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ALGEBRAIC COARSE-GRAINING METHODS IN FRACTURE MECHANICS: TACKLING LOCAL LACK OF CORRELATION USING DOMAIN DECOMPOSITION

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ABSTRACT

In this paper, we propose to couple model order reduction techniques with domain decomposition methods for the solution to parametric problems of fracture. The nonlinear nature of the problems requires the use of a system approximation method to speed-up the assembly of the non-linear opreators. We show that the method efficiently computes a solution faster than a full order model for a given accuracy. The speed-up increases with the problem size.

1 INTRODUCTION

Engineering structural problems are very often characterised by a large ratio between the scale of the structure and the scale at which the phenomena of interest are described. The solution to such problems using super-computers is now at reach, but at a considerable price. However, in engineering design processes, a prohibitively high number of such solutions might be of interest, for different values of design parameters, or to take into account the effect of stochastic phenomena. Consequently, coarse-graining multiscale engineering problem is an issue of tremendous importance in today's computational mechanics. The proper orthogonal decomposition ([5]) is a popular method to produce a relevant basis of few vectors for use in projection-based model order reduction methods. In the case of problems involving localised nonlinearities, such as fracture, it may show poor performance since small perturbations of the input parameters can lead to high variations in the output solution which requires hence the selection of a large number of basis vectors to capture the non-linearity. Kerfriden tackled this drawback by updating the basis vectors "on-the-fly"([4]). This paper aims at addressing those issues by coupling a snapshot-POD reduction technique with a balancing domain decomposition method (described for example in [3]). Domain decomposition methods essentially divides up the main problem into several independent subproblems corresponding to (non-overlapping) subdomains. In the proposed strategy, the subdomains undergoing high nonlinearities can be solved using a full order model while the remaining regions can be solved using a reduced model. Another issue is that for non-linear problems, reducing the size of the

approximation space does not necessarily imply a reduction in the numerical costs. The evaluation of the non-linear operators is done over the entire domain of study and has a high computational cost. A system approximation method is used by only selecting certain points of the domain and reconstructing the non-linear operators using a gappy technique.

2 THE PARTITIONED POD METHOD

2.1 GENERAL PROBLEM STATEMENT

We consider the evolution of a structure described by the partial differential equations of classical continuum mechanics on a bounded spatial domain Ω, over time interval $\mathcal{T} = [0, T]$. We focus on nonlinear constitutive models representing the quasi-static evolution of dissipative phenomena in the structre, such as plasticity or damage. After a classical space (finite element in our examples) and time discretisation ($\mathcal{T}^{\mathrm{h}} = \{t_0, t_1, ..., t_{n_{\mathrm{t}}}\}$ such that $t_0 = 0$ and $t_{n_{\mathrm{t}}} = T$), the following equations of n_{u} spatial unknowns is obtained:

$$\underline{\mathbf{F}}_{\mathbf{int}}\left(\left(\underline{\mathbf{U}}(t_m,\mu)\right)_{m\in[\![0,n]\!]},\mu\right)+\underline{\mathbf{F}}_{\mathbf{ext}}(t_n,\mu)=\underline{\mathbf{0}}\qquad\forall\,(t_n,\mu)\in\mathcal{T}^{\mathrm{h}}\times\mathcal{P}\,. \tag{1}$$

The vector of internal forces, $\underline{\mathbf{F}}_{\mathbf{int}} \in \mathbb{R}^{n_{\mathrm{u}}}$, is a non-linear function of the history of the nodal unknowns $\underline{\mathbf{U}}(t, \mu)$. $\underline{\mathbf{F}}_{\mathbf{ext}} \in \mathbb{R}^{n_{\mathrm{u}}}$ is the vector of external forces.

