Equivariant Operators between some Modules of the Lie Algebra of Vector Fields*

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Abstract

The space $\mathcal{D}_p^k$ of differential operators of order $\leq k$, from the differential forms of degree p of a smooth manifold M into the functions of M, is a module over the Lie algebra of vector fields of M, when it is equipped with the natural Lie derivative. In this paper, we compute all equivariant i.e. intertwining operators $T : \mathcal{D}_p^k \to \mathcal{D}_q^\ell$ and conclude that the preceding modules of differential operators are never isomorphic. We also answer a question of P. Lecomte, who observed that the restriction of some homotopy operator—introduced in [Lec94]—to $\mathcal{D}_p^k$ is equivariant for small values of k and p.

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1 Introduction

Let M be a smooth, Hausdorff, second countable, connected manifold of dimension m.

Denote by $\Omega^p(M)$ the space of differential forms of degree p of M, by N the space $C^\infty(M)$ of smooth functions of M, and by $\mathcal{D}_p^k$ the space of differential operators of order smaller than or equal to k, from $\Omega^p(M)$ into N. If $X \in \text{Vect}(M)$, where $\text{Vect}(M)$ is the Lie algebra of vector fields of M, and $D \in \mathcal{D}_p^k$, the Lie derivative

$$L_X D = L_X \circ D - D \circ L_X \tag{1}$$

is a differential operator of order at most k and so $(\mathcal{D}_p^k, L)$ is a $\text{Vect}(M)$-module.

In this paper, we shall determine all the spaces $\mathcal{I}_{p,q}^{k,\ell}$ of *equivariant* operators from $\mathcal{D}_p^k$ into $\mathcal{D}_q^\ell$, that is all operators $T : \mathcal{D}_p^k \to \mathcal{D}_q^\ell$ such that

$$L_X \circ T = T \circ L_X, \forall X \in \text{Vect}(M). \tag{2}$$

In [LMT96], P. Lecomte, P. Mathonet, and E. Tousset computed all linear equivariant mappings between modules of differential operators acting on densities. It is worth mentioning also similar works by P. Cohen, Yu. Manin, and

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D. Zagier [CMZ97], C. Duval and V. Ovsienko [DO97], and H. Gargoubi and V. Ovsienko [GO96]. We solve here an analogous problem, thus answering a question of P. Lecomte, who noticed that some homotopy operator, which locally coincides—up to a coefficient—with the Koszul differential (see [Lec94]), is equivariant if it is restricted to low order differential operators and asked whether this property still holds for higher orders.

2 Local representation

The formalism described in this section, known in Mechanics as the normal ordering, allows to locally replace a differential operator by a polynomial whose highest order terms are nothing but the principal symbol of the operator. The method can be compared with Operational Calculus. Its systematic use in Differential Geometry is originated in [DWL83]. We try below to carefully develop all chief aspects of this—in the beginning a little bit technical—modus operandi. The reader is referred to [DWL83] for further details.

Consider an open subset U of $\mathbb{R}^m$, two real finite-dimensional vector spaces E and F, and some local, i.e. support preserving, operator

$$O \in \mathcal{L}(C^\infty(U,E), C^\infty(U,F))_{loc}. \tag{3}$$

The operator is fully defined by its values on the products fe, $f \in C^\infty(U), e \in E$. A well-known theorem of J. Peetre (see [Pee60]) states that it has the form

$$(O(fe))\,(x) = \sum_{\alpha\in\mathbb{N}^m} O_{\alpha,x}\left(\partial_x^\alpha(fe)\right) = \sum_{\alpha\in\mathbb{N}^m} O_{\alpha,x}(e)\partial_x^\alpha f \in F,$$

where $x \in U$, $\partial_x^\alpha = \partial_{x^1}^{\alpha^1}\ldots\partial_{x^m}^{\alpha^m}$, and $O_\alpha \in C^\infty(U, \mathcal{L}(E,F))$. Moreover, the coefficients O_α are well-determined by O and the series is locally finite (it is finite if U is relatively compact).

We will symbolize the partial derivative $\partial_x^\alpha f$ by the monomial $\xi^\alpha = \xi_1^{\alpha^1}\ldots \xi_m^{\alpha^m}$ in the components $\xi_1,\ldots,\xi_m$ of some linear form $\xi \in (\mathbb{R}^m)^*$ or—at least mentally—even by $\xi^\alpha f$. The operator O is thus represented by the polynomial

$$\mathbf{O}_x(\xi; e) = \sum_{\alpha\in\mathbb{N}^m} O_{\alpha,x}(e)\xi^\alpha \in F. \tag{4}$$

When identifying the space $Pol(\mathbb{R}^m)^*$ of polynomials of $(\mathbb{R}^m)^*$ with the space $\vee\mathbb{R}^m$ of symmetric contravariant tensors of $\mathbb{R}^m$, the *representative polynomial* $\mathbf{O}$ of the operator O is a member of

$$\mathbf{O} \in C^\infty(U, \vee\mathbb{R}^m \otimes \mathcal{L}(E,F)). \tag{5}$$

Let us emphasize that in equation (4) the form ξ symbolizes the derivatives of O that act on the argument $fe \in C^\infty(U,E)$, while $e \in E$ represents this argument.

