mathbb{N}^*)^{-q}$, $u\in\{0,\ldots,-q-1\}$ and $n\in\{0,\ldots,r_u-1\}$.

Of course, we are obliged to prove that $c_{\vec R}$ ($\vec R\in(\mathbb{N}^*)^{-q}$) is well-defined and skew-symmetric. In other words, if

$$\vec R=\vec r+(n+1-r_u)\ \vec e_{u+1}$$

($\vec{r} \in (\mathbb{N}^*)^{-q},\ u \in \{0, \ldots, -q-1\},\ n \in \{0, \ldots, r_u - 1\}$) and

$$\vec{S} = \vec{s} + (p + 1 - s_v)\vec{e}_{v+1}$$

($\vec{s} \in (\mathbb{N}^*)^{-q},\ v \in \{0, \ldots, -q-1\},\ p \in \{0, \ldots, s_v - 1\}$) verify $\vec{S} = \nu\vec{R}$ for some permutation ν, we must have

$$\frac{1}{p+1}\, a^{\vec{S}}_{\vec{e}_{v+1}} = \operatorname{sign} \nu\, \frac{1}{n+1}\, a^{\vec{R}}_{\vec{e}_{u+1}}. \tag{31}$$

We claim that it's sufficient to show that

$$\frac{1}{r_w}\, a^{\vec{r}}_{\vec{e}_{w+1}} = \frac{1}{r_{w'}}\, a^{\vec{r}}_{\vec{e}_{w'+1}}, \tag{32}$$

for all $\vec{r} \in (\mathbb{N}^*)^{-q}$ and $w, w' \in \{0, \ldots, -q-1\}$. Indeed, it is easily checked that the antisymmetry of the coefficients $a^{\vec{r}}_{\vec{u}}$ in the columns of indexes allows to deduce (31) from (32). To prove (32), suppose $w < w'$ and determine the terms of (26) in $\langle W, \zeta\rangle \langle X, \eta\rangle \langle Y_w, \zeta\rangle \langle Y_{w'}, \eta\rangle \left\langle \vec{Y}, \theta \right\rangle^{\vec{r}-\vec{e}_{w+1}-\vec{e}_{w'+1}}$. Only the rows (L5) and (L7) may contain such terms. The unique interesting term of (L5) of course is $u = w$ i.e.

$$- \langle Y_w, \zeta\rangle\, (W D_{Y_w})(\zeta D_\theta) \left[\langle X, \eta\rangle \sum_{v=0}^{-q-1} \sum_{n=0}^{r_v - 1} \binom{r_v}{n} \frac{1}{n+1}\, a^{\vec{r}+(n+1-r_v)\vec{e}_{v+1}}_{\vec{e}_{v+1}} \right.$$
$$\left. \langle Y_v, \eta\rangle^{r_v - n} \left\langle \vec{Y}, \eta + \theta \right\rangle^{\vec{r}+(n-r_v)\vec{e}_{v+1}} + \langle X, \theta\rangle\, Q_{r_j}(\langle Y_j, \eta\rangle, \langle Y_j, \theta\rangle) \right].$$

The last term of the bracketed expression can't give any contribution. Same conclusion for the terms $v \neq w'$, which produce terms in $\langle W, \eta\rangle$ or $\langle Y_v, \eta\rangle$, and for the terms $v = w',\ n \neq r_{w'} - 1$, which bring out terms in $\langle Y_{w'}, \eta\rangle^{r_{w'}-n}$ with $r_{w'} - n \geq 2$. Consider now the term $v = w',\ n = r_{w'} - 1$. If ζD_θ acts on a factor different from $\langle Y_w, \eta + \theta\rangle^{r_w}$, we can't get a term in $\langle W, \zeta\rangle$. Otherwise, we obtain a unique term of the prescribed type : its coefficient is

$$-r_w\, a^{\vec{r}}_{\vec{e}_{w'+1}}.$$

Let's examine (L7). We necessarily have $u = w',\ n = r_{w'} - 1$. It follows from (27) that this term may be written

$$\langle Y_{w'}, \eta\rangle\ (X D_{Y_{w'}}) \left[\langle W, \zeta\rangle \sum_{i=1}^{-q} a^{\vec{r}}_{\vec{e}_i}\, \langle Y_{i-1}, \zeta\rangle \left\langle \vec{Y}, \eta + \theta \right\rangle^{\vec{r}-\vec{e}_i} + \langle W, \eta + \theta\rangle \ldots \right].$$

Here the terms $i \neq w + 1$ and those in $\langle W, \eta + \theta\rangle$ can be cancelled. Thus, the only term of the specified type has the coefficient

$$r_{w'}\, a^{\vec{r}}_{\vec{e}_{w+1}}.$$

Hence the result (32).

