Big monodromy theorem for abelian varieties over finitely generated fields*

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Abstract

An abelian variety over a field K is said to have *big monodromy*, if the image of the Galois representation on ℓ-torsion points, for almost all primes ℓ, contains the full symplectic group. We prove that all abelian varieties over a finitely generated field K with the endomorphism ring $\mathbb{Z}$ and semistable reduction of toric dimension one at a place of the base field K have big monodromy. We make no assumption on the transcendence degree or on the characteristic of K. This generalizes a recent result of Chris Hall.

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Introduction

It has been known in number theory, since times immemorial that Galois representation attached to the action of the absolute Galois group on torsion points of an abelian group scheme carries a lot of basic arithmetic and geometric information. The first aim which one encounters naturally, while studying such representations is to determine their images in terms of linear algebraic groups. There exists a vast variety of results in the literature concerning computations of Galois representations for abelian varieties defined over number fields and their applications to some classical questions such as Hodge, Tate and Mumford–Tate conjectures, see for example [21], [2]. In this paper we are interested in computing images of Galois representations attached to abelian varieties defined over finitely generated fields in arbitrary characteristic, i.e., to families of abelian varieties.

Let K be a field and denote by G_K its absolute Galois group. Let A/K be an abelian variety and $\ell \neq \mathrm{char}(K)$ a prime number. We denote by $\rho_{A[\ell]} : G_K \longrightarrow \mathrm{Aut}(A[\ell])$ the Galois representation attached to the action of G_K on the ℓ-torsion points of A. We define $\mathcal{M}_K(A[\ell]) := \rho_{A[\ell]}(G_K)$ and call this group *the mod-ℓ monodromy group of A/K*. We fix a polarization and denote by $e_\ell: A[\ell] \times A[\ell] \to \mu_\ell$ the corresponding Weil pairing. Then $\mathcal{M}_K(A[\ell])$ is a subgroup of the group of symplectic similitudes $\mathrm{GSp}(A[\ell], e_\ell)$ of the Weil pairing. We will say that A/K *has big monodromy* if there exists a constant ℓ_0 such that $\mathcal{M}_K(A[\ell])$ contains the symplectic group $\mathrm{Sp}(A[\ell], e_\ell)$, for every prime number $\ell \geq \ell_0$. Note that the property of having big monodromy does not depend on the choice of the polarization.

Certainly, the most prominent result on computing monodromy groups is the classical theorem of Serre (cf. [18], [19]): *If A is an abelian variety over a finitely generated field K of characteristic zero with* $\mathrm{End}(A) = \mathbb{Z}$ *and* $\dim(A) = 2, 6$ *or odd, then A/K has big monodromy.* In this paper we consider monodromies for abelian varieties over finitely generated fields which have been recently investigated by Chris Hall [11], [12]. To simplify notation, we will say that an abelian variety A over a finitely generated field K *is of Hall type*, if $\mathrm{End}(A) = \mathbb{Z}$ and K has a discrete valuation at which A has semistable reduction of toric dimension one.

In the special case, when $K = F(t)$ is a rational function field over another finitely generated field, it has been shown by Hall that certain hyperelliptic Jacobians have big monodromy; namely the Jacobians J_C of hyperelliptic curves C/K with affine equation $C : Y^2 = (X - t)f(X)$, where $f \in F[X]$ is a monic squarefree polynomial of even degree ≥ 4 (cf. [11, Theorem 5.1]). Furthermore, Hall has proved recently [12] the following theorem which in our notation reads: *If K is a* global *field, then every abelian variety A/K of Hall type has big monodromy.* We strengthen these results in our main theorem as follows.

Main Theorem. *[cf. Thm. 3.6] If K is a finitely generated field (of arbitrary characteristic) and A/K is an abelian variety of Hall type, then A/K has big monodromy.*

Our proof of the main theorem follows Hall's proof of [12] to some extent, e.g., we have borrowed a group theory result from [12] (cf. Theorem 3.4). In addition to that we had to apply a substantial quantity of new methods to achieve the extension to all finitely generated fields, such as for instance finite generation properties of fundamental groups of schemes and Galois theory of certain division fields of abelian varieties, which are gathered in Section 2 and Section 3 of the paper. Furthermore, at a technical point in the case $\mathrm{char}(K) = 0$, we perform a tricky reduction argument (described in detail in Section 3) at a place of K whose residue field is a number field. The paper carries an appendix with a self-contained proof of the group theoretical Theorem 3.4 due to Hall, which can be of independent value for the reader.

Theorem A plays an important role in our paper [1], where we make progress on the conjecture of Geyer and Jarden (cf. [9]) on torsion of abelian varieties over large algebraic extensions of finitely generated fields.

As a further application, we combine our monodromy computation with recent results of Ellenberg, Hall and Kowalski in order to obtain the following result on endomorphism rings and simplicity of fibres in certain families of abelian varieties. If K is a finitely generated transcendental extension of another field F and A/K is an abelian variety, then we call A *weakly isotrivial with respect to* F, if there is an abelian variety $B/\widetilde{F}$ and an $\widetilde{K}$-isogeny $B_{\widetilde{K}} \to A_{\widetilde{K}}$.

Corollary. *[cf. Cor. 4.3] Let F be a finitely generated field and $K = F(t)$ the function field of $\mathbb{P}^1/F$. Let A/K be an abelian variety. Let $U \subset \mathbb{P}^1$ be an open subscheme such that A extends to an abelian scheme $\mathcal{A}/U$. For $u \in U(F)$ denote by A_u/F the corresponding special fibre of $\mathcal{A}$. Assume that A is not weakly isotrivial with respect to F and that either of the conditions i) or ii) listed below is satisfied.*

i) A is of Hall type.

ii) $\mathrm{char}(K) = 0$, $\mathrm{End}(A) = \mathbb{Z}$ *and* $\dim(A) = 2, 6$ *or odd.*

Then the sets:

$$X_1 := \{u \in U(F) \mid \quad \mathrm{End}(A_u) \neq \mathbb{Z}\}$$

and

$$X_2 := \{u \in U(F) \mid \quad A_u/F \textit{ is not geometrically simple}\}$$

are finite.

Note that Ellenberg, Elsholtz, Hall and Kowalski proved the statement of the Corollary in the special case when A is the Jacobian variety of the hyperelliptic curve given by the affine equation $Y^2 = (X - t)f(X)$, with $f \in F[X]$ squarefree and monic of even degree ≥ 4 (cf. [6, Theorem 8]). It is the case, where the monodromy of A is known by [11, Theorem 5.1]. We obtain Part (i) of the Corollary as a consequence of the main theorem, our Proposition 4.2 below and also Propositions 4 and 7 of [6]. In order to prove (ii) we use Serre's Theorem [18], [19] instead of the main theorem.

We warmly thank Gerhard Frey, Dieter Geyer, Cornelius Greither and Moshe Jarden for conversations and useful comments on the topic of this paper. The mathematical content of the present work has been much influenced by seminal results of J.-P. Serre contained in [18], [19], [20], [21] and by the inspiring paper [12] of C.Hall. We acknowledge this with pleasure.

1 Notation and background material

In this section we fix notation and gather some background material on Galois representations that is important for the rest of this paper.

Let X be a scheme. For $x \in X$ we denote by $k(x)$ the residue field at x. If X is integral, then $R(X)$ stands for the function field of X, that is, for the residue field at the generic point of X. If X happens to be a scheme of finite type over a base field F, then we often write $F(x)$ instead of $k(x)$ and $F(X)$ instead of $R(X)$.

If K is a field, then we denote by $K_{\rm sep}$ (resp. $\widetilde{K}$) the separable (resp. algebraic) closure of K and by G_K its absolute Galois group. A finitely generated field is by definition a field which is finitely generated over its prime field. For an abelian variety A/K we let $\mathrm{End}_K(A)$ be the ring of all K-endomorphisms of A. We denote by $\mathrm{End}(A) := \mathrm{End}_{\widetilde{K}}(A_{\widetilde{K}})$ the *absolute* endomorphism ring.

If Γ is an object in an abelian category and $n \in \mathbb{Z}$, then $n_\Gamma : \Gamma \to \Gamma$ is the morphism "multiplication by n" and $\Gamma[n]$ is the kernel of n_Γ. Recall that there is an equivalence of categories between the category of finite étale group schemes over K and the category of finite (discrete) G_K-modules, where we attach $\Gamma(K_{\rm sep})$ to a finite étale group scheme Γ/K. For such a finite étale group scheme Γ/K we sometimes write just Γ instead of $\Gamma(K_{\rm sep})$, at least in situations where we are sure that this does not cause any confusion. For example, if A/K is an abelian variety and n an integer coprime to $\mathrm{char}(K)$, then we often write $A[n]$ rather than $A(K_{\rm sep})[n]$. Furthermore we put $A[n^\infty] := \bigcup_{i\in\mathbb{N}} A[n^i]$.