2.2 POD-BASED REDUCED ORDER MODELLING

Let us write that the solution vectors, at any time step and for any value of the parameter, lie in a space of small dimension ("reduced space"), spanned by a few basis vectors $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_{\mathrm{c}}]\!]}$

$$\underline{\mathbf{U}}(t_n,\mu)=\sum_{i=1}^{n_{\mathrm{c}}}\underline{\mathbf{C}}_i\,\alpha_i(t_n,\mu)=\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\,. \tag{2}$$

where $\underline{\underline{\mathbf{C}}}$ is a matrix whose columns are the basis vectors $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_{\mathrm{c}}]\!]}$, and $\underline{\boldsymbol{\alpha}}$ is a vector of reduced state variables $(\alpha_i)_{i\in[\![1,n_{\mathrm{c}}]\!]}$ that depends on time and parameter.To get a basis $(\underline{\mathbf{C}}_i)_{i\in[\![1,n_{\mathrm{c}}]\!]}$ where the solution to the parametric problem of evolution can be obtained accurately, we use the classical snapshot proper orthogonal decomposition method (POD)([5]). This method requires to pre-compute a set of realisations of the parametric problem (i.e. the snapshot), from which a suboptimal reduced basis is extracted.

Injecting this approximation into (1), one obtains an over-constrained problem in the n_{c} reduced state variables $\underline{\boldsymbol{\alpha}}$ ($n_{\mathrm{c}} << n_{\mathrm{u}}$)

$$\underline{\mathbf{F}}_{\mathbf{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}\right)+\underline{\mathbf{F}}_{\mathbf{ext}}=\underline{\mathbf{R}}(\underline{\boldsymbol{\alpha}})\,. \tag{3}$$

A solution to this problem can be obtained by a least-square or Galerkin formulation.

2.3 DOMAIN DECOMPOSITION AND POD-BASED REDUCED ORDER MODELLING

The snapshot-POD method will be applied locally, with a different POD basis for each reduced subdomain. This idea is illustrated in Figure 1, applied on a structure made of damageable lattice bars and put under tension.

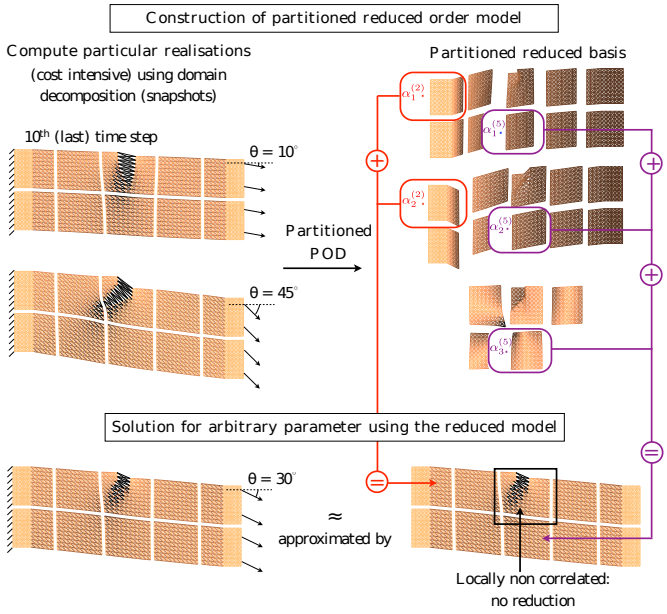


Figure 1: The Partitioned POD

2.4 SYSTEM APPROXIMATION

Despite the reduction of the number of unknowns, $\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}\right)$ is still expensive to evaluate since it is a non-linear quantity that has to be assembled over the entire domain of study. To speed-up the computation, a system approximation is performed by selecting only a few elements of the domain and evaluating the nonlinear operator only on those selected regions. The operator is reconstructed using a second POD basis $(\underline{\mathbf{D}}_i)_{i\in[\![1,n_{\text{d}}]\!]}$ (used in [1, 2]).

$$\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\right) = \sum_{i=1}^{n_{\text{d}}} \underline{\mathbf{D}}_i\,\zeta_i(t_n,\mu) = \underline{\underline{\mathbf{D}}}\,\underline{\boldsymbol{\zeta}}(t_n,\mu)\,. \tag{4}$$

The columns of $\underline{\underline{\mathbf{D}}} \in \mathbb{R}^{n_{\text{u}}} \times \mathbb{R}^{n_{\text{d}}}$ are spatial functions corresponding to a truncated snapshot POD expansion of $\underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}\right)$.

