Imperial College London

Simulation of Shear Deformable Plates using Meshless Maximum Entropy Basis Functions XFEM 2011

J.S. Hale*, P.M. Baiz Villafranca

29th June 2011

Introduction

Aim of research project

Develop a meshless method for the simulation of shear plates using Maximum Entropy meshless basis functions that is free of shear locking

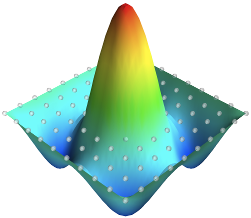


Figure: 6th free vibration mode of SSSS plate

Problem Domain

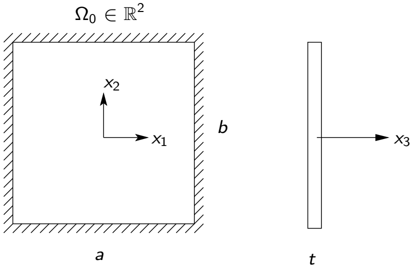
Why plate theories?

Reduce full 3D elasticity equations to 2D problem by making appropriate geometrical, mechanical and kinematical assumptions

Problem Domain

Why plate theories?

Reduce full 3D elasticity equations to 2D problem by making appropriate geometrical, mechanical and kinematical assumptions



Imperial College London

Weak Form

Definition (Reissner-Mindlin Problem)

Find the transverse deflection and rotations $(z_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$ such that:

Weak Form

Definition (Reissner-Mindlin Problem)

Find the transverse deflection and rotations $(z_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$ such that:

$$\begin{aligned} \bar{t}^3 \int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \lambda \bar{t} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta) \; d\Omega \\ = \int_{\Omega_0} p_3 y_3 \; d\Omega \quad \forall (y_3, \eta) \in \mathcal{V}_3 \times \mathcal{R} \end{aligned} \tag{1a}$$

$$\epsilon(\theta) = \frac{1}{2}\left(\nabla\theta + (\nabla\theta)^T\right) \tag{1b}$$

$$L[\epsilon] \equiv D\left[(1-\nu)\epsilon + \nu \operatorname{tr}(\epsilon) I\right] \tag{1c}$$

$$D = \frac{E}{12(1-\nu^2)} \tag{1d}$$

$$\lambda = \frac{Ek}{2(1+\nu)} \tag{1e}$$

Weak Form

Definition (Reissner-Mindlin Problem)

Find the transverse deflection and rotations $(z_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$ such that:

$$\bar{t}^3 \int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \lambda \bar{t} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} p_3 y_3 \; d\Omega \quad \forall (y_3, \eta) \in \mathcal{V}_3 \times \mathcal{R} \tag{1a}$$

$$\epsilon(\theta) = \frac{1}{2}\left(\nabla\theta + (\nabla\theta)^T\right) \tag{1b}$$

$$L[\epsilon] \equiv D\left[(1-\nu)\epsilon + \nu \mathrm{tr}(\epsilon) I\right] \tag{1c}$$

$$D = \frac{E}{12(1-\nu^2)} \tag{1d}$$

$$\lambda = \frac{Ek}{2(1+\nu)} \tag{1e}$$

Locking

Kirchoff classical plate theory and Reissner-Mindlin plate theory both model the same thing! So as $\bar{t} \to 0$ we would expect that the Kirchoff constraint holds.

Locking

Kirchoff classical plate theory and Reissner-Mindlin plate theory both model the same thing! So as $\bar{t} \rightarrow 0$ we would expect that the Kirchoff constraint holds.

Definition (Kirchoff Constraint)

$$\lim_{\bar{t} \rightarrow 0} \nabla z_3 - \theta = 0$$

A naïve Finite Element approach

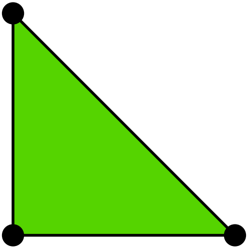


Figure: Use Linear Lagrangian Elements for $V_3h \subset V_3$ and $R_3h \subset R$

