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A meshless method for the Reissner-Mindlin plate equations based on a stabilized mixed weak form using maximum-entropy basis functions

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Introduction

Aim of the research project

Develop a meshless method for the simulation of Reissner-Mindlin plates that is free of shear locking.

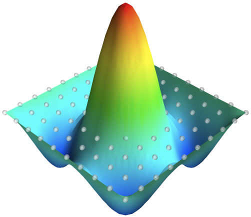


Figure : 6th free vibration mode of SSSS plate

Existing Approaches (FE and Meshless)

- Reduced Integration (Many authors)
- Assumed Natural Strains (ANS) (eg. MITC elements, Bathe)
- Enhanced Assumed Strains (EAS) (Hughes, Simo etc.)
- Discrete Shear Gap Method (DSG) (Bletzinger, Bischoff, Ramm)
- Smoothed Conforming Nodal Integration (SCNI) (Wang and Chen)
- Matching Fields Method (Donning and Liu)
- Direct Application of Mixed Methods (Hale and Baiz)

The Connection

Many of these methods are based on, or have been shown to be equivalent to, mixed variational methods.

Key Features of the Method

Weak Form

Design is based upon on a *Stabilised Mixed Weak Form*, like many successful approaches in the Finite Element literature.

Stabilised Mixed Weak Form (Brezzi and Arnold 1993, Boffi and Lovadina 1997)

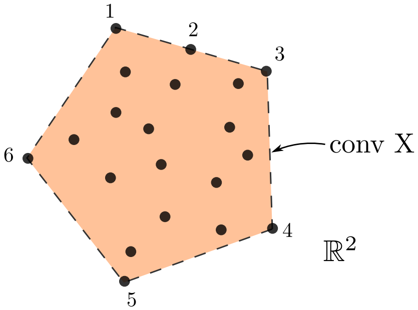
$$a_b(\theta;\eta) + \lambda\alpha a_s(\theta, z_3;\eta, y_3) + (\gamma, \nabla y_3 - \eta)_{L^2} = f(y_3) \quad \text{(1a)}$$

$$(\nabla z_3 - \theta, \psi)_{L^2} - \frac{\bar{t}^2}{\lambda(1-\alpha\bar{t}^2)}(\gamma;\psi)_{L^2} = 0 \quad \text{(1b)}$$

Key Features of the Method

Basis Functions

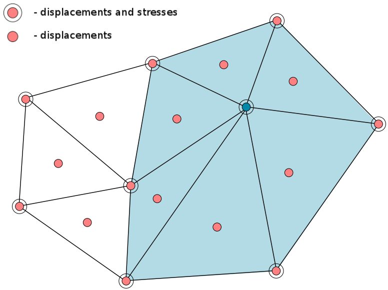
Uses (but is not limited to!) *Maximum-Entropy* Basis Functions which have a weak Kronecker-delta property. On convex node sets boundary conditions can be imposed *directly*.



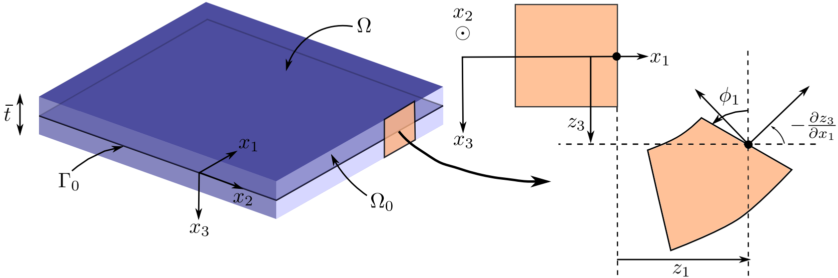
Key Features of the Method

Localised Projection Operator

Shear Stresses are *eliminated* on the 'patch' level using a localised projection operator which leaves a final system of equations in the *displacement unknowns only*.