Note that situation (3) is rather general. Indeed, consider any local linear operator between the spaces of smooth sections of two vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a smooth manifold M,

$$O \in \mathcal{L}\left(\Gamma(\mathcal{E}), \Gamma(\mathcal{F})\right)_{loc}. \tag{6}$$

Its restriction to a domain U of local coordinates of M that is also a common domain of trivialization for $\mathcal{E}$ and $\mathcal{F}$ is obviously of type (3), where E and F are the typical fibers of $\mathcal{E}$ and $\mathcal{F}$ respectively. Thus, the restriction $D|_U$ (or simply D if no confusion is possible) of a differential operator $D \in \mathcal{D}_p^k$ to a chart domain U of M for instance is a local linear operator from $C^\infty(U, \wedge^p(I\!R^m)^*)$ into $C^\infty(U)$, which reads

$$(D(f\omega))(x) = \sum_{|\alpha|\leq k} D_{\alpha,x}(\omega)\partial_x^\alpha f, \tag{7}$$

$f \in C^\infty(U), \omega \in \wedge^p(I\!R^m)^*, x \in U, \mid \alpha \mid = \alpha^1 + \ldots + \alpha^m$, and $D_\alpha \in C^\infty(U, \wedge^p I\!R^m)$.

The principal symbol $\sigma(D)$ of D, defined on U by

$$(\sigma(D))_x(\xi;\omega) = \sum_{|\alpha|=k} D_{\alpha,x}(\omega)\xi^\alpha, \tag{8}$$

where ξ is interpreted as a variable in $T_x^*M \subset T^*M$, is a smooth section of the vector bundle $E_p^k = \vee^k TM \otimes \wedge^p TM \to M$. The space $\mathcal{S}_p^k := \Gamma(E_p^k)$ is the k-th order symbol space and the graded sum $\mathcal{S}_p = \oplus_k \mathcal{S}_p^k$ is the total symbol space.

Set, for the sake of simplicity, $M = I\!R^m$. When mapping $D \in \mathcal{D}_p$ (see (7)) to $\mathbf{D} \in \mathcal{S}_p$ defined by

$$\mathbf{D}_x(\xi;\omega) = \sum_{|\alpha|\leq k} D_{\alpha,x}(\omega)\xi^\alpha \tag{9}$$

(see (8)), we get a vector space isomorphism. This *total symbol map* or *normal ordering map* is exactly the above-mentioned representative polynomial map. Since it does not commute with the Lie derivative of all vector fields but only with the action of the affine subalgebra of $\mathrm{Vect}(I\!R^m)$, generated by constant and linear vector fields, we also call this mapping *affinely equivariant symbol map.*

Note that when starting from situation (6) so that the open subset U in (3) is a domain of local coordinates of a variety M and of common trivialization of two bundles $\mathcal{E}$ and $\mathcal{F}$, the local representative polynomial (4) depends as well on the coordinates as on the trivializations. Only its homogeneous part of highest order—the symbol— has an intrinsic meaning. The polynomial representation of the restrictions of locally differential operators is nevertheless very convenient. It leads to a computing technique that was successfully applied in numerous works (see e.g. [DWGL84], [Bon00], [Lec00], [Mat00], [Pon01], [GP03]) and is worth being described in more details.

Remark first that the derivatives acting on a product have a pleasant symbolization, an observation already made—as others below—in [DWL83]. Indeed, if $f, g \in C^\infty(U)$ then

$$\begin{aligned} \partial_x^\alpha(fg) &= \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!}\partial_x^{\alpha_1} f \partial_x^{\alpha_2} g \\ &\simeq \textstyle\sum_{\alpha_1+\alpha_2=\alpha} \frac{\alpha!}{\alpha_1!\alpha_2!}(\xi^1)^{\alpha_1}(\xi^2)^{\alpha_2} \\ &= (\xi^1+\xi^2)^\alpha, \end{aligned}$$

$\xi^1, \xi^2 \in (I\!R^m)^*$ and $\beta! = \beta^1! \ldots \beta^m!$ for any $\beta \in I\!N^m$. So, working in the context (7), (9), we see that

$$(D(fg\,\omega))(x) \simeq \mathbf{D}_x(\xi^1+\xi^2;\omega). \tag{10}$$

In order to introduce simple and standard notations, we use the same typographical sign O (respectively e), when referring to the operator O (respectively the argument fe of O) and its local polynomial representation $\mathbf{O}$ (respectively the argument e of $\mathbf{O}$). Moreover, we systematically suppress the point $x \in U$ and write

$$O(e) \simeq O(\xi; e) \tag{11}$$

instead of $(O(fe))\,(x)$ is symbolized by $\mathbf{O}_x(\xi; e)$. Let us emphasize that e is of type $fe \in C^\infty(U, E)$ in the left hand side of (11), whereas $e \in E$ in the right hand side. So

$$D(\omega) \simeq D(\xi; \omega), \tag{12}$$

where (the derivatives on) the left $\omega \in C^\infty(U, \wedge^p(\mathbb{R}^m)^*)$ is (respectively are) represented by the right $\omega \in \wedge^p(\mathbb{R}^m)^*$ (respectively by $\xi \in (\mathbb{R}^m)^*$).

When locally representing—or symbolizing—an operator O, it is advantageous to always imagine this operator evaluated on an argument ($e \in C^\infty(U, E)$ written—at least mentally—in the form fe, $f \in C^\infty(U), e \in E$). If the argument is itself a locally differential operator or if the operator is a compound operator, symbolize first the "interior" operator.

Note also that the local polynomial representation canonically extends to multilinear local operators

$$O \in \mathcal{L}_r\left(C^\infty(U, E_1) \times \ldots \times C^\infty(U, E_r), C^\infty(U, F)\right)_{loc}\ \ (r \geq 1)\ ,$$

with self-explaining notations. We then obtain representations that are polynomials in several variables $\xi^1, \ldots, \xi^r \in (\mathbb{R}^m)^*$, the form ξ^i symbolizing the derivatives acting on the i-th argument.