Finally, we actually can define the coefficients $c_{\vec{R}}$ ($\vec{R} \in (\mathbb{N}^*)^{-q}$) by (30). So it's clear from (29) (one immediately sees that the first order terms in η vanish) that

$$\begin{aligned} & T'(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) \\ = \ & \langle X, \theta \rangle\, Q'_{r_j}(\langle Y_j, \eta\rangle, \langle Y_j, \theta\rangle) = \langle X, \theta\rangle \sum_{\substack{0 \le v_j \le r_j \\ |\vec{v}| \ge 2}} b'^{\vec{r}}_{\vec{v}} \left\langle \vec{Y}, \eta \right\rangle^{\vec{v}} \left\langle \vec{Y}, \theta \right\rangle^{\vec{r} - \vec{v}}, \end{aligned} \tag{33}$$

where the coefficients $b'^{\vec{r}}_{\vec{v}} \in \mathbb{R}$ are again skew-symmetric in the columns of indexes.

For the sake of simplicity, we henceforth shall omit superscript $'$. Let's now seek in (26) the terms in $\langle W, \zeta\rangle \langle X, \theta\rangle \langle Y_w, \zeta\rangle$ ($w \in \{0, \ldots, -q-1\}$). Only the rows (L4), (L5) and (L7) have to be examined. In view of (33), all terms of (L4) and (L7) are in $\langle W, \eta\rangle$ or $\langle W, \theta\rangle$. The only convenient term of (L5) reads

$$- \langle W, \zeta\rangle \langle X, \theta\rangle \langle Y_w, \zeta\rangle\, D_{\langle Y_w, \theta\rangle}\, Q_{r_j}(\langle Y_j, \eta\rangle, \langle Y_j, \theta\rangle) = 0.$$

Since $Q_{r_j}(\langle Y_j, \eta\rangle, \langle Y_j, \theta\rangle)$ thus is independant of $\langle Y_w, \theta\rangle$ ($w \in \{0, \ldots, -q-1\}$) and since $\vec{r} \in (\mathbb{N}^*)^{-q}$ implies $|\vec{r}| \ge -q \ge 2$, we have

$$T(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) = b^{\vec{r}}_{\vec{r}} \langle X, \theta\rangle \left\langle \vec{Y}, \eta\right\rangle^{\vec{r}}, \tag{34}$$

so that

$$T^{2,*}(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) = 0. \tag{35}$$

In fact, the degree of $T(\eta, \theta;\ X, \Lambda\, Y_j^{r_j})$ in η is $|\vec{r}| \ge -q$. Consequently, property (35) is satisfied, if $q \le -3$ and if $q = -2$ and $|\vec{r}| \ge 3$. If $q = -2$ and $|\vec{r}| = r_0 + r_1 = 2$, the coefficient $b^{\vec{r}}_{\vec{r}} = b^{1,1}_{1,1}$ vanishes, in view of its antisymmetry, and so (35) is again satisfied.

Now determine the terms of (26) in $\langle W, \theta\rangle, \langle X, \zeta\rangle \langle Y_w, \zeta\rangle \left\langle \vec{Y}, \eta\right\rangle^{\vec{r} - \vec{e}_{w+1}}$ ($w \in \{0, \ldots, -q-1\}$). It follows from (34) and (35) that (L2) and (L7) vanish. Only row (L3) is interesting. We get

$$-r_w b^{\vec{r}}_{\vec{r}} \langle W, \theta\rangle \langle X, \zeta\rangle \langle Y_w, \zeta\rangle \left\langle \vec{Y}, \eta\right\rangle^{\vec{r} - \vec{e}_{w+1}} = 0,$$

which implies that

$$T(\eta, \theta;\ X, \Lambda\, Y_j^{r_j}) = 0. \tag{36}$$

Result (36) means that T vanishes on $\mathrm{Diff}^1 \times \mathcal{E}^{-q-1}$. Therefore equation (13) reads

$$A(T(B, C)) = T([A, B], C) + T(B, [A, C]),$$

if $A \in \mathrm{Diff}^1$, $B \in \mathcal{E}^0$ and $C \in \mathcal{E}^{-q-1}$. In view of lemma 1, we then have

$$T(B, C) = 0,$$

for all $B \in \mathcal{E}^0$ and $C \in \mathcal{E}^{-q-1}$. ■

4. FIRST COHOMOLOGY SPACE

THEOREM 2. *Let M be a smooth connected second-countable Hausdorff manifold of dimension $m \geq 2$. Then*