The Reissner-Mindlin Problem



The Reissner-Mindlin Problem

Displacement Weak Form

Find $(z_3, \theta) \in (\mathcal{V}_3 \times \mathcal{R})$ such that for all $(y_3, \eta) \in (\mathcal{V}_3 \times \mathcal{R})$:

$$\int_{\Omega_0} L\epsilon(\theta) : \epsilon(\eta) \, d\Omega + \lambda \bar{t}^{-2} \int_{\Omega_0} (\nabla z_3 - \theta) \cdot (\nabla y_3 - \eta) \, d\Omega = \int_{\Omega_0} g y_3 \, d\Omega \tag{2}$$

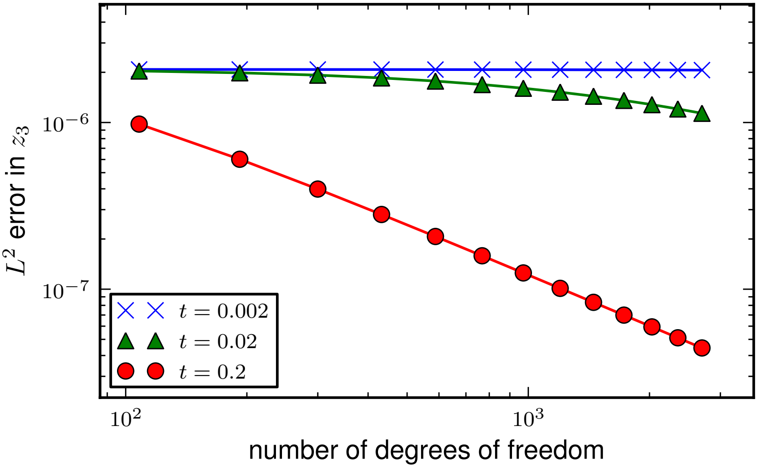
or:

$$a_b(\theta; \eta) + \lambda \bar{t}^{-2} a_s(\theta, z_3; \eta, y_3) = f(y_3) \tag{3}$$

Locking Problem

Whilst this problem is always stable, it is poorly behaved in the thin-plate limit $\bar{t} \to 0$

Shear Locking



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Shear Locking

The Problem

Inability of the basis functions to represent the limiting Kirchhoff mode

$$\nabla z_3 - \eta = 0 \tag{4}$$

A solution?

Move to a mixed weak form

Mixed Weak Form

Treat the shear stresses as an *independent* variational quantity:

$$\gamma = \lambda \bar{t}^{-2}(\nabla z_3 - \theta) \in \mathcal{S} \tag{5}$$

Mixed Weak Form

Find $(z_3, \theta, \gamma) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$:

$$a_b(\theta; \eta) + (\gamma; \nabla y_3 - \eta)_{L^2} = f(y_3) \tag{6a}$$

$$(\nabla z_3 - \theta; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda}(\gamma; \psi)_{L^2} = 0 \tag{6b}$$

Stability Problem

Whilst this problem is well-posed in the thin-plate limit, ensuring stability is no longer straightforward

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Stabilised Mixed Weak Form

Displacement Formulation

Locking as $\bar{t} \to 0$

Mixed Formulation

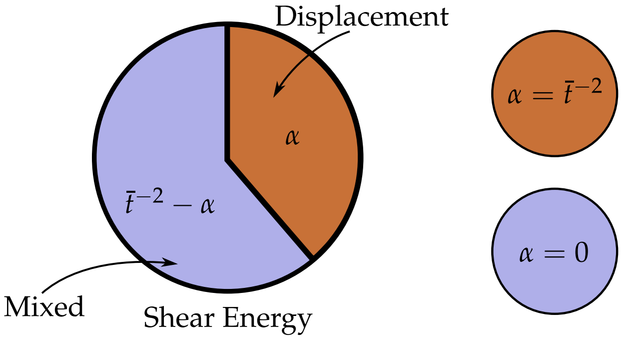
Not necessarily stable

Solution

Combine the displacement and mixed formulation to retain the advantageous properties of both

Stabilised Mixed Weak Form

Split the discrete shear term with a parameter $0 < \alpha < \bar{t}^{-2}$ that is *independent of the plate thickness*:



$$a_s = \alpha a^{\text{displacement}} + (\bar{t}^{-2} - \alpha) a^{\text{mixed}} \qquad (7)$$