Now we look for the local representation of

$$L_X D = L_X \circ D - D \circ L_X. \tag{13}$$

Symbolize first the Lie derivative of a differential p-form $\omega \in C^\infty(U, \wedge^p(\mathbb{R}^m)^*)$ with respect to a vector field $X \in C^\infty(U, \mathbb{R}^m)$, i.e.

$$L_X\omega = i_X(d\omega) + d(i_X\omega),$$

with standard notations. Remember that ω (respectively X) stands for $f\omega$ (respectively gX), $f \in C^\infty(U), \omega \in \wedge^p(\mathbb{R}^m)^*$ (respectively $g \in C^\infty(U), X \in \mathbb{R}^m$). When representing the derivatives ∂_{x^i} acting on ω (respectively X) by ξ_i or—as above-mentioned—better by $\xi_i f$ (respectively ζ_i or $\zeta_i g$) and when exceptionally writing things explicitly, we get

$$d\omega = d(f\omega) = df \wedge \omega = \left(\sum_i \partial_{x^i} f dx^i\right) \wedge \omega \simeq \left(\sum_i \xi_i f dx^i\right) \wedge \omega = f(\xi \wedge \omega) \tag{14}$$

and

$$i_X(d\omega) \simeq fg(\langle X, \xi\rangle \omega - \xi \wedge i_X\omega), \tag{15}$$

where $\langle X, \xi\rangle$ denotes the evaluation of $X \in \mathbb{R}^m$ on $\xi \in (\mathbb{R}^m)^*$. Since (14) and (10) show that

$$d(i_X\omega) \simeq fg\left((\xi + \zeta) \wedge i_X\omega\right), \tag{16}$$

we eventually get from (15) and (16)

$$L_X\omega \simeq fg(\langle X,\xi\rangle\omega + \zeta \wedge i_X\omega). \tag{17}$$

Let us underline that when symbolizing compound operators such as $(L_X D)(\omega) = L_X(D(\omega)) - D(L_X\omega)$, the derivatives acting on ω, D, and X are *traditionally* represented by ξ, η, and ζ respectively (see e.g. [Lec00]). Equations (10), (12), and (17) then yield

$$L_X(D(\omega)) \simeq \langle X, \eta+\xi\rangle D(\xi;\omega),$$

since $D(\omega) \in C^\infty(U)$, and

$$D(L_X\omega) \simeq D(\xi+\zeta; \langle X,\xi\rangle\omega + \zeta\wedge i_X\omega).$$

Finally, in view of (5),

$$(L_X D)(\omega) \simeq \langle X,\eta\rangle D(\xi;\omega) - \langle X,\xi\rangle\,(\tau_\zeta D)\,(\xi;\omega) - D(\xi+\zeta;\zeta\wedge i_X\omega), \tag{18}$$

where $(\tau_\zeta D)\,(\xi;\omega) := D(\xi+\zeta;\omega) - D(\xi;\omega)$ (see (10)) is above all a notation.

3 Locality of equivariant operators

Let U be a domain of local coordinates of M. If $D \in \mathcal{D}^k_p|_U$ ($\mathcal{D}^k_p|_U$ is defined similarly to $\mathcal{D}^k_p$, but for $M = U$), its representation is a polynomial D in $C^\infty(U, E^k_p)$, where $E^k_p = \vee^{\le k}I\!R^m \otimes \wedge^p I\!R^m$, or—equivalently—in the space of smooth sections $\Gamma(\vee^{\le k}TU \otimes \wedge^p TU)$. We denote by $\sigma(D)$ or simply σ the homogeneous part of maximal degree of this polynomial i.e. the principal symbol of the considered differential operator. If L^t is the natural Lie derivative of tensor fields and e_i (respectively ε^i) ($i \in \{1,\ldots,m\}$) is the canonical basis of $I\!R^m$ (respectively $(I\!R^m)^*$), we have, using standard notations,

$$\begin{gathered}(L^t_X(\sigma(D)))\,(\xi;\omega) = \\ \sum_j X^j\partial_{x^j}\left(\sigma(\xi;\omega)\right) - \sum_{jk}\partial_{x^j}X^k\,\xi_k\partial_{\xi_j}\left(\sigma(\xi;\omega)\right) - \sum_{jk}\partial_{x^j}X^k\,\sigma(\xi;\varepsilon^j\wedge i_{e_k}\omega) \simeq \\ \langle X,\eta\rangle\sigma(\xi;\omega) - \langle X,\xi\rangle(\zeta\partial_\xi)\left(\sigma(\xi;\omega)\right) - \sigma(\xi;\zeta\wedge i_X\omega),\end{gathered}$$

where η and ζ are associated to σ and X respectively, and where $\zeta\partial_\xi$ denotes the derivation with respect to ξ in the direction of ζ. If L^{op} is the formerly defined Lie derivative L of differential operators, it follows from (18) that

$$\left(\sigma(L^{op}_X D)\right)(\xi;\omega) = \left(L^t_X(\sigma(D))\right)(\xi;\omega),$$

i.e. that the principal symbol is equivariant with respect to all vector fields.

