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SENSITIVITY ANALYSIS AND SHAPE OPTIMISATION WITH THE ISOGEOMETRIC BOUNDARY ELEMENT METHOD

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ABSTRACT

This paper applies an isogeometric boundary element method (IGABEM) to sensitivity analysis and shape optimisation for linear elastic analysis. In the present work, Non-Uniform Rational B-splines (NURBS) are adopted for the geometry representation and the basis for analysis (utilising the isogeometric concept) thus greatly reducing the gap between Computer Aided Design (CAD) and analysis. It is found that IGABEM is particularly suitable for shape optimisation which is illustrated in the present paper.

1 INTRODUCTION

Isogeometric analysis (IGA) was originally proposed by Hughes et al. [1] to bridge the gap between CAD and analysis. The main advantages of IGA are that mesh generation is either eliminated or greatly reduced and analysis is conducted on the exact geometry at all stages. In IGA, the same basis functions as used in CAD which mostly commonly take the form of NURBS [2] and, more recently, T-splines [3] are used to approximate not only the geometry of the domain, but also the unknown fields. However, much of the work on IGA until present has focused on its use with the finite element method [4] which is based on a domain representation, whereas most computational geometry is based on a boundary representation. To overcome this limitation, Simpson et al. [5] introduced the isogeometric boundary element method (IGABEM), which can be naturally combined with CAD through the boundary integral equation. Hence, analysis can be performed directly from CAD data with minimal preprocessing. This feature is particularly beneficial for shape optimisation where in this paper, we extend the work of [5] to conduct sensitivity analysis and shape optimisation.

2 ISOGEOMETRIC BOUNDARY ELEMENT METHOD

For two dimensional linear elastostatic analysis, the regularised form of the displacement boundary integral equation (DBIE) is

$$\int_\Gamma T_{ij}(\mathbf{x}',\mathbf{x})[u_j(\mathbf{x})-u_j(\mathbf{x}')]\,\mathrm{d}\Gamma(\mathbf{x}) = \int_\Gamma U_{ij}(\mathbf{x}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \quad i,j=1,2 \tag{1}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are field point and source point respectively, u_j and t_j are the components of displacement and traction around the boundary, and U_{ij} and T_{ij} are fundamental solutions. The boundary is discretised by:

$$x_j = \sum_{\alpha}^{n_\alpha} N_\alpha(\xi) P_j^\alpha \tag{2}$$

where N_α are NURBS basis functions, P_j represent control point coordinates. The displacement and traction are also discretised with NURBS basis functions as:

$$u_j = \sum_{\alpha=1}^{n_\alpha} N_\alpha(\xi) d_j^\alpha = N_\alpha(\xi) d_j^\alpha \tag{3}$$

$$t_j = \sum_{\beta=1}^{n_\beta} N_\beta(\xi) q_j^\beta = N_\beta(\xi) q_j^\beta \tag{4}$$

where $N_\alpha(\xi)$ and $N_\beta(\xi)$ also represent NURBS basis functions. d_j^α and q_j^β are the nodal parameters associated with displacement and traction respectively. After inserting Eqn. (3) and (4) into Eqn. (1), the DBIE can be written as

$$\begin{aligned}&\int_\Gamma T_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi))[N_\alpha(\xi) d_j^\alpha - N_\alpha(\xi') d_j^\alpha] J(\xi) \mathrm{d}\xi \\ &= \int_\Gamma U_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi)) N_\beta q_j^\beta(\xi) J(\xi) \mathrm{d}\xi \quad i,j = 1,2\end{aligned} \tag{5}$$

where J is the Jacobian matrix for the transformation between physical space and parametric space.

