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A MESHLESS MAXIMUM-ENTROPY METHOD FOR THE REISSNER-MINDLIN PLATE MODEL BASED ON A MIXED WEAK FORM

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ABSTRACT

Meshless methods, such as the Element Free Galerkin (EFG) method [1], hold various advantages over mesh-based techniques such as robustness in large-deformation problems and high continuity. The Reissner-Mindlin plate model is a particularly popular choice for simulating both thin and moderately thick structures.

It is well known in the Finite Element and Meshless literature that the simplest numerical treatments of the Reissner-Mindlin model lead to shear-locking which in turn produces erroneous results. This is due to the inability of the approximation functions to satisfy the Kirchhoff constraint in the thin-plate limit [2].

A recent advance in the area of meshless approximation schemes are Maximum-Entropy (MaxEnt) approximants [3]. MaxEnt schemes provide a weak Kronecker-delta property on convex node sets which allows the direct imposition of Dirichlet (essential) boundary conditions.

In this work, we derive a shear-locking free meshless method using MaxEnt approximants by considering a mixed weak form. We use a combination of MaxEnt approximants for the displacement variables and rotated Raviart-Thomas-Nedelec elements for the shear strain variables. This results in a saddle-point system which enforces the Kirchhoff constraint in the thin-plate limit. We show the performance of the method for a variety of test problems.

1 REISSNER-MINDLIN PROBLEM

We begin by recalling the Reissner-Mindlin plate problem (Fig 1.) in a normalised weak form: Find the transverse displacements and rotations of the fibres normal to the midplane $(z_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$ such that for all test functions $(y_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$:

$$\int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta)\, d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta)\, d\Omega = \int_{\Omega_0} g y_3 \, d\Omega \tag{1}$$

where the operators L and ϵ are defined as:

$$\epsilon(\mathbf{v}) = \frac{1}{2}\left((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T\right) \qquad L[\epsilon] \equiv D\left[(1-\nu)\epsilon + \nu \operatorname{tr}(\epsilon) I\right] \tag{2}$$

ν is Poisson's ratio, $\bar{t}$ is the plate thickness normalised with respect to in-plane dimension L, g is a loading constant, I is the identity tensor, tr is the trace operator, $D = E/12(1-\nu^2)$ is the bending modulus, $\lambda = E\kappa / 2(1+\nu)$ is the shear modulus and $\kappa = 5/6$ is a shear correction factor. We can write the above formulation as a combination of bilinear and linear forms as [2]:

$$a(z_3, \theta; y_3, \eta) = a_b(\theta; \eta) + \bar{t}^{-2} a_s(z_3, \theta; y_3, \eta) = (g, y_3) \tag{3}$$

The above variational formulation can be discretised using any appropriate PUM in the standard manner by constructing finite dimensional subspaces $\mathcal{V}_{3h} \subset \mathcal{V}_3$ and $\mathcal{R}_h \subset \mathcal{R}$.

It is well known that naive discretisations of the displacement formulation will result in shear locking. This locking phenomenon can be viewed as the inability of the numerical spaces to reproduce the Kirchhoff, or thin-plate limit as the thickness $\bar{t} \to 0$ [2].

A remedy for this problem is to reformulate the original displacement problem in a mixed form where the transverse shear strain is treated as an independent unknown in the variational problem. The mixed formulation is derived by introducing the scaled shear stress [2]:

$$\gamma = \lambda \bar{t}^{-2}(\nabla z_3 - \theta) \in \mathcal{S} \tag{4}$$

into (1) and forming the weak form of equation (4) using test functions $\psi \in \mathcal{S}$ results in the following mixed variational problem: Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ [2]:

$$\begin{aligned} &\int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta)\, d\Omega + \int_{\Omega_0} \gamma \cdot (\nabla y_3 - \eta)\, d\Omega = \int_{\Omega_0} g y_3 \, d\Omega \\ &\int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi \, d\Omega - \frac{\bar{t}^2}{\lambda} \int_{\Omega_0} \gamma \cdot \psi \, d\Omega = 0 \end{aligned} \tag{5}$$

