XFEM 2011
II European Conference on eXtended Finite Element
Cardiff University, United Kingdom
Cardiff, June 29-30 July 1, 2011

Simulation of Shear Deformable Plates using Meshless Maximum Entropy Basis Functions

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1 Introduction

Plate theories such as the Mindlin-Reissner, or shear deformable, and Kirchoff, or classical, have seen varied and wide use throughout engineering practice to simulate the mechanical response of structures that are far larger in their planar dimensions than through their thickness [2]. Meshless methods such as the EFG method [1] have been used to construct approximation spaces for the solution of the plate and shell governing equations.

This extended abstract is designed to give background information to supplement the main presentation. It begins with outlining the more mathematical aspects of the Reissner-Mindlin model, its relationship with the Kirchoff model and the well-known shear locking problem. It then goes on to give a review of existing methods to alleviate locking in the literature for both FE and Meshless methods. Finally an overview of the meshless Maximum Entropy shape functions is given [9, 10]. Initial results to be presented suggest that Maximum Entropy shape functions have the potential to match the performance of MLS based methods whilst being significantly easier to implement.

2 The Reissner-Mindlin Plate Model

The Reissner-Mindlin Plate Model [2] can be derived from the full 3D equations of elasticity by making a set of geometrical and kinematical assumptions. The key reason behind making these assumptions is that we can trade integration over the domain $\Omega \in \mathbb{R}^3$ for a cheaper integration over the domain $\Omega_0 \in \mathbb{R}^2$ representing the mid-plane of the plate. The Reissner-Mindlin assumptions are as follows [2]:

Geometry The elasticity problem domain Ω has one thin dimension, in this case in the x_3 direction. The plate mid-surface is then described by the domain $\Omega_0 \subset \mathbb{R}^2$ and the thickness by a function $t : \Omega_0 \to (0, \infty)$. Therefore the whole domain $\Omega \subset \mathbb{R}^3$ can be written as [2]:

$$\Omega \equiv \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, (x_1, x_2) \in \Omega_0, x_3 \in \left[-t(x_1, x_2)/2, t(x_1, x_2)/2\right]\right\} \tag{1}$$

Mechanics Plane stress assumptions apply $\sigma_{33} = 0$

Kinematics We assume that the displacement vector $u : \Omega \to \mathbb{R}^3$ can be written in the form [2]:

$$u(x_1, x_2, x_3) = (z_1(x_1, x_2) - \theta_1(x_1, x_2)x_3,\, z_2(x_1, x_2) - \theta_2(x_1, x_2)x_3,\, z_3(x_1, x_2)) \tag{2}$$

Weak Form of the Reissner-Mindlin Plate Problem: Find the transverse deflection and rotations $(z_3, \theta) \in \mathcal{V}_3 \times \mathcal{R}$ such that [2]:

$$\bar{t}^3 \int_{\Omega_0} L\varepsilon(\theta) : \varepsilon(\eta)\, d\Omega + \lambda t \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta)\, d\Omega = \int_{\Omega_0} p_3 y_3\, d\Omega \;\; \forall (y_3, \eta) \in \mathcal{V}_3 \times \mathcal{R} \tag{3a}$$

$$L[\varepsilon] \equiv D\left[(1-\nu)\varepsilon + \nu \operatorname{tr}(\varepsilon) I\right] \tag{3b}$$

$$D = \frac{E}{12(1-\nu^2)} \tag{3c}$$

$$\lambda = \frac{Ek}{2(1+\nu)} \tag{3d}$$

where ν is Poisson's ratio, E is Young's modulus and $k = 5/6$ is a shear correction factor accounting for the difference in shear energy produced by the assumed displacement field and the full equations of elasticity. The exact specification of the Dirichlet or essential boundary conditions on Γ_d leads to different function spaces for $\mathcal{V}_3$ and $\mathcal{R}$.

3 Shear Locking in the Thin Plate Limit

The thin plate limit describes the asymptotic behaviour of the Reissner-Mindlin plate model as $t \to 0$. On an intuitive level, we would assume that given that the Reissner-Mindlin theory describes thick plate behaviour, as $\bar{t} \to 0$ the solution for a given $\bar{t}$ $(z_3^{\bar{t}}, \theta^{\bar{t}})$ should tend towards that of a model describing thin plate behaviour, such as the Kirchoff model, with solution (z_3^0, θ^0).