If M is a G_K-module (for example $M = \mu_n$ or $M = A[n]$ where A/K is an abelian variety), then we shall denote the corresponding representation of the Galois group G_K by

$$\rho_M : G_K \to \mathrm{Aut}(M)$$

and define $\mathcal{M}_K(M) := \rho_M(G_K)$. We define $K(M) := K_{\rm sep}^{\ker(\rho_M)}$ to be the fixed field in $K_{\rm sep}$ of the kernel of ρ_M. Then $K(M)/K$ is a Galois extension and $G(K(M)/K) \cong \mathcal{M}_K(M)$.

If R is a commutative ring with 1 (usually $R = \mathbb{F}_\ell$ or $R = \mathbb{Z}_\ell$) and M is a finitely generated free R-module equipped with a non-degenerate alternating bilinear pairing $e : M \times M \to R'$ into a free R'-module of rank 1 (which is a multiplicatively written R-module in our setting below), then we denote by

$$\mathrm{Sp}(M, e) = \{f \in \mathrm{Aut}_R(M) \mid \quad \forall x, y \in M : e(f(x), f(y)) = e(x, y)\}$$

the corresponding symplectic group and by

$$\mathrm{GSp}(M,e) = \{f \in \mathrm{Aut}_R(M) \mid \quad \exists \varepsilon \in R^\times : \forall x,y \in M : e(f(x),f(y)) = \varepsilon e(x,y)\}$$

the corresponding group of symplectic similitudes.

Let n be an integer coprime to $\mathrm{char}(K)$ and ℓ be a prime different from $\mathrm{char}(K)$. Let A/K be an abelian variety. We denote by $A^\vee$ the dual abelian variety and let $e_n : A[n] \times A^\vee[n] \to \mu_n$ and $e_{\ell^\infty} : T_\ell A \times T_\ell A^\vee \to \mathbb{Z}_\ell(1)$ be the corresponding Weil pairings. If $\lambda : A \to A^\vee$ is a polarization, then we deduce Weil pairings $e_n^\lambda : A[n] \times A[n] \to \mu_n$ and $e_{\ell^\infty}^\lambda : T_\ell A \times T_\ell A \to \mathbb{Z}_\ell(1)$ in the obvious way. If ℓ does not divide $\deg(\lambda)$ and if n is coprime to $\deg(\lambda)$, then e_n^λ and $e_{\ell^\infty}^\lambda$ are non-degenerate, alternating, G_K-equivariant pairings. Hence we have representations

$$\rho_{A[n]} : G_K \to \mathrm{GSp}(A[n], e_n^\lambda),$$

$$\rho_{T_\ell A} : G_K \to \mathrm{GSp}(T_\ell A, e_{\ell^\infty}^\lambda)$$

with images $\mathcal{M}_K(A[n]) \subset \mathrm{GSp}(A[n], e_n^\lambda)$ and $\mathcal{M}_K(T_\ell A) \subset \mathrm{GSp}(T_\ell A, e_{\ell^\infty}^\lambda)$. We shall say that an abelian variety (A, λ) over a field K has *big monodromy*, if there is a constant $\ell_0 > \max(\mathrm{char}(K), \deg(\lambda))$ such that $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for every prime number $\ell \geq \ell_0$.

Now let S be a noetherian regular 1-dimensional connected scheme with function field $K = R(S)$ and A/K an abelian variety. Denote by $\mathcal{A} \to S$ the Néron model (cf. [3]) of A. For $s \in S$ let $A_s := \mathcal{A} \times_S \mathrm{Spec}(k(s))$ be the corresponding fibre. Recall that we say that A has *good reduction at s* provided A_s is an abelian variety. In general, we denote by A_s° the connected component of A_s. If T is a maximal torus in A_s°, then $\dim(T)$ does not depend on the choice of T [10, IX.2.1] and we call $\dim(T)$ the *toric dimension* of the reduction A_s of A at s. Finally recall that one says that A has *semi-stable reduction at s*, if A_s° is an extension of an abelian variety by a torus.

We shall also need the following connections between the reduction type of A and properties of the Galois representations attached to A. Let s be a closed point of S. The valuation v attached to s admits an extension to the separable closure K_{sep}; we choose such an extension $\overline{v}$ and denote by $D(\overline{v})$ the corresponding decomposition group. This is the absolute Galois group of the quotient field $K_s = Q(\mathcal{O}_{S,s}^h)$ of the henselization $\mathcal{O}_{S,s}^h$ of the valuation ring $\mathcal{O}_{S,s}$ of v. Hence the results mentioned in [10, I.0.3] for the henselian case carry over to give the following description of $D(\overline{v})$: If $I(\overline{v})$ is the kernel of the canonical map $D(\overline{v}) \to G_{k(s)}$ defined by $\overline{v}$, then $D(\overline{v})/I(\overline{v}) \cong G_{k(s)}$. Let p be the characteristic of the residue field $k(s)$ (p is zero or a prime number). $I(\overline{v})$ has a maximal pro-p subgroup $P(\overline{v})$ ($P(\overline{v}) = 0$ if $p = 0$) and

$$I(\overline{v})/P(\overline{v}) \cong \varprojlim_{n \notin p\mathbb{Z}} \mu_n(k(s)_{\mathrm{sep}}) \cong \prod_{\ell \neq p \text{ prime}} \mathbb{Z}_\ell(1).$$

Hence the maximal pro-ℓ-quotient $I_\ell(\overline{v})$ of $I(\overline{v})$ is isomorphic to $\mathbb{Z}_\ell(1)$, if $\ell \neq p$ is a prime.

Proposition 1.1. *Let $\ell \neq p$ be a prime number. Assume that A has semi-stable reduction at s.*

a) The image $\rho_{A[\ell]}(P(\overline{v})) = \{Id\}$ *and* $\rho_{A[\ell]}(I(\overline{v}))$ *is a cyclic* ℓ*-group.*

b) Let g *be a generator of* $\rho_{A[\ell]}(I(\overline{v}))$*. Then* $(g - Id)^2 = 0$.

c) Assume that ℓ *does not divide the order of the component group of* A_s*. The toric dimension of* A *at* s *is equal to* $2\dim(A) - \dim_{\mathbb{F}_\ell}(\mathrm{Eig}(g,1))$ *if* $\mathrm{Eig}(g,1) = \ker(g - Id)$ *is the eigenspace of* g *at* 1.

Proof. Parts a) and b) are immediate consequences of [10, IX.3.5.2.].

Assume from now on that ℓ does not divide the order of the component group of A_s. This assumption implies $A_s^\circ[\ell] \cong A_s[\ell]$.

As we assumed A to be semi-stable at s, there is an exact sequence

$$0 \to T \to A_s^\circ \to B \to 0$$

where T is a torus and B is an abelian variety and $\dim(T)+\dim(B) = \dim(A_s) = \dim(A)$. Now $\dim_{\mathbb{F}_\ell}(T[\ell]) = \dim(T)$ and $\dim_{\mathbb{F}_\ell}(B[\ell]) = 2\dim(B) = 2\dim(A) - 2\dim(T)$. Taking into account that we have an exact sequence

$$0 \to T[\ell] \to A_s^\circ[\ell] \to B[\ell] \to 0$$

(note that $T(\widetilde{k}) \cong (\widetilde{k}^\times)^{\dim(T)}$ is divisible by ℓ), we find that $\dim_{\mathbb{F}_\ell}(A_s[\ell]) = \dim_{\mathbb{F}_\ell}(A_s^\circ[\ell]) = 2\dim(A) - \dim(T)$. This implies c), because $A_s[\ell] = A[\ell]^{I(\overline{v})}$ ([22, p. 495]) and obviously $A[\ell]^{I(\overline{v})} = \mathrm{Eig}(g,1)$. □

In general, if V is a finite dimensional vector space over $\mathbb{F}_\ell$, and $g \in \mathrm{End}_{\mathbb{F}_\ell}(V)$, then one defines $\mathrm{drop}(g) = \dim(V) - \dim(\mathrm{Eig}(g,1))$. One calls g a *transvection*, if it is unipotent of drop 1. We shall say that an abelian variety A over a field K is *of Hall type*, provided $\mathrm{End}(A) = \mathbb{Z}$ and there is a discrete valuation v on K such that A has semistable reduction of toric dimension 1 at v (i.e. at the maximal ideal of the discrete valuation ring of v). We have thus proved the following

Proposition 1.2. *If* A *is an abelian variety of Hall type over a finitely generated field* K*, then there is a constant* ℓ_0 *such that* $\mathcal{M}_K(A[\ell])$ *contains a transvection for every prime number* $\ell \geq \ell_0$*.*

2 Finiteness properties of division fields

If A is an abelian variety over a field K (of arbitrary characteristic) and $p = \mathrm{char}(K)$, then we denote by $A_{\neq p}$ the group of points in $A(K_{\mathrm{sep}})$ of order prime to p. Then

$$K(A_{\neq p}) = \prod_{\ell \neq p \ prime} K(A[\ell^\infty]) = \bigcup_{n \notin p\mathbb{Z}} K(A[n]).$$

If $p = 0$, then $K(A_{\neq p}) = K(A_{\mathrm{tor}})$. In this section we prove among other things: If K is finitely generated of positive characteristic, then $G(K(A_{\neq p})/K)$ is a finitely generated profinite group.