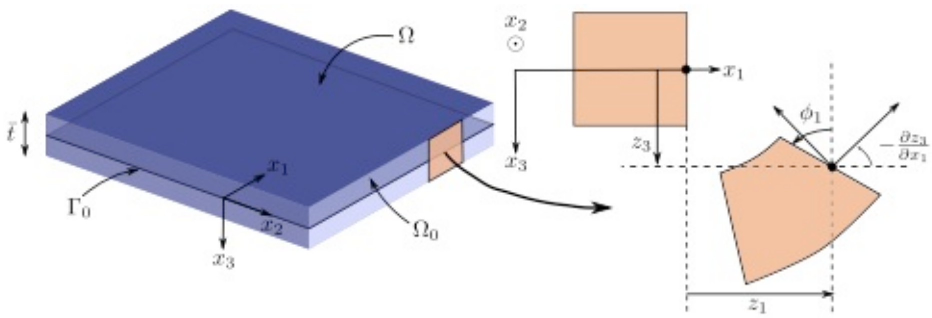

Fig 1. The Reissner-Mindlin Plate problem

3 MAXIMUM ENTROPY BASIS FUNCTIONS

In this work we use Maximum Entropy basis functions to discretise the generalised displacement fields θ and z_3. Maximum Entropy (MaxEnt) basis functions are one of the most recent developments in the construction of meshless approximation schemes [3]. A brief overview of their mathematical formulation and properties is given here.

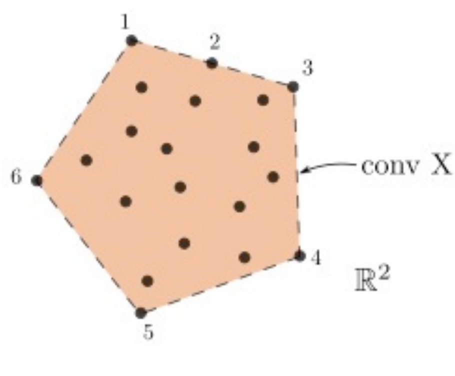

Fig. 2: Node set X and its convex hull $\text{conv}(X)$. Nodes $1,3,4,5,6$ on extreme vertices have the Kronecker delta property and node 2 has a 'weak' Kronecker delta property.

Consider a set of n nodes X. Each node i has a position $x_i \in \mathbb{R}^d$ associated with it. The convex hull of the set of nodes X is denoted $D \equiv \mathrm{conv}(X)$, see Fig 2. For a function $u(x): D \to \mathbb{R}$, the numerical approximation $u^h(x)$ can be written in terms of a set of shape functions $\phi_i : D \to \mathbb{R}$ and values u_i at the nodes X [3]:

$$u^h(x) = \sum_{i=1}^{n} \phi_i(x) u_i \tag{6}$$

Typically we wish the shape functions to satisfy the well known partition of unity condition as well as first order consistency:

$$\forall x \in D, \quad \sum_{i=1}^{N} \phi_i(x) = 1 \qquad \sum_{i=1}^{N} \phi_i(x) x_i = x_i \tag{7}$$

These two conditions alone are not enough to specify a unique approximation scheme. To this end, Shannon's concept of informational entropy is introduced. The Shannon entropy $S(p)$ of a discrete probability distribution with n events x_i with probabilities p_i is [3]:

$$S(p) = -\sum_{i=1}^{n} p_i \ln p_i \tag{8}$$

The principle of maximum entropy was proposed by Jaynes. It states that given a set of testable information about a probability distribution, the least statistically biased distribution is the one that maximises Shannon's measure of informational entropy whilst still satisfying the constraints imposed by the testable information. The maximum entropy principle can be directly applied to the problem of producing a unique approximation scheme by viewing the shape functions ϕ as being directly analogous to a discrete probability distribution. In the words of Sukumar [4], ``the basis function value $\phi_i(x)$ is viewed as being the probability of influence of a node X_i at a point x":

$$p \to \phi \tag{9}$$

This analogy between shape functions and probabilities naturally implies that the shape functions are always positive:

$$\phi_i(x) \geq 0 \quad \forall x \in D, i = 1, \ldots, n \tag{10}$$

Therefore we can find a unique set of shape functions by maximising the entropy $S(\phi)$ subject to the constraints outlined above [3]:

$$\max_{\phi \in \mathbb{R}^d_+} \left(S(\phi) = -\sum_{i=1}^{n} \phi_i \ln \phi_i \right) \qquad \sum_{i=1}^{n} \phi_i = 1 \quad \forall\, x \in D \qquad \sum_{i=1}^{n} \phi_i x_i = x \quad \forall\, x \in D \tag{11}$$