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Stabilised Mixed Weak Form

Mixed Weak Form

Find $(z_3, \theta, \gamma) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$ such that for all $(y_3, \eta, \psi) \in (\mathcal{V}_3 \times \mathcal{R} \times \mathcal{S})$:

$$a_b(\theta; \eta) + (\gamma; \nabla y_3 - \eta)_{L^2} = f(y_3) \tag{8a}$$

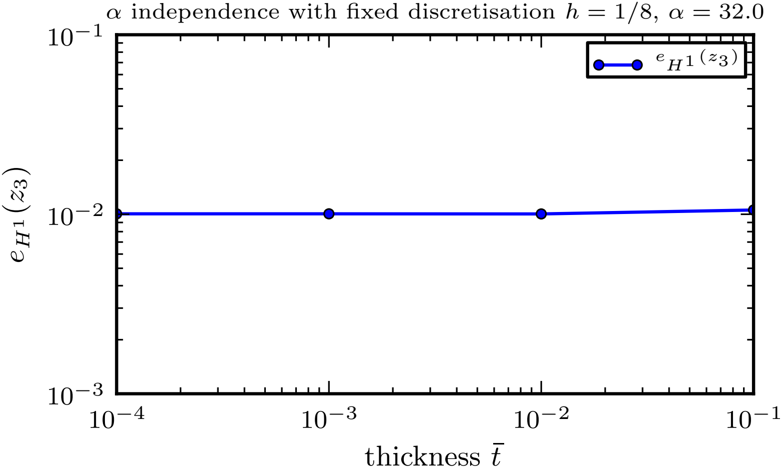
$$(\nabla z_3 - \theta; \psi)_{L^2} - \frac{\bar{t}^2}{\lambda}(\gamma; \psi)_{L^2} = 0 \tag{8b}$$

Stabilised Mixed Weak Form (Brezzi and Arnold 1993, Boffi and Lovadina 1997)

$$a_b(\theta; \eta) + \lambda \alpha a_s(\theta, z_3; \eta, y_3) + (\gamma, \nabla y_3 - \eta)_{L^2} = f(y_3) \tag{9a}$$

$$(\nabla z_3 - \theta, \psi)_{L^2} - \frac{\bar{t}^2}{\lambda(1 - \alpha \bar{t}^2)}(\gamma; \psi)_{L^2} = 0 \tag{9b}$$

α independence



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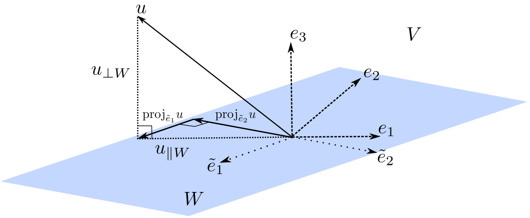
Eliminating the Stress Unknowns

- Find a (cheap) way of eliminating the extra unknowns associated with the shear-stress variables

$$\gamma_h = \frac{\lambda(1-\alpha\bar{t}^2)}{\bar{t}^2}\Pi_h(\nabla z_{3h} - \theta_h, \psi_h) \quad (10)$$

Figure : The Projection Π_h represents a softening of the energy associated with the shear term



Eliminating the Stress Unknowns

- We use a version of a technique proposed by Ortiz, Puso and Sukumar for the Incompressible-Elasticity/Stokes' flow problem which they call the "Volume-Averaged Nodal Pressure" technique.
- A more general name might be the "Local Patch Projection" technique.

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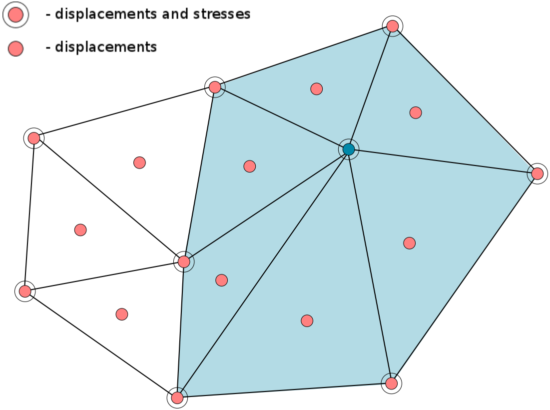
Eliminating the Stress Unknowns



Eliminating the Stress Unknowns

For one component of shear (for simplicity):

$$(z_{3,x} - \theta_1, \psi_{13})_{L^2} - \frac{\bar{t}^2}{\lambda(1 - \alpha \bar{t}^2)} (\gamma_{13}; \psi_{13})_{L^2} = 0 \qquad (11)$$

Substitute in meshfree and FE basis, perform row-sum (mass-lumping) and rearrange to give nodal shear unknown for a node a. Integration is performed over local domain Ω_a:

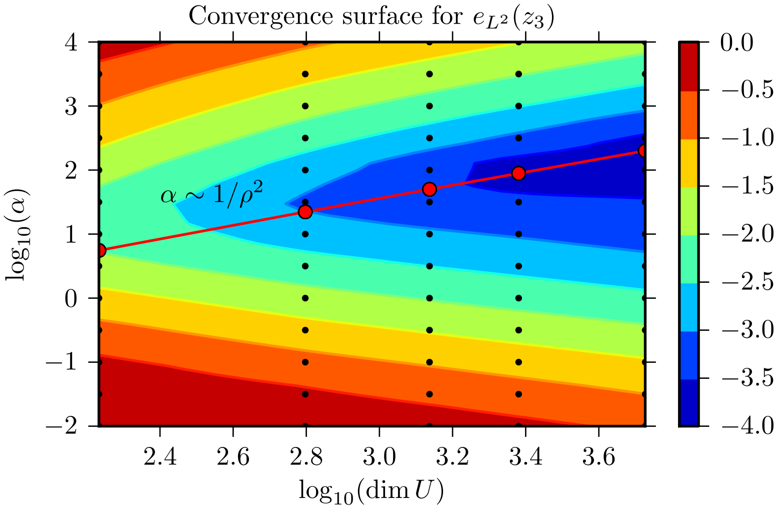
$$\gamma_{13a} = \sum_{i=1}^{N} \frac{\int_{\Omega_a} N_a \begin{Bmatrix} -\phi_i & \phi_{i,x} \end{Bmatrix} \; d\Omega}{\int_{\Omega_a} N_a \; d\Omega} \begin{Bmatrix} \phi_i \\ z_{3i} \end{Bmatrix} \qquad (12)$$

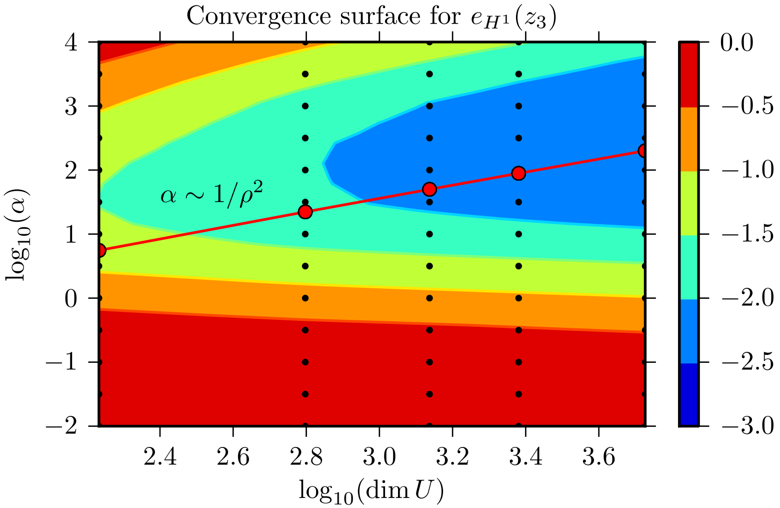
Choosing α

The *dimensionally consistent* choice for α is length^{-2}. In the FE literature typically this paramemeter has been chosen as either h^{-1} or h^{-2} where h is the local mesh size.

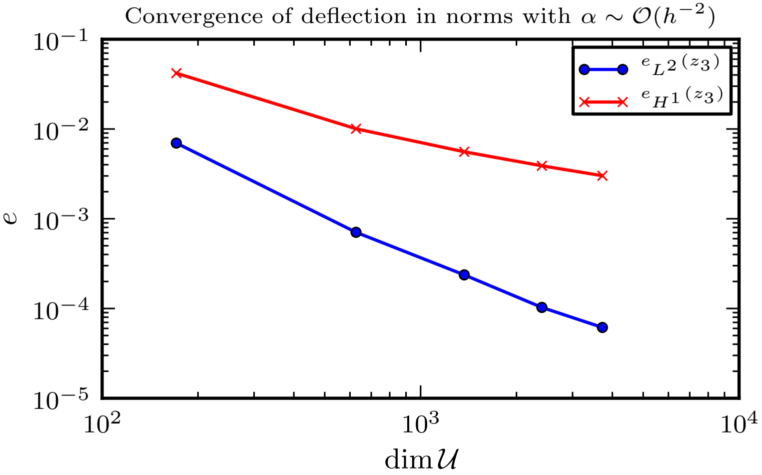
Meshless methods

A sensible place to start would be ρ^{-2} where ρ is the local support size.