(i) $T \in \Lambda^1_{alt}(\mathcal{E})_{-1,\,loc} \cap \ker\theta \cap \ker\partial$ *implies* $T = 0$,
(ii) $T \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc} \cap \ker\partial$ *with* $q \leq -2$, *implies* $T = 0$.

Thus

(j) $H^1(\ker\theta) = 0$ *and* $H^1_{alt}(\mathcal{E})_{-1,\,loc} \approx H^1_{DR}(M)$,
(jj) $H^1_{alt}(\mathcal{E})_{q,\,loc} = 0$, *if* $q \leq -2$.

Proof. We need only prove (i) and (ii) (see (1) and theorem 1). In view of (i) and (ii), it may be interesting to verify that all coboundaries vanish. This indeed is an immediate consequence of the definition

$$\Lambda^0_{alt}(\mathcal{E})_{q,\,loc} = \mathcal{E}^q \quad (q \in \mathbb{Z}).$$

Now let $T \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc} \cap \ker\partial$ ($q \leq -1$; if $q = -1$, we also suppose that $T \in \ker\theta$). Observe that $T(B) = 0$, if $b \leq -q-2$. It thus is sufficient to prove that $T(B) = 0$, if $b \leq p$ ($p \geq -q-2$), implies $T(B) = 0$, if $b \leq p+1$.

Use the equation $\partial T(A,B) = 0$ i.e.

$$[A, T(B)] - (-1)^{ab}\,[B, T(A)] - T([A,B]) = 0, \tag{37}$$

with $a = -1$ and $b = p+1$. Since it is trivial for $p = -q-2$, we first consider the case $p > -q-2$. It then reads (see (3))

$$i_A\, T(B) = 0,$$

so that

$$T(B) = 0.$$

If $q = -1$ and $p = -q-2 = -1$, we have to verify that $b = 0$ implies $T(B) = 0$ i.e. that $T \in \ker\theta$.
If $q \leq -2$ and $p = -q-2 \geq 0$, write equation (37) with $a = 0$ and $b = p+1 = -q-1$. Since it reads

$$A(T(B)) = T([A,B]),$$

it follows from lemma 1 that

$$T(B) = 0. \qquad \blacksquare$$

5. SECOND COHOMOLOGY SPACE

THEOREM 3. *Let M be a smooth connected second-countable Hausdorff manifold of dimension $m \geq 3$. Then*

(i) $H^2(\ker\theta) = 0$ and $H^2_{alt}(\mathcal{E})_{-1,\,loc} \approx H^2_{DR}(M)$,
(ii) $H^2_{alt}(\mathcal{E})_{q,\,loc} = 0$, if $q \leq -2$.

Proof. Let $T \in \Lambda^2_{alt}(\mathcal{E})_{q,\,loc} \cap \ker\partial$ ($q \leq -1$; if $q = -1$, we also suppose that $T \in \ker\theta$). Here $T(B,C) = 0$, if $b{+}c \leq -q{-}2$. We shall demonstrate that $T(B,C) = 0$, if $b{+}c \leq p$ ($p \geq -q-2$), implies $T(B,C) = 0$, if $b{+}c \leq p{+}1$ – modulo a coboundary. We have to examine the degrees $(k, p{-}k{+}1)$ with $-1 \leq k \leq p{-}k{+}1$ i.e. $-1 \leq k \leq (p{+}1)/2$.

(1) First consider the case $p = -q-2$, $k = 0$ i.e. $(k, p-k+1) = (0, p+1) = (0, -q-1)$. If $q = -1$, we need only observe that $T \in \ker\theta$. If $q \leq -2$, it follows from lemma 3 (at least *if M is a contractible chart domain* $(*)$) that $T(B,C) = 0$, if $b = 0$ and $c = -q-1$ (modulo a coboundary).