In this section, a *function field of n variables* over a field F will be a finitely generated field extension E/F of transcendence degree n. As usual we call such a function field E/F of n variables *separable* if it has a separating transcendency base. The following Lemma is an easy consequence of [8, Proposition 3.1].

Lemma 2.1. *(cf. [8]) Let F be a field and K/F a function field of one variable. Assume that K/F is separable. Let $p = \mathrm{char}(F)$. Let A/K be an abelian variety. Let F' be the algebraic closure of F in $K(A_{\neq p})$. Then $G(K(A_{\neq p})/F'K)$ is a finitely generated profinite group.*

Lemma 2.2. *Let (K, v) be a discrete valued field, A/K an abelian variety with good reduction at v, n an integer coprime to the residue characteristic of v, $L = K(A[n])$ and w an extension of v to L. Denote the residue field of v (resp. w) by $k(v)$ (resp. $k(w)$). Let $A_v/k(v)$ be the reduction of A at v. Then $k(w) = k(v)(A_v[n])$.*

Proof. Let R be the valuation ring of v and $S = \mathrm{Spec}(R)$. Let $\mathcal{A} \to S$ be an abelian scheme with generic fibre A. Then $A_v = \mathcal{A} \times_S \mathrm{Spec}(k(v))$, $\mathcal{A}[n]$ is a finite étale group scheme over S, and if T is the normalization of S in L, then the restriction map $r : A[n](L) \cong \mathcal{A}[n](T) \to A_v[n](k(w))$ is bijective (cf. [22]). The assertion follows easily from that. □

Definition 2.3. *We shall say in the sequel that a field K has property $\mathcal{F}$, if $G(K'(A_{\neq p})/K')$ is a* finitely generated *profinite group for every finite separable extension K'/K and every abelian variety A/K'.*

Proposition 2.4. *Let F be a field that has property $\mathcal{F}$. Let $p = \mathrm{char}(F)$. Let K be a function field over F. Assume that K/F is separable. Then K has property $\mathcal{F}$.*

Proof. By a routine induction on $\mathrm{trdeg}(K/F)$ it is enough to prove the proposition in the special case where K/F is a function field in *one* variable. We may thus assume $\mathrm{trdeg}(K/F) = 1$ and we have to show that $G(K'(A_{\neq p})/K')$ is finitely generated for every finite separable extension K'/K and every abelian variety A/K'. But if K'/K is a finite separable extension, then K'/F is a separable function field of one variable again. Hence it is enough to prove that $G(K(A_{\neq p})/K)$ is finitely generated for every abelian variety A/K.

Let A/K be an abelian variety. Let F_0 be the algebraic closure of F in K. Then K/F_0 is a regular extension. Let C/F_0 be a smooth curve with function field K and such that A has good reduction at all points of C. There is a finite Galois extension F_1/F_0 such that $C(F_1) \neq \emptyset$. If we put $K_1 := F_1K$, then K_1/F_1 is regular. Furthermore there is an exact sequence

$$1 \to G(K_1(A_{\neq p})/K_1) \to G(K(A_{\neq p})/K) \to G(K_1/K)$$

and $G(K_1/K)$ is finite. If we prove that $G(K_1(A_{\neq p})/K_1)$ is finitely generated, then it follows that $G(K(A_{\neq p})/K)$ is finitely generated as well. Hence we may assume that $K_1 = K$, i.e. that K/F is regular and that $C(F) \neq \emptyset$.

Choose a point $c \in C(F)$ and denote by A_c/F the (good) reduction of A at c. As in Lemma 2.1 denote by F' the algebraic closure of F in $K(A_{\neq p})$.

Claim. $F' \subset F(A_{c,\neq p})$.

Let $x \in F'$. Then x is algebraic over F and $x \in K(A[n])$ for some n which is coprime to p. If F_n denotes the algebraic closure of F in $K(A[n])$, then $x \in F_n$. Let w be the extension to $K(A[n])$ of the valuation attached to c. Then $k(w) = F(A_c[n])$ by Lemma 2.2. Obviously $F_n \subset k(w)$. Hence $x \in F(A_c[n]) \subset F(A_{c,\neq p})$. This finishes the proof of the claim.

The profinite group $G(F(A_{c,\neq p})/F)$ is finitely generated, because F has property $\mathcal{F}$ by assumption. Hence its quotient $G(F'/F)$ is finitely generated as well. Note that $G(F'K/K) = G(F'/F)$. On the other hand $G(K(A_{\neq p})/F'K)$ is finitely generated by Lemma 2.1. From the exact sequence

$$1 \to G(K(A_{\neq p})/F'K) \to G(K(A_{\neq p})/K) \to G(F'K/K) \to 1$$

we see that $G(K(A_{\neq p})/K)$ is finitely generated as desired. $\square$

Corollary 2.5. *Let K be a finitely generated field* of positive characteristic *or K be a function field over an algebraically closed field of arbitrary characteristic. Then K has property $\mathcal{F}$. In particular $G(K(A_{\neq p})/K)$ is finitely generated for every abelian variety A/K.*

Proof. In both cases K is a function field over a perfect field F which has property $\mathcal{F}$. The assertion hence follows from Proposition 2.4 $\square$

Remark 2.6. *A finitely generated field K of characteristic zero does* not *have property $\mathcal{F}$. In fact, if A/K is a principally polarized abelian variety, then by the existence of the Weil pairing $K(A_{\mathrm{tor}}) \supset K(\mu_\infty)$, and plainly $G(K(\mu_\infty)/K)$ is* not *finitely generated, when K is a finitely generated extension of $\mathbb{Q}$.*

3 Monodromy Computations

Let K be a field and A/K an abelian variety. We begin with the question whether $A[\ell]$ is a simple G_K-module for sufficiently large ℓ.

Proposition 3.1. *Let A be an abelian variety over a finitely generated field K. Assume that $\mathrm{End}_K(A) = \mathbb{Z}$. Then there is a constant ℓ_0 such that $A[\ell]$ is a simple $\mathbb{F}_\ell[G_K]$-module for all primes $\ell \geq \ell_0$.*

In the cases we need to consider, this proposition is a consequence of the following classical result (cf. [7, p. 118, p. 204], [24], [25],[15]).

Theorem 3.2. (Faltings, Zarhin) *Let K be a finitely generated field and A/K an abelian variety. Then there is a constant $\ell_0 > \mathrm{char}(K)$ such that the $\mathbb{F}_\ell[G_K]$-module $A[\ell]$ is semisimple and the canonical map $\mathrm{End}_K(A) \otimes \mathbb{F}_\ell \to \mathrm{End}_{\mathbb{F}_\ell}(A[\ell])$ is injective with image $\mathrm{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$ for all primes $\ell \geq \ell_0$.*

Proof of Proposition 3.1. By Theorem 3.2 there is a constant ℓ_0 such that $A[\ell]$ is a semisimple $\mathbb{F}_\ell[G_K]$-module with $\mathrm{End}_{\mathbb{F}_\ell[G_K]}(A[\ell]) = \mathbb{F}_\ell Id$ for every prime

$\ell \geq \ell_0$. This is only possible if $A[\ell]$ is a simple $\mathbb{F}_\ell[G_K]$-module for all primes $\ell \geq \ell_0$. □

We need some notation in order to explain a theorem of Raynaud that will be of importance later. Let $E/\mathbb{F}_p$ be a finite field extension with $|E| = p^d$ and $F/\mathbb{F}_p$ an algebraic extension. Denote by $\mathrm{Emb}(E, \widetilde{F})$ the set of all embeddings $E \to \widetilde{F}$. Let $\chi : E^\times \to \widetilde{F}$ be a character. If $i \in \mathrm{Emb}(E, \widetilde{F})$ is one such embedding, then there is a unique function $e : \mathrm{Emb}(E, \widetilde{F}) \to \{0, \cdots, p-1\}$ such that

$$\chi = \prod_{j \in \mathrm{Emb}(E,\widetilde{F})} (j|E^\times)^{e(j)},$$

and such that $e(j) < p - 1$ for some $j \in \mathrm{Emb}(E, \widetilde{F})$. We define $\mathrm{amp}(\chi) := \max(e(j) : j \in \mathrm{Emb}(E, \widetilde{F}))$ to be the *amplitude of the character* χ. Let $\rho : E^\times \to \mathrm{Aut}_{\mathbb{F}_p}(V)$ be a representation of $E^\times$ on a finite dimensional $\mathbb{F}_p$-vector space V. If V is a *simple* $\mathbb{F}_p[E^\times]$-module, then there is a finite field F_V with $|F_V| = |V|$ and a structure of 1-dimensional F_V-vector space on V such that ρ factors through a character $\chi_\rho : E^\times \to F_V^\times$. We then define $\mathrm{amp}(V) := \mathrm{amp}(\rho) := \mathrm{amp}(\chi_\rho)$. In general we define $\mathrm{amp}(V) := \mathrm{amp}(\rho) := \max(\mathrm{amp}(V_i) : i = 1, \cdots, t)$ where $\{V_1, \cdots, V_t\}$ is the set of Jordan-Hölder quotients of V to be the *amplitude of the representation* ρ.

