Dealing with interfaces in partitioned model order reduction for application to nonlinear problems

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Abstract

We propose a reduced order modelling technique based on a partitioning of the domain of study in the context of parametric nonlinear problems. A formulation of the reduction of the displacement and of the interface tractions linking subdomains to each others will be performed in a FETI context.

Keywords: Model Order Reduction, Domain decomposition method, Nonlinear mechanics

1. Introduction

Simulating fracture in realistic engineering components is computationally expensive. In the context of early-stage design, or reverse engineering, such simulations might need to be performed for a large range of material and geometric parameters, which makes the solution to the parametric problem of fracture unaffordable.

Model order reduction, such as the proper orthogonal decomposition (POD) (see for instance [1, 2, 3, 4, 5]), is one way to reduce significantly the computational time by reducing the number of spatial unknowns. The solution is searched for in a reduced space spanned by a few well-chosen basis vectors only. In the context of solid mechanics involving structural softening, the strong topological changes in the zone where damage localises are extremely sensitive to variations of the parameters, which requires reduced spaces of prohibitively large dimensions in order to approximate the solution with a sufficiently high degree of accuracy. Introduced in [6], partitioned POD is an alternative to global model order reduction that essentially divides up the problem into smaller regions. Each region can then be tackled using a reduced model of appropriate size, if at all, depending on the local material non-linearities in the region. The continuity and equilibrium of the solution over the whole domain is ensured by solving the interface problem iteratively via a Krylov solver. Note that different ideas of local reduced basis had been proposed in [7, 8, 9, 10, 11]. In [6], the uncoupling of each region is achieved by making use of the balancing domain decomposition method (BDD, [12, 13]). In the BDD, the continuity of the solution between each subdomain is insured by introducing a unique displacement field on the interface between subdomains, and looking for the equilibrium of the reaction forces iteratively. In this context, the interface equilibrium equations cannot be projected in the Lagrange reduced spaces associated to each of the subdomains, which limits the efficiency of the method.

The dual Schur-based domain decomposition, known as FETI [6] insures continuity by enforcing the continuity of interface tractions a priori, while looking for the kinematic continuity in an iterative manner. This allows the partitioned model order reduction approach to be formulated in a more natural and efficient manner, allowing for a reduction of the interface problem, and therefore resulting in an increased speed-up. We will also demonstrate how a relevant approximation of the interface tractions can be constructed in order to speed-up the Krylov solver.

2. General problem statement

We consider the evolution of a structure described by the partial differential equations of classical continuum mechanics on a bounded spatial domain Ω, over time interval $\mathcal{T} = [0, T]$. We focus on nonlinear constitutive models representing the quasi-static evolution of dissipative phenomena in the structure, such as plasticity or damage. After a classical space (finite element in our examples) and time discretisation ($\mathcal{T}^{\mathrm{h}} = \{t_0, t_1, ..., t_{n_{\mathrm{t}}}\}$ such that $t_0 = 0$ and $t_{n_{\mathrm{t}}} = T$), the following equations of n_{u} spatial unknowns is obtained:

$$\underline{\mathbf{F}}_{\mathbf{int}}\left(\left(\underline{\mathbf{U}}(t_m,\mu)\right)_{m\in[\![0,n]\!]},\mu\right)+\underline{\mathbf{F}}_{\mathbf{ext}}(t_n,\mu)=\underline{\mathbf{0}}\qquad\forall\,(t_n,\mu)\in\mathcal{T}^{\mathrm{h}}\times\mathcal{P}\,. \tag{1}$$

The vector of internal forces, $\underline{\mathbf{F}}_{\mathbf{int}} \in \mathbb{R}^{n_{\mathrm{u}}}$, is a non-linear function of the history of the nodal unknowns $\underline{\mathbf{U}}(t, \mu)$. $\underline{\mathbf{F}}_{\mathbf{ext}} \in \mathbb{R}^{n_{\mathrm{u}}}$ is the vector of external forces.