Convergence surface for $e_{L^2}(z_3)$
$\alpha \sim 1/\rho^2$
$\log_{10}(\alpha)$
$\log_{10}(\dim U)$
4
3
2
1
0
−1
−2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0


Convergence surface for $e_{H^1}(z_3)$
$\alpha \sim 1/\rho^2$
$\log_{10}(\alpha)$
$\log_{10}(\dim U)$
4
3
2
1
0
−1
−2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0

Results - Convergence



Results - Surface Plots

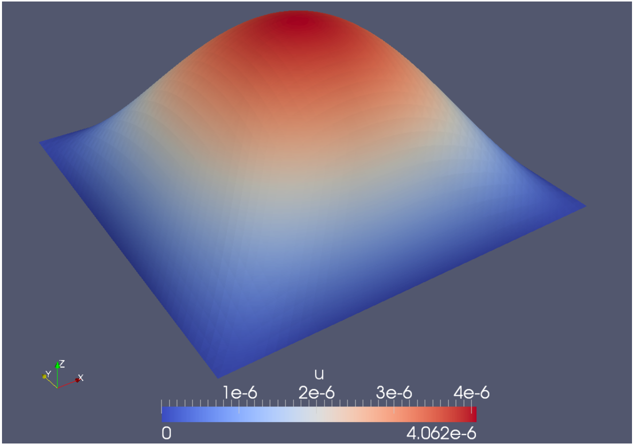


Figure : Displacement z_{3h} of SSSS plate on 12×12 node field + 'bubbles', $t = 10^{-4}$, $\alpha = 120$

Results - Surface Plots

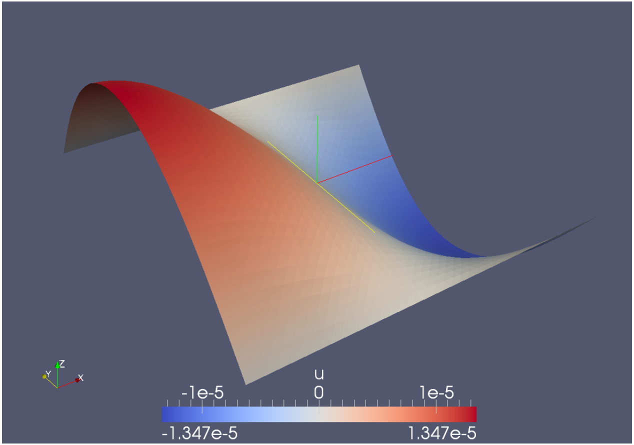


Figure : Rotation component θ_1 of SSSS plate on 12×12 node field + 'bubbles', $t = 10^{-4}$, $\alpha = 120$

Summary

A method:

- using (but not limited to) Maximum-Entropy basis functions for the Reissner-Mindlin plate problem that is *free of shear-locking*
- based on a stabilised mixed weak form
- where secondary stress are eliminated from the system of equations *a priori* using "Local Patch Projection" technique

Possible future work:

- Extension to Naghdi Shell model
- Investigate locking-free PUM enriched methods

Thanks for listening.

LBB Stability Conditions

Theorem (LBB Stability)

The discretised mixed problem is uniquely solvable if there exists two positive constants α_h and β_h such that:

$$a_b(\eta_h; \eta_h) \geq \alpha_h \|\eta_h\|^2_{\mathcal{R}_h} \quad \forall \eta_h \in \mathcal{K}_h \tag{13a}$$

$$\inf_{\psi_h \in \mathcal{S}_h} \sup_{(\eta_h, y_{3h}) \in (\mathcal{R}_h \times \mathcal{V}_{3h})} \frac{((\nabla y_{3h} - \eta_h), \psi_h)_{L^2}}{(\|\eta_h\|_{\mathcal{R}_h} + \|y_{3h}\|_{\mathcal{V}_{3h}})\|\psi_h\|_{\mathcal{S}'_h}} \geq \beta_h \tag{13b}$$

LBB Stability Conditions

The Problem

- To satisfy the second condition 13b make displacement spaces $\mathcal{R}_h \times \mathcal{V}_{3h}$ 'rich' with respect to the shear space $\mathcal{S}_h$
- If $\mathcal{R}_h \times \mathcal{V}_{3h}$ is too 'rich' then the first condition 13a may fail as $\mathcal{K}_h$ grows.
- *Balancing these two competing requirements makes the design of a stable formulation difficult.*