Algebraic coarse-graining methods in fracture mechanics: tackling local lack of correlation using domain decomposition

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Outline

1. Introduction
   - Why model order reduction?
   - A straigthforward solution?
2. Partitioned POD method
   - Domain decomposition methods
   - System approximation
3. Results & Conclusion

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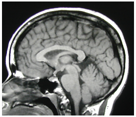
Non-linear expensive simulations

- Problems depending on microscale phenomena $\Longrightarrow$ requires very fine mesh: expensive simulations
- **Surgical simulation**: real-time brain surgery simulation



- **Aeronautics**: advanced early-stage design

Projection-based model order reduction

We want to solve a parametrised mechanical problem:

$$\underbrace{\underline{\mathbf{F}}_{\text{int}}(\underline{\mathbf{U}}(\lambda), \lambda)}_{\text{Non-linear}} + \underline{\mathbf{F}}_{\text{ext}}(\lambda) = \underline{\mathbf{0}} \quad (1)$$

We are interested in the solution $\underline{\mathbf{U}}(\lambda)$ for many different values of λ.

Projection-based model order reduction assumption:

Solutions $\underline{\mathbf{U}}(\lambda)$ for different parameters λ are contained in a space of small dimension $span((\underline{\mathbf{C}}_i)_{i \in [\![1, n_c]\!]})$

Proper Orthogonal Decomposition (POD)

Look for $\underline{\mathbf{U}}$ as $\underline{\mathbf{U}} = \underline{\underline{\mathbf{C}}}\,\underline{\alpha}$. Where does the basis $\underline{\underline{\mathbf{C}}}$ comes from?

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- In the Galerkin framework: $\underline{\underline{\mathbf{C}}}^T \underline{\mathbf{F}}_{\mathbf{int}} \left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha} \right) + \underline{\underline{\mathbf{C}}}^T \underline{\mathbf{F}}_{\mathbf{ext}} = 0$

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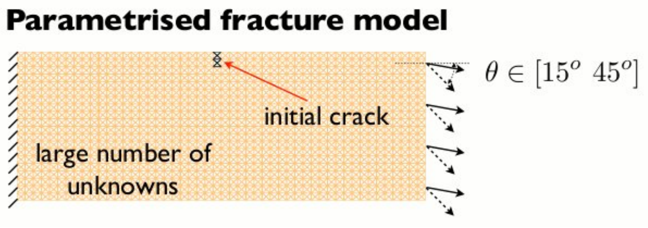
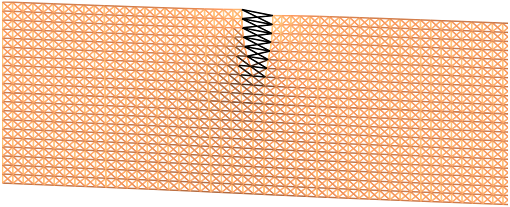
Example

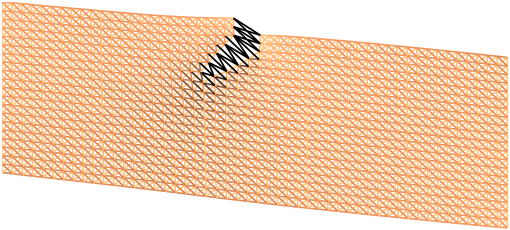

Parametrised fracture model
initial crack
large number of
unknowns
$\theta \in [15^o \; 45^o]$

Snapshots

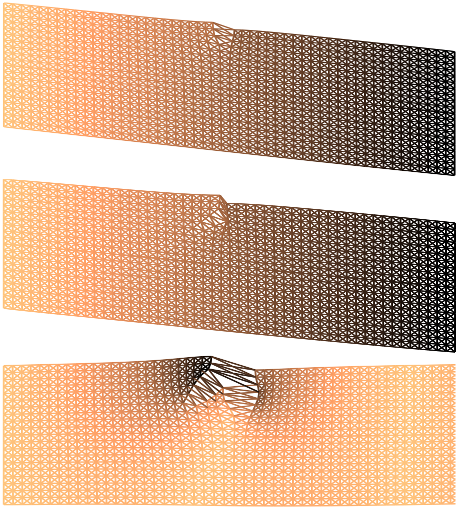
15 degrees:



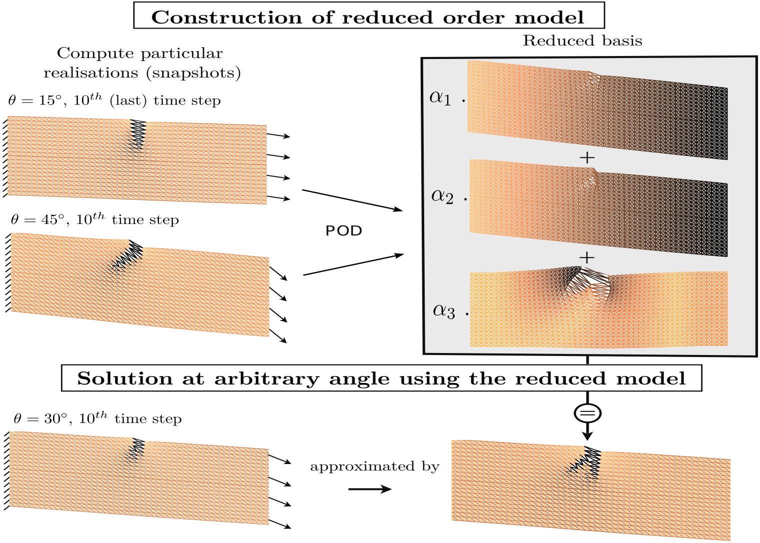
45 degrees:



First 3 modes of the POD basis



Fracture not well captured



What can we do?

Idea: juste divide up the domain and select regions that are "reducible"

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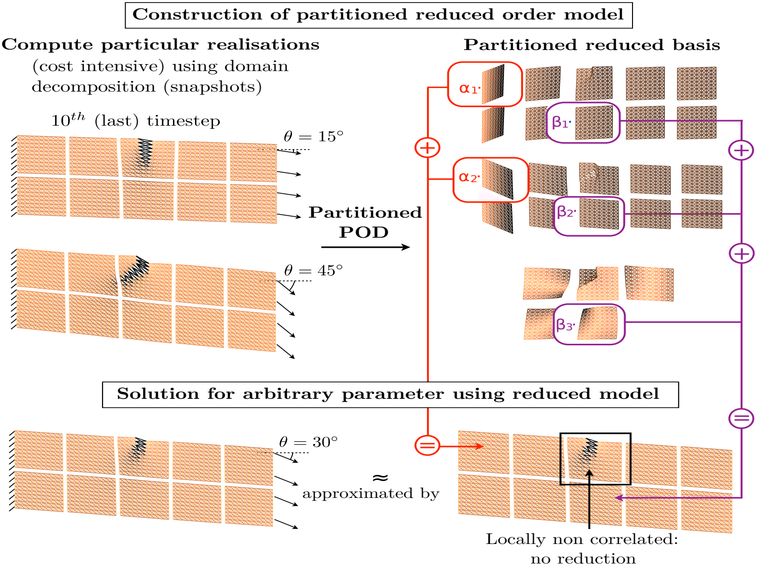
Construction of partitioned reduced order model
Compute particular realisations
(cost intensive) using domain
decomposition (snapshots)
10^{th} (last) timestep
$\theta = 15°$
$\theta = 45°$
Partitioned
POD
Partitioned reduced basis
$\alpha_1 \cdot$
$\beta_1 \cdot$
$\alpha_2 \cdot$
$\beta_2 \cdot$
$\beta_3 \cdot$
Solution for arbitrary parameter using reduced model
$\theta = 30°$
≈
approximated by
Locally non correlated:
no reduction

Is that good enough?