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3. POD-based reduced order modelling

Let us write that the solution vectors, at any time step and for any value of the parameter, lie in a space of small dimension ("reduced space"), spanned by a few basis vectors $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_c]\!]}$

$$\underline{\mathbf{U}}(t_n,\mu) = \sum_{i=1}^{n_c} \underline{\mathbf{C}}_i\, \alpha_i(t_n,\mu) = \underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\,. \tag{2}$$

where $\underline{\underline{\mathbf{C}}}$ is a matrix whose columns are the basis vectors $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_c]\!]}$, and $\underline{\boldsymbol{\alpha}}$ is a vector of reduced state variables $(\alpha_i)_{i\in[\![1,n_c]\!]}$ that depends on time and parameter.To get a basis $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_c]\!]}$ where the solution to the parametric problem of evolution can be obtained accurately, we use the classical snapshot proper orthogonal decomposition method (POD)([1]). This method requires to pre-compute a set of realisations of the parametric problem (i.e. the snapshot), from which a suboptimal reduced basis is extracted. Substituting this approximation into (1), one obtains a minimisation problem in the n_c reduced state variables $\underline{\boldsymbol{\alpha}}$ ($n_c << n_u$)

$$\underline{\boldsymbol{\alpha}} = \underset{\underline{\boldsymbol{\alpha}}^\star}{\operatorname{argmin}} \| \underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}^\star\right) + \underline{\mathbf{F}}_{\mathbf{ext}} \| \tag{3}$$

A solution to this problem can be obtained by a least-square or Galerkin formulation.

4. System approximation

Despite the reduction of the number of unknowns, $\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}\right)$ is still expensive to evaluate since it is a non-linear quantity that has to be assembled over the entire domain of study. To speed-up the computation, a system approximation is performed by selecting only a few elements of the domain and evaluating the nonlinear operator only on those selected regions. The operator is reconstructed using a second POD basis $(\underline{\mathbf{D}}_i)_{i\in[\![1,n_d]\!]}$ (used in [5, 14]).

$$\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\right) = \sum_{i=1}^{n_d} \underline{\mathbf{D}}_i\, \zeta_i(t_n,\mu) = \underline{\underline{\mathbf{D}}}\,\underline{\boldsymbol{\zeta}}(t_n,\mu)\,. \tag{4}$$

The columns of $\underline{\underline{\mathbf{D}}} \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_d}$ are spatial functions corresponding to a truncated snapshot POD expansion of $\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}\right)$. Static coefficients $\underline{\boldsymbol{\zeta}}$ are found by expressing that at any point $(t_n,\mu) \in \mathcal{P}$, the interpolation is optimal for a limited number n_{sa} of degrees of freedom:

$$\underline{\boldsymbol{\zeta}}(t_n,\mu) = \underset{\underline{\boldsymbol{\zeta}}^\star}{\operatorname{argmin}} \| \underline{\underline{\mathbf{D}}}\,\underline{\boldsymbol{\zeta}}^\star(t_n,\mu) - \underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\right) \|_{\underline{\underline{\mathbf{P}}}} \tag{5}$$

$\underline{\underline{\mathbf{P}}}$ is a boolean diagonal operator with very few non-zero values $n_{sa} \geq n_d$ corresponding to the evaluation points of the interpolation. $\|\underline{\mathbf{X}}\|_{\underline{\underline{\mathbf{P}}}} = \sqrt{\underline{\mathbf{X}}^T \underline{\underline{\mathbf{P}}}\,\underline{\mathbf{X}}}$ is the semi-norm associated with $\underline{\underline{\mathbf{P}}}$.

5. Domain decomposition

The snapshot-POD method will be applied locally, with a different POD basis for each reduced subdomain. This idea is illustrated in Figure 1, applied on a structure made of damageable lattice bars and put under tension. After linearisation of equation (1) by a Newton-Raphson scheme, one obtains successive systems of the form:

$$\underline{\underline{\mathbf{K}}}\,\underline{\mathbf{U}} = \underline{\mathbf{F}}\,. \tag{6}$$

Domain Ω is now split into non-overlapping subdomains $(\Omega^{(e)})_{e\in[\![1,n_e]\!]}$ such that $\bigcup_{e\in[\![1,n_e]\!]} \Omega^{(e)} = \Omega$. We later refer to the subdomain set $[\![1,n_e]\!]$ as $\mathcal{E}$ for simplicity. Let $\underline{\mathbf{U}}^{(e)}$ be the vector of local degrees of freedom corresponding to $\Omega^{(e)}$ and $\underline{\underline{\mathbf{K}}}^{(e)}$ the corresponding local stiffness matrix. The local equilibrium on $\Omega^{(e)}$ reads