We are now prepared to prove the following lemma:

Lemma 1 *Every equivariant operator $T \in \mathcal{I}^{k,\ell}_{p,q}$ is local.*

Proof. It suffices to show that the family $\mathcal{L}^k_p = \{L_X : \mathcal{D}^k_p \longrightarrow \mathcal{D}^k_p \mid X \in \mathrm{Vect}(M)\}$ is closed with respect to locally finite sums and is locally transitive (lt); this means that each point of M has some neighborhood Ω, such that for every open subset $\omega \subset \Omega$ and every $D \in \mathcal{D}_p^k$ with compact support in ω, D can be decomposed into Lie derivatives,

$$D = \sum_{i=1}^{n} L_{X_i} D_i \quad (D_i \in \mathcal{D}_p^k,\ \mathrm{supp}\, X_i \subset \omega,\ \mathrm{supp}\, D_i \subset \omega), \tag{19}$$

with n independent of D and Ω. Indeed, proposition 3 of [DWL81] states that $\mathcal{L}_p^k$ is then globally transitive i.e. that (19) holds for every open subset ω of M and every $D \in \mathcal{D}_p^k$ with support in ω. Take now a differential operator $D \in \mathcal{D}_p^k$ that vanishes in an open subset V of M and let x_0 be an arbitrary point of V. Since $\mathrm{supp}\, D \subset \omega = M \backslash \omega_0$ for some neighborhood ω_0 of x_0, D has form (19) and $T(D)(x_0) = \sum_{i=1}^n \left(L_{X_i}(T(D_i))\right)(x_0) = 0$.

In order to confirm local transitivity (LT), let us point out that example 12 of [DWL81] shows that the family $\mathcal{L}_s^r = \{L_X : \Gamma(\otimes_s^r M) \longrightarrow \Gamma(\otimes_s^r M) \mid X \in \mathrm{Vect}(M)\}$ is lt, if

$$s - r \neq m. \tag{20}$$

Moreover, the domains U of the charts of M that correspond to cubes in $\mathbb{R}^m$, play the role of the neighborhoods Ω in the definition of LT.

LT of $\mathcal{L}_p^k$ may now be proved as follows. If $x \in M$, take $\Omega = U$ and if $D \in \mathcal{D}_p^k$ is a differential operator with compact support in some open subset ω of U, note that its principal symbol $\sigma(D)$ is a contravariant tensor field with compact support in ω. Hence condition (20) is satisfied and

$$\sigma(D) = \sum_{i=1}^{n} L_{X_i}^t \sigma(D_i) = \sigma\left(\sum_{i=1}^{n} L_{X_i}^{op} D_i\right)$$

($D_i \in \mathcal{D}_p^k$; $\mathrm{supp}\, X_i$, $\mathrm{supp}\, D_i \subset \omega$; n independent of D and U). Thus $D - \sum_{i=1}^n L_{X_i}^{op} D_i \in \mathcal{D}_p^{k-1}$ and we conclude by iteration. ∎

4 Local expression of the equivariance equation

Let U be a connected, relatively compact domain of local coordinates of M.

Recall that if $D \in \mathcal{D}_p^k|_U$, its representation is a polynomial $D \in C^\infty(U, E_p^k)$, where $E_p^k = \vee^{\leq k} \mathbb{R}^m \otimes \wedge^p \mathbb{R}^m$.

We identify $\mathcal{D}_p^k|_U$ with $C^\infty(U, E_p^k)$. Thus, if $T \in \mathcal{I}_{p,q}^{k,\ell}$, the restriction $T|_U$ is a local operator from $C^\infty(U, E_p^k)$ into $C^\infty(U, E_q^\ell)$, with representation $T(\eta; D)(\xi; \omega)$ $(\eta, \xi \in (\mathbb{R}^m)^*, D \in E_p^k, \omega \in \wedge^q (\mathbb{R}^m)^*)$.

Equation (18) shows that equivariance condition (2),

$$\left(L_X\left(T(D)\right)\right)(\omega) - \left(T\left(L_X D\right)\right)(\omega) = 0, \forall X \in \mathrm{Vect}(M), D \in \mathcal{D}_p^k, \omega \in \Omega^q(M), \tag{21}$$

locally reads

$$\begin{aligned}(X.T)(\eta; D)(\xi; \omega) &- \langle X, \eta\rangle \left((\tau_\zeta T)(\eta; D)\right)(\xi; \omega) \\ &-\langle X, \xi\rangle \left(\tau_\zeta(T(\eta; D))\right)(\xi; \omega) + T(\eta + \zeta; X\tau_\zeta D)(\xi; \omega) \\ &-T(\eta; D)(\xi + \zeta; \zeta \wedge i_X \omega) + T(\eta + \zeta; D(\cdot + \zeta; \zeta \wedge i_X \cdot))(\xi; \omega) = 0,\end{aligned} \tag{22}$$

where $X.T$ is obtained by derivation of the coefficients of T in the direction of X. It is worth emphasizing that we do not symbolize the derivatives acting on the coefficients of T. Indeed, the left hand side of equation (21) vanishes as a locally differential operator in the variables X, D, ω, so that equation (22) is a polynomial identity in the variables ζ, η, ξ but would not be a polynomial identity in the linear form possibly symbolizing the derivatives on T. Since the polynomial in the left hand side of (22) vanishes identically, its terms of fixed degree in ζ vanish identically. Remark also that the argument D in $T(\eta; D)$ is a member of $E^k_p = \vee^{\leq k}\mathbb{R}^m \otimes \wedge^p \mathbb{R}^m$, i.e. a polynomial of degree k in $\xi \in (\mathbb{R}^m)^*$ with coefficients in the linear forms in $\omega \in \wedge^p(\mathbb{R}^m)^*$. So the argument $X\tau_\zeta D$ in equation (22) is the polynomial defined by $(X\tau_\zeta D)\,(\xi;\omega) = \langle X, \xi\rangle\,(\tau_\zeta D)\,(\xi;\omega)$.

Now take in equation (22) the terms of degree 0 in ζ:

$$(X.T)(\eta; D)(\xi; \omega) = 0.$$

This means that the coefficients of T are constant.