(2) In order to study the case $k = -1$ i.e. $(k, p-k+1) = (-1, p+2)$, we define $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ ($\cap \ker\theta$, if $q = -1$) by setting

$$\begin{aligned} S(C) &= 0, \text{ if } c \neq p+2 \text{ and} \\ S(C)(u_0, \ldots, u_{p+q+2}) &= T(C, u_0)(u_1, \ldots, u_{p+q+2}) \in N, \text{ if } c = p+2, \end{aligned}$$

where $u_0, \ldots, u_{p+q+2} \in N$. We claim that $S(C) \in \mathcal{E}^{p+q+2} = A^{p+q+2}(N)_{loc,\,n.c.}$, if $c = p+2$. Indeed, $S(C)$ obviously is a multilinear local mapping of $N \times \cdots \times N$ into N. To prove the antisymmetry and the vanishing on constants, write the equation

$$\begin{aligned} \partial T(A,B,C) &= [A, T(B,C)] - (-1)^{ab}\,[B, T(A,C)] + (-1)^{(a+b)c}\,[C, T(A,B)] \\ &\quad - T([A,B],C) + (-1)^{bc}\,T([A,C],B) - (-1)^{a(b+c)}\,T([B,C],A) = 0, \end{aligned} \quad (38)$$

with $a = b = -1$ and $c = p+2$. Notice that it's trivial when $p = -q-2$. If $p > -q-2$, it reads (we use the more convenient notations u and v instead of A and B)

$$i_v\, T(C,u) = -i_u\, T(C,v),$$

so that $S(C)$ is skew-symmetric and vanishing on $1 \in N$. If $p = -q-2$, we need only check that $S(C)(1) = T(C,1) = 0$. Since $I_1 T = T(.,1) \in \Lambda^1_{alt}(\mathcal{E})_{q-1,\,loc} \cap \ker\partial$ (see remark 1), it's clear from theorem 2 that $S(C)(1) = 0$. Finally, $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ ($\cap \ker\theta$, if $q = -1$) as claimed.

Set now

$$T' = T + (-1)^{q+1}\,\partial S.$$

If $p = -q-2$, the cohomologous 2-cocycle T' still vanishes on $\mathcal{E}^0 \times \mathcal{E}^{-q-1}$. For $q = -1$, this is obvious. For $q \leq -2$, it is coming from the definition : $S(C) = 0$ unless $c = p{+}2$ ($= -q \geq 2$). The cocycle T' also continues to vanish when $b{+}c \leq p$ ($p \geq -q-2$). In fact, if $b \geq p{+}2$ or $c \geq p{+}2$, one has $c \leq -2$ or $b \leq -2$ respectively and $T'(B,C) = 0$.

Otherwise, $b < p+2,\ c < p+2$ and $b+c \leq p$, so that $T'(B,C) = (-1)^{q+1}\,\partial S(B,C) = 0$. Moreover, from the definition of S it is clear that

$$T'(C,u) = 0,$$

if $c = p+2\ (\geq 1)$ and $u \in N$.

(3) Let us now look at the degrees $(k, p-k+1)$, where $0 \leq k \leq p-k+1$ i.e. $0 \leq k \leq (p+1)/2$. We shall use equation (38) with $a=-1,\ b=k$ and $c = p-k+1$. Observe that it is trivial for $p = -q-2$. If $p > -q-2$, writing again u instead of A and T instead of T', we get $(a+c \leq p;\ a+b \leq p$, since $p \geq -1;\ a+b+c = p)$

$$i_u\, T(B,C) = 0.$$

For $p = -q-2$, as $(0, -q-1)$ has already been considered, we just have to check the degrees $(k, -k-q-1)$ with $1 \leq k \leq -(q+1)/2$. Such k only exist if $q \leq -3$. Take $a = 0,\ b = k$ and $c = -k-q-1$ and note that equation (38) then reads $(a+c = p-k+1 \leq p;\ a+b = k \leq -(q+1)/2 \leq -q-2 = p;\ (b+c, a) = (-q-1, 0))$

$$A(T(B,C)) = T([A,B],C) + T(B,[A,C]).$$

By lemma 1, this implies that

$$T(B,C) = 0.$$

This completes the induction and the proof of theorem 3, part (i). In view of $(*)$, part (ii) is only valid on a contractible chart domain. Thus, if $T \in \Lambda^2_{alt}(\mathcal{E})_{q,\,loc} \cap \ker \partial$ $(q \leq -2)$, if $(U_i)_{i\in\mathbb{N}}$ is a contractible covering of M by chart domains and if $i \in \mathbb{N}$, there is $S_i \in \Lambda^1_{alt}(\mathcal{E}_{U_i})_{q,\,loc}$ such that