$$\underline{\underline{\mathbf{K}}}^{(e)}\underline{\mathbf{U}}^{(e)} = \underline{\mathbf{F}}^{(e)} + \underline{\underline{\mathbf{t}}}^{(e)^T}\,\underline{\boldsymbol{\lambda}}^{(e)}\,, \tag{7}$$

where $\underline{\boldsymbol{\lambda}}^{(e)}$ corresponds to the contact forces applied on the boundary of $\Omega^{(e)}$, and $\underline{\underline{\mathbf{t}}}^{(e)}$ is the trace operator that extracts boundary degrees of freedom ($\underline{\underline{\mathbf{t}}}^{(e)^T}\,\underline{\boldsymbol{\lambda}}^{(e)}$ is hence a vector with non-zero values only on the boundary of $\Omega^{(e)}$).

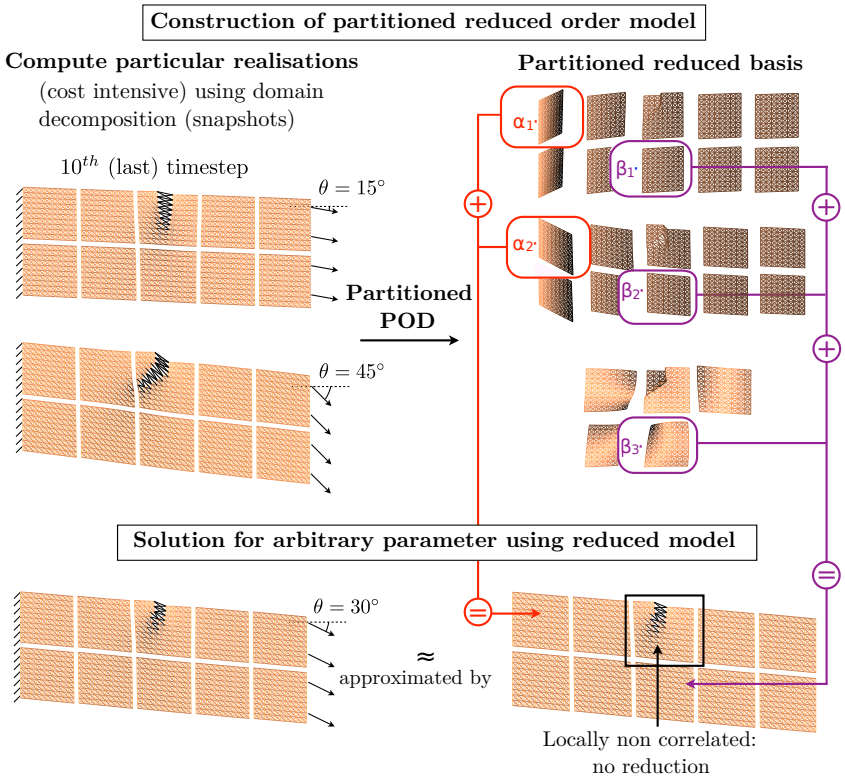


Figure 1: The Partitioned POD

5.1. Standard FETI formulation

We explain briefly here the FETI method. The reader is invited to have a look at [13] for more details about the method or at [15] for an general overview of domain decomposition methods. The FETI method is based on the definition of a unique vector of contact forces $\underline{\boldsymbol{\lambda}}$, which guarantees the compatibility of the local $\underline{\boldsymbol{\lambda}}^{(e)}$ with each other. The continuity of the solution at the interfaces between subdomains can be expressed as:

$$\sum_{e \in \mathcal{E}} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{U}}}^{(e)} = \underline{\mathbf{0}}\,, \tag{8}$$

where $\underline{\underline{\mathbf{B}}}^{(e)}$ is a signed boolean operator. The local contact forces can be retrieved from the global one through:

$$\underline{\boldsymbol{\lambda}}^{(e)} = \underline{\underline{\mathbf{B}}}^{(e)^T} \underline{\boldsymbol{\lambda}} \tag{9}$$