For the piecewise integration, we must transfer the quantities of every element into the local coordinates system $\hat{\xi} \in [-1, 1]$

$$\begin{aligned}&\left\{ \sum_e^{N_e} \int_{-1}^{+1} T_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi))[N_\alpha(\xi) - N_\alpha(\xi')] J(\xi) \hat{J}^e(\hat{\xi}) \mathrm{d}\hat{\xi} \right\} d_j^\alpha \\ &= \left\{ \sum_e^{N_e} \int_{-1}^{+1} U_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi)) N_\beta(\xi) J(\xi) \hat{J}^e(\hat{\xi}) \mathrm{d}\hat{\xi} \right\} q_j^\beta \quad i,j = 1,2\end{aligned} \tag{6}$$

In matrix form, this can be written as follows

$$\mathbf{Hu} = \mathbf{Gt} \tag{7}$$

where $\mathbf{u}$ and $\mathbf{t}$ are vectors containing the nodal parameters of the displacement and traction respectively.

3 SHAPE SENSITIVITY ANALYSIS WITH IGABEM

We transfer Eqn. (1) into the material coordinate ξ as

$$\begin{aligned}&\int_\Gamma T_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi))[u_j(\mathbf{x}(\xi)) - u_j(\mathbf{x}'(\xi'))] J(\xi) \mathrm{d}\xi \\ &= \int_\Gamma U_{ij}(\mathbf{x}'(\xi'), \mathbf{x}(\xi)) t_j(\mathbf{x}(\xi)) J(\xi) \mathrm{d}\xi \qquad i,j = 1,2\end{aligned} \tag{8}$$

This equation can be simplified as

$$\int_\Gamma T_{ij}(u_j - u'_j)J\mathrm{d}\xi = \int_\Gamma U_{ij}t_jJ\mathrm{d}\xi \qquad i,j = 1,2 \tag{9}$$

which, when differentiated with respect to the design variables t_m and noting that ξ is independent of t_m, gives

$$\begin{aligned}&\int_\Gamma (T_{ij,m}J + T_{ij}J_{,m})(u_j - u'_j)\mathrm{d}\xi + \int_\Gamma (T_{ij}J)(u_{j,m} - u'_{j,m})\mathrm{d}\xi \\ &= \int_\Gamma (U_{ij,m}J + U_{ij}J_{,m})t_j\mathrm{d}\xi + \int_\Gamma (U_{ij}J)t_{j,m}\mathrm{d}\xi \qquad i,j = 1,2\end{aligned} \tag{10}$$

In addition to the approximation of the displacement and traction given by Eqn. (3) and (4), the shape derivatives are also discretised with NURBS basis functions as

$$u_{j,m} = \sum_{\alpha=1}^{n_\alpha} N_\alpha(\xi)d^\alpha_{j,m} = N_\alpha(\xi)d^\alpha_{j,m} \tag{11}$$

$$t_{j,m} = \sum_{\beta=1}^{n_\beta} N_\beta(\xi)q^\beta_{j,m} = N_\beta(\xi)q^\beta_{j,m} \tag{12}$$

where $d^\alpha_{j,m}$ and $q^\beta_{j,m}$ are the nodal parameters associated with displacement sensitivities and traction sensitivities respectively. After discretisation,

$$\begin{aligned}&\int_\Gamma (T_{ij,m}J + T_{ij}J_{,m})(N_\alpha d^\alpha_j - N'_\alpha d^\alpha_j)\mathrm{d}\xi + \int_\Gamma (T_{ij}J)(N_\alpha d^\alpha_{j,m} - N'_\alpha d^\alpha_{j,m})\mathrm{d}\xi \\ &= \int_\Gamma (U_{ij,m}J + U_{ij}J_{,m})N_\beta q^\beta_j\mathrm{d}\xi + \int_\Gamma (U_{ij}J)N_\beta q^\beta_{j,m}\mathrm{d}\xi \qquad i,j = 1,2\end{aligned} \tag{13}$$

Taking the nodal parameters out of the integrand, the above equation can be written as

$$\begin{aligned}&\left\{\int_\Gamma (T_{ij,m}J + T_{ij}J_{,m})(N_\alpha - N'_\alpha)\mathrm{d}\xi\right\} d^\alpha_j + \left\{\int_\Gamma (T_{ij}J)(N_\alpha - N'_\alpha)\mathrm{d}\xi\right\} d^\alpha_{j,m} \\ &= \left\{\int_\Gamma (U_{ij,m}J + U_{ij}J_{,m})N_\beta\mathrm{d}\xi\right\} q^\beta_j + \left\{\int_\Gamma (U_{ij}J)N_\beta\mathrm{d}\xi\right\} q^\beta_{j,m} \qquad i,j = 1,2\end{aligned} \tag{14}$$