Theorem 3.3. (Raynaud [16], [17, p. 277]) *Let A be an abelian variety over a number field K. Let v be a place of K with residue characteristic p. Let e be the ramification index of $v|\mathbb{Q}$. Let w be an extension of v to $K(A[p])$. Let I be the inertia group of $w|v$ and P the p-Sylow subgroup of I. Let $C \subset I$ be a subgroup that maps isomorphically onto I/P. Then there is a finite extension $E/\mathbb{F}_p$ and a surjective homomorphism $E^\times \to C$ such that the resulting representation*

$$\rho : E^\times \to C \to \mathrm{Aut}_{\mathbb{F}_p}(A[p])$$

has amplitude $\mathrm{amp}(\rho) \leq e$.

The technical heart of our monodromy computations is the following group theoretical result, which can be extracted from the work of C. Hall [11], [12].

Theorem 3.4. *Let $\ell > 2$ be a prime, let (V, e_V) be a finite-dimensional symplectic space over $\mathbb{F}_\ell$ and M a subgroup of $\Gamma := \mathrm{GSp}(V, e_V)$. Assume that M contains a transvection and that V is a simple $\mathbb{F}_\ell[M]$-module. Denote by R the subgroup of M generated by the transvections in M.*

a) Then there is a non-zero symplectic subspace $W \subset V$, which is a simple $\mathbb{F}_\ell[R]$-module, such that the following properties hold true:

  i) Let $H = \mathrm{Stab}_M(W)$. There is an orthogonal direct sum decomposition $V = \bigoplus_{g \in M/H} gW$. In particular $|M/H| \leq \dim(V)$.

  ii) $R \cong \prod_{g \in M/H} \mathrm{Sp}(W)$ and $N_\Gamma(R) \cong \prod_{g \in M/H} \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$.

  iii) $R \subset M \subset N_\Gamma(R)$.

Denote by $\varphi : N_\Gamma(R) \to \mathrm{Sym}(M/H)$ the projection.

b) Let $e \in \mathbb{N}$. Let $E/\mathbb{F}_\ell$ be a finite extension and $\rho : E^\times \to M \subset \mathrm{GSp}(V, e_V)$ a homomorphism such that the corresponding representation of $E^\times$ on V has amplitude $\mathrm{amp}(\rho) \leq e$. If $\ell > \dim(V)e + 1$, then $\varphi(\rho(E^\times)) = \{1\}$.

Hall's proof in [11], [12] addresses a slightly less general situation. We will present a self-contained proof of Theorem 3.4 in the Appendix.

Remark 3.5. *Assume that in the situation of Theorem 3.4 the module V is a simple $\mathbb{F}_\ell[\ker(\varphi) \cap M]$-module. Then V is in particular a simple $\mathbb{F}_\ell[\ker(\varphi)]$-module and $\ker(\varphi) = \prod_{g \in M/H} \mathrm{GSp}(W)$. This is only possible if $M = H$, $V = W$ and $R = \mathrm{Sp}(V, e) \subset M$.*

We now state the main result of this section.

Theorem 3.6. *Let K be a finitely generated field. Let (A, λ) be a polarized abelian variety over K of Hall type. Then (A, λ) has big monodromy.*

The case where K is a global field is due to Hall (cf. [12]) and we follow his line of proof to some extent, but we need a lot of additional arguments in order to make things work in the more general situation. The proof will occupy almost the rest of this section.

There is a constant $\ell_0 > \max(\deg(\lambda), \mathrm{char}(K))$ such that the following holds true for all primes $\ell \geq \ell_0$:

1. The subgroup $\mathcal{M}_K(A[\ell])$ of $\mathrm{GSp}(A[\ell], e_\ell^\lambda)$ contains a transvection. Denote by R_ℓ the subgroup of $\mathcal{M}_K(A[\ell])$ generated by the transvections in $\mathcal{M}_K(A[\ell])$ (cf. Proposition 1.2).
2. $A[\ell]$ is a simple $\mathbb{F}_\ell[G_K]$-module (cf. Proposition 3.1).

Now Hall's group theory result (cf. Theorem 3.4) gives - for every prime $\ell \geq \ell_0$ - a non-zero symplectic subspace $W_\ell \subset A[\ell]$, which is simple as a $\mathbb{F}_\ell[R_\ell]$-module such that the properties i), ii) and iii) of Theorem 3.4 are satisfied. Let H_ℓ be the stabilizer of W_ℓ under the action of $\mathcal{M}_K(A[\ell])$. Define $M_\ell := \mathcal{M}_K(A[\ell])$ and $\Gamma_\ell := \mathrm{GSp}(A[\ell], e_\ell^\lambda)$. Then

$$\prod_{M_\ell/H_\ell} \mathrm{Sp}(W_\ell, e_\ell^\lambda) \cong R_\ell \subset M_\ell \subset N_{\Gamma_\ell}(R_\ell) = \prod_{M_\ell/H_\ell} \mathrm{Sp}(W_\ell, e_\ell^\lambda) \rtimes \mathrm{Sym}(M_\ell/H_\ell),$$

and we denote by $\varphi_\ell : N_{\Gamma_\ell}(R_\ell) \to \mathrm{Sym}(M_\ell/H_\ell)$ the projection. We have the following property (cf. Remark 3.5):

If $A[\ell]$ is a simple $\mathbb{F}_\ell[\ker(\varphi_\ell) \cap M_\ell]$-module for some prime $\ell \geq \ell_0$, then $M_\ell = H_\ell$, $W_\ell = A[\ell]$ and $M_\ell \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for this prime ℓ.

We denote by N_ℓ the fixed field inside K_{sep} of the preimage $\rho_{A[\ell]}^{-1}(M_\ell \cap \ker(\varphi_\ell))$, where $\rho_{A[\ell]} : G_K \to \Gamma_\ell$ is the mod-ℓ representation attached to A. Then N_ℓ is an intermediate field of $K(A[\ell])/K$ which is Galois over K, and $G(N_\ell/K)$ is isomorphic to the subgroup $\varphi_\ell(M_\ell)$ of $\mathrm{Sym}(M_\ell/H_\ell)$. In particular $[N_\ell : K] \leq (2\dim(A))!$ is bounded independently of ℓ. If we denote by $N := \prod_{\ell \geq \ell_0 \text{ prime}} N_\ell$ the corresponding composite field, then $G_N = \bigcap_{\ell \geq \ell_0 \text{ prime}} G_{N_\ell}$. Hence the following property holds true.

If $A[\ell]$ is simple as a $\mathbb{F}_\ell[G_N]$-module for some prime $\ell \geq \ell_0$, then $M_\ell \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for this prime ℓ. $(*)$

Proof of Theorem 3.6 in the special case $\mathrm{char}(K) > 0$. If $\mathrm{char}(K) > 0$, then the Galois group $G(K(A_{\neq p})/K)$ ($p := \mathrm{char}(K)$) is finitely generated, because K then has property $\mathcal{F}$ by Corollary 2.5. Furthermore N_ℓ is an intermediate field of $K(A_{\neq p})/K$ which is Galois over K and with $[N_\ell : K]$ bounded independently of ℓ. Hence N/K must be finite. In particular N is finitely generated. A second application of the result of Faltings and Zarhin (cf. Proposition 3.1) yields a constant $\ell_1 \geq \ell_0$ such that $A[\ell]$ is a simple $\mathbb{F}_\ell[G_N]$-module for all primes $\ell \geq \ell_0$. Hence A has big monodromy by $(*)$. □

To finish the proof of Theorem 3.6 we assume for the rest of the proof that $\mathrm{char}(K) = 0$. We shall prove that N/K is finite also in that case, but the proof of this fact is more complicated, because now K is *not* $\mathcal{F}$-finite (cf. Remark 2.6). We briefly sketch the main steps in the proof, before we go into the details: The first and hardest step is to show that the algebraic closure L of $\mathbb{Q}$ in N is a *finite* extension of $\mathbb{Q}$. In order to achieve this we will construct a finite extension $L'/\mathbb{Q}$ such that some L'-rational "place" of KL' splits up completely into L'-rational "places" of $N_\ell L'$ for every sufficiently large prime ℓ. We use this to show that $G(NL/KL) \cong G(NL_{\mathrm{sep}}/KL_{\mathrm{sep}})$ and the fact that the latter group can be proved to be finite, because KL_{sep} is $\mathcal{F}$-finite (unlike K itself). This suffices to prove that N/K is finite. Once we know this, we shall proceed as in the positive characteristic case above.