Locking: Numerical Demonstration

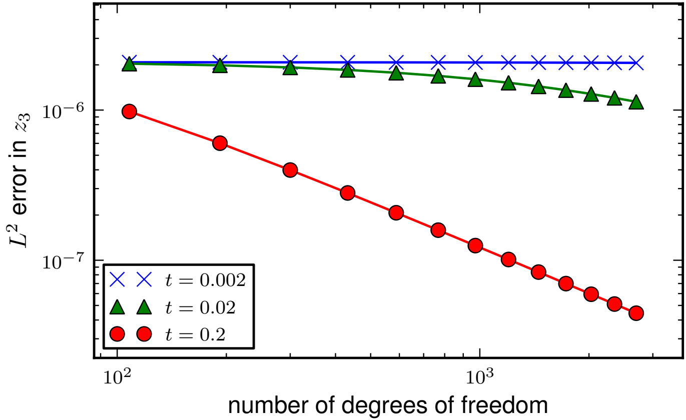


Figure: Locking in L^2 norm using Linear Lagrangian elements

Existing work

1. Increase polynomial order

Take MLS approximation:

$$u_h(x) = p^T(x) \cdot a(x)$$

Increase monomial order m of $p^T(x)$ until locking is alleviated. Typically $m > 3$.

Imperial College London

Existing work

1. Increase polynomial order

Take MLS approximation:

$$u_h(x) = p^T(x) \cdot a(x)$$

Increase monomial order m of $p^T(x)$ until locking is alleviated. Typically $m > 3$.

- Advantages: Simple to implement, eliminates locking acceptably well.

Existing work

1. Increase polynomial order

Take MLS approximation:

$$u_h(x) = p^T(x) \cdot a(x)$$

Increase monomial order m of $p^T(x)$ until locking is alleviated. Typically $m > 3$.

- Advantages: Simple to implement, eliminates locking acceptably well.
- Disadvantages: MLS has no Kronecker delta property. Very large support sizes needed making high order basis expensive. Never totally eliminates locking.

Existing work

2. Matching fields approach

Approximate rotations using derivatives of basis functions used to approximate transverse displacements (Donning and Liu 1998). Satisfies Kirchoff's constraint exactly.

Existing work

2. Matching fields approach

Approximate rotations using derivatives of basis functions used to approximate transverse displacements (Donning and Liu 1998). Satisfies Kirchoff's constraint exactly.

- Advantages: Simple to implement, does 'work'

Existing work

2. Matching fields approach

Approximate rotations using derivatives of basis functions used to approximate transverse displacements (Donning and Liu 1998). Satisfies Kirchoff's constraint exactly.

- Advantages: Simple to implement, does 'work'
- Disadvantages: Very ill-conditioned stiffness matrix, approximation is rank-deficient, rotations no longer approximated with PU (Tiago and Leitao 2005)

Existing work

3. Reduced integration

Stabilised Conforming Nodal Integration method (Wang and Chen 2004). Uses stabilisation with cell smoothing technique to construct locking free formulation.

Existing work

3. Reduced integration

Stabilised Conforming Nodal Integration method (Wang and Chen 2004). Uses stabilisation with cell smoothing technique to construct locking free formulation.

- Advantages: Eliminates locking. Solid theoretical approach.

Existing work

3. Reduced integration

Stabilised Conforming Nodal Integration method (Wang and Chen 2004). Uses stabilisation with cell smoothing technique to construct locking free formulation.

- Advantages: Eliminates locking. Solid theoretical approach.
- Disadvantages: Must use stabilisation method to ensure convergence. Second order consistency required. Some cell structure (Voronoi/Delaunay) still required. Comparatively complicated.

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

Features

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

Features

- Easy imposition of Dirichlet boundary conditions

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

Features

- Easy imposition of Dirichlet boundary conditions
- Elimination of shear locking in the thin plate limit

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

Features

- Easy imposition of Dirichlet boundary conditions
- Elimination of shear locking in the thin plate limit
- Only first-order consistency required

This work

Description

A combined Meshless-FE approach using Meshless Maximum Entropy approximants and Nédélec's/Rotated Raviart-Thomas Finite Elements to eliminate shear locking.