For the piecewise integration, we must transfer the quantities of every element into the local coordinates system $\hat{\xi} \in [-1,1]$

$$\begin{aligned}&\sum_e^{N_e}\left\{\int_{-1}^{+1} (T_{ij,m}J + T_{ij}J_{,m})(N_\alpha - N'_\alpha)\hat{J}^e\mathrm{d}\hat{\xi}\right\} d^\alpha_j + \sum_e^{N_e}\left\{\int_{-1}^{+1} (T_{ij}J)(N_\alpha - N'_\alpha)\hat{J}^e\mathrm{d}\hat{\xi}\right\} d^\alpha_{j,m} \\ &= \sum_e^{N_e}\left\{\int_{-1}^{+1} (U_{ij,m}J + U_{ij}J_{,m})N_\beta\hat{J}^e\mathrm{d}\hat{\xi}\right\} q^\beta_j + \sum_e^{N_e}\left\{\int_{-1}^{+1} (U_{ij}J)N_\beta\hat{J}^e\mathrm{d}\hat{\xi}\right\} q^\beta_{j,m} \qquad i,j = 1,2\end{aligned} \tag{15}$$

where $\hat{J}^e$ is the Jacobian matrix for the transformation between local coordinates and material coordinates. This can be written in compact form

$$\mathbf{H}_{,m}\mathbf{u} + \mathbf{H}\mathbf{u}_{,m} = \mathbf{G}_{,m}\mathbf{t} + \mathbf{G}\mathbf{t}_{,m} \tag{16}$$

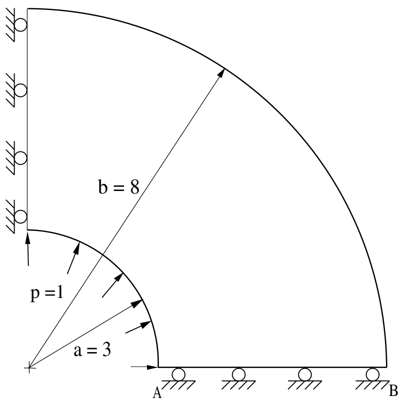

Figure 1: the pressure cylinder

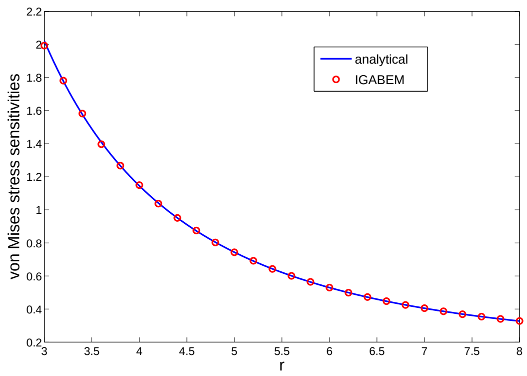

Figure 2: von Mises stress sensitivities on edge AB

4 NUMERICAL EXAMPLES

For the pressure cylinder shown in Fig. (1), the Young's modulus $E = 1$ and Poisson ration $\nu = 0.3$. The design variable is b, which varies from 3 to 10. The optimisation objective is to reduce the area subjected to the constraint that the von Mises stress should be below the yield stress. A series of points on the cylinder boundary are chosen as the monitor points. The von Mises stress sensitivities along edge AB are shown in Fig. (2), which indicates that IGABEM sensitivity analysis result agrees with the analytical solution very well. Based on the shape derivatives, the optimisation can be completed with a gradient based optimisation solver. In our case we choose the globally convergent version of the method of moving asymptotes (GCMMA) [6]. Finally, The optimal value of b is 5.2895.

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