We now go into the details. Let F be the algebraic closure of $\mathbb{Q}$ in K. Then F is a number field. Let S be a smooth affine F-variety with function field K such that A extends to an abelian scheme $\mathcal{A}$ over S with generic fibre A (i.e. such that A has good reduction along S). Let S_ℓ be the normalization of S in N_ℓ and let S'_ℓ be the normalization of S_ℓ in $K(A[\ell])$. Then $S'_\ell \to S_\ell \to S$ are finite étale covers. (Note that $\mathrm{char}(F(s)) = 0$ for every point $s \in S$.) In particular S'_ℓ and S_ℓ are smooth F-schemes. (Compare the diagram below.)

Fix a geometric point $P \in S(F_{\mathrm{sep}})$ and denote by $A_P := \mathcal{A} \times_S \mathrm{Spec}(F(P))$ the corresponding special fibre of $\mathcal{A}$. Then A_P is an abelian variety over the number field $F(P)$. Fix for every $\ell \geq \ell_0$ a geometric point $Q_\ell \in S_\ell(F_{\mathrm{sep}})$ over P and a geometric point $Q'_\ell \in S'_\ell(F_{\mathrm{sep}})$ over Q_ℓ. Then $F(Q'_\ell)/F(Q_\ell)$ and $F(Q_\ell)/F(P)$ are finite extensions of number fields. Note that $F(Q'_\ell) = F(P)(A_P[\ell])$ by Lemma 2.2. Denote by $\mathcal{O}$ (resp. $\mathcal{O}_\ell$, resp. $\mathcal{O}'_\ell$) the integral closure of $\mathbb{Z}$ in $F(P)$ (resp. in $F(Q_\ell)$, resp. in $F(Q'_\ell)$). For every prime $\ell \geq \ell_0$ we have the following diagram on the level of schemes

$$
\begin{array}{ccccccc}
 & & \mathrm{Spec}(K(A[\ell])) & \longrightarrow & \mathrm{Spec}(N_\ell) & \longrightarrow & \mathrm{Spec}(K) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & S'_\ell & \longrightarrow & S_\ell & \longrightarrow & S \\
 & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Spec}(F(P)(A_P[\ell])) & = & \mathrm{Spec}(F(Q'_\ell)) & \longrightarrow & \mathrm{Spec}(F(Q_\ell)) & \longrightarrow & \mathrm{Spec}(F(P)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Spec}(\mathcal{O}'_\ell) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_\ell) & \xrightarrow{f_\ell} & \mathrm{Spec}(\mathcal{O})
\end{array}
$$

We now study the ramification of prime ideals $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$ in the extension $F(Q_\ell)/F(P)$. Let $\mathbb{P}_{\mathrm{bad}}$ be the (finite) set of primes $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ where $A_P/F(P)$ has bad reduction.

Lemma 3.7. *There is a constant $\ell_2 \geq \ell_0$ with the following property: For every prime number $\ell \geq \ell_2$ the map $f_\ell : \mathrm{Spec}(\mathcal{O}_\ell) \to \mathrm{Spec}(\mathcal{O})$ is étale at every point $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$ outside of $\mathbb{P}_{\mathrm{bad}}$.*

Proof. Let $\ell_2 := \max(\ell_0, (2\dim(A))![F(P) : \mathbb{Q}] + 2)$.

Now let $\ell \geq \ell_2$ be a prime number. Let $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$ be an arbitrary prime ideal with $\mathfrak{m} \notin \mathbb{P}_{\mathrm{bad}}$. We have to show that $\mathfrak{m}$ is unramified in $F(Q_\ell)$. Let $p = \mathrm{char}(\mathcal{O}/\mathfrak{m})$ be the residue characteristic of $\mathfrak{m}$.

If $p \neq \ell$, then $\mathfrak{m}$ is unramified even in $F(Q'_\ell) = F(P)(A_P[\ell])$.

We can hence assume that $\boxed{p = \ell}$. Let $\mathfrak{m}_\ell \in \mathrm{Spec}(\mathcal{O}_\ell)$ be a point over $\mathfrak{m}$ and $\mathfrak{m}'_\ell \in \mathrm{Spec}(\mathcal{O}'_\ell)$ a point over $\mathfrak{m}_\ell$. Let $D(\mathfrak{m}'_\ell)$ (resp. $D(\mathfrak{m}_\ell)$) be the decomposition group of $\mathfrak{m}'_\ell/F(P)$ (resp. of $\mathfrak{m}_\ell/F(P)$) and $I(\mathfrak{m}'_\ell)$ (resp. $I(\mathfrak{m}_\ell)$) the corresponding inertia group. Let $P(\mathfrak{m}'_\ell)$ (resp. $P(\mathfrak{m}_\ell)$) be the (unique) p-Sylow subgroup of $I(\mathfrak{m}'_\ell)$ (resp. $I(\mathfrak{m}_\ell)$).

We have the following commutative diagram on the level of groups:

$$
\begin{array}{ccccc}
\prod_{M_\ell/H_\ell} \mathrm{GSp}(W_\ell) & \hookrightarrow & N_{\Gamma_\ell}(M_\ell) & \twoheadrightarrow & Sym(M_\ell/H_\ell) \\
\uparrow & & \uparrow & & \uparrow \\
M_\ell \cap ker(\varphi_\ell) & \hookrightarrow & M_\ell & \twoheadrightarrow & \varphi_\ell(M_\ell) \\
\| & & \| & & \| \\
G(K(A[\ell])/N_\ell) & \hookrightarrow & G(K(A[\ell])/K) & \twoheadrightarrow & G(N_\ell/K) \\
\uparrow & & \uparrow & & \uparrow \\
G(F(Q'_\ell)/F(Q_\ell)) & \hookrightarrow & G(F(Q'_\ell)/F(P)) & \twoheadrightarrow & G(F(Q_\ell)/F(P)) \\
 & & \uparrow & & \uparrow \\
 & & D(\mathfrak{m}'_\ell) & \twoheadrightarrow & D(\mathfrak{m}_\ell) \\
 & & \uparrow & & \uparrow \\
 & & I(\mathfrak{m}'_\ell) & \twoheadrightarrow & I(\mathfrak{m}_\ell) \\
 & & \uparrow & & \uparrow \\
 & & P(\mathfrak{m}'_\ell) & \twoheadrightarrow & P(\mathfrak{m}_\ell)
\end{array}
$$

We have to prove that the image of $I(\mathfrak{m}'_\ell)$ in $\mathrm{Sym}(M_\ell/H_\ell)$ by the maps in the diagram is $\{1\}$. Now $p = \ell > (2\dim(A))!$ due to our choice of ℓ_2 and $|\mathrm{Sym}(M_\ell/H_\ell)| \leq (2\dim(A))!$, hence $P(\mathfrak{m}'_\ell)$ maps to $\{1\}$ in $\mathrm{Sym}(M_\ell/H_\ell)$. In particular, $P(\mathfrak{m}_\ell) = \{1\}$. Consider the tame ramification group $I_t = I(\mathfrak{m}'_\ell)/P(\mathfrak{m}'_\ell)$. It is a cyclic group of order prime to p. Choose a subgroup $C \subset I(\mathfrak{m}'_\ell)$ that maps isomorphically onto I_t under the projection. It is enough to show that C maps to $\{1\}$ in $\mathrm{Sym}(M_\ell/H_\ell)$.

By Raynaud's theorem (cf. Theorem 3.3) there is a finite extension $E/\mathbb{F}_p$ and an epimorphism $E^\times \to C$ such that the resulting representation

$$E^\times \to C \to \mathrm{Aut}(A_P[\ell]) = \mathrm{Aut}(A[\ell])$$

has amplitude $\leq e$, where e is the ramification index of $\mathfrak{m}$ over $\mathbb{Q}$. Clearly $e \leq [F(P) : \mathbb{Q}]$. By part b) of Theorem 3.4, the image of $E^\times$ in $\mathrm{Sym}(M_\ell/H_\ell)$ is $\{1\}$. Hence the image of C in $\mathrm{Sym}(M_\ell/H_\ell)$ is $\{1\}$ as desired. □

Lemma 3.8. *Let L be the algebraic closure of F in N. Then L/F is a finite extension.*

Proof. Let $L' := \prod_{\ell \geq \ell_0 \text{ prime}} F(Q_\ell)$. For every prime $\ell \geq \ell_2$ the Galois extension of number fields $F(Q_\ell)/F(P)$ is unramified outside $\mathbb{P}_{\mathrm{bad}}$ by Lemma 3.7. Furthermore $[F(Q_\ell) : F(P)] \leq (2\dim(A))!$ for every prime $\ell \geq \ell_2$. The Theorem of Hermite-Minkowski (cf. [14], p. 122) implies that $\prod_{\ell \geq \ell_2 \text{ prime}} F(Q_\ell)$ is a *finite* extension of $F(P)$. This in turn implies that L'/F is a finite extension. It is thus enough to show that $L \subset L'$.