The terms of degree 1 lead to the equation

$$\begin{aligned}&\langle X, \eta\rangle(\zeta\partial_\eta)T(\eta; D)(\xi; \omega)\\ &\quad -T(\eta; X(\zeta\partial_\xi)D)(\xi; \omega) - T(\eta; D(\cdot\,; \zeta \wedge i_X\cdot))(\xi; \omega)\\ &\quad +\langle X, \xi\rangle(\zeta\partial_\xi)T(\eta; D)(\xi; \omega) + T(\eta; D)(\xi; \zeta \wedge i_X\omega) = 0,\end{aligned}$$

which, if ρ denotes the natural action of $gl(m, \mathbb{R})$, may be rewritten

$$\rho(X \otimes \zeta)\,(T(\eta; D)(\xi; \omega)) = 0.$$

Note that $T(\eta; D)(\xi; \omega)$ is completely characterized by $T(\eta; Y^r \otimes (X_1 \wedge \ldots \wedge X_p))(\xi; \nu^1 \wedge \ldots \wedge \nu^q)$ ($Y, X_i \in \mathbb{R}^m$, $\nu^j \in (\mathbb{R}^m)^*$, $r \in \{0, \ldots, k\}$). This last expression is a polynomial in Y, X_i and in η, ξ, ν^j. It thus follows from the description of invariant polynomials under the action of $gl(m, \mathbb{R})$ (see [Wey46]), that it is a polynomial $\mathcal{T}_r(\langle Y, \xi\rangle, \langle Y, \eta\rangle, \langle Y, \nu^j\rangle, \langle X_i, \xi\rangle, \langle X_i, \eta\rangle, \langle X_i, \nu^j\rangle)$ in the evaluations of the vectors on the linear forms.

In order to determine the most general structure of $\mathcal{T}_r$, observe that this polynomial is homogeneous of degree r in Y and degree 1 in the X_i's and the ν^j's, and that furthermore it is antisymmetric in the X_i's and the ν^j's. It follows from the skew-symmetry in the ν^j's, that Y is evaluated on at most one ν^j, so that $q \leq p + 1$, and from the skew-symmetry in the X_i's, that ξ and η are evaluated on at most one X_i, so that $q \geq p - 2$. Finally, q is $p - 2$, $p - 1$, p or $p + 1$. In order to simplify notations, set $\Lambda = X_1 \wedge \ldots \wedge X_p$, $\omega = \nu^1 \wedge \ldots \wedge \nu^q$, $u = \langle Y, \xi\rangle$ and $v = \langle Y, \eta\rangle$. The following possible forms of the terms of $\mathcal{T}_r$ and the corresponding conditions on p and q are immediate consequences of the preceding observations.

term type		condition	term form
(1) no $\langle Y, \nu^j\rangle$			
(1.1) no $\langle X_i, \xi\rangle$	no $\langle X_i, \eta\rangle$	$q = p$	$v^s u^{r-s}\langle \Lambda, \omega\rangle$
(1.2) one $\langle X_i, \xi\rangle$	no $\langle X_i, \eta\rangle$	$q = p - 1$	$v^s u^{r-s}\langle i_\xi\Lambda, \omega\rangle$
(1.3) no $\langle X_i, \xi\rangle$	one $\langle X_i, \eta\rangle$	$q = p - 1$	$v^s u^{r-s}\langle i_\eta\Lambda, \omega\rangle$
(1.4) one $\langle X_i, \xi\rangle$	one $\langle X_i, \eta\rangle$	$q = p - 2$	$v^s u^{r-s}\langle i_\xi i_\eta\Lambda, \omega\rangle$

(2) one $\langle Y, \nu^j\rangle$

$$
\begin{array}{lllll}
(2.1)\ \text{no}\ \langle X_i, \xi\rangle & \text{no}\ \langle X_i, \eta\rangle & q = p+1 & v^s u^{r-s-1}\langle Y\wedge\Lambda, \omega\rangle & \\
(2.2)\ \text{one}\ \langle X_i, \xi\rangle & \text{no}\ \langle X_i, \eta\rangle & q = p & v^s u^{r-s-1}\langle Y\wedge i_\xi\Lambda, \omega\rangle & \quad (23)\\
(2.3)\ \text{no}\ \langle X_i, \xi\rangle & \text{one}\ \langle X_i, \eta\rangle & q = p & v^s u^{r-s-1}\langle Y\wedge i_\eta\Lambda, \omega\rangle & \\
(2.4)\ \text{one}\ \langle X_i, \xi\rangle & \text{one}\ \langle X_i, \eta\rangle & q = p-1 & v^s u^{r-s-1}\langle Y\wedge i_\xi i_\eta\Lambda, \omega\rangle &
\end{array}
$$

Remark 1 These structures of $\mathcal{T}_r$ show that a priori equivariant operators are mappings $T : \mathcal{D}^k_p \longrightarrow \mathcal{D}^{k+1}_{p-2}$, $T : \mathcal{D}^k_p \longrightarrow \mathcal{D}^{k+1}_{p-1}$, $T : \mathcal{D}^k_p \longrightarrow \mathcal{D}^k_p$ or $T : \mathcal{D}^k_p \longrightarrow \mathcal{D}^{k-1}_{p+1}$.