Equations (7) are solved for each subdomain $\Omega^{(e)}$ with the use of a generalised inverse since the stiffness matrix $\underline{\underline{\mathbf{K}}}^{(e)}$ will be singular if the domain $\Omega^{(e)}$ does not intersect any Dirichlet boundary condition. The generalised inverse is noted $\underline{\underline{\mathbf{K}}}^{(e)^+}$ and verifies the property $\underline{\underline{\mathbf{K}}}^{(e)}\,\underline{\underline{\mathbf{K}}}^{(e)^+}\,\underline{\mathbf{y}} = \underline{\mathbf{y}}$ for $\underline{\mathbf{y}} \in \text{range}(\underline{\underline{\mathbf{K}}}^{(e)})$. The solution to (7) is then expressed as:

$$\underline{\mathbf{U}}^{(e)} = \underline{\underline{\mathbf{K}}}^{(e)^+} \left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)}\right) + \underline{\mathbf{R}}^{(e)} \underline{\boldsymbol{\alpha}}^{(e)}\,. \tag{10}$$

$\underline{\mathbf{R}}^{(e)}$ is a basis of the kernel of $\underline{\underline{\mathbf{K}}}^{(e)}$, which represents the rigid body motion of the subdomain (e). Of course, this holds only if $\left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)}\right)$ belongs to the range of $\underline{\underline{\mathbf{K}}}^{(e)}$, which is insured through the following equation:

$$\underline{\mathbf{R}}^{(e)^T} \left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)}\right) = \underline{\mathbf{0}}. \tag{11}$$

Local equilibrium (10) is condensed on the interface of (e) with the use of the trace operator $\underline{\underline{\mathbf{t}}}^{(e)}$ and then left multiplied by the signed boolean operator $\underline{\underline{\mathbf{B}}}^{(e)}$. Summing over all subdomains has the effect of cancelling out the displacement (thanks to equation (8)) and we obtain a mixed system depending only on the tractions forces $\underline{\boldsymbol{\lambda}}$ and the coefficients of the rigid

body modes associated:

$$\begin{bmatrix} \underline{\underline{\mathbf{S_d}}} & \underline{\underline{\mathbf{G}}} \\ \underline{\underline{\mathbf{G}}}^T & \underline{\underline{\mathbf{0}}} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\lambda}} \\ \underline{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} -\underline{\mathbf{F}} \\ -\underline{\mathbf{e}} \end{bmatrix} \quad \text{with} \quad \begin{cases} \underline{\underline{\mathbf{S_d}}} = \sum_{e \in \mathcal{E}} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{K}}}^{(e)^+} \underline{\underline{\mathbf{t}}}^{(e)^T} \underline{\underline{\mathbf{B}}}^{(e)^T} \\ \underline{\underline{\mathbf{G}}} = \left(\dots \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{R}}}^{(e)} \dots \right) \\ \underline{\mathbf{F}} = \sum_{e \in \mathcal{E}} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{K}}}^{(e)^+} \underline{\mathbf{F}}^{(e)} \\ \underline{\mathbf{e}} = \left(\dots \underline{\mathbf{F}}^{(e)^T} \underline{\underline{\mathbf{t}}}^{(e)^T} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{R}}}^{(e)} \dots \right)^T \end{cases} \tag{12}$$

The displacement can then be computed from equation (10). In practice, this system is solved iteratively with a projected preconditioned conjugate gradient.

5.2. Reduced FETI formulation

In this section, we develop a reduced formulation of the FETI method presented above. We assume that appropriate snapshots have been computed and that corresponding POD bases are hence at hand. For simplicity, the system approximation technique presented in section 4 won't be included in the formulation of this paper.

5.2.1. Reduction of the displacement

The displacement is now expressed in the following form:

$$\underline{\underline{\mathbf{U}}}^{(e)} = \underline{\underline{\mathbf{C}}}^{(e)} \underline{\boldsymbol{\beta}}^{(e)} + \underline{\underline{\mathbf{R}}}^{(e)} \underline{\boldsymbol{\alpha}}^{(e)}, \tag{13}$$

where $\underline{\underline{\mathbf{R}}}^{(e)}$ is a basis of the kernel of $\underline{\underline{\mathbf{K}}}^{(e)}$ (as defined in section 5.1), and $\underline{\underline{\mathbf{C}}}^{(e)}$ is a basis of the displacement that is chosen orthogonal to the kernel $\underline{\underline{\mathbf{R}}}^{(e)}$ (i.e. $\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{R}}}^{(e)} = \underline{\underline{\mathbf{0}}}$). In this way, the displacement is explicitly described as a direct sum of a body deformation ($\underline{\underline{\mathbf{C}}}^{(e)} \underline{\boldsymbol{\beta}}^{(e)}$) and a rigid body motion ($\underline{\underline{\mathbf{R}}}^{(e)} \underline{\boldsymbol{\alpha}}^{(e)}$). This means also that a reduction will only be performed on the deformations of the body but not on the rigid body motions. Those can be considered irreducible and all the reduction will be concentrated onto the deformations. Substituting equation (13) into equation (7), and using Galerkin orthogonality one obtains the following equation:

$$\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \underline{\boldsymbol{\beta}}^{(e)} = \underline{\underline{\mathbf{C}}}^{(e)^T} \left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)} \right), \tag{14}$$

where we used the fact that $\underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{R}}}^{(e)} \underline{\boldsymbol{\alpha}}^{(e)} = \underline{\mathbf{0}}$. $\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)}$ is invertible since $\underline{\underline{\mathbf{C}}}$ has been chosen in a space orthogonal to the kernel of $\underline{\underline{\mathbf{K}}}^{(e)}$. We hence have $\underline{\boldsymbol{\beta}}^{(e)} = \left(\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \right)^{-1} \underline{\underline{\mathbf{C}}}^{(e)^T} \left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)} \right)$ and we obtain a reduced version of equation (10):

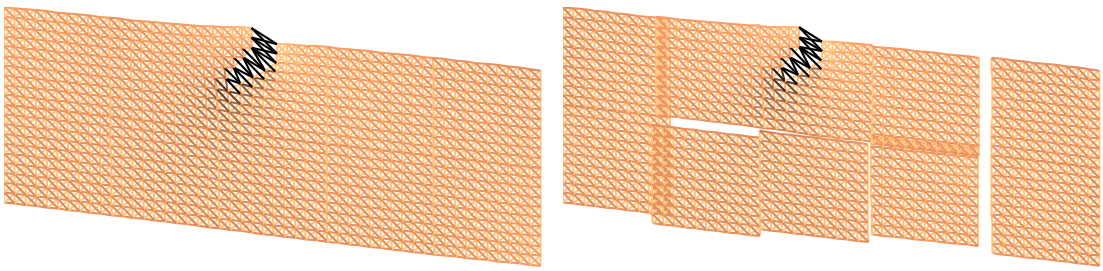
$$\underline{\underline{\mathbf{U}}}^{(e)} = \underline{\underline{\mathbf{C}}}^{(e)} \left(\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \right)^{-1} \underline{\underline{\mathbf{C}}}^{(e)^T} \left(\underline{\mathbf{F}}^{(e)} + \underline{\boldsymbol{\lambda}}^{(e)} \right) + \underline{\underline{\mathbf{R}}}^{(e)} \underline{\boldsymbol{\alpha}}^{(e)}. \tag{15}$$

From there, we deduce the global reduced interface problem in a similar fashion than for equation (12):

$$\begin{bmatrix} \underline{\underline{\mathbf{S}}}_{\mathrm{d,r}} & \underline{\underline{\mathbf{G}}} \\ \underline{\underline{\mathbf{G}}}^T & \underline{\underline{\mathbf{0}}} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\lambda}} \\ \underline{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} -\underline{\mathbf{F}}_{\mathrm{r}} \\ -\underline{\mathbf{e}} \end{bmatrix} \quad \text{with} \quad \begin{cases} \underline{\underline{\mathbf{S}}}_{\mathrm{r}} = \sum_{e \in \mathcal{E}} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \left(\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \right)^{-1} \underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{t}}}^{(e)^T} \underline{\underline{\mathbf{B}}}^{(e)^T} \\ \underline{\underline{\mathbf{G}}} = \left(\dots \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{R}}}^{(e)} \dots \right) \\ \underline{\mathbf{F}}_{\mathrm{r}} = \sum_{e \in \mathcal{E}} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \left(\underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\underline{\mathbf{K}}}^{(e)} \underline{\underline{\mathbf{C}}}^{(e)} \right)^{-1} \underline{\underline{\mathbf{C}}}^{(e)^T} \underline{\mathbf{F}}^{(e)} \\ \underline{\mathbf{e}} = \left(\dots \underline{\mathbf{F}}^{(e)^T} \underline{\underline{\mathbf{t}}}^{(e)^T} \underline{\underline{\mathbf{B}}}^{(e)} \underline{\underline{\mathbf{t}}}^{(e)} \underline{\underline{\mathbf{R}}}^{(e)} \dots \right)^T \end{cases} \tag{16}$$

5.2.2. Reduction of the traction forces

Reducing the displacement only, while keeping all the traction forces leads to a problem that is not well defined with too many continuity conditions on the interfaces between subdomains. This means that equation (16) may not have a solution or at least that the convergence of the Krylov solver will be very slow. The contact forces $\underline{\boldsymbol{\lambda}}_b$ need hence to be reduced to obtain a well defined problem. Furthermore, this will have the effect of speeding up the solving of the system since it will have fewer unknowns. However, the drawback will be that the strict continuity of the displacement between two neighbouring subdomains won't be guaranteed anymore. See Figure 2.