Recall that $K = F(S)$ is the function field of the F-variety S and S_ℓ is the normalization of S in the finite Galois extension N_ℓ/K. Denote by $\hat{S}$ the normalization of S in N and by $h_\ell : \hat{S} \to S_\ell$ the canonical projection. The canonical morphism $\hat{S} \to S$ is surjective, hence there is a point $\hat{P} \in \hat{S}(F_{\text{sep}})$ over P. The point $h_\ell(\hat{P}) \in S_\ell(F_{\text{sep}})$ lies over P. Hence $h_\ell(\hat{P})$ is conjugate to Q_ℓ under the action of $G(N_\ell/K)$. This implies that $F(h_\ell(\hat{P})) = F(Q_\ell)$. For every $\ell \geq \ell_0$ there is a diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(N) & \longrightarrow & \mathrm{Spec}(N_\ell) & \longrightarrow & \mathrm{Spec}(K) \\
\uparrow & & \uparrow & & \uparrow \\
\hat{S} & \xrightarrow{h_\ell} & S_\ell & \longrightarrow & S \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(F(\hat{P})) & \longrightarrow & \mathrm{Spec}(F(Q_\ell)) & \longrightarrow & \mathrm{Spec}(F(P))
\end{array}$$

where the morphisms $S_\ell \to S$ are étale covers and $N = \prod_{\ell \geq \ell_0} N_\ell$. It follows that $F(\hat{P}) = \prod_{\ell \geq \ell_0} F(Q_\ell) = L'$. On the other hand L is the algebraic closure of F in N, hence $\hat{S}$ is a scheme over L. This implies that L is a subfield of $F(\hat{P})$. Hence in fact $L \subset L'$ as desired. ☐

End of the proof of Theorem 3.6 in the case $\mathrm{char}(K) = 0$. We have an isomorphism $G(NL_{\text{sep}}/KL_{\text{sep}}) \cong G(N/KL)$, because N/L and KL/L are regular extensions. The field KL_{sep} is $\mathcal{F}$-finite by Corollary 2.5. Hence the profinite group $G(KL_{\text{sep}}(A_{\text{tor}})/KL_{\text{sep}})$ is finitely generated. As $NL_{\text{sep}} \subset KL_{\text{sep}}(A_{\text{tor}})$, $G(NL_{\text{sep}}/KL_{\text{sep}})$ must be finitely generated as well. Furthermore $NL_{\text{sep}} = \prod_{\ell \geq \ell_0} N_\ell L_{\text{sep}}$ where $[N_\ell L_{\text{sep}} : KL_{\text{sep}}]$ is bounded independently from ℓ. Hence $G(NL_{\text{sep}}/KL_{\text{sep}})$ is finite and this implies that N/KL is a finite extension. On the other hand it follows from Lemma 3.8 that KL/K is finite. Hence N/K is a *finite* extension. Consequently N is finitely generated, because K is finitely generated. Proposition 3.1 yields a constant $\ell_3 > \ell_0$ such that $A[\ell]$ is a simple $\mathbb{F}_\ell(G_N)$-module for every prime $\ell \geq \ell_3$. Hence A/K has big monodromy by (∗), as desired. ☐

4 Applications

In this section we apply our methods to prove a generalization of a result of Ellenberg, Elsholz, Hall and Kowalski on endomorphism rings and simplicity of fibres in certain families of abelian varieties (cf. [6, Theorem 8]).

Proposition 4.1. *Let K be a field and (A, λ) a polarized abelian variety over K with big monodromy. Let L/K be a finite extension. Then the following properties hold.*

a) There is a constant $\ell_0 \geq \max(\mathrm{char}(K), \deg(\lambda))$ such that $\mathcal{M}_L(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for every prime number $\ell \geq \ell_0$.

b) A is geometrically simple.

Proof. Part a). Let E_0 be the maximal separable extension of K in L and E/K a finite Galois extension containing E_0. By our assumption there is a constant $\ell_0 > \max(\deg(\lambda), \mathrm{char}(K), 5)$ such that $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for every prime $\ell \geq \ell_0$. For $\ell \geq \ell_0$ let K_ℓ be the fixed field of $\mathrm{Sp}(A[\ell], e_\ell^\lambda)$ in $K(A[\ell])/K$. Then $\mathcal{M}_{K_\ell}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ and $\mathcal{M}_{EK_\ell}(A[\ell])$ is a normal subgroup of $\mathcal{M}_{K_\ell}(A[\ell])$ of index $\leq [E : K]$. Put $\ell_1 := \max(\ell_0, [E : K] + 1)$. Then

$$|\mathcal{M}_{EK_\ell}(A[\ell])| \geq \frac{1}{[E : K]} |\mathrm{Sp}(A[\ell], e_\ell^\lambda)| > 2$$

for all primes $\ell \geq \ell_1$. On the other hand the only proper normal subgroups of $\mathrm{Sp}(A[\ell], e_\ell^\lambda)$ are $\{\pm 1\}$ and the trivial group (cf. [21, p. 53]). Hence

$$\mathcal{M}_{E_0}(A[\ell]) \supset \mathcal{M}_E(A[\ell]) \supset \mathcal{M}_{EK_\ell}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$$

for all primes $\ell \geq \ell_1$. As L/E_0 is purely inseparable, we find

$$\mathcal{M}_L(A_L[\ell]) = \mathcal{M}_{E_0}(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$$

for all primes $\ell \geq \ell_1$ as desired.

Part b). Let $A_1, A_2/\widetilde{K}$ be abelian varieties and $f : A_{\widetilde{K}} \to A_1 \times A_2$ an isogeny. Then A_1, A_2 and f are defined over some finite extension L/K. Hence there is an $\mathbb{F}_\ell[G_L]$-module isomorphism $A[\ell] \cong A_1[\ell] \times A_2[\ell]$ for every prime $\ell > \deg(f)$. By Part a) $\mathcal{M}_L(A[\ell]) \supset \mathrm{Sp}(A[\ell]), e_\ell^\lambda)$ for all sufficiently large primes ℓ. Hence $A[\ell]$ is a simple $\mathbb{F}_\ell[\mathcal{M}_L(A[\ell])]$-module and in particular a simple $\mathbb{F}_\ell(G_L)$-module for all sufficiently large primes ℓ. This is only possible if $A_1 = 0$ or $A_2 = 0$. □

Let F be a finitely generated field and K/F a finitely generated transcendental field extension and A/K an abelian variety. We say that A/K is *weakly isotrivial with respect to F*, if there is an abelian variety $B/\widetilde{F}$ and a $\widetilde{K}$-isogeny $B_{\widetilde{K}} \to A_{\widetilde{K}}$.

Proposition 4.2. *Let F be a finitely generated field, K/F a finitely generated separable transcendental field extension and (A, λ) a polarized abelian variety over K. Assume that A/K has big monodromy and that A/K is* not *weakly isotrivial with respect to F. Define $K' := F_{\mathrm{sep}}K$. Then there is a constant* $\ell_0 \geq \max(\mathrm{char}(K), \deg(\lambda))$ *such that* $\mathcal{M}_{K'}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ *for every prime number $\ell \geq \ell_0$.*

Proof. Let $\ell_0 \geq \max(\deg(\lambda), \mathrm{char}(K), 5)$ be a constant such that $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for every prime $\ell \geq \ell_0$. Let $\ell \geq \ell_0$ be a prime number. Then we have

$$\mathcal{M}_{K'}(A[\ell]) \subset \mathrm{Sp}(A[\ell], e_\ell^\lambda) \subset \mathcal{M}_K(A[\ell]),$$

because K' contains μ_ℓ. Furthermore, $\mathcal{M}_{K'}(A[\ell])$ is a normal subgroup of $\mathcal{M}_K(A[\ell])$, because K'/K is Galois. It follows that $\mathcal{M}_{K'}(A[\ell])$ is *normal* in $\mathrm{Sp}(A[\ell], e_\ell^\lambda)$.

The only proper normal subgroups in $\mathrm{Sp}(A[\ell], e_\ell^\lambda)$ are $\{1\}$ and $\{\pm 1\}$ (cf. [21, p. 53]), because $\ell \geq 5$. Hence either $\mathcal{M}_{K'}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ or $|\mathcal{M}_{K'}(A[\ell])| \leq 2$.

Let Λ be the set of prime numbers $\ell \geq \ell_0$ where $|\mathcal{M}_{K'}(A[\ell])| \leq 2$. We claim that Λ is *finite*.