Features

- Easy imposition of Dirichlet boundary conditions
- Elimination of shear locking in the thin plate limit
- Only first-order consistency required
- Potential for straightforward blending with FE

Mixed Weak Form

Treat the shear stress γ as an independent variable:

$$\gamma = \lambda \bar{t}(\nabla z_3 - \theta) \in \mathcal{S}$$

Definition (Mixed Reissner Mindlin Problem)

Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that:

Mixed Weak Form

Treat the shear stress γ as an independent variable:

$$\gamma = \lambda \bar{t}(\nabla z_3 - \theta) \in \mathcal{S}$$

Definition (Mixed Reissner Mindlin Problem)

Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that:

$$\bar{t}^3 \int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \int_{\Omega_0} \gamma \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} p_3 y_3 \; d\Omega \tag{2a}$$

$$\lambda \bar{t} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi \; d\Omega - \int_{\Omega_0} \gamma \cdot \psi \; d\Omega = 0 \quad \forall (y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S}) \tag{2b}$$

Imperial College London

Mixed Weak Form

Treat the shear stress γ as an independent variable:

$$\gamma = \lambda \bar{t}(\nabla z_3 - \theta) \in \mathcal{S}$$

Definition (Mixed Reissner Mindlin Problem)

Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that:

$$\bar{t}^3 \int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \int_{\Omega_0} \gamma \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} p_3 y_3 \; d\Omega \tag{2a}$$

$$\lambda \bar{t} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi \; d\Omega - \int_{\Omega_0} \gamma \cdot \psi \; d\Omega = 0 \quad \forall (y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S}) \tag{2b}$$

Mixed Weak Form

Treat the shear stress γ as an independent variable:

$$\gamma = \lambda \bar{t}(\nabla z_3 - \theta) \in \mathcal{S}$$

Definition (Mixed Reissner Mindlin Problem)

Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that:

$$\bar{t}^3 \int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \; d\Omega + \int_{\Omega_0} \gamma \cdot (\nabla y_3 - \eta) \; d\Omega = \int_{\Omega_0} p_3 y_3 \; d\Omega \tag{2a}$$

$$\lambda \bar{t} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi \; d\Omega - \int_{\Omega_0} \gamma \cdot \psi \; d\Omega = 0 \quad \forall (y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S}) \tag{2b}$$

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

Lemma

For fixed $\bar{t}$:

$$\mathcal{S} = H(\mathrm{curl}, \Omega_0)$$

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

Lemma

For fixed $\bar{t}$:

$$\mathcal{S} = H(\text{curl}, \Omega_0)$$

Definition

$$H(\text{curl}, \Omega_0) := \left\{\gamma \in [L^2(\Omega_0)]^2 \mid \text{curl}\,\gamma \in L^2(\Omega_0)\right\}$$

$$\text{curl}(\gamma) = \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}$$

Imperial College London

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

Lemma

For fixed $\bar{t}$:

$$\mathcal{S} = H(\text{curl}, \Omega_0)$$

Definition

$$H(\text{curl}, \Omega_0) := \left\{ \gamma \in [L^2(\Omega_0)]^2 \mid \text{curl}\,\gamma \in L^2(\Omega_0) \right\}$$

$$\text{curl}(\gamma) = \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}$$

Imperial College London

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

Lemma

For fixed $\bar{t}$:

$$\mathcal{S} = H(\text{curl}, \Omega_0)$$

Definition

$$H(\text{curl}, \Omega_0) := \left\{ \gamma \in [L^2(\Omega_0)]^2 \mid \text{curl}\,\gamma \in L^2(\Omega_0) \right\}$$

$$\text{curl}(\gamma) = \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}$$

The space $\mathcal{S}$

On first inspection we see that the weak form contains no derivatives of γ or ψ. We therefore might assume that $\mathcal{S} = [L^2(\Omega_0)]^2$.

Lemma

For fixed $\bar{t}$:

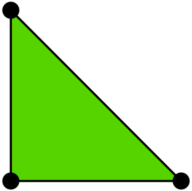
$$\mathcal{S} = H(\text{curl}, \Omega_0)$$

Definition

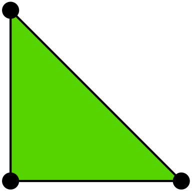
$$H(\text{curl}, \Omega_0) := \left\{ \gamma \in [L^2(\Omega_0)]^2 \mid \text{curl}\, \gamma \in L^2(\Omega_0) \right\}$$

$$\text{curl}(\gamma) = \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}$$

The space $\mathcal{S}_h$



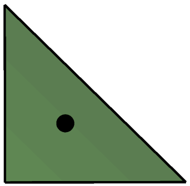
(a) Element for γ_{xz}



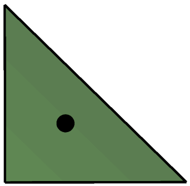
(b) Element for γ_{yz}

We cannot use continuous C^0 Lagrangian elements (or indeed standard meshless basis functions) to approximate each component ie. $\gamma_h \in [H^1_h(\Omega_0)]^2$. This approximation would be too continuous.