(a) Solution with a basis $\underline{\underline{\mathbf{\Lambda}}}$ enriched enough

(b) Solution with a basis $\underline{\underline{\mathbf{\Lambda}}}$ too small. The contact forces are poorly approximated by $\underline{\underline{\mathbf{\Lambda}}}$: the solution is now obviously discontinuous.

Figure 2: A damageable lattice structure is put under tension in incremental time steps. A crack at its top-centre propagates. The domain is divided into 10 subdomains and solved with a reduction of the contact forces. A basis $\underline{\underline{\mathbf{\Lambda}}}$ of different sizes is used.

The global traction force vector $\underline{\boldsymbol{\lambda}}$ is expressed as $\underline{\boldsymbol{\lambda}} = \underline{\underline{\mathbf{\Lambda}}}\,\underline{\boldsymbol{\gamma}}$. $\underline{\underline{\mathbf{\Lambda}}}$ can be ordered in a block structure with each block representing a reduced basis of the traction forces at each interface between a set of subdomains (we assume there are n_b of them):

$$\underline{\underline{\mathbf{\Lambda}}} = \begin{bmatrix} \underline{\underline{\mathbf{\Lambda}}}^1 & & & \\ & \underline{\underline{\mathbf{\Lambda}}}^2 & & \\ & & \ddots & \\ & & & \underline{\underline{\mathbf{\Lambda}}}^{n_\mathrm{b}} \end{bmatrix}. \tag{17}$$

$\underline{\boldsymbol{\gamma}}$ is ordered in the same way:

$$\underline{\boldsymbol{\gamma}} = \begin{bmatrix} \underline{\boldsymbol{\gamma}}^{1\,T} & \underline{\boldsymbol{\gamma}}^{2\,T} & \cdots & \underline{\boldsymbol{\gamma}}^{n_\mathrm{b}\,T} \end{bmatrix}^T. \tag{18}$$

In this fashion, $\underline{\underline{\mathbf{\Lambda}}}^i\,\underline{\boldsymbol{\gamma}}^i$ is the approximation of the traction forces onto interface number i. Now, substituting $\underline{\boldsymbol{\lambda}}$ by $\underline{\underline{\mathbf{\Lambda}}}\,\underline{\boldsymbol{\gamma}}$ in equations (15) and (11), and using Galerkin orthogonality for the upper equations, we obtain the fully reduced symmetric interface problem:

$$\begin{bmatrix} \underline{\underline{\mathbf{\Lambda}}}^T \underline{\underline{\mathbf{S_d}}}\,\underline{\underline{\mathbf{\Lambda}}} & \underline{\underline{\mathbf{\Lambda}}}^T \underline{\underline{\mathbf{G}}} \\ \left(\underline{\underline{\mathbf{\Lambda}}}^T \underline{\underline{\mathbf{G}}}\right)^T & \underline{\underline{\mathbf{0}}} \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\gamma}} \\ \underline{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} -\underline{\underline{\mathbf{\Lambda}}}^T \underline{\mathbf{F}} \\ -\underline{\mathbf{e}} \end{bmatrix} \tag{19}$$

This system is now of potentially much smaller dimension than system (16) for problems that are large enough. The solution algorithm of this problem is hence expected to run much faster. Work need to be done to understand which dimension the POD basis $\underline{\underline{\mathbf{\Lambda}}}$ should have to guaranty a reasonable accuracy of the solution.

6. Conclusions

In this paper, we proposed a reduced formulation of the well-known FETI method. Reductions of both the displacement (which results in a solution obtained on the local problems) and of the contact forces (which results in a solution obtained faster on the coarse (or interface) problem) have been formulated. Further work need to be done to understand how to choose the relative sizes of the different POD bases used to obtain a solution with an error smaller than a predefined tolerance.

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