For every $\ell \in \Lambda$ we have $[K'(A[\ell]) : K'] \leq 2$. Furthermore $G(K'(A_{\neq p})/K')$ is profinitely generated, where $p = \mathrm{char}(K)$. To see this note that

$$G(K'(A_{\neq p})/K') = G(\widetilde{F}K'(A_{\neq p})/\widetilde{F}K')$$

because $\widetilde{F}/F_{\mathrm{sep}}$ is purely inseparable and use Corollary 2.5. Hence $N := \prod_{\ell\in\Lambda} K'(A[\ell])$ is a *finite* extension of K'. In particular N/F_{sep} is a finitely generated regular extension. A/K must be geometrically simple by our assumption that A/K has big monodromy (cf. Proposition 4.1). In particular A_N is simple. Hence the assumption that A is not weakly isotrivial with respect to F implies that the Chow trace $\mathrm{Tr}_{N/F_{\mathrm{sep}}}(A_N)$ is zero. It follows by the Mordell-Lang-Néron theorem (cf. [4, Theorem 2.1]) that $A(N)$ is a finitely generated $\mathbb{Z}$-module. In particular the torsion group $A(N)_{\mathrm{tor}}$ is finite. On the other hand, $A(N)$ contains a non-trivial ℓ-torsion point for every $\ell \in \Lambda$. It follows that Λ is in fact finite.

Thus, after replacing ℓ_0 by a bigger constant, we see that we have the equality $\mathcal{M}_{K'}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for all primes $\ell \geq \ell_0$. □

Corollary 4.3. *Let F be a finitely generated field and $K = F(t)$ the function field of $\mathbb{P}^1/F$. Let A/K be a polarized abelian variety. Let $U \subset \mathbb{P}^1$ be an open subscheme such that A extends to an abelian scheme $\mathcal{A}/U$. For $u \in U(F)$ denote by A_u/F the corresponding special fibre of $\mathcal{A}$. Assume that A is not weakly isotrivial with respect to F and that either condition i) or ii) is satisfied.*

i) A is of Hall type.

ii) $\mathrm{char}(K) = 0$, $\mathrm{End}(A) = \mathbb{Z}$ and $\dim(A) = 2, 6$ or odd.

Then the sets:

$$X_1 := \{u \in U(F) \mid \quad \mathrm{End}(A_u) \neq \mathbb{Z}\}$$

and

$$X_2 := \{u \in U(F) \mid \quad A_u/F \textit{ is not geometrically simple}\}$$

are finite.

Proof. The abelian variety A/K has big monodromy. In case i) this follows by Theorem 3.6. In case ii) this is a well-known theorem of Serre, cf. [18], [19].) Define $K' := F_{\mathrm{sep}}K$. As A/K is not weakly isotrivial with respect to F by assumption, Proposition 4.2 implies that there is a constant $\ell_0 > \mathrm{char}(K)$ such that $\mathcal{M}_{K'}(A[\ell]) = \mathrm{Sp}(A[\ell], e_\ell^\lambda)$ for all primes $\ell \geq \ell_0$. Hence $A_{K'}/K'$ has big monodromy. Now Propositions 4 and 7 of [6] imply the assertion. Note that the notion of "big monodromy" in the paper [6] is slightly different from ours. □

A Appendix. Proof of Theorem 3.4

The aim of this Appendix is to provide a self-contained proof of Theorem 3.4, which was first proven in the papers [11] and [12]. We have also taken advantage of the exposition in [13].

Let $\ell > 2$ be a prime number, let (V, e) be a finite-dimensional symplectic space over $\mathbb{F}_\ell$ and $\Gamma = \mathrm{GSp}(V, e)$. In what follows M will be a subgroup of Γ which contains a transvection, such that V is a simple $\mathbb{F}_\ell[M]$-module.

Remark A.1.

- *For a set $U \subset V$, we will denote by $\langle U \rangle$ the vector space generated by U in V.*
- *For a vector $u \in V$ and a scalar $\lambda \in \mathbb{F}_\ell$, we denote by $T_u[\lambda] \in \Gamma$ the morphism $v \mapsto v + \lambda e(v, u)u$. For each transvection $\tau \in \Gamma$ there exist $u \neq 0$, $\lambda \neq 0$ such that $\tau = T_u[\lambda]$, and $\langle u \rangle = \ker(\tau - \mathrm{Id})$. If this is the case we will say that $\langle u \rangle$ is the direction of τ. Each nonzero vector in $\langle u \rangle$ shall be called a* direction vector *of τ.*
- *Given a group $G \subset \Gamma$, we will denote by $L(G)$ the set of vectors $u \in V$ such that there exists a transvection in G with direction vector u.*
- *We will say that a group $G \subset \Gamma$ fixes a vector space W if $\{g(w) : g \in G, w \in W\} \subset W$.*

The proof of Part iii) of Theorem 3.4 is quite simple and is based on the following observation.

Lemma A.2. *Let $G \subseteq \mathrm{GSp}(V)$ be a subgroup and R the subgroup of G generated by the transvections in G. Then for all $g \in G$, $r \in R$, $grg^{-1} \in R$.*

Proof. Note that if $T = T_v[\lambda] \in G$ is a transvection, then $gT_v[\lambda]g^{-1} = T_{gv}[\lambda]$ is also a transvection, which belongs to G, therefore also to R. Now if we have an element of R, say $T_1 \circ \cdots \circ T_k$ for certain transvections $T_1, \dots, T_k$, then $g(T_1 \circ \cdots \circ T_k)g^{-1} = (gT_1g^{-1}) \circ \cdots \circ (gT_kg^{-1})$ is the composition of transvections of G, therefore an element of R. □

Part i) of Theorem 3.4 is essentially Lemma 3.2 of [11]. Before proceeding to prove it, note the following elementary facts.

Lemma A.3. *Let G be a group that acts irreducibly on V, and let $W \subset V$ a nonzero vector space. Then $V = \sum_{g \in G} gW$.*

Proof. Let S be the set $S = \{g(w) : g \in G, w \in W\}$. Consider the vector space $\langle S \rangle$. This vector space is fixed by G, hence since G acts irreducibly on V it must coincide with V. □

Lemma A.4. *Let W be a vector subspace of V, and assume that it is fixed by a transvection $T = T_u[\lambda]$. Then either $u \in W$ or $u \in W^\perp$.*

Proof. Recall that, for all $v \in V$, $T(v) = v + \lambda e(v,u)u$. If $u \notin W$, the only way for T to fix W is that $e(w,u) = 0$ for all $w \in W$. □

Proof of Theorem 3.4, i)

Consider the action of R on V. The first step is to fix one simple nonzero R-submodule W contained in V (This always exists because V is finite-dimensional as an $\mathbb{F}_\ell$-vector space).

By Lemma A.3, we know that $V = \sum_{g \in M} gW$. Moreover, for $g_1, g_2 \in M$ it holds that $g_1W = g_2W$ if and only if $g_1H = g_2H$. Therefore we can write $V = \sum_{g \in M/H} gW$, where H is the stabilizer of W in M. The proof of i) boils down to prove that the sum is direct and orthogonal, that is, if $g_1H \neq g_2H$, then $g_1W \cap g_2W = 0$ and $g_1W \subset (g_2W)^\perp$. Equivalently, we will prove that for any $g \in M$, if $gW \neq W$, then $gW \cap W = 0$ and $gW \perp W$.

The first claim, namely $gW \neq W$ implies $gW \cap W = 0$ is easy. The key point is to note that for each $g \in M$, gW is also fixed by R. Take $r \in R$, $gw \in gW$. Then $rgw = g(g^{-1}rg)w \in gW$ since $g^{-1}rg \in R$ by Lemma A.2 and hence fixes W. Now it follows that $W \cap gW$ is fixed by R, and thus is an R-subrepresentation of W. But W is an simple R-module, hence since $W \cap gW \neq W$, it must follow that $gW \cap W = 0$.

To prove that $gW \neq W$ implies $gW \perp W$, we need to make first the following very important observation.

Claim A.5. *The set $L(M) \cap W$ generates W.*

Proof of Claim A.5. First let us see that $L(M) \cap W$ is nontrivial. Since any transvection in M fixes W by definition of W, it follows by Lemma A.4 that either its direction vector belongs to W, or else it is orthogonal to W, in which case the transvection acts trivially on W. But it cannot happen that all transvections in M act trivially on W. For, if a transvection T acts trivially on W, then for all $g \in M$, gTg^{-1} acts trivially on gW. But since $R = gRg^{-1}$ (because of Lemma A.2), then if all R acts trivially on W, it also acts trivially on gW. Now recall that $V = \sum_{g \in M} gW$. Then R would act trivially on V. But R contains at least a transvection, and this does not act trivially on V. We have a contradiction.