- Speed-up actually poor
- Equation "$\underline{\underline{\mathbf{C}}}^T\,\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right) + \underline{\underline{\mathbf{C}}}^T\,\underline{\mathbf{F}}_{\mathbf{ext}} = 0$" quicker to solve but $\underline{\underline{\mathbf{C}}}^T\,\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)$ still expensive to evaluate
- Need to do something more $\Longrightarrow$ system approximation

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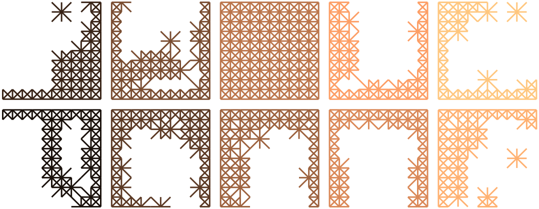
Idea

- Integrate only over some nodes of the domain
- Reconstruct the operators using a second POD basis

“Gappy“ technique

Originally used to reconstruct images



- $\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)$ approximated by $\widetilde{\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)} = \underline{\underline{\mathbf{D}}}\,\underline{\beta}$

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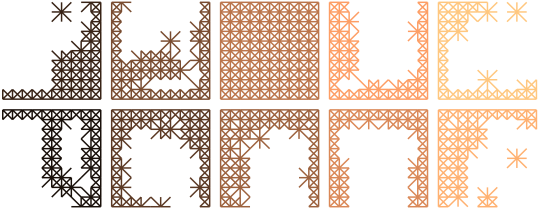

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- $\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)$ is evaluated exactly only on a few selected nodes: $\widehat{\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)}$

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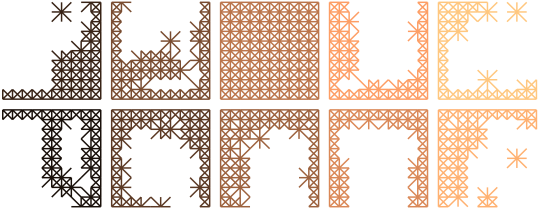