It is interesting to rewrite (22) in terms of the $\mathcal{T}_r$'s ($r \in \{0, \ldots, k\}$). To simplify notations, we set

$$
\begin{array}{llll}
\lambda = \langle X, \xi\rangle & \mu = \langle X, \eta\rangle & \nu = \langle X, \zeta\rangle & \pi^j = \langle X, \nu^j\rangle \\
u = \alpha_0 = \langle Y, \xi\rangle & v = \beta_0 = \langle Y, \eta\rangle & \gamma_0 = \langle Y, \zeta\rangle & \delta^j_0 = \langle Y, \nu^j\rangle \\
\alpha_i = \langle X_i, \xi\rangle & \beta_i = \langle X_i, \eta\rangle & \gamma_i = \langle X_i, \zeta\rangle & \delta^j_i = \langle X_i, \nu^j\rangle
\end{array}
$$

Substitute in (22), $Y^r \otimes (X_1 \wedge \ldots \wedge X_p)$ and $\nu^1 \wedge \ldots \wedge \nu^q$ to D and ω respectively, and use subscript $\iota \in \{0, \ldots, p\}$. Note that:

- Taylor expansion gives

$$T(\eta; D)(\xi + \zeta; \omega) = \sum \frac{1}{a!}(\zeta\partial_\xi)^a T(\eta; D)(\xi; \omega) = \sum \frac{1}{a!}(\zeta\partial_\xi)^a \mathcal{T}_r(\alpha_\iota, \beta_\iota, \delta^j_\iota)$$

- the computation of the successive directional derivatives $(\zeta\partial_\xi)^a$ of the composite function $\mathcal{T}_r(\alpha_\iota, \beta_\iota, \delta^j_\iota)$ leads to the same rule as the computation of the successive powers of a sum of $p + 1$ real terms:

$$
\begin{aligned}
&(\zeta\partial_\xi)^a \mathcal{T}_r(\alpha_\iota, \beta_\iota, \delta^j_\iota) \\
&\quad = \sum_{\varrho_0+\ldots+\varrho_p=a} \frac{a!}{\varrho_0! \ldots \varrho_p!}\, \gamma_0^{\varrho_0} \ldots \gamma_p^{\varrho_p} \left(\partial^{\varrho_0}_{\alpha_0} \ldots \partial^{\varrho_p}_{\alpha_p} \mathcal{T}_r\right)(\alpha_\iota, \beta_\iota, \delta^j_\iota)
\end{aligned}
$$

- $$\tau_\zeta Y^r = \sum_{a=0}^{r-1} \binom{r}{a} \gamma_0^{r-a} Y^a$$

- $$XY^a = \frac{1}{a+1}(X\partial_Y)Y^{a+1}$$

- $$
\begin{aligned}
&(X_1 \wedge \ldots \wedge X_p)(\zeta \wedge i_X \cdot) \\
&\quad = X \wedge i_\zeta(X_1 \wedge \ldots \wedge X_p) \\
&\quad = \sum_{b=1}^{p} \gamma_b (X\partial_{X_b})(X_1 \wedge \ldots \wedge X_p) \qquad (24)
\end{aligned}
$$

Transform now the terms of (22) in conformity with the preceding hints. If $(\partial\mathcal{T})(\cdot) = (\partial\mathcal{T})(\alpha_\iota, \cdot, \delta_\iota^j)$ and if $d_r(\xi)$ denotes the degree of $\mathcal{T}_r$ in ξ, the local equivariance equation (22) finally reads in terms of the $\mathcal{T}_r$'s:

$$\begin{aligned}
&\boldsymbol{\mu}\left(\mathcal{T}_r(\beta_\iota+\boldsymbol{\gamma_\iota})-\mathcal{T}_r(\beta_\iota)\right)\\
&\quad+\boldsymbol{\lambda}\sum_{a=1}^{d_r(\xi)}\sum_{\varrho_0+\ldots+\varrho_p=a}\frac{1}{\varrho_0!\ldots\varrho_p!}\,\boldsymbol{\gamma_0^{\varrho_0}\ldots\gamma_p^{\varrho_p}}\left(\partial^{\varrho_0}_{\alpha_0}\ldots\partial^{\varrho_p}_{\alpha_p}\mathcal{T}_r\right)(\beta_\iota)\\
&\quad+\sum_{b=1}^{q}\boldsymbol{\pi^b}\sum_{a=0}^{d_r(\xi)}\sum_{\iota=0}^{p}\boldsymbol{\gamma_\iota}\\
&\quad\sum_{\varrho_0+\ldots+\varrho_p=a}\frac{1}{\varrho_0!\ldots\varrho_p!}\,\boldsymbol{\gamma_0^{\varrho_0}\ldots\gamma_p^{\varrho_p}}\left(\partial^{\varrho_0}_{\alpha_0}\ldots\partial^{\varrho_p}_{\alpha_p}\partial_{\delta^b_\iota}\mathcal{T}_r\right)(\beta_\iota)\\
&\quad-\sum_{a=0}^{r-1}\binom{r}{a}\frac{1}{a+1}\,\boldsymbol{\gamma_0^{r-a}}\ \big[\,\boldsymbol{\lambda}\left(\partial_{\alpha_0}\mathcal{T}_{a+1}\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\\
&\qquad\qquad+\boldsymbol{(\mu+\nu)}\left(\partial_{\beta_0}\mathcal{T}_{a+1}\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\\
&\qquad\qquad+\ \textstyle\sum_{j=1}^{q}\boldsymbol{\pi^j}\left(\partial_{\delta^j_0}\mathcal{T}_{a+1}\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\,\big]\\
&\quad-\sum_{b=1}^{p}\boldsymbol{\gamma_b}\sum_{a=0}^{r}\binom{r}{a}\boldsymbol{\gamma_0^{r-a}}\ \big[\,\boldsymbol{\lambda}\left(\partial_{\alpha_b}\mathcal{T}_a\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\\
&\qquad\qquad+\boldsymbol{(\mu+\nu)}\left(\partial_{\beta_b}\mathcal{T}_a\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\\
&\qquad\qquad+\ \textstyle\sum_{j=1}^{q}\boldsymbol{\pi^j}\left(\partial_{\delta^j_b}\mathcal{T}_a\right)(\beta_\iota+\boldsymbol{\gamma_\iota})\,\big]=0,
\end{aligned}\tag{25}$$

for each $r \in \{0, \ldots, k\}$.