$$T|_{U_i} = \partial S_i,$$

where $\mathcal{E}_{U_i}$ denotes the space $\mathcal{E}$ associated to U_i and $T|_{U_i}$ the restriction of T to U_i. Since we then have

$$S_i|_{U_i\cap U_j} - S_j|_{U_i\cap U_j} \in \Lambda^1_{alt}(\mathcal{E}_{U_i\cap U_j})_{q,\,loc} \cap \ker \partial,$$

it follows from theorem 2 that

$$S_i|_{U_i\cap U_j} = S_j|_{U_i\cap U_j} \quad \text{for each } i,j \in \mathbb{N}.$$

This means that there exists $S \in \Lambda^1_{alt}(\mathcal{E})_{q,\,loc}$ such that $S|_{U_i} = S_i$ and so

$$T|_{U_i} = \partial S_i = (\partial S)|_{U_i} \quad \text{for all } i \in \mathbb{N}.$$

■

6. THIRD COHOMOLOGY SPACE

THEOREM 4. *Let M be a smooth connected second-countable Hausdorff manifold of dimension $m \geq 3$. Then*

$$H^3(\ker\theta) = 0 \ \textit{ and } \ H^3_{alt}(\mathcal{E})_{-1,\,loc} \approx H^3_{DR}(M).$$

Proof. Let $T \in \Lambda^3_{alt}(\mathcal{E})_{-1,\,loc} \cap \ker\theta \cap \ker\partial$. As $I_1 T \in \wedge^2_{alt}(\mathcal{E})_{-2,\,loc} \cap \ker\partial$ (see remark 1), there is $S \in \wedge^1_{alt}(\mathcal{E})_{-2,\,loc}$ such that $I_1 T = \partial S$ (see theorem 3). We claim that there exists a lifting $\hat{S}$ of S i.e. a cochain $\hat{S} \in \Lambda^2_{alt}(\mathcal{E})_{-1,\,loc} \cap \ker\theta$ which verifies $I_1\,\hat{S} = S$. Indeed, $\hat{S}$ defined by

$$\hat{S}(A,B) = \begin{cases} S(A).B, \text{ if } a \geq 1,\ b = -1, \\ (-1)^{b+1}\,\hat{S}(B,A), \text{ if } a = -1,\ b \geq 1, \\ 0, \text{ otherwise}, \end{cases}$$

has the required properties, so that $I_1\,\partial\hat{S} = \partial S$ (see again remark 1). Thus, the cohomologous cocycle

$$T' = T - \partial\hat{S}$$

vanishes on constants (recall that we note n.c.).

Subsequently, we write as usual T for T'. Since T is n.c. and $T(B,C,D) = 0$ when $b+c+d \leq -1$, we once more proceed by induction. We shall show that T is n.c. and $T(B,C,D) = 0$, if $b+c+d \leq p$ ($p \geq -1$), imply – modulo a coboundary – T is n.c. and $T(B,C,D) = 0$, if $b+c+d \leq p+1$. It of course is enough to examine the triplets $(b,c,d) = (r,s,p-r-s+1)$ with $-1 \leq r \leq s \leq p-r-s+1$.

(1) First consider the degrees $(-1,s,p-s+2)$ where $-1 \leq s \leq p-s+2$. Define $S \in \Lambda^2_{alt}(\mathcal{E})_{-1,\,loc} \cap \ker\theta$ by setting,

(i) if $(a,b) = (-1,p+3)$ or $(a,b) = (p+3,-1)$,

$$S(A,B)(u_0,\ldots,u_{p+1}) = \frac{1}{p+3}\sum_{\nu\in\Pi_{p+2}} \operatorname{sign}\nu\; T(A,B,u_{\nu_0})(u_{\nu_1},\ldots,u_{\nu_{p+1}}) \in N,$$

where Π_{p+2} denotes the set of all cyclical permutations of $p+2$ objects,

(ii) if $(a,b) = (s,p-s+2)$ with $0 \leq s \leq p+2$,

$$S(A,B)(u_0,\ldots,u_{p+1}) = T(A,B,u_0)(u_1,\ldots,u_{p+1}) \in N,$$

(iii) otherwise,

$$S(A,B) = 0.$$

Let's verify that $S(A,B) \in \mathcal{E}^{a+b-1}$. It is obvious that $S(A,B)$ is multilinear and local. Since T is n.c. by assumption, $S(A,B)$ is n.c. too. We still have to check skew symmetry. In (i), remember the antisymmetry of $T(A,B,u_{\nu_0})$. In (ii), we consider the cocycle-equation