The objective function $-S(\phi)$ is strictly convex on $\mathbb{R}^N_+$ and the two constraints are affine, so the above problem can be solved using standard duality methods from the field of convex optimisation.

The primary advantage of the MaxEnt scheme over the widely used MLS scheme is that it produces shape functions with a weak Kronecker-delta property [3]. This makes the imposition of Dirichlet boundary conditions trivial as in the Finite Element method without resorting to modified variational forms [3].

4 ROTATED RAVIART-THOMAS-NEDELEC ELEMENTS

To discretise the shear strain field γ we use rotated Raviart-Thomas-Nedelec elements (sometimes referred to as edge elements). We make this choice as the shear strain field lies in the Sobelov space $H(\text{rot},\Omega)$ which is the space of functions with square integrable rot [2]. We discretise the shear strains on a reference triangular element $\hat{K}$ as:

$$\gamma_h(\hat{x}_1,\hat{x}_2) = \sum_{i=1}^{3} \mathbf{N}_i \gamma_i = \left[\begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \end{pmatrix} \begin{pmatrix} \hat{x}_2 \\ 1-\hat{x}_1 \end{pmatrix} \begin{pmatrix} 1-\hat{x}_2 \\ \hat{x}_1 \end{pmatrix} \right] \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix} = \mathbf{N}_\gamma \gamma \quad \forall (\hat{x}_1,\hat{x}_2) \in \hat{K} \tag{12}$$

4 RESULTS

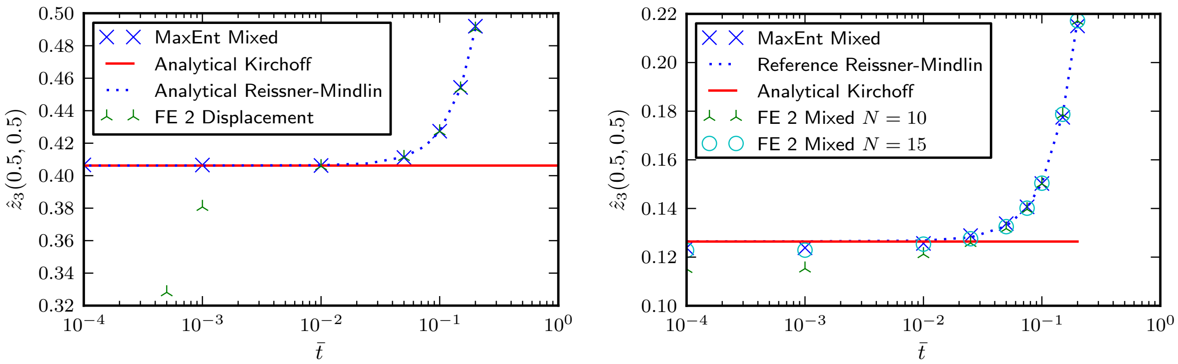

Fig 3: Left: Normalised deflection of center point of SSSS plate for various methods. Right: Normalised deflection of center point of CCCC plate for various methods.

In Fig. 3 we show some results for square geometry plates with uniform loading under simply supported and fully clamped boundary conditions using a variety of analytical and numerical solution techniques. We show the analytical Reissner Mindlin and Kirchhoff solutions for both problems. The correct behaviour of any numerical method based upon the Reissner assumptions should be to track the Reissner analytical solution before converging to the Kirchhoff solution in the thin plate limit. Typically displacement based numerical methods converge to zero in any displacement based formulation (ie. FE 2 Displacement). We can clearly see that the proposed MaxEnt Mixed method correctly converges to the Kirchhoff solution in the thin plate limit for both sets of boundary conditions.

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