Remark 2 In the following we skip a few borderline cases: we suppose that dimension m is not only $\geq \sup(p, q)$ but also $\geq \inf\{p+2, q+3\}$.

We claim that (25) is a polynomial identity in the independent variables $\lambda, \ldots, \delta_p^q$. Indeed, since as well the vectors X, Y, X_i as the forms ξ, η, ζ, ν^j are arbitrary, take $X = e_1, Y = e_2, X_i = e_{i+2}$ (e_k: canonical basis of $\mathbb{R}^m$) and let the $p+2$ first components of the preceding forms in the dual basis of e_k vary in $\mathbb{R}$, if $inf\{p+2, q+3\} = p+2$; proceed similarly but exchange roles of vectors and forms, if $inf\{p+2, q+3\} = q+3$.

When seeking in (25) the terms of degree 1 in ν and γ_0, and of degree 0 in $\gamma_1, \ldots, \gamma_p$ (in the sequel we shall denote these terms by $(\nu)^1\gamma_0^1\gamma_1^0\ldots\gamma_p^0$), we get

$$\partial_{\beta_0}\mathcal{T}_r = 0, \tag{26}$$

where $\mathcal{T}_0$ is a priori independent of Y and $\beta_0 = \langle Y, \eta\rangle$. The terms in $(\nu)^1\gamma_0^0\gamma_1^0$ $\ldots\gamma_i^1\ldots\gamma_p^0$ ($i \in \{1, \ldots, p\}$, $p > 0$), $(\lambda)^1\gamma_0^2\gamma_1^0\ldots\gamma_p^0$, $(\lambda)^1\gamma_0^1\gamma_1^0\ldots\gamma_i^1\ldots\gamma_p^0$ ($i \in \{1, \ldots, p\}$, $p > 0$), $(\pi^j)^1\gamma_0^2\gamma_1^0\ldots\gamma_p^0$ ($j \in \{1, \ldots, q\}$, $q > 0$) and $(\pi^j)^1\gamma_0^1\gamma_1^0\ldots\gamma_i^1$ $\ldots\gamma_p^0$ ($i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, q\}$, $p > 0$, $q > 0$), read:

$$\partial_{\beta_i}\mathcal{T}_r = 0, \tag{27}$$

$$\partial^2_{\alpha_0}\mathcal{T}_r - r\partial_{\alpha_0}\mathcal{T}_{r-1} = 0, \tag{28}$$

$$\partial_{\alpha_0\alpha_i}\mathcal{T}_r - r\partial_{\alpha_i}\mathcal{T}_{r-1} = 0, \tag{29}$$

$$2\partial_{\alpha_0\delta_0^j}\mathcal{T}_r - r\partial_{\delta_0^j}\mathcal{T}_{r-1} = 0 \tag{30}$$

resp.

$$\partial_{\alpha_i\delta_0^j}\mathcal{T}_r + \partial_{\alpha_0\delta_i^j}\mathcal{T}_r - r\partial_{\delta_i^j}\mathcal{T}_{r-1} = 0. \tag{31}$$

These partial equations will allow to compute all equivariant operators.

5 Determination of the equivariant operators

Proposition 1 *Equivariant operators $T \in \mathcal{I}_{p,q}^{k,\ell}$ are mappings*

$$T : \mathcal{D}_p^k \longrightarrow \mathcal{D}_{p-1}^{k+1},\ T : \mathcal{D}_p^k \longrightarrow \mathcal{D}_p^k \text{ or } T : \mathcal{D}_p^k \longrightarrow \mathcal{D}_{p+1}^{k-1}.$$

Proof. If $T \in \mathcal{I}_{p,p-2}^{k,k+1}$, it follows from (23) and (26) that $\mathcal{T}_r = c_r\alpha_0^r \det(\alpha_i, \beta_i, \delta_i^j)$ ($r \in \{0, \ldots, k\}, c_r \in I\!R$) and from (27) that $\mathcal{T}_r = 0$ ($r \in \{0, \ldots, k\}$). Hence the result (see remark 1). ∎

5.1 Case $q = p + 1$

Proposition 2 *All spaces $\mathcal{I}_{p,p+1}^{k,k-1}$ vanish, except*

(i) the spaces $\mathcal{I}_{p,p+1}^{1,0}$ with bases defined by equation (34), and
(ii) the space $\mathcal{I}_{0,1}^{2,1}$ with basis defined by equation (35).

Proof. Equations (23) and (26) show that $\mathcal{T}_r = c_r\alpha_0^{r-1}\det(\delta_\iota^j)(r \in \{0, \ldots, k\}$, $c_r \in I\!R$). Moreover, equations (28) and (30) yield

$$(r-1)c_r = rc_{r-1}\ (r \in \{3, \ldots, k\}), \tag{32}$$

$$2(r-1)c_r = rc_{r-1}\ (r \in \{2, \ldots, k\}). \tag{33}$$

If $k \geq 3$, (32) and (33) imply that each invariant vanishes. If $k = 2$, (33) confirms that $c_1 = c_2 = c$ ($c \in I\!R$). If in addition $p > 0$, (31), written for $r = 2$, gives $c = 0$. If $(k, p) = (2, 0)$ or $k = 1$, all local invariants have the form $\mathcal{T}_0 = 0, \mathcal{T}_1 = c\langle Y, \nu\rangle, \mathcal{T}_2 = c\langle Y, \xi\rangle\langle Y, \nu\rangle$ resp. $\mathcal{T}_0 = 0, \mathcal{T}_1 = c\det(\delta_\iota^j)$.

In [Lec94], P. Lecomte introduced—in a more general framework—some homotopy operator K for the dual d^* of the de Rham differential d.