We begin by re-writing Reissner-Mindlin plate problem in the following equivalent bilinear form with loading p_3 scaled with $\bar{t}^3$ to ensure the problem is well behaved as $\bar{t} \to 0$: Find the transverse deflection and rotations $(z_3, \theta) \in (\mathcal{V}_3, \mathcal{R})$ such that [2]:

$$a(\theta, \eta) + \lambda \bar{t}^{-2} b(\nabla z_3 - \theta, \nabla y_3 - \eta) = f(y_3) \;\; \forall (y_3, \eta) \in \mathcal{V}_3 \times \mathcal{R} \tag{4}$$

The locking problem can be demonstrated as follows [2]. The obvious (but naïve) choice is to introduce the approximation space for the solution (z_3^h, θ^h) of the piecewise linear Lagrangian finite elements P_1 such that $z_3^h \in \mathcal{V}_3^h \subset \mathcal{V}_3$ and $\theta^h \in \mathcal{R}^h \subset \mathcal{R}$. The clearest case is if we apply hard clamped boundary conditions on Γ_d such that $z_3 = 0\, \forall x \in \Gamma_d$ and $\theta = 0\, \forall x \in \Gamma_d$. We know that in the thin plate limit $\bar{t} \to 0$ that $\nabla \theta_h = z_{3h}$. If $\theta_h = 0$ on the boundary, then using P_1 elements $\nabla \theta_h$ must be zero everywhere in Ω. Thus $z_{3h} = 0$ everywhere in Ω also, and our finite element problem has converged to $(0, 0)$. This behaviour is known as *shear locking*.

4 Alleviating Locking

The vast majority of successful treatments of locking in the finite element literature are treated through the application of mixed variational formulations. Popular techniques that can be viewed as mixed formulations include (but are not limited to) reduced integration methods and the Mixed Interpolation of Tensorial Components (MITC) elements. In mixed formulations the shear stress vector $\gamma = (\gamma_{xz}, \gamma_{yz})$ is treated as an independent variable [2]:

$$\gamma = \lambda \bar{t}^{-2} (\nabla z_3 - \theta) \tag{5}$$

giving the equivalent mixed problem as: Find the transverse deflection, rotations and transverse shear stresses $(z_3, \theta, \gamma) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S})$ such that [2]:

$$a(\theta, \eta) + b(\gamma, \nabla y_3 - \eta) = f(y_3) \tag{6a}$$

$$\lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot \psi\, d\Omega - \int_{\Omega_0} \gamma \cdot \psi\, d\Omega = 0 \quad \forall (y_3, \eta, \psi) \in (\mathcal{V}_3, \mathcal{R}, \mathcal{S}) \tag{6b}$$

The above mixed formulation is known to be well-behaved in the limiting case as $\bar{t} \to 0$, however at the cost of introducing two extra variables representing the transverse shear stresses into the problem.

The MITC family [3] of elements works by defining a discrete *reduction* operator that takes values in the discretised rotation space $\mathcal{R}_h \subset \mathcal{R}$ and maps them into the discretised shear space $\mathcal{S}_h$ [3]:

$$R_h : \mathcal{R}_h \to \mathcal{S}_h \tag{7}$$

so the original weak form can be re-cast in terms of the displacement variables only: Find the transverse deflection and rotations $(z_3, \theta) \in (\mathcal{V}_{3h}, \mathcal{R}_h)$ such that [3]:

$$a(\theta_h, \eta_h) + \lambda \bar{t}^{-2} b(\nabla z_{3h} - R_h(\theta_h), \nabla y_{3h} - R_h(\eta_h)) = f(y_3) \;\; \forall (y_{3h}, \eta_h) \in \mathcal{V}_{3h} \times \mathcal{R}_h \tag{8}$$

In theory the operator R_h alleviates the shear locking by allowing the Kirchoff constraint to be satisfied for choices other than $\theta = 0$ and $z_3 = 0$ as $\bar{t} \to 0$. In practice this is achieved by a cleverly chosen 'tying' of the two transverse shear strains to the usual nodal displacements at the edge of the element [3].
In meshless methods a variety of approaches has been undertaken to eliminate locking in Mindlin-Reissner plates. This overview is by no means exhaustive:

Higher Order Monomial Basis This approach is amongst the most widely used eg. [4, 5] to eliminate locking. By increasing the monomial basis to higher orders $m > 2$ in the approximation, locking is gradually eliminated because the displacement and rotation space $\mathcal{V}_3 \times \mathcal{R}$ can better approximate the Kirchoff constraint $\nabla z_{3h} = \theta_h$. However, locking is typically only eliminated satisfactorily by using monomial basis of order $m > 4$ [4]. The higher order basis can be introduced either intrinsically with meshless shape functions, or extrinsically, using PU concepts.