Hence $L(M) \cap W$ is non zero. But now observe that this set is fixed by the action of R, since the elements of M bring direction vectors into direction vectors. Therefore the vector space $\langle L(M) \cap W \rangle \subset W$ is fixed by the action of R. Since we are assuming W is an simple R-module, it follows that $\langle L(M) \cap W \rangle = W$ □

Now we are able to prove that if $gW \neq W$, then $gW \subset W^\perp$. Because of the previous claim, it suffices to show that, for any nonzero vector $w \in W$ which is the direction vector of a transvection in M, say T, $w \in (gW)^\perp$. Now recall that, since T fixes gW, by Lemma A.4 either $w \in gW$ or $w \in (gW)^\perp$. But $gW \cap W = 0$, so $w \in (gW)^\perp$. □

Before proving Part ii) of Theorem 3.4, we will introduce some notation.

Definition A.6. *Let $g \in M$. We will denote by R_g the subgroup of R generated* *by the transvections that act non-trivially on gW.*

The following lemma is Lemma 7 of [12].

Lemma A.7. *Let $g_1, g_2 \in M$ with $g_1H \neq g_2H$. Then the commutator $[R_{g_1}, R_{g_2}]$ is trivial.*

Proof. For $i = 1, 2$, let $T_i \in R_{g_i}$ be a transvection. We will see that they commute. By Lemma A.4 applied to g_iW, either T_i acts trivially on g_iW or its direction vector, say u_i, belongs to g_iW. By definition of R_{g_i} we have the second possibility. But because of Part i) of Theorem 3.4, for each $g \in M$ such that $g_iW \neq gW$, $g_iW \cap gW = 0$, hence $u_i \notin gW$. Therefore again by Lemma A.4 applied now to gW, it follows that T_i acts trivially on gW. Therefore T_1 and T_2 commute on each gW, since at least one of them acts trivially on it. Since $V = \bigoplus_{g \in M/H} gW$, it follows that they commute on all V. □

Proof of Theorem 3.4, ii).

Let $M/H = \{g_1H, \ldots, g_sH\}$, with $g_1 = \mathrm{Id}$. Define the map

$$P : \prod_{i=1}^{s} R_{g_i} \to R$$
$$(r_1, r_2, \ldots, r_s) \mapsto r_1 \cdot r_2 \cdot \cdots \cdot r_s.$$

Since by Lemma A.7 elements from the different R_{g_i} commute, this map is a group homomorphism. Let us see that it is also an isomorphism.

Assume that $r_1 \cdot r_2 \cdot \cdots \cdot r_s = \mathrm{Id}$, and that there is a certain r_j which is not the identity matrix. Then r_j must act nontrivially on a certain vector $v \in V$. Since the elements of R_{g_j} act trivially on the elements of g_iW for $i \neq j$ and $V = \bigoplus_{i=1}^{s} g_iW$, we can assume that $v \in g_jW$. But then the remaining r_i with $i \neq j$ act trivially on v and on $r_j(v)$. Therefore $\mathrm{Id}(v) = r_1 \cdot \cdots \cdot r_s(v) = r_j(v) \neq v$, which is a contradiction. To prove surjectivity, it suffices to note that each transvection T of M belongs to one of the R_{g_i}, (hence each element of R can be generated by elements of $\cup_i R_{g_i}$). And this holds because, since T fixes all the g_iW, the direction vector of T must either belong to g_iW or be orthogonal to it because of Lemma A.4, and since $V = \oplus_{i=1}^{s} g_iW$ it cannot be orthogonal to all the g_iW. Therefore we get that $R \simeq \prod_{i=1}^{s} R_{g_i}$.

Now we are going to apply the following result [23, Main Theorem]:

Theorem A.8. *Suppose $G \subset \mathrm{GL}(n, k)$ is an irreducible group generated by transvections. Suppose also that k is a finite field of characteristic $\ell > 2$, and that $n > 2$. Then G is conjugate in $\mathrm{GL}(n, k)$ to one of the groups $\mathrm{SL}(n, k_0)$, $\mathrm{Sp}(n, k_0)$ or $\mathrm{SU}(n, k_0)$, where k_0 is a subfield of k.*

Note that, if $n = 2$, the result is also true and well known (cf. [5, Section 252]).

Now R_{g_1} is generated by transvections, and acts irreducibly on W (because R acts irreducibly on W, and R_{g_1} is the group generated by all those transvections in M that act nontrivially on W). Therefore R_{g_1} is conjugated to $\mathrm{Sp}(W)$. Since all R_{g_i} are conjugated to R_{g_1}, the same holds for them. Therefore we have the isomorphism $R \simeq \prod_{i=1}^{s} \mathrm{Sp}(W)$.

Finally, we can view $H_1 = \prod_{i=1}^{s} \mathrm{GSp}(W) \simeq \prod_{i=1}^{s} \mathrm{GSp}(g_i W)$ as the subgroup of Γ fixing each $g_i W$ and, fixing a symplectic basis on each $g_i W$, we can view $H_2 = \mathrm{Sym}(M/H)$ as the subgroup of Γ that permutes the $g_i W$ by bringing the fixed symplectic basis of each $g_i W$ into the fixed symplectic basis of another $g_j W$. The group generated by H_1 and H_2 inside Γ, which is the group of elements of Γ that permute the $g_i W$, is the semidirect product $H_1 \rtimes H_2$.

Recall that $N_\Gamma(R) = \{g \in \Gamma : gRg^{-1} = R\}$. Note that $g \in N_\Gamma(R)$ if and only if for all transvections $T \in M$, $gTg^{-1} \in R$. Now, if $T = T_v[\lambda]$, it holds that $gTg^{-1} = T_{g(v)}[\lambda]$, and this transvection belongs to R if and only if it is a transvection of M, that is to say, if and only if $g(v) \in L(M)$. Therefore $g \in N_\Gamma(R)$ if and only if $g(L(M)) = L(M)$. Now since R is isomorphic to $\prod_{i=1}^{s} \mathrm{Sp}(g_i W)$, $L(M)$ is the disjoint union of the $g_i W$. And moreover, if W is an R-module and $g \in N_\Gamma(R)$, then R fixes gW. Therefore, if W is an simple R-module, then $gW \neq W$ implies that $gW \cap W = 0$. Thus if $g \in N_\Gamma(R)$, then g permutes the $g_i W$. In other words, $N_\Gamma(R) \subset \prod_{i=1}^{s} \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$. Reciprocally, each element of $\prod_{i=1}^{s} \mathrm{GSp}(W) \rtimes \mathrm{Sym}(G/H)$ carries elements of $\bigcup_i g_i W$ in elements of $\bigcup_i g_i W$, that is to say, carries $L(M)$ into $L(M)$, and therefore belongs to $N_\Gamma(R)$. □

This completes the proof of Part a) of Theorem 3.4.

Proof of Part b) of Theorem 3.4. Recall that (V, e) is a symplectic space over $\mathbb{F}_\ell$ and M a subgroup of $\Gamma := \mathrm{GSp}(V, e)$. M contains a transvection and V is a simple $\mathbb{F}_\ell[M]$-module by assumption. Furthermore R is the subgroup of M generated by the transvections in M, $0 \neq W \subset V$ is a simple $\mathbb{F}_\ell[R]$-module and $H = \mathrm{Stab}_M(W)$. We already proved that there is an orthogonal direct sum decomposition $V = \bigoplus_{g \in M/H} gW$. Furthermore $R \cong \prod_{g \in M/H} \mathrm{Sp}(W)$, $N_\Gamma(R) \cong \prod_{g \in M/H} \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$ and $R \subset M \subset N_\Gamma(R)$. Denote by $\varphi : N_\Gamma(R) \to \mathrm{Sym}(M/H)$ the projection.

Let $E/\mathbb{F}_\ell$ be a finite extension and $\rho : E^\times \to M \subset \mathrm{GL}(V)$ a representation of amplitude $\mathrm{amp}(\rho) \leq e$. Assume that $\ell > e \dim(V) + 1$. We have to prove that $\varphi(\rho(E^\times)) = \{1\}$.

Define $S := \ker(\varphi \circ \rho) \subset E^\times$. Then $[E^\times : S] \leq |M/H| \leq \dim(V)$, and this implies $e[E^\times : S] < \ell - 1$. Furthermore

$$\rho(S) \subset \ker(\varphi) \cong \prod_{g \in M/H} \mathrm{GSp}(gW).$$

Obviously $\rho(S)$ commutes with the centre

$$Z(\ker(\rho)) \cong \prod_{g \in M/H} \mathbb{F}_\ell^\times Id_{gW}$$

of $\ker(\rho)$. Now by [12, Lemma 3] $\rho(E^\times)$ commutes with $Z(\ker(\rho))$, because $e[E^\times : S] < \ell - 1$. It can easily be seen that the centralizer of $Z(\ker(\rho))$ in

$N_\Gamma(R)$ is equal to $\ker(\varphi) \cong \prod_{g\in M/H} \mathrm{GSp}(gW)$. Hence $\rho(E^\times) \subset \ker(\varphi)$ and this implies $\varphi \circ \rho(E^\times) = \{1\}$. □

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