$$\partial T(A,B,C,D) = 0, \tag{39}$$

with $a = s, b = p - s + 2, c = -1$ and $d = -1$. Noticing that $0 \leq s \leq p + 2$ and $p \geq 0$ (for $p = -1$, the problem fades away) and substituting u and v for C and D, we get

$$i_u\, T(A, B, v) = -i_v\, T(A, B, u).$$

Hence $S(A, B) \in \mathcal{E}^{a+b-1}$ and $S \in \Lambda^2_{alt}(\mathcal{E})_{-1,\, loc} \cap \ker \theta$.

Set now

$$T' = T - \partial S.$$

The cocycle T' still vanishes on constants (see remark 1 and definition of S). It also continues to vanish if the total degree of its arguments is less than or equal to p. Indeed, if $a + b + c \leq p$, one easily verifies that $\partial S(A, B, C) = 0$ (see (38)), since $S(A, B) \neq 0$ only if $a + b = p + 2$.

In addition, $T'(A, B, C) = 0$ for $(a, b, c) = (-1, s, p - s + 2)$ with $-1 \leq s \leq p - s + 2$. Distinguish the cases $s \geq 0$ and $s = -1$ (systematically denote functions by $u, v, \ldots$ rather than by $A, B, \ldots$).

(I) $s \geq 0$

According to (38),

$$T'(u, B, C) = T(u, B, C) + (-1)^{p+1}\, i_u\, S(B, C) = 0.$$

Straightforward.

(II) $s = -1$

Here

$$T'(u, v, C) = T(C, u, v) - i_v\, S(C, u) - i_u\, S(C, v). \tag{40}$$

First observe that if $a = b = c = -1$ and $d = p + 3$ with $p \geq 0$, equation (39) yields

$$i_v\, T(D, u, w) + i_u\, T(D, v, w) = -i_w\, T(D, u, v). \tag{41}$$

From the definition of S, we obtain

$$\begin{aligned} &S(C, u)(v, u_1, \ldots, u_{p+1}) + S(C, v)(u, u_1, \ldots, u_{p+1}) \\ &= \frac{1}{p+3} \sum_{k=1}^{p+2} (-1)^{(k-1)(p+1)} \Big[T(C, u, \underset{(1)}{u_{p-k+3}})(\underset{(2)}{u_{p-k+4}}, \ldots, u_{p+1}, \underset{(k)}{v}, u_1, \ldots, \underset{(p+2)}{u_{p-k+2}}) \\ &\quad + T(C, v, \underset{(1)}{u_{p-k+3}})(\underset{(2)}{u_{p-k+4}}, \ldots, u_{p+1}, \underset{(k)}{u}, u_1, \ldots, \underset{(p+2)}{u_{p-k+2}}) \Big]. \end{aligned} \tag{42}$$

Now remark that the term $k = 1$ of the r.h.s. of (42) is

$$2T(C, u, v)(u_1, \ldots, u_{p+1}). \tag{43}$$

(IIa) If $p = -1$, (40), (42) and (43) imply

$$T'(u, v, C) = 0.$$

(IIb) Suppose henceforth $p \geq 0$. We can write more simply any term $k \neq 1$ of the r.h.s. of (42). Using the skew symmetry of the values of T to transfer v resp. u from position (k) to position (2) and applying (41), we see that each term $k \neq 1$ equals

$$T(C, u, v)(u_1, \ldots, u_{p+1}). \tag{44}$$

Then by (40), (42), (43) and (44),

$$T'(u, v, C) = 0.$$

(2) Finally examine the remaining degrees i.e. the triplets $(r, s, p - r - s + 1)$ with $0 \leq r \leq s \leq p - r - s + 1$. Presently realize that $r + s \leq \frac{2}{3}(p + 1) \leq p + 1$. If $p = -1$, we necessarily have $r = s = 0$ and the unique degree to consider is $(0, 0, 0)$. This case however is trivial, since $T' \in \ker \theta$. If $p \geq 0$, we need only observe that equation (39), written for $a = -1,\ b = r,\ c = s$ and $d = p - r - s + 1$, reads

$$i_u\, T'(B, C, D) = 0.$$

This completes the proof of theorem 4. ■

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