The space $\mathcal{S}_h$



(c) Element for γ_{xz}



(d) Element for γ_{yz}

We cannot use discontinuous C^{-1} Lagrangian elements to approximate each component ie. $\gamma_h \in [L^2_h(\Omega_0)]^2$. This approximation would be too discontinuous

Nédélec's Element of First Kind

We need continuity for γ only in the tangential directions $\hat{\tau}_i$ to the edges $\hat{e}_i$. This element is a vector-valued element with three vector valued shape functions.

Nédélec's Element of First Kind

We need continuity for γ only in the tangential directions $\hat{\tau}_i$ to the edges $\hat{e}_i$. This element is a vector-valued element with three vector valued shape functions.

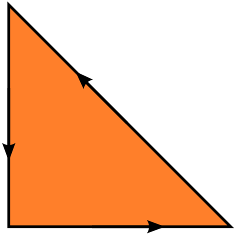


Figure: Nedelec Element of First Kind on reference element $\hat{K}$. Also known as Rotated Raviart-Thomas Element.

Nédélec's Element of First Kind

Construction

- Set of polynomials: $\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \hat{x}_2 \\ -\hat{x}_1 \end{bmatrix} \right\rangle$
- Degrees of freedom: $\int_{\hat{e}_i} (\hat{\gamma} \cdot \hat{t}) \, d\hat{s}$

Performing all the calculations gives the shape functions on a reference element $\hat{K}$.

Nédélec's Element of First Kind on $\hat{K}$

$$\gamma = \left[\begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 \end{pmatrix} \begin{pmatrix} -\hat{x}_2 \\ \hat{x}_1 - 1 \end{pmatrix} \begin{pmatrix} 1 - \hat{x}_2 \\ \hat{x}_1 \end{pmatrix} \right] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\boldsymbol{\gamma} = \mathbf{N}_\gamma \mathbf{a}$$

Constructing $\mathcal{R}_h$ and $\mathcal{V}_{3h}$ with MaxEnt

To construct approximations for θ and z_3 we use Maximum Entropy (MaxEnt) meshless basis functions (Arroyo, Ortiz 2006) (Sukumar, Wright 2007)

Advantages

- 'Weak' Kronecker Delta property on convex node sets
- Seamless and straightforward blending with Finite Elements

Disadvantages

- Trickier implementation
- Non-trivial (although possible) to extend to second-order or higher intrinsic consistency

Constructing $\mathcal{R}_h$ and $\mathcal{V}_{3h}$ with MaxEnt

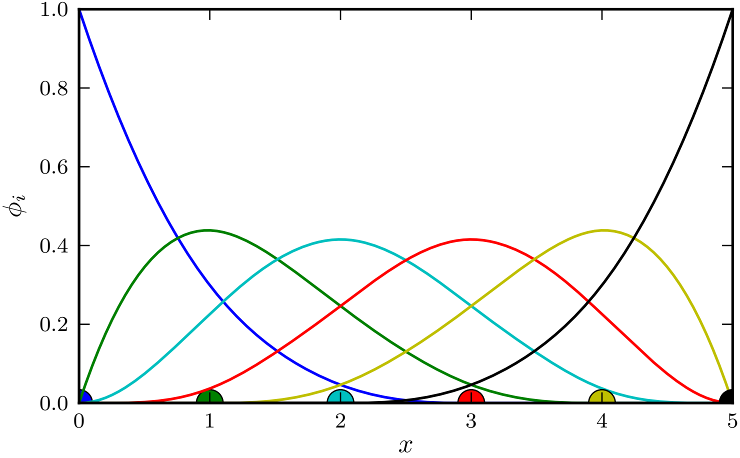


Figure: MaxEnt basis functions for 6 evenly distributed nodes

Discrete Formulation

$$\left[\begin{array}{cc|c} \mathbf{K}_b & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \mathbf{C} \\ \hline \multicolumn{2}{c|}{\mathbf{C}^T} & -\mathbf{V} \end{array}\right] \left\{\begin{array}{c} \theta \\ \hline z_3 \\ \hline \gamma \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline f_{z3} \\ \hline 0 \end{array}\right\} \tag{3a}$$

$$\mathbf{K}_b = \bar{t}^3 \int_{\Omega_0} \mathbf{B}_b^T \mathbf{D}_b \mathbf{B}_b \, d\Omega \tag{3b}$$