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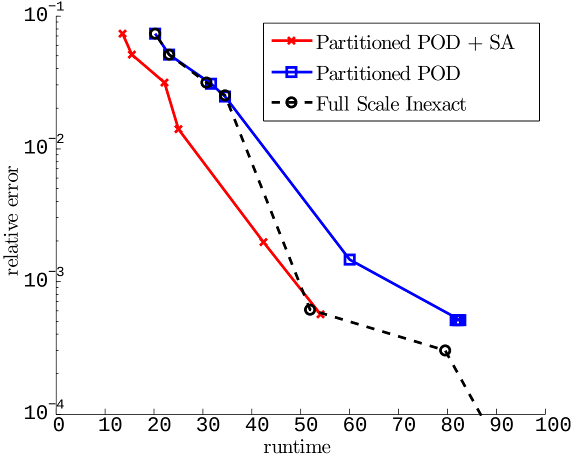
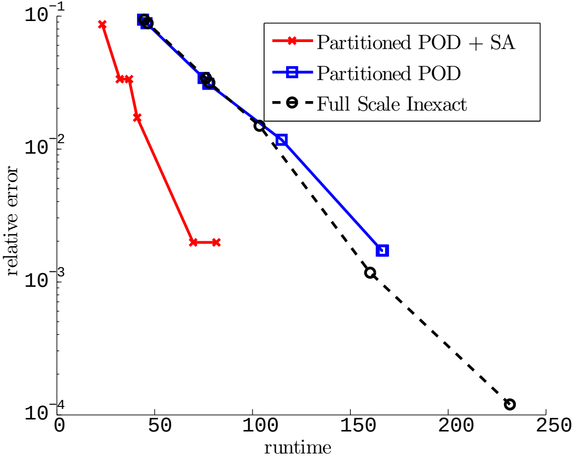
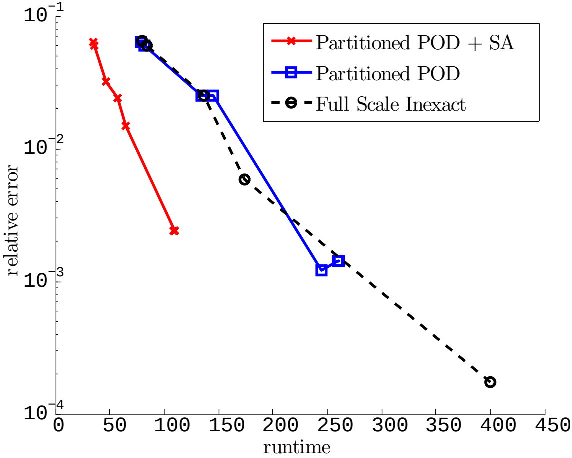
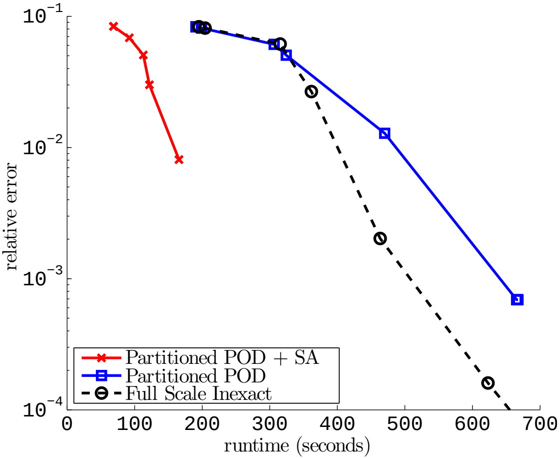
- $\underline{\mathbf{F}}_{\mathrm{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)$ is evaluated exactly only on a few selected nodes: $\widehat{\underline{\mathbf{F}}_{\mathrm{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)}$
- $\underline{\beta}$ found through: $\min_{\underline{\beta}} \left\| \widehat{\underline{\underline{\mathbf{D}}}}\,\underline{\beta} - \widehat{\underline{\mathbf{F}}_{\mathrm{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\alpha}\right)} \right\|_2$

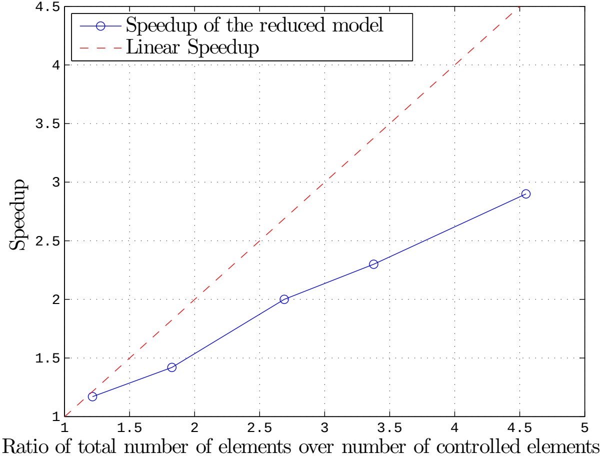
Outline


Partitioned POD + SA
Partitioned POD
Full Scale Inexact
relative error
runtime
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0 10 20 30 40 50 60 70 80 90 100


10^-1
10^-2
10^-3
10^-4
relative error
0
50
100
150
200
250
runtime
Partitioned POD + SA
Partitioned POD
Full Scale Inexact


Partitioned POD + SA
Partitioned POD
Full Scale Inexact
relative error
runtime
10^-1
10^-2
10^-3
10^-4
0
50
100
150
200
250
300
350
400
450


relative error
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
100
200
300
400
500
600
700
runtime (seconds)
Partitioned POD + SA
Partitioned POD
Full Scale Inexact


Speedup of the reduced model
Linear Speedup
Speedup
Ratio of total number of elements over number of controlled elements
4.5
4
3.5
3
2.5
2
1.5
1
1
1.5
2
2.5
3
3.5
4
4.5
5

Thank you for your attention!