Static coefficients $\underline{\boldsymbol{\zeta}}$ are found by expressing that at any point $(t_n,\mu) \in \mathcal{P}$, the interpolation is optimal for a limited number n_{sa} of degrees of freedom:

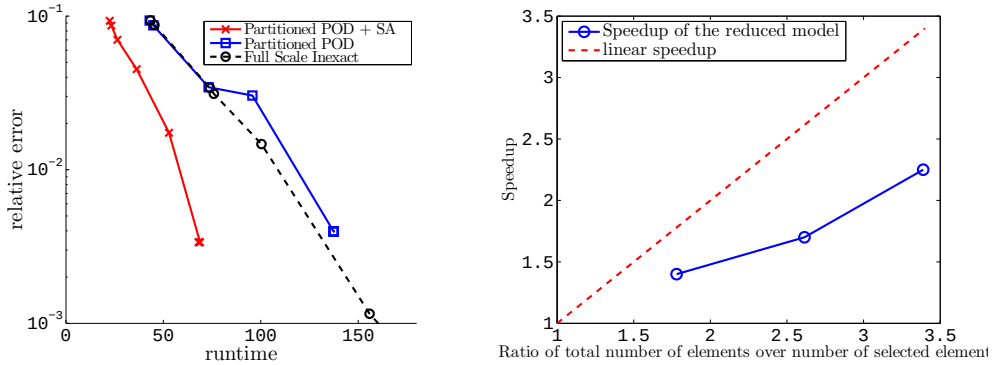
$$\underline{\boldsymbol{\zeta}}(t_n,\mu) = \underset{\underline{\boldsymbol{\zeta}}^\star}{\text{argmin}} \|\underline{\underline{\mathbf{D}}}\,\underline{\boldsymbol{\zeta}}^\star(t_n,\mu) - \underline{\mathbf{F}}_{\text{int}}\left(\underline{\underline{\mathbf{C}}}\,\underline{\boldsymbol{\alpha}}(t_n,\mu)\right)\|_{\underline{\underline{\mathbf{P}}}} \tag{5}$$

$\underline{\underline{\mathbf{P}}}$ is a boolean diagonal operator with very few non-zero values $n_{\text{sa}} \geq n_{\text{d}}$ corresponding to the evaluation points of the interpolation. $\|\underline{\mathbf{X}}\|_{\underline{\underline{\mathbf{P}}}} = \sqrt{\underline{\mathbf{X}}^T\underline{\underline{\mathbf{P}}}\,\underline{\mathbf{X}}}$ is the semi-norm associated with $\underline{\underline{\mathbf{P}}}$.

3 RESULTS

Results obtained from the problem illustrated in Figure 1 are shown in Figure 2. Using different tolerances in the non-linear solvers, the full scale model is compared to the Partitioned POD model and the Partitioned POD model using system approximation in Figure 2(a). The variation of the speedup

with the problem size is shown in Figure 2(b). It can be seen that the POD reduced model with system approximation is able to provide a solution at cheaper costs for a given accuracy. The number of vectors of the POD bases (influencing the number of elements that are skipped) is essentially independent of the problem size. Hence, the speedup increases with the problem size and is proportional to the ratio of the total number of elements over the number of selected elements for system approximation.



(a) Relative error for the different models using 256 nodes per subdomain

(b) Speedup relative to the size of the problem

Figure 2: Performance of the different models for a load angle of $40°$

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