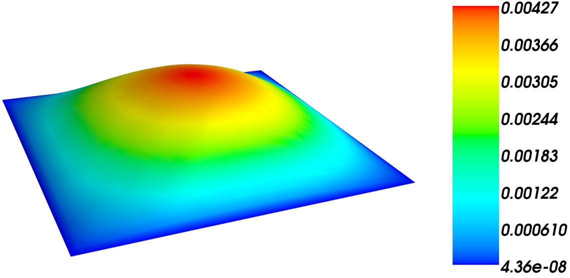
$$\mathbf{C} = \int_{\Omega_0} \mathbf{N}_\gamma^T \mathbf{D}_s \mathbf{B}_s \, d\Omega \tag{3c}$$

$$\mathbf{V} = \frac{1}{\bar{t}} \int_{\Omega_0} \mathbf{N}_\gamma^T \mathbf{N}_\gamma \, d\Omega \tag{3d}$$

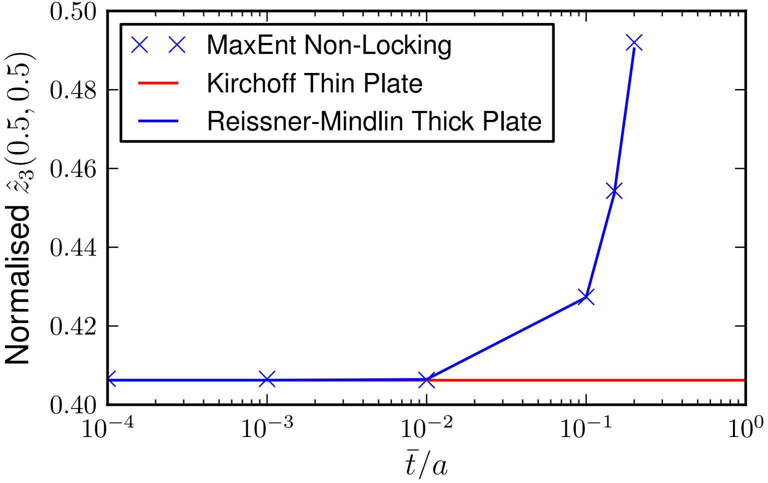
$$f_{z3} = p_3 \int_{\Omega_0} \phi^T \, d\Omega \tag{3e}$$

Imperial College London

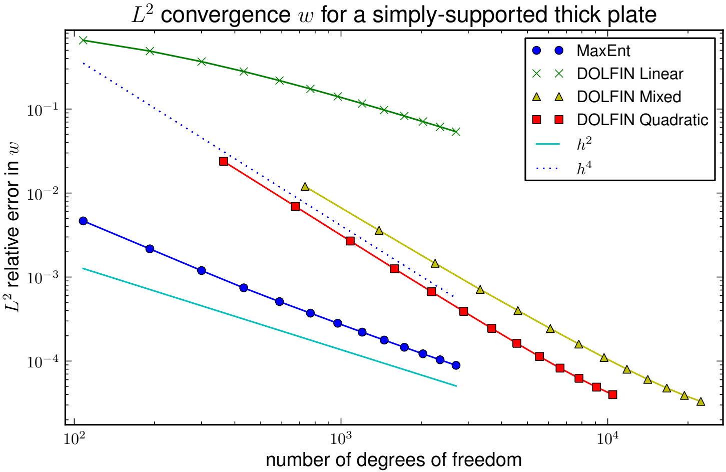
Results - Bending



Results - No locking



Results - Convergence



Thanks for listening. Questions?