Matching Rotation and Displacement Spaces This approach was originally introduced in [6]. By approximating the displacement space $\mathcal{V}_3$ using the meshless shape functions ϕ_i and the rotation space $\mathcal{R}$ using the derivatives of the shape functions $\nabla \phi_i$ then the Kirchoff constraint is met *exactly*. However as pointed out in [7] using the derivatives of the shape functions to approximate $\mathcal{R}$ produces a linearly dependent stiffness matrix.

Mixed Methods It is possible to construct the spaces $(\mathcal{V}_h, \mathcal{R}_h, \mathcal{S}_h)$ are constructed using MLS shape functions solve for each field directly. This eliminates locking, however it produces non-positive definite system matrices with more unknowns than a displacement approach.

Nodal Integration Techniques This technique can be seen as a form of reduced integration, and therefore shares some of the same issues such as spurious modes. These can be alleviated with techniques such as curvature smoothing [8].

5 Maximum Entropy Basis Functions

Maximum Entropy (MaxEnt) basis functions are one of the most recent developments in the construction of meshless approximation schemes [9, 10]. A brief overview of their mathematical formulation and properties is given here.
Consider a set of n nodes X. Each node i has a position $x_i \in \mathbb{R}^d$ associated with it. The convex hull of the set of nodes X is denoted $D \equiv \text{conv}(X)$. For a function $u(x) : D \to \mathbb{R}$, the numerical approximation $u^h(x)$ can be written in terms of a set of shape functions $\phi_i : D \to \mathbb{R}$ and values u_i at the nodes X [10]:

$$u^h(x) = \sum_{i=1}^{n} \phi_i(x) u_i \tag{9}$$

Typically we wish the shape functions to satisfy the well known partition of unity condition as well as first order consistency:

$$\forall x \in D, \quad \sum_{i=1}^{N} \phi_i(x) = 1 \qquad \sum_{i=1}^{N} \phi_i(x) x_i = x_i \tag{10}$$

These two conditions alone are not enough to specify a unique approximation scheme. To this end, Shannon's concept of informational entropy is introduced. The Shannon entropy $S(p)$ of a discrete probability distribution with n events x_i with probabilities p_i is [ref]:

$$S(p) = -\sum_{i=1}^{n} p_i \ln p_i \tag{11}$$

The principle of maximum entropy was proposed by Jaynes;

$$p \longleftrightarrow \phi \tag{12}$$

This analogy naturally implies that the shape functions are always positive:

$$\phi_i(x) \geq 0 \;\; \forall x \in D, i = 1, \ldots, n \tag{13}$$

Therefore we can find a unique set of shape functions by maximising the entropy $S(\phi)$ subject to the constraints outlined above [9]:

$$\max_{\phi \in \mathbb{R}^d_+} \left(S(\phi) = -\sum_{i=1}^{n} \phi_i \ln \phi_i \right) \qquad \sum_{i=1}^{n} \phi_i = 1 \quad \forall\, x \in D \qquad \sum_{i=1}^{n} \phi_i x_i = x \quad \forall\, x \in D \tag{14}$$

The objective function $-S(\phi)$ is strictly convex on $\mathbb{R}^N_+$ and the two constraints are affine, so the above problem can be solved using standard duality methods from the field of convex optimisation [10].
The above problem creates globally supported shape functions. To compute compactly supported shape functions we use a more general form of entropy measure that includes a prior distribution w_i that estimates ϕ_i [10]:

$$H(\phi, w) = -\sum_{i=1}^{n} p_i \ln \left(\frac{\phi_i}{w_i} \right) \tag{15}$$

The prior distribution can take the form of cardinal spline functions or Radial Basis Functions (RBF) such as the Gaussian associated with each node.
The primary advantage of the MaxEnt scheme over standard MLS scheme is that it produces shape functions with a weak Kronecker-delta property [10]. This makes the imposition of Dirichlet boundary conditions trivial as in the Finite Element method without resorting to modified variational forms [12].

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