It is well-known that each $D' \in \mathcal{D}_p^1$ admits a global decomposition $D' = \sum\langle\Lambda, L_X\cdot\rangle + \sum\langle\Omega, \cdot\rangle$, where the sums are finite, where Λ and Ω are antisymmetric contravariant p-tensors on M, and where X denotes a vector field of M. Similarly, any differential operator $D'' \in \mathcal{D}_0^2$ may be written $D'' = \sum \Lambda\, L_X \circ L_Y + \sum \Omega\, L_Z + \sum \Theta$, with $\Lambda, \Omega, \Theta \in N$ and $X, Y, Z \in \mathrm{Vect}(M)$.

We easily verify that

$$K|_{\mathcal{D}_p^1}(D') = \frac{1}{1+p}\sum\langle\Lambda, i_X\cdot\rangle \tag{34}$$

and that

$$K|_{\mathcal{D}_0^2}(D'') = \frac{1}{2}\sum \Lambda\,(i_X L_Y + L_X i_Y) + \sum \Omega\, i_Z. \tag{35}$$

For the first formula for instance, it suffices to remember that

$$\langle \Lambda, L_X\omega\rangle \simeq \langle X,\xi\rangle(X_1\wedge\ldots\wedge X_p)(\omega) + (X_1\wedge\ldots\wedge X_p)(\zeta\wedge i_X\omega),$$

with unmistakable notations (see section 2), and to use the local representation of K (see [Lec94]).

These results imply that—in the implicated cases—the homotopy operator is independent of the local coordinates and the partition of unity involved in its construction. Furthermore, the r.h.s. of (34) and (35) is independent of the chosen decomposition of D' and D'' respectively. It is now obvious that

$$K:\mathcal{D}^1_p\to\mathcal{D}^0_{p+1} \text{ and } K:\mathcal{D}^2_0\to\mathcal{D}^1_1$$

are equivariant operators.

Finally, the spaces $\mathcal{I}^{1,0}_{p,p+1}$ and $\mathcal{I}^{2,1}_{0,1}$ are generated by K. ∎

5.2 Case $q = p - 1$ $(p > 0)$

Proposition 3 *The spaces* $\mathcal{I}^{k,k+1}_{p,p-1}$ *are one-dimensional vector spaces with basis* d^*.

Proof. Equations (23), (26), and (27) tell us that $\mathcal{T}_r = c_r\alpha_0^r\det(\alpha_i,\delta_i^j)$ $(r\in\{0,\ldots,k\}, c_r\in\mathbb{R})$ and equation (29) brings out that $c_r = c$ $(r\in\{0,\ldots,k\}, c\in\mathbb{R})$.

Since obviously $d^*\in\mathcal{I}^{k,k+1}_{p,p-1}$, the conclusion follows. ∎

5.3 Case $q = p$

Proposition 4 *The spaces* $\mathcal{I}^{k,k}_{p,p}$ *are one-dimensional, except* $\mathcal{I}^{k,k}_{0,0}$ $(k>0)$ *and* $\mathcal{I}^{1,1}_{p,p}$ $(p>0)$ *that have dimension* 2*. Possible bases are* id, (id, I_0) *resp.* (id, d^*K), *where* I_0 *is defined by* $I_0 : D \to D(1)id$ (1 *stands for the constant function* $x\to 1$).

Proof. (i) Look first at the case $p=0$. Equations (23), (26), and (28) show that $\mathcal{T}_0 = c_0$ $(c_0\in\mathbb{R})$ and $\mathcal{T}_r = c\alpha_0^r$ $(r\in\{1,\ldots,k\}, c\in\mathbb{R})$.

If $k=0$, the space of invariants is generated by id. Otherwise, dimension of $\mathcal{I}^{k,k}_{0,0}$ is 2 and the invariants id and I_0 form a basis.

(ii) If $p>0$, it follows from equations (23), (24), (26), and (27) that

$$\mathcal{T}_r = c_r\alpha_0^r\det(\delta_i^j) + d_r\alpha_0^{r-1}\sum_{n=1}^{p}\alpha_n\det{}_{n0}(\delta_i^j)\ \ (r\in\{0,\ldots,k\}, c_r, d_r\in\mathbb{R}).$$

In this expression, $\det_{n0}(\delta_i^j)$ denotes the determinant $\det(\delta_i^j)$, where the line $(\delta_n^1,\ldots,\delta_n^p)$ has been replaced by $(\delta_0^1,\ldots,\delta_0^p)$. When exploiting equations (29) and (30), you find $(r-1)d_r = rd_{r-1}$ $(r\in\{2,\ldots,k\})$ resp. $2(r-1)d_r = rd_{r-1}$ $(r\in\{2,\ldots,k\})$, so that $d_r = 0$ $(r\in\{0,\ldots,k\})$, if $k\geq 2$. Apply now equation (31). If $k\geq 2$, this condition shows that

$$\mathcal{T}_r = c\alpha_0^r\det(\delta_i^j)\ \ (r\in\{0,\ldots,k\}, c\in\mathbb{R})$$

and if $k = 1$, it entails that $c_1 - c_0 + d_1 = 0$.

If $k = 0$ or $k \geq 2$, invariants are thus generated by id. In the case $k = 1$, dimension is 2 and $id, d^*K \in \mathcal{I}^{1,1}_{p,p}$ are possible generators. ∎

Remark 3 Equation (34) reveals that $(d^*K)(D) = (1/(1+p)) \sum \langle \Lambda, \, i_X d\cdot \rangle$, if $D = \sum \langle \Lambda, L_X \cdot \rangle + \sum \langle \Omega, \cdot \rangle$.

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