Extra Slides

Mechanical Assumptions

Plane Stress

$$\sigma_{33} = 0$$

Imperial College London

Kinematical Assumptions

Figure: TODO: Graphic showing displacement of Mindlin plate

We can write the displacement vector $u : \Omega \rightarrow \mathbb{R}^3$ in the form:

$$u(x_1, x_2, x_3) = \begin{Bmatrix} z_1(x_1, x_2) - \theta_1(x_1, x_2)x_3 \\ z_2(x_1, x_2) - \theta_2(x_1, x_2)x_3 \\ z_3(x_1, x_2) \end{Bmatrix}$$

Locking

Substitute in our Galerkin subspaces $\mathcal{V}_{3h} \subset H^1_0(\Omega)$ and $\mathcal{R}_{3h} \subset H^1_0(\Omega)$ built using our linear polynomial Lagrangian elements:

$$\sum_{i=1}^{n} \nabla N_i \; z_{3i} - \sum_{i=1}^{n} N_i \; \theta_i = 0$$

Problem!

$$\nabla N_i = \mathrm{const.} \quad \forall x$$

but we have specified clamped boundary conditions $\mathcal{R}_{3h} \subset H^1_0(\Omega)$. The only way the above can be satisfied is if:

$$z_{3i} = \theta_i = 0 \quad \forall x \in \Omega$$

What is curl?

Take a two component vector field $\gamma = (\gamma_1, \gamma_2)$. The curl operator can be defined as:

$$\text{curl}(\gamma) = \frac{\partial \gamma_2}{\partial x_1} - \frac{\partial \gamma_1}{\partial x_2}$$

In $\mathbb{R}^2$ curl is directly related to the familiar div operator. Setting:

$$\mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{curl}\,\gamma \equiv \text{div}\,\mathbf{R}\gamma$$

The curl operator crops up frequently in electromagnetic problems.

What is $H(\mathrm{curl}, \Omega_0)$?

It is the Sobelov space of functions with square integrable curl:

$$H(\mathrm{curl}, \Omega_0) := \left\{ \gamma \in [L^2(\Omega_0)]^2 \mid \mathrm{curl}\, \gamma \in L^2(\Omega_0) \right\} \tag{4}$$

So as in all conforming Galerkin methods we must construct a conforming subspace:

$$\mathcal{S}_h \subset \mathcal{S} \tag{5}$$

This requires continuity in the tangential component of γ across the element edges. Problem: Standard Lagrangian elements are too continuous.

Maximum Entropy

A quick reminder from elementary statistics: A discrete probability distribution associates N outcomes $\mathbf{X} = \{X_1, X_2, X_3, \dots, X_N\}^T$ with N probabilities $\mathbf{p} = \{p_1, p_2, p_3, \dots, p_N\}^T$.

Maximum Entropy

A quick reminder from elementary statistics: A discrete probability distribution associates N outcomes $\mathbf{X} = \{X_1, X_2, X_3, \ldots, X_N\}^T$ with N probabilities $\mathbf{p} = \{p_1, p_2, p_3, \ldots, p_N\}^T$.

Definition

The **least statistically biased** discrete probability distribution is the one that **maximises** Shannon's informational entropy (1948) $H(\mathbf{p})$:

$$H(\mathbf{p}) = \langle - \ln p \rangle = - \sum_{i=1}^{n} p_i \ln p_i \qquad (6)$$

and satisfies all prior known information about the probability distribution.

Imperial College London

Coin Example 1

I have a coin with two outcomes $\mathbf{X} = \{X_1, X_2\}^T$ associated with two probabilities $\mathbf{p} = \{p_1, p_2\}^T$. I have no other testable information.

Coin Example 1

I have a coin with two outcomes $\mathbf{X} = \{X_1, X_2\}^T$ associated with two probabilities $\mathbf{p} = \{p_1, p_2\}^T$. I have no other testable information.

What are the least biased probabilities $\mathbf{p}^*$ according to Shannon's measure of informational entropy? I know that:

Coin Example 1

I have a coin with two outcomes $\mathbf{X} = \{X_1, X_2\}^T$ associated with two probabilities $\mathbf{p} = \{p_1, p_2\}^T$. I have no other testable information.

What are the least biased probabilities $\mathbf{p}^*$ according to Shannon's measure of informational entropy? I know that:

$$p_1 + p_2 = 1 \tag{7a}$$

$$p_1 \geq 0,\ \ p_2 \geq 0 \tag{7b}$$

Imperial College London

Coin Example 1

I have a coin with two outcomes $\mathbf{X} = \{X_1, X_2\}^T$ associated with two probabilities $\mathbf{p} = \{p_1, p_2\}^T$. I have no other testable information.
What are the least biased probabilities $\mathbf{p}^*$ according to Shannon's measure of informational entropy? I know that:

$$p_1 + p_2 = 1 \tag{7a}$$

$$p_1 \geq 0,\ p_2 \geq 0 \tag{7b}$$

and that:

$$\mathbf{p}^* = \arg\max_{p_1, p_2} \ -\sum_{i=1}^{n} p_i \ln p_i \tag{7c}$$

Coin Example 2

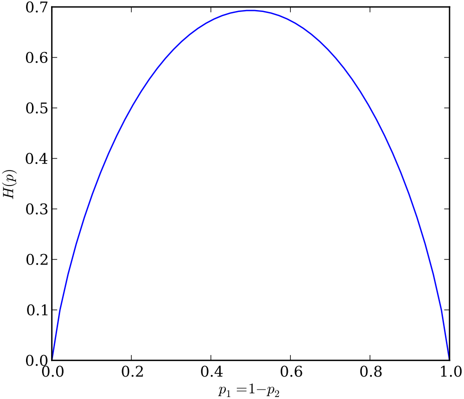


Figure: Maximum $\mathbf{p}^* = \{0.5, 0.5\}$

The Connection

Probability and Basis Functions

Arroyo and Ortiz[a] as well as Sukumar and Wright[b] had the idea to view the basis functions ϕ_i as a **discrete probability distribution** p_i:

$$\phi_i \Longleftrightarrow p_i \tag{8}$$

[a] Local maximum-entropy approximation schemes: a seamless bridge between finite elements and meshfree methods, M. Arroyo M. Ortiz 2006 DOI: 10.1002/nme.1534

[b] Overview and construction of meshfree basis functions: From moving least squares to maximum entropy approximants N. Sukumar, R. W. Wright 2007 DOI: 10.1002/nme.1885

So in other words, the shape function $\phi_i(x)$ is the probability that the approximate solution $u_h(x)$ takes the value of the ith solution node.

Constraints

Probabilities can only be positive thus we have:

$$\phi_i \geq 0 \quad \forall i \tag{9}$$

Constraints

Probabilities can only be positive thus we have:

$$\phi_i \geq 0 \quad \forall i \tag{9}$$

We want to construct a partition of unity:

$$\sum_{i \in \mathcal{S}} \phi_i(\mathbf{x}) = 1 \tag{10}$$

Constraints

Probabilities can only be positive thus we have:

$$\phi_i \geq 0 \quad \forall i \tag{9}$$

We want to construct a partition of unity:

$$\sum_{i \in \mathcal{S}} \phi_i(\mathbf{x}) = 1 \tag{10}$$

We also want to be able to interpolate linear functions exactly:

$$\sum_{i \in \mathcal{S}} \phi_i(\mathbf{x})\mathbf{x}_i = \mathbf{x} \tag{11}$$

Imperial College London

Weighting functions

Finally, we want the nodes nearest to the point $\mathbf{x}$ to have a greater influence over the approximation than those further away.

Imperial College London

Weighting functions

Finally, we want the nodes nearest to the point $\mathbf{x}$ to have a greater influence over the approximation than those further away.

We introduce a weighting function w_i for each node i.

Weighting functions

Finally, we want the nodes nearest to the point $\mathbf{x}$ to have a greater influence over the approximation than those further away.

We introduce a weighting function w_i for each node i.

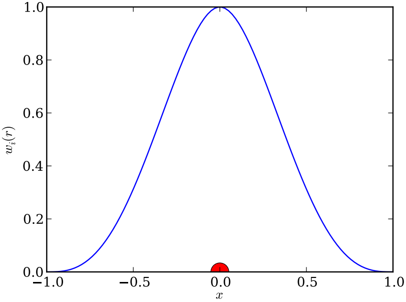


Figure: Quartic Spline Weight Function $w(r) = 1 - 6r^2 + 8r^3 - 3r^4$

Convex Optimisation Problem

We can show that the above is a constrained convex optimisation problem, and thus there is a unique minimum ϕ^*:

$$\phi^* = \arg \min_{\phi \in \mathbb{R}^n_+} \sum_{i \in \mathcal{S}} \phi_i \ln \left(\frac{\phi_i}{w_i} \right) \tag{12a}$$

subject to the constraints:

$$\sum_{i \in \mathcal{S}} \phi_i(\mathbf{x}) = 1 \tag{12b}$$

$$\sum_{i \in \mathcal{S}} \phi_i(\mathbf{x}) \mathbf{x}_i = \mathbf{x} \tag{12c}$$